Associative submanifolds in Joyce's generalised Kummer constructions

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Abstract

This article constructs examples of associative submanifolds in G_2–manifolds obtained by resolving G_2–orbifolds using Joyce's generalised Kummer construction. As the G_2–manifolds approach the G_2–orbifolds, the volume of the associative submanifolds tends to zero. This partially verifies a prediction due to Halverson and Morrison.

1 Introduction

The Teichmüller space

$$\mathscr{T}(Y) := \{\phi \in \Omega^3(Y) : \phi \text{ is a torsion-free } G_2\text{–structure}\}/\mathrm{Diff}_0(Y)$$

of torsion-free G_2–structures on a closed 7–manifold Y is a smooth manifold of dimension $b^3(Y)$ [Joy96a, Theorem C]. The G_2 period map $\Pi\colon \mathscr{T} \to \mathrm{H}^3_{\mathrm{dR}}(Y) \oplus \mathrm{H}^4_{\mathrm{dR}}(Y)$ defined by

$$\Pi(\phi \cdot \mathrm{Diff}_0(Y)) := ([\phi], [\psi]) \quad \text{with} \quad \psi := *_\phi \phi$$

is a Lagrangian immersion [Joy96b, Lemma 1.1.3].[1] It is constrained by the following inequalities [Joy96b, Lemma 1.1.2; HL82, §IV.2.A and §IV.2.B]:

(1) $\int_Y \alpha \wedge \alpha \wedge \phi < 0$ for every non-zero $[\alpha] \in \mathrm{H}^2_{\mathrm{dR}}(Y)$ if $\pi_1(Y)$ is finite.

(2) $\int_Y p_1(V) \wedge \phi = -\frac{1}{4\pi^2}\mathrm{YM}(A) < 0$ for every vector bundle V which admits a non-flat G_2–instanton A; in particular, for $V = TY$ unless Y is covered by T^7. Here $p_1(V)$ denotes the 1^{st} Pontryagin class of V [MS74, §15], and $\mathrm{YM}(A) := \frac{1}{2}\int_Y |F_A|^2$ is the Yang–Mills energy.

(3) $\int_P \phi = \mathrm{vol}(P) > 0$ for every associative submanifold $P \looparrowright Y$.

[1] Whether or not Π is an embedding is an open question.

(4) $\int_Q \psi = \mathrm{vol}(Q) > 0$ for every coassociative submanifold $Q \hookrightarrow Y$.

These should be compared with the inequalities cutting out the Kähler cone of a Calabi–Yau 3–fold; see Wilson [Wil92].

By analogy with Calabi–Yau 3–folds, Halverson and Morrison [HM16, §3] suggest that the above inequalities completely characterise the ideal boundary of $\mathscr{T}(Y)$. Of course, making this precise is complicated by the fact that the notions of G_2–instanton and (co)associative submanifold depend on the G_2–structure ϕ. The situation would be improved if there were invariants whose non-vanishing guaranteed the existence of G_2–instantons and (co)associative submanifolds as suggested by Donaldson and Thomas [DT98, §3]. However, their construction is fraught with enormous difficulty [DS11; Joy18; Hay17; Wal17; DW19].

A more down to earth problem is to exhibit concrete examples of degenerating families of G_2–manifolds which admit G_2–instantons whose Yang–Mills energies tend to zero [Wal13] or which admit (co)associative submanifolds whose volumes tend to zero. The purpose of this article is to present examples of the latter in G_2–manifolds arising from Joyce's generalised Kummer construction. Although these examples had been anticipated (e.g. by Halverson and Morrison [HM16, §6.2]), their rigorous construction has only recently become possible due to the work of Platt [Pla22].

Remark 1.1. Of course, there are already numerous examples of closed associative submanifolds in the literature.

(1) Joyce [Joy96b, §4.2; Joy00, §12.6] has constructed (co)associative submanifolds in generalised Kummer constructions as fixed-point sets of involutions.

(2) Corti, Haskins, Nordström, and Pacini [CHNP15, §5.5 and §7.2.2] have constructed associative submanifolds in twisted connected sums using rigid holomorphic curves and special Lagrangians in asymptotically cylindrical Calabi–Yau 3–folds.

(3) In the physics literature, Braun, Del Zotto, Halverson, Larfors, Morrison, and Schäfer-Nameki [BDHLMS18, §4.4] have proposed a construction of infinitely many associative submanifolds in certain twisted connected sums. An important ingredient in the proof of this conjecture will be the gluing theorem for associative submanifolds in twisted connected sums proved by Bera [Ber22]—analogous to [SW15]. Building on [BDHLMS18], Acharya, Braun, Svanes, and Valandro [ABSV19, §2.2 and §4.2] have constructed infinitely many associative submanifolds in certain G_2–orbifolds (without using any analytic methods).

(4) Lotay [Lot12], Kawai [Kaw15], and Ball and Madnick [BM20] have produced a wealth of examples of associative submanifolds in S^7, the squashed S^7, and the Berger space with their nearly parallel G_2–structures.

The novelty of the examples discussed in the present article is that their volumes tend to zero as the ambient G_2–manifolds degenerate. ♣

2 Joyce's generalised Kummer construction

The generalised Kummer construction is a method, developed by Joyce [Joy96a; Joy96b], to produce G_2–manifolds by desingularising certain closed flat G_2–orbifolds (Y_0, ϕ_0). Besides a rather delicate singular perturbation theory it relies on the fact that the hyperkähler 4–orbifolds $\mathbf{H}/\Gamma$, obtained as quotients of the quaternions $\mathbf{H}$ by a finite subgroup $\Gamma < \mathrm{Sp}(1)$, can be desingularised by hyperkähler 4–manifolds. The following model spaces feature prominently throughout this article.

Example 2.1 (model spaces). Let X be a hyperkähler 4–orbifold with hyperkähler form

$$\boldsymbol{\omega} \in (\operatorname{Im}\mathbf{H})^* \otimes \Omega^2(X).$$

Denote by $\mathrm{vol} \in \Omega^3(\operatorname{Im}\mathbf{H})$ and $\mathbf{1} \in \Omega^1(\operatorname{Im}\mathbf{H}) \otimes \operatorname{Im}\mathbf{H}$ the volume form and the tautological 1–form respectively.

(1) The 3–form

$$\mathrm{vol} - \langle \mathbf{1} \wedge \boldsymbol{\omega} \rangle \in \Omega^3(\operatorname{Im}\mathbf{H} \times X) \tag{2.2}$$

defines a torsion-free G_2–structure on $\operatorname{Im}\mathbf{H} \times X$. Here $\langle \cdot \wedge \cdot \rangle$ is induced by the wedge product on forms and the duality pairing between $\operatorname{Im}\mathbf{H}$ and $(\operatorname{Im}\mathbf{H})^*$. The corresponding Riemannian metric and the cross-product on $\operatorname{Im}\mathbf{H} \times X$ recover the Riemannian metric and the hypercomplex structure $\mathbf{I} \in (\operatorname{Im}\mathbf{H})^* \otimes \Gamma(\mathrm{End}(TX))$ on X.

(2) Let $G < \mathrm{SO}(\operatorname{Im}\mathbf{H}) \ltimes \operatorname{Im}\mathbf{H}$ be a Bieberbach group; that is: discrete, cocompact, and torsion-free. Let $\rho\colon G \to \mathrm{Isom}(X)$ be a homomorphism. Suppose that $\boldsymbol{\omega}$ is G–invariant; that is: for every $(R, t) \in G$

$$\big(R^* \otimes \rho(R, t)^*\big)\boldsymbol{\omega} = \boldsymbol{\omega}.$$

Set

$$Y := (\operatorname{Im}\mathbf{H} \times X)/G.$$

The G_2–structure (2.2) descends to a G_2–structure

$$\phi \in \Omega^3(Y).$$

The canonical projection $p\colon Y \to B := \operatorname{Im}\mathbf{H}/G$ is a flat fibre bundle whose fibres are coassociative submanifolds diffeomorphic to X; cf. [Bar19, §3.4]. ♠

Remark 2.3 (Classification of Bieberbach groups). If $G < \mathrm{SO}(\operatorname{Im}\mathbf{H}) \ltimes \operatorname{Im}\mathbf{H}$ is a Bieberbach group, then $\Lambda := G \cap \operatorname{Im}\mathbf{H} < \operatorname{Im}\mathbf{H}$ is a lattice of full rank and $H := G/\Lambda < \mathrm{SO}(\Lambda) \times (\operatorname{Im}\mathbf{H}/\Lambda)$ is isomorphic to either $\mathbf{1}$, C_2, C_3, C_4, C_6, or C_2^2; cf. [HW35; CR03; Szc12, §3.3]. More precisely, G is among the following:

(1) Λ is arbitrary and $G = \Lambda$.

(C_2) $\Lambda = \langle \lambda_1, \lambda_2, \lambda_3 \rangle$ with
$$\langle \lambda_1, \lambda_2 \rangle = \langle \lambda_1, \lambda_3 \rangle = 0.$$
G is generated by Λ and $(R_2, \frac{1}{2}\lambda_1)$ with $R_2 \in \mathrm{SO}(\Lambda)$ as in (2.5).

(C_3) $\Lambda = \langle \lambda_1, \lambda_2, \lambda_3 \rangle$ with
$$\langle \lambda_1, \lambda_2 \rangle = \langle \lambda_1, \lambda_3 \rangle = 0 \quad \text{and} \quad |\lambda_2|^2 = |\lambda_3|^2 = -2\langle \lambda_2, \lambda_3 \rangle. \tag{2.4}$$
G is generated by Λ and $(R_3, \frac{1}{3}\lambda_1)$ with $R_3 \in \mathrm{SO}(\Lambda)$ as in (2.5).

(C_4) $\Lambda = \langle \lambda_1, \lambda_2, \lambda_3 \rangle$ with
$$\langle \lambda_1, \lambda_2 \rangle = \langle \lambda_2, \lambda_3 \rangle = \langle \lambda_3, \lambda_1 \rangle = 0 \quad \text{and} \quad |\lambda_2|^2 = |\lambda_3|^2.$$
G is generated by Λ and $(R_4, \frac{1}{4}\lambda_1)$ with $R_4 \in \mathrm{SO}(\Lambda)$ as in (2.5).

(C_6) $\Lambda = \langle \lambda_1, \lambda_2, \lambda_3 \rangle$ with (2.4). G is generated by Λ and $(R_6, \frac{1}{6}\lambda_1)$ with $R_6 \in \mathrm{SO}(\Lambda)$ as in (2.5).

(C_2^2) $\Lambda = \langle \lambda_1, \lambda_2, \lambda_3 \rangle$ with
$$\langle \lambda_1, \lambda_2 \rangle = \langle \lambda_2, \lambda_3 \rangle = \langle \lambda_3, \lambda_1 \rangle = 0.$$
G is generated by Λ, $(R_+, \frac{1}{2}(\lambda_1 + \lambda_2))$, and $(R_-, \frac{1}{2}(\lambda_2 + \lambda_3))$ with $R_\pm \in \mathrm{SO}(\Lambda)$ as in (2.5).

Here $R_2, R_3, R_4, R_6, R_\pm \in \mathrm{GL}_3(\mathbf{Z})$ are defined by
$$\begin{gathered} R_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad R_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad R_4 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \\ R_6 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad R_\pm := \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \mp 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{gathered} \tag{2.5}$$

$\mathrm{GL}_3(\mathbf{Z})$ is identified with $\mathrm{GL}(\Lambda)$ by the choice of generators of Λ. ♣

The version of the generalised Kummer construction considered in this article desingularises closed flat G_2–orbifolds (Y_0, ϕ_0) whose singularities are modelled on Example 2.1 with $X := \mathbf{H}/\Gamma$ for finite $\Gamma < \mathrm{Sp}(1)$. This requires a choice of the following data; cf. [Joyoo, §11.4.1].

Definition 2.6. Let (Y_0, ϕ_0) be a flat G_2–orbifold. Denote the connected components of the singular set of Y_0 by S_α $(\alpha \in A)$. **Resolution data** $\mathfrak{R} = (\Gamma_\alpha, G_\alpha, \rho_\alpha; R_\alpha, \jmath_\alpha; \hat{X}_\alpha, \hat{\boldsymbol{\omega}}_\alpha, \hat{\rho}_\alpha, \tau_\alpha)_{\alpha \in A}$ for (Y_0, ϕ_0) consist of the following for every $\alpha \in A$:

(1) A finite subgroup $\Gamma_\alpha < \mathrm{Sp}(1) < \mathrm{SO}(\mathbf{H})$, a Bieberbach group $G_\alpha < \mathrm{SO}(\mathrm{Im}\,\mathbf{H}) \ltimes \mathrm{Im}\,\mathbf{H}$, and a homomorphism $\rho_\alpha \colon G_\alpha \to N_{\mathrm{SO}(\mathbf{H})}(\Gamma_\alpha) \hookrightarrow \mathrm{Isom}(\mathbf{H}/\Gamma_\alpha)$ as in Example 2.1 (2) with $X := \mathbf{H}/\Gamma_\alpha$ and its canonical hyperkähler form $\boldsymbol{\omega}$. Here $N_G(H)$ denotes the normaliser of $H < G$.

Denote by (Y_α, ϕ_α) the model space associated with $\mathbf{H}/\Gamma_\alpha$, $\boldsymbol{\omega}$, G_α, and ρ_α.

(2) $R_\alpha > 0$ defining the open set

$$U_\alpha \coloneqq \left(\operatorname{Im}\mathbf{H} \times (B_{2R_\alpha}(0)/\Gamma_\alpha)\right) / G_\alpha \subset Y_\alpha,$$

and an open embedding $\jmath_\alpha \colon U_\alpha \hookrightarrow Y_0$ satisfying $S_\alpha \subset \operatorname{im} \jmath_\alpha$ and

$$\jmath_\alpha^* \phi_0 = \phi_\alpha.$$

(3) A hyperkähler 4–manifold $\hat{X}_\alpha$ with hyperkähler form $\hat{\boldsymbol{\omega}}_\alpha \in (\operatorname{Im}\mathbf{H})^* \otimes \Omega^2(\hat{X}_\alpha)$, a homomorphism $\hat{\rho}_\alpha \colon G_\alpha \to \operatorname{Diff}(\hat{X}_\alpha)$ with respect to which $\hat{\boldsymbol{\omega}}_\alpha$ is G_α–invariant (in the sense of Example 2.1 (2)), a compact subset $K_\alpha \subset \hat{X}_\alpha$, and a G_α–equivariant open embedding $\tau_\alpha \colon \hat{X}_\alpha \backslash K_\alpha \hookrightarrow \mathbf{H}/\Gamma_\alpha$ with $(\mathbf{H}\backslash B_{R_\alpha}(0))/\Gamma \subset \operatorname{im} \tau_\alpha$ and

$$|\nabla^k(\tau_* \hat{\boldsymbol{\omega}}_\alpha - \boldsymbol{\omega})| = O(r^{-4-k}) \tag{2.7}$$

for every $k \in \mathbf{N}_0$. •

Remark 2.8 (ADE classification of finite subgroups of $\mathrm{Sp}(1)$). Klein [Kle93] classified the (nontrivial) finite subgroups $\Gamma < \mathrm{Sp}(1)$. They obey an ADE classification. Γ is isomorphic to either:

(A_k) a cyclic group C_{k+1},

(D_k) a dicyclic group Dic_{k-2},

(E_6) the binary tetrahedral group $2T$,

(E_7) the binary octahedral group $2O$, or

(E_8) the binary icosahedral group $2I$. ♣

Remark 2.9. Whether or not the data in Definition 2.6 (1) and (2) exists is a property of a neighborhood of the singular set of Y_0. If it does exist, then it is essentially unique. The data in Definition 2.6 (3) involves a choice. ♣

Remark 2.10. There are many examples of closed flat G_2–orbifolds admitting resolution data in the above sense; see [Joy96b, §3; Joy00, §12; Bar06, §3; Rei17, §5.3.4 and §5.3.5]. They arise from certain crystallographic groups $G < G_2 \ltimes \mathbf{R}^7$. It would be interesting to classify these (possibly computer-aided) to grasp the full scope of Joyce's generalised Kummer construction. Partial results have been obtained by Barrett [Bar06, §3.2], and Reidegeld [Rei17, Theorem 5.3.1] proved that the only possibilities for Γ_α in Definition 2.6 (1) are C_2, C_3, C_4, C_6, Dic_2, Dic_3, and $2T$. ♣

Remark 2.11 (scaling resolution data). For every $(t_\alpha) \in (0,1]^A$ the data $\hat{\boldsymbol{\omega}}_\alpha$ and τ_α in Definition 2.6 (3) can be replaced with $t_\alpha^2 \hat{\boldsymbol{\omega}}_\alpha$ and $t_\alpha \tau_\alpha$. ♣

The following two lengthy remarks help to find resolution data $\mathfrak{R}$ with certain properties.

Remark 2.12 (Gibbons–Hawking construction of A_k ALE spaces). Let $k \in \mathbf{N}$. Consider the subgroup $C_k \hookrightarrow \mathrm{Sp}(1)$ generated by right multiplication with $e^{2\pi i/k}$. (Of course, i can be replaced by $\hat{\xi} \in S^2 \subset \operatorname{Im}\mathbf{H}$ throughout.) The A_k ALE hyperkähler 4–manifolds used to resolve $\mathbf{H}/C_k$ can be understood concretely using the Gibbons–Hawking construction [GH78; GRG97, §3.5].

(1) Let

$$\zeta \in \Delta \coloneqq \mathrm{Sym}^k_0(\mathrm{Im}\,\mathbf{H}) \coloneqq \big\{[\zeta_1,\ldots,\zeta_k] \in (\mathrm{Im}\,\mathbf{H}^k)/S_k : \zeta_1+\cdots+\zeta_k = 0\big\}.$$

Set $Z \coloneqq \{\zeta_1,\ldots,\zeta_k\}$ and $B \coloneqq \mathrm{Im}\,\mathbf{H}\backslash Z$. The function $V_\zeta \in C^\infty(B)$ defined by

$$V_\zeta(q) \coloneqq \sum_{a=1}^{k} \frac{1}{2|q-\zeta_a|}$$

is harmonic and

$$[*\mathrm{d}V_\zeta] \in \mathrm{im}\big(\mathrm{H}^2(B, 2\pi\mathbf{Z}) \to \mathrm{H}^2_{\mathrm{dR}}(B)\big).$$

Therefore, there is a U(1)–principal bundle $p_\zeta\colon X^\circ_\zeta \to B$ and a connection 1–form $i\theta_\zeta \in \Omega^1(X^\circ_\zeta, i\mathbf{R})$ with

$$\mathrm{d}\theta_\zeta = -p^*_\zeta(*\mathrm{d}V_\zeta). \tag{2.13}$$

Indeed, p_ζ is determined by V_ζ up to isomorphism. The Euclidean inner product on $\mathrm{Im}\,\mathbf{H}$ defines

$$\sigma \in (\mathrm{Im}\,\mathbf{H})^* \otimes \Omega^1(\mathrm{Im}\,\mathbf{H}).$$

X°_ζ is an incomplete hyperkähler manifold with hyperkähler form $\boldsymbol{\omega}_\zeta$ defined by

$$\boldsymbol{\omega}_\zeta \coloneqq \theta_\zeta \wedge p^*_\zeta\sigma + p^*_\zeta(V_\zeta \cdot *\sigma).$$

(2) The map $p_0\colon (\mathbf{H}\backslash\{0\})/C_k \to B$ defined by

$$p_0([x]) \coloneqq \frac{xix^*}{2k}$$

is a U(1)–principal bundle with $[x]\cdot e^{i\alpha} \coloneqq [xe^{i\alpha/k}]$. The connection 1–form $i\theta_0$ defined by

$$\theta_0([x,v]) \coloneqq \frac{\langle xi, v\rangle}{k|x|^2}$$

satisfies (2.13). Therefore, $X^\circ_0 = (\mathbf{H}\backslash\{0\})/\Gamma$. A straightforward (but slightly tedious) computation reveals that $\boldsymbol{\omega}_0$ agrees with the standard hyperkähler form on $(\mathbf{H}\backslash\{0\})/\Gamma$. As a consequence, X°_ζ can be extended to a complete hyperkähler orbifold X_ζ by adding $\#Z$ points. If

$$\zeta \in \Delta^\circ \coloneqq \big\{[\zeta_1,\ldots,\zeta_k] \in \Delta : \zeta_1,\ldots,\zeta_k \text{ are pairwise distinct}\big\},$$

then X_ζ is a manifold. Since

$$|\nabla^k(V_\zeta - V_0)\circ p_0| = O(|x|^{-4-k}) \tag{2.14}$$

for every $k \in \mathbf{N}_0$, the asymptotic decay condition (2.7) holds.

(3) Let $\zeta \in \Delta^\circ$. Denote by $\mathbf{I}_\zeta \in (\operatorname{Im}\mathbf{H})^* \otimes \Gamma(\operatorname{End}(TX_\zeta))$ the hypercomplex structure induced by $\boldsymbol{\omega}_\zeta$. If

$$\ell = \left\{\hat{\xi}t + \eta : t \in [a,b]\right\} \subset \operatorname{Im}\mathbf{H}$$

with $\eta \in \operatorname{Im}\mathbf{H}$, $[a,b] \subset \mathbf{R}$, and $\hat{\xi} \in S^2 \subset \operatorname{Im}\mathbf{H}$ is a segment satisfying $\partial\ell \subset Z$ and $\ell^\circ \subset B$, then

$$\Sigma_\ell \coloneqq p_\zeta^{-1}(\ell) \subset X_\zeta$$

is $I_{\zeta,\hat{\xi}}$–holomorphic with

$$I_{\zeta,\hat{\xi}} \coloneqq \langle \mathbf{I}_\zeta, \hat{\xi}\rangle \in \Gamma(\operatorname{End}(TX_\zeta))$$

and $\Sigma_\ell \cong S^2$. $\mathrm{H}_2(X_\zeta, \mathbf{Z})$ is generated by the homology classes of these curves. In fact, X_ζ retracts to a tree of these curves.

(4) Identify $(\operatorname{Im}\mathbf{H})^* = \operatorname{Im}\mathbf{H}$. The canonical hyperkähler form on $\mathbf{H}$ can be written as $\boldsymbol{\omega} = -\frac{1}{2}\mathrm{d}q \wedge \mathrm{d}\bar{q} \in \operatorname{Im}\mathbf{H} \otimes \Omega^+(\mathbf{H})$.[2] Define $\Lambda^+\colon \mathrm{SO}(\mathbf{H}) \to \mathrm{SO}(\operatorname{Im}\mathbf{H})$ by requiring that

$$\big((\Lambda^+R)^* \otimes R^*\big)(\mathrm{d}q \wedge \mathrm{d}\bar{q}) = \mathrm{d}q \wedge \mathrm{d}\bar{q}.$$

Define $\alpha\colon N_{\mathrm{SO}(\mathbf{H})}(\Gamma) \to \{\pm 1\}$ by

$$\alpha(R) \coloneqq \begin{cases} 1 & \text{if } R \in Z_{\mathrm{SO}(\mathbf{H})}(\Gamma) \\ -1 & \text{otherwise.} \end{cases}$$

Let $\zeta \in \Delta^\circ$. Let ℓ be as in (3). If $R \in N_{\mathrm{SO}(\mathbf{H})}(\Gamma)$ satisfies $\alpha(R)\Lambda^+R(\zeta) = \zeta$ and $\alpha(R)\Lambda^+R(\ell) = \ell$, then it lifts to an isometry $\hat{R} \in \operatorname{Diff}(X_\zeta)$ satisfying

$$\big((\Lambda^+R)^* \otimes \hat{R}^*\big)\boldsymbol{\omega}_\zeta = \boldsymbol{\omega}_\zeta \quad \text{and} \quad \hat{R}(\Sigma_\ell) = \Sigma_\ell. \qquad \clubsuit$$

Remark 2.15 (Kronheimer's construction of ALE spaces). Let $\Gamma < \mathrm{Sp}(1)$ be a finite subgroup—not necessarily cyclic. The ALE hyperkähler 4–manifolds asymptotic to $\mathbf{H}/\Gamma$ can be understood using the work of Kronheimer [Kro89b; Kro89a]. This is rather more involved than Remark 2.12 and summarised in the following. (This is only used for Example 4.6 and might be skipped at the reader's discretion.)

(1) Denote by $\mathbf{C}[\Gamma] = \operatorname{Map}(\Gamma, \mathbf{C})$ the regular representation of Γ equipped with the standard Γ–invariant Hermitian inner product. Set

$$S \coloneqq (\mathbf{H} \otimes_{\mathbf{R}} \mathfrak{u}(\mathbf{C}[\Gamma]))^\Gamma \quad \text{and} \quad G \coloneqq \mathbf{P}U(\mathbf{C}[\Gamma])^\Gamma.$$

The adjoint action of G on S has a distinguished hyperkähler moment map

$$\mu\colon S \to (\operatorname{Im}\mathbf{H})^* \otimes \mathfrak{g}^*.$$

[2] Indeed, $-\frac{1}{2}\mathrm{d}q \wedge \mathrm{d}\bar{q} = i \otimes (\mathrm{d}q_0 \wedge \mathrm{d}q_1 + \mathrm{d}q_2 \wedge \mathrm{d}q_3) + j \otimes (\mathrm{d}q_0 \wedge \mathrm{d}q_2 + \mathrm{d}q_3 \wedge \mathrm{d}q_1) + k \otimes (\mathrm{d}q_0 \wedge \mathrm{d}q_3 + \mathrm{d}q_1 \wedge \mathrm{d}q_2)$.

Denote by $\mathfrak{z}^* \subset \mathfrak{g}^*$ the annihilator of $[\mathfrak{g}, \mathfrak{g}]$ or, equivalently, the dual of the centre $\mathfrak{z}$ of $\mathfrak{g}$. For every

$$\boldsymbol{\zeta} \in \tilde{\Delta} := (\operatorname{Im} \mathbf{H})^* \otimes \mathfrak{z}^*$$

the hyperkähler quotient

$$X_{\boldsymbol{\zeta}} := S /\!\!/\!\!/_{\boldsymbol{\zeta}} G := \mu^{-1}(\boldsymbol{\zeta})/G$$

is an ALE hyperkähler 4–orbifold asymptotic to $\mathbf{H}/\Gamma$.

(2) Remark 2.8 associates a Dynkin diagram with Γ. According to the McKay correspondence [McK81], the non-trivial irreducible complex representations $R_1, \ldots, R_r$ of Γ correspond to the vertices of this diagram. Denote by Φ the corresponding root system.

(3) [Kro89b, §2] defines an isomorphism $\tau^* \colon \mathfrak{z}^* \cong (\mathbf{R}\Phi)^*$. Therefore, every root $\theta \in \Phi$ defines a hypersurface $D_\theta := \ker \theta \subset \mathfrak{z}^*$. If

$$\boldsymbol{\zeta} \in \tilde{\Delta}^\circ := \tilde{\Delta} \backslash D \quad \text{with} \quad D := \bigcup_{\theta \in \Phi} (\operatorname{Im} \mathbf{H})^* \otimes D_\theta,$$

then $X_{\boldsymbol{\zeta}}$ is a manifold [Kro89b, Proposition 2.8].

(4) Let $\boldsymbol{\zeta} \in \tilde{\Delta}^\circ$ and $\theta \in \Phi$. [Kro89b, §4] defines another isomorphism $\sigma \colon \mathfrak{z}^* \cong (\mathbf{R}\Phi)^*$. [Kro89b, Propostion 4.1] shows that $\sigma(\boldsymbol{\zeta})\theta \neq 0$. Define $\xi = \xi(\theta) \in \operatorname{Im} \mathbf{H}$ by $\langle \xi, \cdot \rangle := \sigma(\boldsymbol{\zeta})\theta$ and set $\hat{\xi} := \xi/|\xi|$. Suppose that θ cannot be decomposed as $\theta = \theta_1 + \theta_2$ with $\theta_1, \theta_2 \in \Phi$ and $\theta(\xi_1) = \theta(\xi_2)$. There is an $I_{\boldsymbol{\zeta},\hat{\xi}}$–holomorphic curve

$$\Sigma_\theta \subset X_{\boldsymbol{\zeta}}$$

with $\Sigma_\theta \cong S^2$. (If θ can be decomposed, then Σ_θ is nodal.) $\mathrm{H}_2(X_{\boldsymbol{\zeta}})$ is generated by the homology classes of these curves. In fact, $X_{\boldsymbol{\zeta}}$ retracts to a tree of these curves. This identifies $\mathrm{H}_2(X_{\boldsymbol{\zeta}})$ with the root lattice $\mathbf{Z}\Phi$.

(5) $N_{\mathrm{SO}(\mathbf{H})}(\Gamma)$ acts on Γ by conjugation; that is: there is a homomorphism $C \colon N_{\mathrm{SO}(H)}(\Gamma) \to \operatorname{Aut}(\Gamma)$ such that for every $R \in N_{\mathrm{SO}(H)}(\Gamma)$, $g \in \Gamma$, and $x \in \mathbf{H}$

$$RgR^{-1}x = C_R(g)x.$$

Identify $\operatorname{Aut}(\Gamma) \subset \mathrm{U}(\mathbf{C}[\Gamma])$. Denote by $\operatorname{Ad} \colon \mathrm{U}(\mathbf{C}[\Gamma]) \to \mathfrak{pu}(\mathbf{C}[\Gamma])^*$ the coadjoint action. Ad_{C_R} acts on Φ.

The hyperkähler moment map satisfies

$$\mu \circ (R \otimes \operatorname{Ad}_{C_R}) = (\Lambda^+ R \otimes \operatorname{Ad}_{C_R}) \circ \mu.$$

Let $\boldsymbol{\zeta} \in \tilde{\Delta}^\circ$. Let $\theta \in \Phi$ be as in (4). If $R \in N_{\mathrm{SO}(\mathbf{H})}(\Gamma)$ satisfies $(\Lambda^+ R \otimes \operatorname{Ad}_{C_R})\boldsymbol{\zeta} = \boldsymbol{\zeta}$ and Ad_{C_R} preserves θ, then it lifts to an isometry $\hat{R} \in \operatorname{Diff}(X_{\boldsymbol{\zeta}})$ satisfying

$$\big((\Lambda^+ R)^* \otimes \hat{R}^*\big)\boldsymbol{\omega}_{\boldsymbol{\zeta}} = \boldsymbol{\omega}_{\boldsymbol{\zeta}} \quad \text{and} \quad \hat{R}(\Sigma_\theta) = \Sigma_\theta.$$

Denote by W the Weyl group of Φ. Every $\sigma \in W$ induces a hyperkähler isometry $\hat{\sigma} \colon X_{\boldsymbol{\zeta}} \cong X_{\sigma(\boldsymbol{\zeta})}$ satisfying $\hat{\sigma}(\Sigma_\theta) = \Sigma_{\sigma(\theta)}$. In particular, $\tilde{\Delta}$ and $\tilde{\Delta}^\circ$ can be replaced with

$$\Delta := \tilde{\Delta}/W \quad \text{and} \quad \Delta^\circ := \tilde{\Delta}^\circ/W.$$

Of course, for $\Gamma = C_k$ the above parallels Remark 2.12. ♣

The generalised Kummer construction proceeds by constructing an approximate resolution and correcting it via singular perturbation theory.

Definition 2.16 (approximate resolution). Let (Y_0, ϕ_0) be a flat G_2–orbifold together with resolution data $\mathfrak{R}$. Let $t \in (0, 1]$. Set

$$Y_0^\circ := Y_0 \backslash \bigcup_{\alpha \in A} \jmath_\alpha \Big((\operatorname{Im} \mathbf{H} \times (\overline{B}_{R_\alpha}(0)/\Gamma_\alpha)) / G_\alpha \Big).$$

For $\alpha \in A$ denote by

$$(\hat{Y}_{\alpha,t}, \hat{\phi}_{\alpha,t})$$

the model space associated with $\hat{X}_\alpha$, $t^2 \hat{\boldsymbol{\omega}}_\alpha$, G_α, and $\hat{\rho}_\alpha$. Set

$$\begin{aligned} \hat{Y}_t^\circ &:= \coprod_{\alpha \in A} \hat{Y}_{\alpha,t}^\circ \quad \text{with} \quad \hat{Y}_{\alpha,t}^\circ := \Big(\operatorname{Im} \mathbf{H} \times \big(K_\alpha \cup (t\tau_\alpha)^{-1} (B_{2R_\alpha}(0)/\Gamma_\alpha) \big) \Big) \Big/ G_\alpha, \\ \hat{V}_t &:= \coprod_{\alpha \in A} \hat{V}_{\alpha,t} \quad \text{with} \quad \hat{V}_{\alpha,t} := \Big(\operatorname{Im} \mathbf{H} \times (t\tau_\alpha)^{-1} \big((B_{2R_\alpha}(0) \backslash \overline{B}_{R_\alpha}(0))/\Gamma_\alpha \big) \Big) \Big/ G_\alpha, \quad \text{and} \\ V &:= \coprod_{\alpha \in A} V_\alpha \quad \text{with} \quad V_\alpha := \Big(\operatorname{Im} \mathbf{H} \times \big((B_{2R_\alpha}(0) \backslash \overline{B}_{R_\alpha}(0))/\Gamma_\alpha \big) \Big) \Big/ G_\alpha. \end{aligned}$$

Denote by $f \colon \hat{V}_t \to V$ the diffeomorphism induced by $\jmath_\alpha$ and $t\tau_\alpha$ ($\alpha \in A$). Denote by Y_t the 7–manifold obtained by gluing $\hat{Y}_t^\circ$ and Y_0° along f:

$$Y_t := \hat{Y}_t^\circ \cup_f Y_0^\circ.$$

A cut-and-paste procedure (whose details are swept under the rug here, but can be found in [Joy96b, Proof of Theorem 2.2.1; Joy00, §11.5.3]) produces a closed 3–form

$$\tilde{\phi}_t \in \Omega^3(Y_t)$$

which agrees with $\hat{\phi}_{\alpha,t}$ on $\hat{Y}_\alpha^\circ \backslash \hat{V}_{\alpha,t}$ ($\alpha \in A$) and with ϕ_0 on $Y_0^\circ \backslash V$; moreover: if t is sufficiently small, then $\tilde{\phi}_t$ defines a G_2–structure on Y_t. •

Remark 2.17. Since $\hat{X}_\alpha$ retracts to a compact subset, there are canonical maps

$$\upsilon_\alpha \colon H_\bullet(\hat{Y}_{\alpha,t}, \mathbf{Z}) \cong H_\bullet(\hat{Y}_{\alpha,t}^\circ, \mathbf{Z}) \to H_\bullet(Y_t, \mathbf{Z}).$$ ♣

Remark 2.18 ($\tilde{\phi}_t$ vs. $t^{-3}\tilde{\phi}_t$). As t tends to zero, the Riemannian metric $\tilde{g}_t$ associated with $\tilde{\phi}_t$ degenerates quite severely: $\|R_{\tilde{g}_t}\|_{L^\infty} \sim t^{-2}$ and $\operatorname{inj}(\tilde{g}_t) \sim t^{-1}$. To ameliorate this it can be convenient to pass to the Riemannian metric $t^{-2}\tilde{g}_t$ associated with $t^{-3}\tilde{\phi}_t$. This is at the expense of the diameter and volume of $(Y_t, t^{-2}\tilde{g}_t)$ tending to ∞. For the purposes of the present article this is mostly harmless. ♣

The following refinement of Joyce's existence theorem for torsion-free G_2–structures [Joy96a, Theorem B; Joy00, Theorems G1 and G2] is crucial.

Theorem 2.19 (Platt [Pla22, Theorem 4.58]). *Let $\mathfrak{R}$ be resolution data for a closed flat G_2–orbifold (Y_0, ϕ_0). Let $\alpha \in (0, 1/16)$. There are $T_0 = T_0(\mathfrak{R})$, $c = c(\mathfrak{R}, \alpha) > 0$ and for every $t \in (0, T_0)$ there is a torsion-free G_2–structure $\phi_t \in \Omega^3(Y_t)$ with $[\phi_t] = [\tilde{\phi}_t] \in \mathrm{H}^3_{\mathrm{dR}}(Y_t)$ satisfying*

$$\|t^{-3}(\phi_t - \tilde{\phi}_t)\|_{C^{1,\alpha}} \leqslant ct^{5/2}.$$

Here $\|-\|_{C^{1,\alpha}}$ is with respect to $t^{-2}\tilde{g}_t$.

Remark 2.20 (K–equivariant generalised Kummer construction). Let (Y_0, ϕ_0) be a closed flat G_2–orbifold. Let K be a group. Let $\lambda\colon\ K \to \mathrm{Diff}(Y_0)$ be a homomorphism with respect to which ϕ_0 is K–invariant. K acts on the singular set of Y_0 and, therefore, on A. K–**equivariant resolution data** for $(Y_0, \phi_0; \lambda)$ consist of resolution data $\mathfrak{R} = (\Gamma_\alpha, G_\alpha, \rho_\alpha; R_\alpha, \jmath_\alpha; \hat{X}_\alpha, \hat{\boldsymbol{\omega}}_\alpha, \hat{\rho}_\alpha, \tau_\alpha)_{\alpha \in A}$ for (Y_0, ϕ_0) with the property that for every $\alpha \in A$ and $g \in K$

$$\Gamma_{g\alpha} = \Gamma_\alpha, \quad G_{g\alpha} = G_\alpha, \quad \rho_{g\alpha} = \rho_\alpha, \quad \text{and} \quad R_{g\alpha} = R_\alpha,$$

and of the following additional data for every $\alpha \in A$:

(1) A pair of homomorphisms $\lambda_\alpha\colon\ K \to N_{\mathrm{SO}(\mathrm{Im}\,\mathbf{H}) \ltimes \mathrm{Im}\,\mathbf{H}}(G_\alpha) < \mathrm{SO}(\mathrm{Im}\,\mathbf{H}) \ltimes \mathrm{Im}\,\mathbf{H}$ and $\kappa_\alpha\colon\ K \to N_{N_{\mathrm{SO}(\mathbf{H})}(\Gamma_\alpha)}(\rho_\alpha(G_\alpha)) \hookrightarrow \mathrm{Isom}(\mathbf{H}/\Gamma_\alpha)$ such that for every $g \in K$

$$\lambda(g) \circ \jmath_\alpha = \jmath_{g\alpha} \circ [\lambda_\alpha(g) \times \kappa_\alpha(g)].$$

Here $[\lambda_\alpha(g) \times \kappa_\alpha(g)]$ denotes the induced isometry of $U_\alpha = U_{g\alpha}$.

(2) A homomorphism $\hat{\kappa}_\alpha\colon\ K \to N_{\mathrm{Diff}(\hat{X}_\alpha)}(\hat{\rho}_\alpha(G_\alpha))$ such that $\hat{\boldsymbol{\omega}}_\alpha$ is K–invariant with respect to λ_α and $\hat{\kappa}_\alpha$ (in the sense of Example 2.1 (2)) and τ_α is K–equivariant with respect to $\hat{\kappa}_\alpha$.

The approximate resolution in Definition 2.16 can be done so that λ and $(\lambda_\alpha, \kappa_\alpha)_{\alpha \in A}$ lift to a homomorphism $\lambda_t\colon\ K \to \mathrm{Diff}(Y_t)$ with respect to which $\tilde{\phi}_t$ is K–invariant. In this situation, $\tilde{\phi}_t$ constructed by Theorem 2.19 is K–invariant. ♣

3 Perturbing Morse–Bott families of associative submanifolds

This section lays the technical foundation for the construction of the examples in Section 4. Throughout, let Y be a 7–manifold with a G_2–structure $\phi \in \Omega^3(Y)$. Set

$$\psi := *\phi \in \Omega^4(Y).$$

Encode the torsion of ϕ as the section $\tau \in \Gamma(\mathfrak{gl}(TY))$ defined by

$$\nabla_v \psi =: \tau(v)^\flat \wedge \phi.$$

Here $-^\flat\colon\ TY \to T^*Y$ denotes the isomorphism induced by the Riemannian metric.

Definition 3.1. A closed oriented 3–dimensional immersed submanifold $P \looparrowright Y$ is **(ϕ–)associative** if

$$\phi|_P > 0 \quad \text{and} \quad (i_v\psi)|_P = 0 \quad \text{for every} \quad v \in NP$$

or, equivalently, if it is ϕ–(semi-)calibrated; that is: $\phi_P = \mathrm{vol}_P$ [HL82, Theorem 1.6]. •

Example 3.2. Assume the situation of Example 2.1 (2). Let $\hat{\xi} \in S^2 \subset \operatorname{Im}\mathbf{H}$, $L > 0$, and $\Sigma \subset X$. Suppose that Σ is a closed $I_{\hat{\xi}}$–holomorphic curve with $I_{\hat{\xi}} := \langle \mathbf{I}, \hat{\xi} \rangle$, $\xi := L\hat{\xi} \in \Lambda < G$ is primitive, $\mathbf{Z}\xi < G$ is normal, and, for every $g \in G$, $\rho(g)(\Sigma) = \Sigma$. In this situation, for every

$$[\eta] \in M/H \quad \text{with} \quad M := (\operatorname{Im}\mathbf{H}/\mathbf{R}\xi)/(\Lambda/\mathbf{Z}\xi) \cong T^2 \quad \text{and} \quad H := G/\Lambda$$

the submanifold

$$P_{[\eta]} := \big((\mathbf{R}\xi + \eta) \times \Sigma\big) \big/ \mathbf{Z}\xi \looparrowright Y$$

is diffeomorphic to the mapping torus T_μ of $\mu := \rho(\xi)|_\Sigma \in \operatorname{Diff}(\Sigma)$. By direct inspection of (2.2), $P_{[\eta]}$ is associative. ♠

Remark 3.3. $\mathbf{Z}\xi < G$ is normal if and only if ξ is an eigenvector of every $R \in G \cap \mathrm{SO}(\operatorname{Im}\mathbf{H})$. Direct inspection of Remark 2.3 reveals the following possibilities (without loss of generality):

(1) $H \cong \mathbf{1}$ and $\xi \in \Lambda$ is any primitive element.

(C_2^+) $H \cong C_2$ and $\xi = \lambda_1$. The orbifold M/H has 4 singularities: each with isotropy C_2.

(C_2^-) $H \cong C_2$ and $\xi = \lambda_2$. M/H is diffeomorphic to the Klein bottle $\mathbf{R}P^2\#\mathbf{R}P^2$.

(C_3) $H \cong C_3$ and $\xi = \lambda_1$. The orbifold M/H has 3 singularities: each with isotropy C_3.

(C_4) $H \cong C_4$ and $\xi = \lambda_1$. The orbifold M/H has 3 singularities: two with isotropy C_4, one with isotropy C_2.

(C_6) $H \cong C_6$ and $\xi = \lambda_1$. The orbifold M/H has 3 singularities: one with isotropy C_6, one with isotropy C_3, one with isotropy C_2.

(C_2^2) $H \cong C_2^2$ and $\xi = \lambda_1$. The orbifold M/H has 2 singularities: each with isotropy C_2.

The construction method summarised in Proposition 4.1 hinges upon understanding the singularities of M/H. This is foreshadowed in Remark 3.7 (2). ♣

Remark 3.4. The examples discussed in Section 4 are based on Example 3.2 with $\mu = \mathrm{id}_\Sigma$; in particular, $P_{[\eta]}$ is diffeomorphic to $S^1 \times \Sigma$. ♣

Let $\beta \in \mathrm{H}_3(Y, \mathbf{Z})$. Denote by $\mathcal{S} = \mathcal{S}(Y)$ the orbifold of closed connected oriented 3–dimensional immersed submanifolds $P \looparrowright Y$ with $[P] = \beta$; cf. [KM97, §44]. Define $\delta\Upsilon = \delta\Upsilon^\psi \in \Omega^1(\mathcal{S})$ by

$$(\delta\Upsilon)_P(v) := \int_P i_v\psi \quad \text{for} \quad v \in T_P\mathcal{S} = \Gamma(NP).$$

By construction, if $\langle[\phi], \beta\rangle > 0$, then $P \in \mathcal{S}$ is a zero of $\delta\Upsilon$ if and only if P is associative.[3]

If $\mathrm{d}\psi = 0$, then $\delta\Upsilon$ is closed; indeed: there is a covering map $\pi\colon \tilde{\mathcal{S}} \to \mathcal{S}$ such that $\pi^*\delta\Upsilon$ is exact. The covering map π is the principal covering map associated with the sweep-out homomorphism

$$\mathrm{sweep}\colon \pi_1(\mathcal{S}) \to \mathrm{H}_4(Y).$$

[3]An analogous statement holds with the orientation of P reversed if $\langle[\phi], \beta\rangle < 0$. $\delta\Upsilon$ has no zeros if $\langle[\phi], \beta\rangle = 0$.

More concretely: choose $P_0 \in \mathscr{S}$ and denote by $\tilde{S}$ the set of equivalence classes $[P, Q]$ of pairs consisting of $P \in \mathscr{S}$ and a 4–chain Q satisfying $\partial Q = P - P_0$ with respect to the equivalence relation $\sim$ defined by

$$(P_1, Q_1) \sim (P_2, Q_2) \iff \big(P_1 = P_2 \quad \text{and} \quad [Q_1 - Q_2] = 0 \in \mathrm{H}_4(Y, \mathbf{Z})\big).$$

$\tilde{\mathscr{S}}$ admits a unique smooth structure such that the canonical projection map $\pi\colon \tilde{S} \to S$ is a smooth covering map. By Cartan's formula and Stokes' theorem, $\Upsilon = \Upsilon^{\psi} \in C^{\infty}(\tilde{\mathscr{S}})$ defined by

$$\Upsilon([P, Q]) := \int_Q \psi \tag{3.5}$$

satisfies

$$\mathrm{d}\Upsilon = \pi^*(\delta\Upsilon).$$

The fundamental strategy of this article is to find associatives submanifolds within a suitably constructed family of submanifolds; that is: a smooth map $\mathbf{P}\colon M \to \mathscr{S}$. Here, a priori, M is an arbitrary orbifold; but in Section 4 it arises from Remark 3.3. Of course, if $\mathbf{P}\colon M \to \mathscr{S}$ is a smooth map, then zeros of $\mathbf{P}^*(\delta\Upsilon)$ need not correspond to associative submanifolds. However, the following trivial observation turns out to be helpful.

Lemma 3.6. *Suppose that $\langle[\phi], \beta\rangle > 0$. Let $\mathbf{P}\colon M \to \mathscr{S}$ be a smooth map. If $\mathbf{P}$ is **transverse** to* $\ker \delta\Upsilon$ *at $x \in M$; that is: if*

$$\ker(\delta\Upsilon)_{\mathbf{P}(x)} + \operatorname{im} T_x\mathbf{P} = T_{\mathbf{P}(x)}\mathscr{S},$$

then $\mathbf{P}(x)$ is associative if and only if x is a zero of $\mathbf{P}^*(\delta\Upsilon)$. ■

Remark 3.7. Lemma 3.6 is particularly useful if there is a mechanism that forces $\mathbf{P}^*(\delta\Upsilon)$ to have zeros; e.g.:

(1) If M is closed, then $\mathbf{P}^*(\delta\Upsilon)$ has $\chi(M)$ zeros (counted with signs and multiplicities).

(2) If there is a finite group H acting on M and $\mathbf{P}^*(\delta\Upsilon)$ is H–invariant, then every isolated fixed-point is a zero.

(3) If M is closed and $\mathbf{P}^*(\delta\Upsilon)$ is exact, then it has at least two zeros (indeed: at least three unless M is homeomorphic to a sphere). By (the proof of) Poincaré's Lemma, $\mathbf{P}^*(\delta\Upsilon)$ is exact if and only if the composite homomorphism

$$\pi_1(M) \xrightarrow{\pi_1(\mathbf{P})} \pi_1(\mathscr{S}) \xrightarrow{\text{sweep}} \mathrm{H}_4(Y) \xrightarrow{\langle -, [\psi]\rangle} \mathbf{R}$$

vanishes. Indeed, if this homomorphism vanishes, then (3.5) is independent of the choice of the 4–chain Q and defines a primitive of $\mathbf{P}^*(\delta\Upsilon)$. ♣

The deformation theory of associative submanifolds is quite well-behaved. Here is a summary of the salient points.

Definition 3.8. A **tubular neighborhood** of $P \in \mathscr{S}$ is an open immersion $\jmath\colon U \hookrightarrow Y$ extending $P \hookrightarrow Y$ with $U \subset NP$ an open neighborhood of the zero section in NP satisfying $t \cdot U \subset U$ for every $t \in [0, 1]$. •

Let $\jmath\colon U \hookrightarrow Y$ be a tubular neighborhood of $P \in \mathcal{S}$. Define $\mathbf{Q} = \mathbf{Q}_\jmath\colon \Gamma(U) \to \mathcal{S}$ by

$$\mathbf{Q}(v) := \jmath(\Gamma_v) \quad \text{with} \quad \Gamma_v := \operatorname{im} v \subset NP.$$

This map is (the inverse of) a chart of $\mathcal{S}$. Since $\Gamma(U) \subset \Gamma(NP)$ is open, $\Omega^1(\Gamma(U))$ can be identified with $C^\infty(\Gamma(U), \Gamma(NP)^*)$. Therefore, it makes sense to Taylor expand $\mathbf{Q}^*(\delta\Upsilon)$. If P is associative, then the zeroth order term vanishes and the first order term is independent of $\jmath$.

Definition 3.9. Let $P \in \mathcal{S}$ be associative. Define $\gamma\colon \operatorname{Hom}(TP, NP) \to NP$ by

$$\langle \gamma(v \cdot u^\flat), w\rangle := \phi(u, v, w).$$

Denote by $\tau^\perp \in \Gamma(\mathfrak{gl}(NP))$ the restriction of $\tau \in \Gamma(\mathfrak{gl}(TY))$. The **Fueter operator** $D = D_P\colon \Gamma(NP) \to \Gamma(NP)$ associated with P is defined by

$$D := -\gamma\nabla + \tau^\perp. \qquad \bullet$$

Proposition 3.10 (McLean [McL98, §5], Akbulut and Salur [AS08, Theorem 6], Gayet [Gay14, Theorem 2.1], Joyce [Joy18, Theorem 2.12]). *Let $P \in \mathcal{S}$ be associative. Let $\jmath\colon U \hookrightarrow Y$ be a tubular neighborhood of P. There are a constant $c = c(\jmath) > 0$ and a smooth map $\mathcal{N} = \mathcal{N}_\jmath \in C^\infty(\Gamma(U), \Gamma(NP))$ such that*

$$\langle \mathbf{Q}^*(\delta\Upsilon)(v), w\rangle = \langle Dv + \mathcal{N}(v), w\rangle_{L^2}$$

and

$$\|\mathcal{N}(v) - \mathcal{N}(w)\|_{C^{0,\alpha}} \leqslant c(\|v\|_{C^{1,\alpha}} + \|w\|_{C^{1,\alpha}})\|v - w\|_{C^{1,\alpha}}.$$

Here $\langle\cdot,\cdot\rangle$ denotes the pairing between $\Gamma(NP)^*$ and $\Gamma(NP)$; and, as explained above, $\mathbf{Q}^*(\delta\Upsilon) \in C^\infty(\Gamma(U), \Gamma(NP)^*)$.

Remark 3.11. If ψ is closed, then D is self-adjoint; indeed, it corresponds to the Hessian of Υ; cf. [Joy18, Lemma 2.13] ♣

Proof of Proposition 3.10. To ease notation, set $f := \mathbf{Q}^*(\delta\Upsilon)$. Since

$$\langle f(v), w\rangle = \int_{\Gamma_v} i_w \jmath^*\psi,$$

$T_u f\colon T_u\Gamma(U) = \Gamma(NP) \to \Gamma(NP)^*$ satisfies

$$\langle T_u f(v), w\rangle = \int_{\Gamma_u} \mathcal{L}_v i_w \jmath^*\psi.$$

Since

$$f(v) = T_0 f(v) + \underbrace{\int_0^1 (T_{tv} f - T_0 f)(v)\,\mathrm{d}t}_{=:\langle \mathcal{N}(v), -\rangle_{L^2}},$$

it remains to identify $T_0 f$ as D and estimate $\mathcal{N}(v)$.

Choose a frame (e_1, e_2, e_3) on U which restricts to a positive orthonormal frame on Γ_{tu} for every $t \in [0,1]$. Denote by ∇ the Levi-Civita connection of $\jmath^* g$ on U. To ease notation, henceforth suppress $\jmath$. Since ∇ is torsion-free,

$$(3.12)\qquad \begin{aligned}(\mathscr{L}_v i_w \psi)(e_1, e_2, e_3) &= \psi(\nabla_v w, e_1, e_2, e_3) + \langle \tau v, w\rangle \phi(e_1, e_2, e_3) \\ &\quad + \psi(w, \nabla_{e_1} v, e_2, e_3) + \psi(w, e_1, \nabla_{e_2} v, e_3) + \psi(w, e_1, e_2, \nabla_{e_3} v).\end{aligned}$$

A moment's thought derives the asserted estimate on $\mathscr{N}$ from this; cf. [MS12, Remark 3.5.5].

Since P is associative, on $P = \Gamma_0$, the first term in (3.12) vanishes and the second equals $\langle \tau^\perp v, w\rangle$. To digest the second line of (3.12), define the cross-product $-\times -\colon TY \otimes TY \to TY$ and the associator $[-,-,-]\colon TY \otimes TY \otimes TY \to TY$ by

$$\langle u \times v, w\rangle \coloneqq \phi(u, v, w) \quad \text{and} \quad \psi(u, v, w, x) \coloneqq \langle [u, v, w], x\rangle.$$

These are related by

$$[u, v, w] = (u \times v) \times w + \langle v, w\rangle u - \langle u, w\rangle v;$$

cf. [SW17, §4]. Therefore,

$$\psi(w, \nabla_{e_i} v, e_j, e_k) = -\langle w, (e_j \times e_k) \times \nabla_{e_i} v\rangle.$$

Since P is associative, $e_i \times e_j = \sum_{k=1}^3 \varepsilon_{ij}{}^k e_k$. Here $\varepsilon_{ij}{}^k$ is the Levi-Civita symbol: if (i, j, k) is a permutation of $(1, 2, 3)$, then it is the sign of this permutation; otherwise it vanishes. Therefore, the second line of (3.12) is

$$-\sum_{a=1}^{3} \langle e_a \times \nabla_{e_a} v, w\rangle = -\langle \gamma \nabla v, w\rangle. \qquad \blacksquare$$

In Example 3.2, the operator D, governing the infinitesimal deformation theory of $P = P_{[\eta]}$, can be understood rather concretely.

Example 3.13. Assume the situation of Example 3.2 with $\mu = \mathrm{id}_\Sigma$. Evidently,

$$TP_{[\eta]} = \mathbf{R}\xi \oplus T\Sigma \quad \text{and} \quad NP_{[\eta]} = (\mathbf{R}\xi)^\perp \oplus N\Sigma.$$

Direct inspection reveals that $\gamma(-\cdot \xi^\flat)$ defines a complex structure i on $(\mathbf{R}\xi)^\perp$ and agrees with $-I_\xi$ on $N\Sigma$; moreover, for $\zeta \cdot v^\flat \in \mathrm{Hom}(T\Sigma, (\mathbf{R}\xi)^\perp)$

$$\gamma(\zeta \cdot v^\flat) = I_\zeta v \in N\Sigma.$$

A moment's thought shows that

$$\gamma(\zeta \cdot v^\flat I_\xi) = I_\xi \gamma(\zeta \cdot v^\flat) = \gamma(i\zeta \cdot v^\flat).$$

Denote by $\overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, (\mathbf{R}\xi)^\perp) \subset \mathrm{Hom}(T\Sigma, (\mathbf{R}\xi)^\perp)$ the subspace of complex anti-linear maps. The restriction of γ to $\mathrm{Hom}(T\Sigma, (\mathbf{R}\xi)^\perp)$ is the composition of a complex linear isomorphism

$$\kappa\colon \overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma, (\mathbf{R}\xi)^\perp) \cong N\Sigma$$

and the projection $(-)^{0,1}\colon\ \mathrm{Hom}(T\Sigma,(\mathbf{R}\xi)^\perp)\to\overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma,(\mathbf{R}\xi)^\perp)$ defined by $A^{0,1}:=\frac{1}{2}(A+IAI)$ Therefore,

$$D=D_{P_{[\eta]}}=(-i\oplus I_\xi)\cdot\partial_\xi-\begin{pmatrix}0&\bar\partial^*\kappa^*\\ \kappa\bar\partial&0\end{pmatrix}$$

with ∂_ξ denoting the derivative along ξ and the Cauchy–Riemann operator $\bar\partial\colon\ C^\infty(\Sigma,(\mathbf{R}\xi)^\perp)\to\Gamma\big(\overline{\mathrm{Hom}}_{\mathbf{C}}(T\Sigma,(\mathbf{R}\xi)^\perp)\big)$ defined by

$$(\bar\partial f)(v):=(\mathrm{d}f)^{0,1}(v)=\frac{1}{2}(\nabla_v f+i\nabla_{I_\xi v}f)$$

and $\bar\partial^*$ denoting its formal adjoint. In particular,

$$\ker D_{P_{[\eta]}}\cong(\mathbf{R}\xi)^\perp\oplus\mathrm{H}^{0,1}(\Sigma,(\mathbf{R}\xi)^\perp).\qquad\spadesuit$$

If $\operatorname{coker}D=0$, then P is **unobstructed** and stable under perturbations of the G_2–structure ϕ. In Example 3.2, $P_{[\eta]}$ is *never* unobstructed. However, the entire family of $P_{[\eta]}$ parametrised by $[\eta]\in M$ does satisfy the following property if $\Sigma=S^2$ because $(\mathbf{R}\xi)^\perp=T_{[\eta]}M$.[4]

Definition 3.14. A smooth map $\mathbf{P}\colon\ M\to\mathcal{S}$ is a **Morse–Bott** family of (ϕ–)associative submanifolds if it is an immersion and for every $x\in M$

$$(\delta\Upsilon)_{\mathbf{P}(x)}=0\quad\text{and}\quad\ker D_{\mathbf{P}(x)}=\operatorname{im}T_x\mathbf{P}.\qquad\bullet$$

Informally, this condition asserts that $\mathbf{P}$ integrates every infinitesimal deformation. Unfortunately, Morse–Bott families of ϕ–associative submanifolds are not stable under small deformations of the G_2–structure; however, $\mathbf{P}$ being transverse to $\ker\delta\Upsilon$ (as in Lemma 3.6) is. Most of the remainder of this section is devoted to establishing this. This requires a family version of the discussion preceding Proposition 3.10. In a sense this is standard, but: since the application in Section 4 is carried out very close to the degenerate limit, some caution and precision is advised.

Henceforth, the choice of G_2–structure $\phi\in\Omega^3(Y)$ made at the beginning of this section shall be undone.

Definition 3.15. Let $\mathbf{P}_0\colon\ M\to\mathcal{S}$ be a smooth map. Consider the fibre bundle

$$p\colon\ \underline{\mathbf{P}}_0:=\coprod_{x\in M}\mathbf{P}_0(x)\hookrightarrow M\times Y\to M.$$

(1) The **normal bundle** of $\mathbf{P}_0$ is the vector bundle

$$q\colon\ N\mathbf{P}_0:=\coprod_{x\in M}N\mathbf{P}_0(x)\to\underline{\mathbf{P}}_0.$$

There is a canonical isomorphism $N\mathbf{P}_0\cong N\underline{\mathbf{P}}_0:=T(M\times Y)|_{\underline{\mathbf{P}}_0}/T\underline{\mathbf{P}}_0$.

[4] If $P_{[\eta]}$ is multiply covering, then the underlying embedded associative submanifold might be unobstructed; see Remark 4.2.

(2) A **tubular neighborhood** of $\mathbf{P}_0$ is a tubular neighborhood $\jmath\colon \mathbf{U} \hookrightarrow M \times Y$ of $\underline{\mathbf{P}}_0$ with $\mathrm{pr}_M \circ \jmath = p \circ q$. In particular, for every $x \in M$, $\jmath$ induces a tubular neighborhood $\jmath_x\colon U_x \hookrightarrow Y$ of $\mathbf{P}_0(x)$.

(3) The derivative of $\mathbf{P}_0 \in C^\infty(M, \mathscr{S})$ is a section $T\mathbf{P} \in \mathrm{Hom}(TM, \mathbf{P}_0^* T\mathscr{S})$. Therefore, differentiation defines a section $T \in \Gamma(\mathbf{E})$ of the vector bundle

$$r\colon \mathbf{E} := \coprod_{\mathbf{P}_0 \in C^\infty(M,\mathscr{S})} \Gamma\big(\mathrm{Hom}(TM, \mathbf{P}_0^* T\mathscr{S})\big) \to C^\infty(M, \mathscr{S}).$$

Let $\jmath\colon \mathbf{U} \hookrightarrow M \times Y$ be a tubular neighborhood of $\mathbf{P}_0$. The map $\mathbf{Q}_\jmath\colon \Gamma(\mathbf{U}) \to C^\infty(M, \mathscr{S})$ defined by

$$\mathbf{Q}_\jmath(v)(x) := \mathbf{Q}_{\jmath_x}(v) \quad \text{with} \quad v_x := v|_{\mathbf{P}_0(x)}$$

is (the inverse of) a chart on $C^\infty(M, \mathscr{S})$. Within this chart $\mathbf{E}$ is trivialised and T is identified with a smooth map $\mathbf{T} = \mathbf{T}_\jmath \in C^\infty\Big(\Gamma(\mathbf{U}), \Gamma\big(\mathrm{Hom}(p^*TM, N\mathbf{P}_0)\big)\Big)$; that is: the diagram

$$\begin{array}{ccc} \Gamma(\mathbf{U}) \times \Gamma\big(\mathrm{Hom}(p^*TM, N\mathbf{P}_0)\big) & \longrightarrow & \mathbf{E} \\ {\scriptstyle (\mathrm{id},\mathbf{T})}\uparrow\downarrow & & {\scriptstyle r}\downarrow\uparrow{\scriptstyle T} \\ \Gamma(\mathbf{U}) & \xrightarrow{\quad \mathbf{Q} \quad} & C^\infty(M, \mathscr{S}) \end{array}$$

commutes. (See Figure 1 and Remark 3.16.)

Henceforth, suppose that ϕ_0 is a G_2–structure and that $\mathbf{P}_0(x)$ is ϕ_0–associative for every $x \in M$.

(4) Define $D = D_{\mathbf{P}_0}\colon \Gamma(N\mathbf{P}_0) \to \Gamma(N\mathbf{P}_0)$ by

$$(Dv)|_{\mathbf{P}_0(x)} := D_{\mathbf{P}_0(x)}(v|_{\mathbf{P}_0(x)}).$$

Set

$$\mathscr{V} := \{v \in \Gamma(N\mathbf{P}_0) : v|_{\mathbf{P}_0(x)} \perp_{L^2} \mathrm{im}\, T_x\mathbf{P}_0 \text{ for every } x \in M\}$$

with $\perp_{L^2}$ denoting L^2 orthogonality. Denote by $D^\perp = D^\perp_{\mathbf{P}_0}\colon \mathscr{V} \to \mathscr{V}$ the map induced by D and projection onto $\mathscr{V}$.

(5) Let $\jmath\colon \mathbf{U} \hookrightarrow M \times Y$ be a tubular neighborhood of $\mathbf{P}_0$. Define $\mathscr{N} = \mathscr{N}_\jmath \in C^\infty(\Gamma(\mathbf{U}), \Gamma(N\mathbf{P}_0))$ by

$$(\mathscr{N} v)|_{\mathbf{P}_0(x)} := \mathscr{N}_{\jmath_x}(v)|_{\mathbf{P}_0(x)}.$$ ●

Remark 3.16. The upcoming Proposition 3.19 constructs a perturbation $\mathbf{P} := \mathbf{Q}_\jmath(v)$ of $\mathbf{P}_0$. To establish one of the desired properties of $\mathbf{P}$, it is necessary to compare the derivatives $T\mathbf{P}$ and $T\mathbf{P}_0$. The purpose of the map $\mathbf{T}$ is to enable this. ♣

Example 3.17. In the situation of Example 3.2 with $\mu = \mathrm{id}_\Sigma$, $M = T^2$, $\underline{\mathbf{P}}_0 = T^2 \times (S^1 \times \Sigma)$ and $N\mathbf{P}_0 = TT^2 \oplus N\Sigma$. $D_{\mathbf{P}_0(x)}$ and $\mathscr{N}_{\jmath_x}$—for a suitable choice of $\jmath$ and with respect to suitable identifications—are independent of $x \in T^2$. ♠

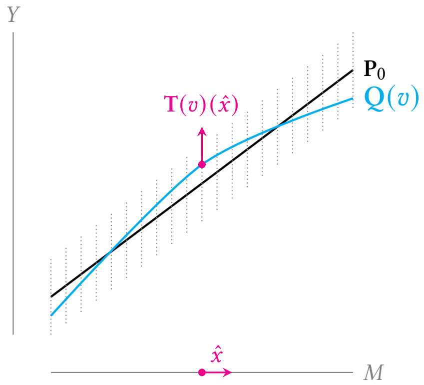

Figure 1: A sketch of the situation of Definition 3.15 (3).

Definition 3.18. Let $\mathbf{P}_0 \colon M \to \mathcal{S}$ be a smooth map. Suppose that Riemannian metrics on M and Y are given. This induces a Euclidean inner product and an orthogonal covariant derivative ∇ on $N\mathbf{P}_0 \to \underline{\mathbf{P}}_0$, and an Ehresmann connection on $p \colon \underline{\mathbf{P}}_0 \to M$. Denote by $\nabla^{1,0}$ and $\nabla^{0,1}$ the restriction of ∇ to the horizontal and vertical directions defined by the Ehresmann connection respectively. Denote by

$$\mathfrak{P} := \coprod_{x \in M} C^\infty([0,1], p^{-1}(x))$$

the set of vertical paths in $p \colon \underline{\mathbf{P}}_0 \to M$. Denote by $\mathfrak{P}^+ \subset \mathfrak{P}$ the subset of non-constant paths. For $\alpha \in (0,1)$ set

$$[v]_{C^0C^{0,\alpha}} := \sup_{\gamma \in \mathfrak{P}^+} \frac{|\mathrm{tra}_\gamma(v(\gamma(0))) - v(\gamma(1))|}{\ell(\gamma)^\alpha} \quad \text{and} \quad \|v\|_{C^0C^{0,\alpha}} := \|v\|_{C^0} + [v]_{C^0C^{0,\alpha}}$$

with $\ell(\gamma)$ denoting the length of γ and tra_γ denoting parallel transport along γ. For $k, \ell \in \mathbf{N}_0$, $\alpha \in (0,1)$ define the norm $\|-\|_{C^kC^{\ell,\alpha}}$ on $\Gamma(N\mathbf{P}_0)$ by

$$\|v\|_{C^kC^{\ell,\alpha}} := \sum_{m=0}^{k} \sum_{n=0}^{\ell} \|(\nabla^{1,0})^m (\nabla^{0,1})^n v\|_{C^0C^{0,\alpha}}.$$ •

Proposition 3.19. *Let $\alpha \in (0,1)$, $\beta, \gamma, c_1, c_2, c_3, c_4, c_5, R > 0$. If $\beta > 2\gamma$, then there are constants $T = T(\alpha, \beta, \gamma, c_1, c_2, c_3, c_4, c_5, R) > 0$ and $c_v = c_v(\alpha, \beta, \gamma, c_1, c_2, c_3) > 0$ with the following significance. Let $\phi_0, \phi \in \Omega^3(Y)$ be two G_2–structures on Y. Let $\mathbf{P}_0 \colon M \to \mathcal{S}$ be a Morse–Bott family of ϕ_0–associative submanifolds. Let $\jmath \colon \mathbf{U} \hookrightarrow M \times Y$ be a tubular neighborhood of $\mathbf{P}_0$. Let $t \in (0,T)$. Suppose that:*

(1) $B_R(0) \subset U_x$.

(2) $\|\jmath^*(\phi - \phi_0)\|_{C^{1,\alpha}(\mathbf{U})} \leqslant c_1 t^\beta$.

(3) *$D^\perp \colon \mathscr{V} \to \mathscr{V}$ is bijective and*

$$\|v\|_{C^1C^{1,\alpha}} \leqslant c_2 t^{-\gamma} \|D^\perp v\|_{C^1C^{0,\alpha}}.$$

(4) $\mathcal{N} \in C^\infty(\Gamma(\mathbf{U}), \Gamma(N\mathbf{P}_0))$ *satisfies*

$$\|\mathcal{N}(v) - \mathcal{N}(w)\|_{C^1C^{0,\alpha}} \leqslant c_3(\|v\|_{C^1C^{1,\alpha}} + \|w\|_{C^1C^{1,\alpha}})\|v - w\|_{C^1C^{1,\alpha}}.$$

(5) *For every* $\hat{x} \in TM$ *and* $v \in \Gamma(\mathbf{U})$

$$|\hat{x}| \leqslant c_4\|\mathbf{T}(0)(\hat{x})\|_{C^0} \quad \text{and} \quad \|\mathbf{T}(v) - \mathbf{T}(0)\|_{C^0} \leqslant c_5\|v\|_{C^1C^{1,\alpha}}.$$

In this situation, there is a $v \in \Gamma(\mathbf{U}) \subset \Gamma(N\mathbf{P}_0)$ *with* $\|v\|_{C^1C^{1,\alpha}} \leqslant c_v t^{\beta-\gamma}$ *such that the map*

$$\mathbf{P} := \mathbf{Q}_J(v) \colon M \to \mathcal{S} \tag{3.20}$$

is transverse to $\ker \delta\Upsilon^\psi$ *(as in Lemma 3.6). Moreover, if* H *is a finite group acting on* M *and* Y*,* ϕ_0 *and* ϕ *are* H*–invariant, and* J *and* $\mathbf{P}_0$ *are* H*–equivariant, then* $\mathbf{P}$ *is* H*–equivariant.*

Remark 3.21. The condition (3) can be understood as a quantification of the Morse–Bott condition. ♣

Proof of Proposition 3.19. To ease notation, define $f_0, f \in C^\infty(\Gamma(\mathbf{U}), \Gamma(N\mathbf{P}_0))$ by

$$\langle f_0(v)|_{\mathbf{P}_0(x)}, w\rangle_{L^2} := \langle \mathbf{Q}_{J_x}^*(\delta\Upsilon^{\psi_0})(v), w\rangle \quad \text{and} \quad \langle f(v)|_{\mathbf{P}_0(x)}, w\rangle_{L^2} := \langle \mathbf{Q}_{J_x}^*(\delta\Upsilon^{\psi})(v), w\rangle.$$

Denote by $(-)^\perp$ the projection onto $\mathcal{V}$. For every $v \in \mathcal{V}$

$$(D^\perp)^{-1} f(v)^\perp = v + \underbrace{(D^\perp)^{-1}(\mathcal{N}(v) + f(v) - f_0(v))^\perp}_{:=E(v)}.$$

By (2), (3), and (4), there is a constant $c_E = c_E(\alpha, \beta, \gamma, c_1, c_2, c_3) > 0$ such that for every $r \in (0, R)$ and $v, w \in \overline{B}_r(0) \subset C^1C^{1,\alpha}\Gamma(N\mathbf{P}_0)$

$$\|E(0)\|_{C^1C^{1,\alpha}} \leqslant c_E t^{\beta-\gamma} \quad \text{and}$$
$$\|E(v) - E(w)\|_{C^1C^{1,\alpha}} \leqslant c_E(r + t^\beta)t^{-\gamma}\|v - w\|_{C^1C^{1,\alpha}}.$$

Therefore, $-E$ defines a contraction on $\overline{B}_r(0) \subset C^1C^{1,\alpha}\Gamma(N\mathbf{P}_0)$ provided

$$c_E(r + t^\beta)t^{-\gamma} < 1 \quad \text{and} \quad c_E t^{\beta-\gamma} + c_E(r + t^\beta)t^{-\gamma} r \leqslant r.$$

These can be seen to hold for $r := 2c_E t^{\beta-\gamma}$ and $t \leqslant T \ll 1$ because $\beta > 2\gamma$. Denote by $v \in \overline{B}_r(0) \subset C^1C^{1,\alpha}\Gamma(N\mathbf{P}_0)$ the unique solution of

$$f(v)^\perp = 0.$$

By elliptic regularity, $v \in \Gamma(\mathbf{U})$.

It remains to prove that $\mathbf{P}$ defined by (3.20) is transverse to $\ker \delta\Upsilon^\psi$; that is: for every $x \in M$

$$\ker(\delta\Upsilon^\psi)_{\mathbf{P}(x)} + \operatorname{im} T_x\mathbf{P} = T_{\mathbf{P}(x)}\mathcal{S},$$

or, equivalently,

$$f_x(v) = 0 \quad \text{or} \quad f_x(v) \notin (\operatorname{im} \mathbf{T}_x(v))^\perp.$$

Here the subscript x indicates restriction to $\mathbf{P}_0(x)$. By construction, $f_x(v) \in \operatorname{im} \mathbf{T}_x(0)$. Therefore, the hypothesis is satisfied by (5) provided $t \leqslant T \ll 1$.

Evidently, this construction preserves H–equivariance. ■

In Example 3.2 with $\Sigma = S^2$, the following gives the required estimate on D.

Situation 3.22. Let X be a compact oriented Riemannian manifold. Let V be a Euclidean vector bundle over X. Let $A\colon \Gamma(V) \to \Gamma(V)$ be a formally self-adjoint linear elliptic differential operator of first order. Denote by $\pi\colon \Gamma(V) \to \ker A$ the L^2 orthogonal projection onto $\ker A$. Let $L > 0$. Define $\Pi\colon \Gamma((\mathbf{R}/L\mathbf{Z}) \times X, V) \to \ker A$ by

$$\Pi s := \fint_0^L \pi(i_t^* s)\, \mathrm{d}t$$

with $i_t(x) := (t, x)$. ×

Remark 3.23. In the situation of Example 3.2 with $\mu = \mathrm{id}_\Sigma$, according to Example 3.13

$$D_{P_{[\eta]}} = (-i \oplus I_\xi) \cdot (\partial_\xi + A) \quad \text{with} \quad A := \begin{pmatrix} 0 & i\bar\partial^*\kappa^* \\ -\kappa\bar\partial i & 0 \end{pmatrix}. \qquad \clubsuit$$

Proposition 3.24. *In Situation 3.22, for every $\alpha \in (0,1)$ there is a constant $c = c(A,\alpha) > 0$ such that for every $s \in \Gamma((\mathbf{R}/L\mathbf{Z}) \times X, V)$*

$$\|s\|_{C^{1,\alpha}} \leqslant c\big((L+1)\|(\partial_t + A)s\|_{C^{0,\alpha}} + \|\Pi s\|_{L^\infty}\big).$$

Proof. By interior Schauder estimates

$$\|s\|_{C^{1,\alpha}} \leqslant c_1(\|(\partial_t + A)s\|_{C^{0,\alpha}} + \|s\|_{L^\infty});$$

see, e.g., [Kic06, §3]. Define $\hat\pi\colon \Gamma((\mathbf{R}/L\mathbf{Z}) \times X, V) \to \Gamma((\mathbf{R}/L\mathbf{Z}) \times X, V)$ by

$$(\hat\pi s)(t,x) := (\pi(i_t^* s))(x).$$

A contradiction argument proves that

$$\|(\mathbf{1} - \hat\pi)s\|_{L^\infty} \leqslant c_2\|(\partial_t + A)(\mathbf{1} - \hat\pi)s\|_{C^{0,\alpha}} \leqslant c_2\big(\|(\partial_t + A)s\|_{C^{0,\alpha}} + \|(\partial_t + A)\hat\pi s\|_{C^{0,\alpha}}\big);$$

cf. [Wal13, Proof of Proposition 8.5]. As a consequence of the fundamental theorem of calculus

$$\|\hat\pi s\|_{L^\infty} \leqslant L\|\partial_t \hat\pi s\|_{L^\infty} + \|\Pi s\|_{L^\infty} = L\|(\partial_t + A)\hat\pi s\|_{L^\infty} + \|\Pi s\|_{L^\infty}.$$

Therefore,

$$\|s\|_{L^\infty} \leqslant c_2\|(\partial_t + A)s\|_{C^{0,\alpha}} + (c_2 + L)\|(\partial_t + A)\hat\pi s\|_{L^\infty} + \|\Pi s\|_{L^\infty}.$$

Since A is formally self-adjoint,

$$\hat\pi(\partial_t + A) = (\partial_t + A)\hat\pi.$$

Therefore,

$$\|(\partial_t + A)\hat\pi s\|_{C^{0,\alpha}} \leqslant c_3\|(\partial_t + A)s\|_{C^{0,\alpha}}.$$

The above observations combine to the asserted estimate with $c = c_1(c_2 + 1)(c_3 + 1)$. ■

4 Examples

The purpose of this section is to construct the associative submanifolds whose existence was promised in Section 1. These associative submanifolds are diffeomorphic to $S^1 \times S^2$, have not appeared in the literature (known to the authors) so far, and—most importantly—their volumes tend to zero as the ambient G_2–manifolds degenerate. In the following, some examples are exhibited. These are certainly not exhaustive; cf. Remark 4.7.

Here is a construction technique based on Proposition 3.19 and Remark 3.7 (2).

Proposition 4.1. *Let $\mathfrak{R} = (\Gamma_\alpha, G_\alpha, \rho_\alpha; R_\alpha, \jmath_\alpha; \hat{X}_\alpha, \hat{\boldsymbol{\omega}}_\alpha, \hat{\rho}_\alpha, \tau_\alpha)_{\alpha\in A}$ be resolution data for a closed flat G_2–orbifold (Y_0, ϕ_0). Denote by $(Y_t, \phi_t)_{t\in(0,T_0)}$ the family of closed G_2–manifolds obtained from the generalised Kummer construction discussed in Section 2. Let $\star \in A$, $\hat{\xi} \in S^2 \subset \operatorname{Im}\mathbf{H}$, $L > 0$, and $\Sigma \subset X_\star$. Set $\xi := L\hat{\xi}$, $\Lambda_\star := G_\star \cap \operatorname{Im}\mathbf{H} < \operatorname{Im}\mathbf{H}$, $M_\star := (\operatorname{Im}\mathbf{H}/\mathbf{R}\xi)/(\Lambda_\star/\mathbf{Z}\xi)$, and $H_\star := G_\star/\Lambda_\star$. Denote by $\mathbf{I}_\star$ the hypercomplex structure on $X_\star$. Suppose that:*

1. *Σ is a closed $I_{\star,\hat{\xi}}$–holomorphic curve. $\Sigma \cong S^2$.*
2. *$\xi \in \Lambda_\star$ is primitive. $\mathbf{Z}\xi < G_\star$ is normal.*
3. *$\rho_\star(g)(\Sigma) = \Sigma$ for every $g \in G_\star$, and $\rho(\xi)|_\Sigma = \mathrm{id}_\Sigma$.*

Denote by n_f the number of singularities of the orbifold $M_\star/H_\star$ (see Remark 3.3). In this situation, there is a constant $T_1 \in (0, T_0]$ and for every $t \in (0, T_1)$ there are at least n_f distinct associative submanifolds in (Y_t, ϕ_t) representing the homology class $\beta := \upsilon_\star([P_{[0]}]) \in \mathrm{H}_3(Y_t, \mathbf{Z})$ with $\upsilon_\star$ as in Remark 2.17 and $P_{[0]} \subset \hat{Y}_{\star,t}$ as in Example 3.2.

Proof. For every $[\eta] \in M_\star/H_\star$ and $t \ll 1$, Example 3.2 constructs a $t^{-3}\tilde{\phi}_t$–associative submanifold $P_{[\eta]} \hookrightarrow \hat{Y}^\circ_{\star,t}\backslash\hat{V}_{\star,t}$. This defines an $H_\star$–invariant Morse–Bott family $\mathbf{P}_0 \colon M_\star \to \mathscr{S}$ of $t^{-3}\tilde{\phi}_t$–associative submanifolds; see Example 3.13. With respect to $t^{-2}\tilde{g}_t$ these submanifolds are isometric to $(\mathbf{R}/t^{-1}L\mathbf{Z}) \times \Sigma$.

The hypotheses of Proposition 3.19 are satisfied for the choices $\phi_0 = t^{-3}\tilde{\phi}_t$, $\phi = t^{-3}\phi_t$, $\alpha \in (0, 1/16)$, $\beta = 5/2$, $\gamma = 1$, and choices of $c_1, c_2, c_3, c_4, c_5, R > 0$ which shall not be specified (because of their secondary importance): (1) holds for $0 < R \ll 1$. (2) holds by Theorem 2.19. Because of Remark 2.18 (the proof of) Proposition 3.10 implies (4). For a suitable choice of $\boldsymbol{J}$, (5) holds with respect to $t^{-2}\tilde{g}_t$ by direct inspection; cf. Example 3.17. It remains to verify (3). As pointed out in Example 3.13, $N\mathbf{P}_0([\eta]) = T_{[\eta]}M_\star \oplus N\Sigma$. Therefore, $N\mathbf{P}_0 \cong TM_\star \oplus N\Sigma \to \underline{\mathbf{P}}_0 \cong (\mathbf{R}/t^{-1}L\mathbf{Z}) \times \Sigma \times M_\star$. By Remark 3.23, $D_{\mathbf{P}_0([\eta])}$ is as in Situation 3.22. By definition, $v \in \mathscr{V}$ if $v|_{\mathbf{P}_0([\eta])} \perp_{L^2} \operatorname{im} T_{[\eta]}\mathbf{P}_0 = T_{[\eta]}M_\star$ for every $[\eta] \in M_\star$. Since $\Sigma \cong S^2$, $\ker A = T_{[\eta]}M_\star$ and the preceding condition is equivalent to $\Pi(v|_{\mathbf{P}_0([\eta])}) = 0$. Therefore, Proposition 3.24 implies (3) with respect to $t^{-2}\tilde{g}_t$.

For $t \in (0, T_{1/2})$ the resulting $H_\star$–invariant map $\mathbf{P} \colon M_\star \to \mathscr{S}$ is transverse to $\ker \delta\Upsilon$ (as in Lemma 3.6). By Remark 3.7 (2), every isolated fixed-point of the action of $H_\star$ on $M_\star$ is a zero of $\mathbf{P}^*(\delta\Upsilon^{\phi_t})$. If $t < T_1 \ll T_{1/2}$, then these map to n_f pairwise distinct elements of $\mathscr{S}$. By Lemma 3.6, each one of these is a ϕ_t–associative submanifold. ∎

Remark 4.2. If $x \in M_\star$ corresponds to an orbifold point $[x] \in M_\star/H_\star$, then $P_0 := \mathbf{P}_0(x)$ and $\mathbf{P}(x)$ are multiply covering and their deck transformation group contain the isotropy group Γ of $[x]$. The embedded associative submanifold $\check{P}_0 := P_0/\Gamma$ is unobstructed; indeed:

$$\ker D_{\check{P}_0} = (\ker D_{P_0})^\Gamma = (T_x M_\star)^\Gamma = 0.$$

This can be used to give a somewhat simpler proof of most of Proposition 4.1 avoiding the use of Proposition 3.19. ♣

Example 4.3. Joyce [Joy96b, Examples 4, 5, 6] constructs 7 examples of closed flat G_2–orbifolds (Y_0, ϕ_0) whose singular set has components S_α ($\alpha \in A$). A is a disjoint union $A = A^0 \amalg A^1$ with $A^1 \neq \varnothing$. For $\alpha \in A^0$, S_α is isometric to $T^3 := \mathbf{R}^3/\mathbf{Z}^3$. For $\alpha \in A^1$, S_α is isometric to T^3/C_2. Here is a more precise description. For $\alpha \in A$ set $\Gamma_\alpha := C_2$. For $\alpha \in A^0$ set $G_\alpha := \Lambda = \langle i, j, k\rangle < \operatorname{Im}\mathbf{H}$ and denote by $\rho_\alpha \colon G_\alpha \to \operatorname{Isom}(\mathbf{H}/\Gamma_\alpha)$ the trivial homomorphism. For $\alpha \in A^1$ let $G_\alpha < \mathrm{SO}(\operatorname{Im}\mathbf{H}) \ltimes \operatorname{Im}\mathbf{H}$ be generated by Λ and $(R_2, \frac{i}{2})$ with R_2 as in (2.5), and define $\rho_\alpha \colon G_\alpha \to G_\alpha/\Lambda \to N_{\mathrm{SO}(\mathbf{H})}(\Gamma_\alpha) \hookrightarrow \operatorname{Isom}(\mathbf{H}/\Gamma_\alpha)$ by

$$\rho_\alpha\big(R_2, \tfrac{i}{2}\big)[q] := [-iqi].$$

For every $\alpha \in A$ there is an open embedding $\jmath_\alpha \colon \big(\operatorname{Im}\mathbf{H} \times (B_{R_\alpha}(0)/\Gamma_\alpha)\big)/G_\alpha \hookrightarrow Y_0$ as in Definition 2.6 (2).

These can be extended to resolution data $\mathfrak{R}$ for (Y_0, ϕ_0) with the aid of the Gibbons–Hawking construction discussed in Remark 2.12. According to Remark 2.12 (2), $(X_\mathbf{0}, \boldsymbol{\omega}_\mathbf{0})$ is $\mathbf{H}/C_2$ with the standard hyperkähler form. If $\boldsymbol{\zeta} = [\zeta, -\zeta] \in \Delta^\circ$, then $(X_{\boldsymbol{\zeta}}, \boldsymbol{\omega}_{\boldsymbol{\zeta}})$ is a hyperkähler manifold and Remark 2.12 (2) provides $\tau \colon X_{\boldsymbol{\zeta}} \backslash K_{\boldsymbol{\zeta}} \to X_\mathbf{0}$. Therefore, completing the resolution data for $\alpha \in A^0$ amounts to a choice of $\zeta_\alpha \in \Delta^\circ$

For $\alpha \in A^1$ the situation is slightly complicated by the fact that ρ_α is non-trivial. The involution $R(q) := -iqi$ lies in $Z_{\mathrm{SO}(\mathbf{H})}(\Gamma_\alpha)$ and $\Lambda^+ R = R_2$. By Remark 2.12 (4), requiring that R lifts to $X_{\boldsymbol{\zeta}}$ imposes the constraint that $\boldsymbol{\zeta}_\alpha \in (\Delta^\circ)^{R_2}$. Therefore, completing the resolution data for $\alpha \in A^1$ amounts to a choice of $\boldsymbol{\zeta}_\alpha \in (\Delta^\circ)^{R_2}$. If

$$\boldsymbol{\zeta} \in \big\{[\zeta, -\zeta] \in \Delta^\circ : \zeta \in \mathbf{R}i\big\} \subset (\Delta^\circ)^{R_2},$$

then the segment joining ζ and $-\zeta$ lifts to an I_i–holomorphic curve $\Sigma \cong S^2$. Therefore, for the corresponding choices of $\mathfrak{R}$, Proposition 4.1 with $\hat{\xi} = i$ and $L = 1$ exhibits 4 associative submanifolds in (Y_t, ϕ_t) for every $t \in (0, T_1)$. ♠

Example 4.4. Joyce [Joy96b, Examples 15, 16] constructs two examples of closed flat G_2–orbifolds (Y_0, ϕ_0) whose singular set has components S_α ($\alpha \in A$). A is a disjoint union $A = A^0 \amalg A^1$ with $A^1 = \{\star\}$. The situation is analogous to that in Example 4.3 except that $\Gamma_\star := C_3$.

Completing the resolution data for $\star$ amounts to a choice of $\boldsymbol{\zeta}_\star \in (\Delta^\circ)^{R_2}$ with R_2 as in (2.5). If

$$\boldsymbol{\zeta} \in \big\{[\zeta_1, \zeta_2, \zeta_3] \in \Delta^\circ : \zeta_1, \zeta_2, \zeta_3 \in \mathbf{R}i\big\} \subset (\Delta^\circ)^{R_2}$$

and ζ_2 is contained in the segment joining ζ_1 and ζ_2, then the segment joining ζ_1 and ζ_2 and the segment joining ζ_2 and ζ_3 lift to $I_{\star,i}$–holomorphic curves $\Sigma_1, \Sigma_2 \cong S^2$ and Proposition 4.1 (2) holds. Therefore, for the corresponding choices of $\mathfrak{R}$, Proposition 4.1 with $\hat{\xi} = i$ and $L = 1$ exhibits $8 = 2 \cdot 4$ associative submanifolds in (Y_t, ϕ_t) for every $t \in (0, T_1)$. ♠

Example 4.5. Reidegeld [Rei17, §5.3.4] constructs an example of a closed flat G_2–orbifold (Y_0, ϕ_0) whose singular set has 16 components S_α $(\alpha \in A)$. For every $\alpha \in A$, S_α is isometric to T^3/C_2^2. Here is a more precise description. For $\alpha \in A$ set $\Gamma_\alpha \coloneqq C_2$, let $G_\alpha < \mathrm{SO}(\mathrm{Im}\,\mathbf{H}) \ltimes \mathrm{Im}\,\mathbf{H}$ be generated by $\Lambda \coloneqq \langle i, j, k \rangle$, $(R_+, \frac{i+k}{2})$, and $(R_-, \frac{j}{2})$ with $R_\pm$ as in (2.5), define $\rho_\alpha \colon G_\alpha \to G_\alpha/\Lambda \to N_{\mathrm{SO}(\mathbf{H})}(\Gamma_\alpha) \hookrightarrow \mathrm{Isom}(\mathbf{H}/\Gamma_\alpha)$ by

$$\rho_\alpha\big(R_+, \tfrac{i+k}{2}\big)[q] \coloneqq [iqi] \quad \text{and} \quad \rho_\alpha\big(R_-, \tfrac{j}{2}\big)[q] \coloneqq [jqj].$$

These act on $\mathrm{Im}\,\mathbf{H}$ as R_+ and R_-. For every $\alpha \in A$ there is an open embedding $\jmath_\alpha \colon \big(\mathrm{Im}\,\mathbf{H} \times (B_{R_\alpha}(0)/\Gamma_\alpha)\big)/G_\alpha \hookrightarrow Y_0$ as in Definition 2.6 (2).

Completing the resolution data for $\alpha \in A$ amounts to a choice of $\boldsymbol{\zeta}_\alpha \in (\Delta^\circ)^{R_+,R_-}$. If

$$\boldsymbol{\zeta} \in \big\{[\zeta, -\zeta] \in \Delta^\circ : \zeta \in \mathbf{R}i\big\} \subset (\Delta^\circ)^{R_+,R_-},$$

then the segment joining ζ and $-\zeta$ lifts to an I_i–holomorphic curve $\Sigma \cong S^2$. Therefore, for every corresponding choice of $\mathfrak{R}$, Proposition 4.1 with $\hat{\xi} = i$ and $L = 1$ exhibits (up to 16 times) 2 associative submanifolds in (Y_t, ϕ_t) for every $t \in (0, T_1)$. ♠

Example 4.6. Here is an example that involves non-cyclic Γ and requires the use of Remark 2.15. Reidegeld [Rei17, §5.3.4] constructs an example of a closed flat G_2–orbifold (Y_0, ϕ_0) whose singular set has 7 components S_α $(\alpha \in A)$. The situation is analogous to that in Example 4.5 except that $A = A^0 \amalg A^1 \amalg A^1$ and

$$\Gamma_\alpha \coloneqq \begin{cases} C_2 & \text{if } \alpha \in A^0 \\ C_4 & \text{if } \alpha \in A^1 \\ \mathrm{Dic}_2 & \text{if } \alpha \in A^2. \end{cases}$$

Completing the resolution data for $\alpha \in A^0$ is identical to Example 4.5. Completing the resolution data for $\alpha \in A^1$ amounts to a choice of $\boldsymbol{\zeta}_\alpha \in (\Delta^\circ)^{R_+,-R_-}$. The minus sign arises because $\alpha(R) = -1$ for $R(q) = jqj$. If

$$\boldsymbol{\zeta} \in \big\{[\zeta_1, \zeta_2, \zeta_3, \zeta_4] \in \Delta^\circ : \zeta_1, \zeta_2, \zeta_2, \zeta_4 \in \mathbf{R}i\big\} \subset (\Delta^\circ)^{R_+,-R_-}$$

then $X_{\boldsymbol{\zeta}}$ contains three $I_{\boldsymbol{\zeta},i}$–holomorphic curves $\Sigma_a \cong S^2$ with $\hat{\xi}_a = \zeta_a/|\zeta_a|$ $(a = 1, 2)$. Therefore, for the corresponding choices of $\mathfrak{R}$, Proposition 4.1 with $\hat{\xi}_a = i$ and $L = 1$ exhibits $6 = 3 \cdot 2$ associative submanifolds in (Y_t, ϕ_t) for every $t \in (0, T_1)$.

To understand the situation for $\alpha \in A^2$, recall that the D_4 root system is

$$\Phi = \big\{\pm e_a \pm e_b \in \mathbf{R}^4 : a \neq b \in \{1, 2, 3, 4\}\big\}.$$

The standard choice of simple roots is

$$\theta_1 \coloneqq e_1 - e_2, \quad \theta_2 \coloneqq e_2 - e_3, \quad \theta_3 \coloneqq e_3 - e_4, \quad \text{and} \quad \theta_4 \coloneqq e_3 + e_4.$$

The Weyl group $W = S^4 \ltimes C_2^3$ acts by permuting and flipping the signs on an even number of the coordinates of $\mathbf{R}^4$. Therefore,

$$\Delta^\circ = \big\{[\zeta_1, \zeta_2, \zeta_3, \zeta_4] \in (\mathrm{Im}\,\mathbf{H} \otimes \mathbf{R}^4)/W : \zeta_a \neq \pm\zeta_b \text{ for } a \neq b \in \{1, 2, 3, 4\}\big\}.$$

The automorphism group of the Dynkin diagram is S^3, which fixes θ_2 and permutes $\theta_1, \theta_3, \theta_4$. Since $\mathrm{Dic}_2 = \langle i, j\rangle < \mathrm{Sp}(1)$, for $R(q) = iqi$ and $R(q) = jqj$, $C_R \in \mathrm{Aut}(\Gamma)$ is inner and, therefore, Ad_{C_R} acts trivially on Φ.

If

$$\zeta \in \left\{[\zeta_1, \zeta_2, \zeta_3, \zeta_4] \in \Delta^\circ : \zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbf{R}i\right\} \subset (\Delta^\circ)^{R_+, R_-},$$

then, by Remark 2.15 (4), X_ζ contains 4 $I_{\zeta,i}$–holomorphic curves $\Sigma_a \cong S^2$ ($a \in \{1, 2, 3, 4\}$). Therefore, for the corresponding choices of $\mathfrak{R}$, Proposition 4.1 with $\hat{\xi} = i$ and $L = 1$ exhibits $8 = 4 \cdot 2$ associative submanifolds in (Y_t, ϕ_t) for every $t \in (0, T_1)$. ♠

Remark 4.7 (Homework assignment). Reidegeld [Rei17, §5.3.4 and §5.3.5] constructed further examples of closed flat G_2–orbifolds (Y_0, ϕ_0) whose singular sets are isometric to T^3, T^3/C_2, and T^3/C_2^2 and whose transverse singularity models are $\mathbf{H}/\Gamma$ with $\Gamma \in \{C_2, C_3, C_4, C_6, \mathrm{Dic}_2, \mathrm{Dic}_3, 2T\}$. The reader might enjoy analysing these examples with the methods used above. ♣

Here is a construction technique based on Proposition 3.19 and Remark 3.7 (3).

Proposition 4.8. *Let $\mathfrak{R} = (\Gamma_\alpha, G_\alpha, \rho_\alpha; R_\alpha, j_\alpha; \hat{X}_\alpha, \hat{\boldsymbol{\omega}}_\alpha, \hat{\rho}_\alpha, \tau_\alpha; \lambda_\alpha, \kappa_\alpha, \hat{\kappa}_\alpha)_{\alpha\in A}$ be K–equivariant resolution data for a closed flat G_2–orbifold (Y_0, ϕ_0) together with a homomorphism $\lambda\colon K \to \mathrm{Diff}(Y_0)$ with respect to which ϕ_0 is K–invariant. Denote by $(Y_t, \phi_t)_{t\in(0,T_0)}$ the family of closed G_2–manifolds obtained from the K–equivariant generalised Kummer construction discussed in Remark 2.20. Let $\star \in A$, $\hat{\xi} \in S^2 \subset \mathrm{Im}\,\mathbf{H}$, $L > 0$, and $\Sigma \subset X_\star$. Set $\xi \coloneqq L\hat{\xi}$, $\Lambda_\star \coloneqq G_\star \cap \mathrm{Im}\,\mathbf{H} < \mathrm{Im}\,\mathbf{H}$, and $M_\star \coloneqq (\mathrm{Im}\,\mathbf{H}/\mathbf{R}\xi)/(\Lambda_\star/\mathbf{Z}\xi)$. Denote by $\mathbf{I}_\star$ the hypercomplex structure on $X_\star$. Suppose that (1), (2), and (3) in Proposition 4.1 hold; and moreover:*

(4) *$g\star = \star$ for every $g \in K$, $\kappa_\star(K) < N_{\mathrm{SO}(\mathrm{Im}\,\mathbf{H})\ltimes\mathrm{Im}\,\mathbf{H}}(\mathbf{Z}\xi)$, and $\hat{\kappa}_\star(g)(\Sigma) = \Sigma$ for every $g \in K$.*

(5) $\mathrm{Hom}(\pi_1(M_\star), \mathbf{R})^K = 0$.

In this situation, there is a constant $T_1 \in (0, T_0]$ and for every $t \in (0, T_1)$ there are at least 3 distinct associative submanifolds in (Y_t, ϕ_t) representing the homology class $\beta \coloneqq v_\star([P_{[0]}]) \in \mathrm{H}_3(Y_t, \mathbf{Z})$ with $v_\star$ as in Remark 2.17 and $P_{[0]} \subset \hat{Y}_{\star,t}$ as in Example 3.2.

Proof. The proof is very similar to that of Proposition 4.1. The additional hypothesis (4) guarantees that K acts on $M_\star$ and that the map $\mathbf{P}_0$ is K–equivariant. Therefore, $\mathbf{P}$ is K–equivariant as well. According to Remark 3.7 (3), the obstruction to $\mathbf{P}^*(\delta\Upsilon^{\phi_t})$ being exact is the composite homomorphism

$$\pi_1(M_\star) \xrightarrow{\pi_1(\mathbf{P})} \pi_1(\mathscr{S}) \xrightarrow{\text{sweep}} \mathrm{H}_4(Y_t) \xrightarrow{\langle -, [\psi_t]\rangle} \mathbf{R}.$$

The first two homomorphisms are manifestly K–equivariant. The third homomorphism is K–equivariant because ψ_t is K–invariant (see Remark 2.20). By (5), the composition vanishes. Therefore, $\mathbf{P}^*(\delta\Upsilon^{\phi_t})$ is exact. Since $M_\star \not\cong S^2$, $\mathbf{P}^*(\delta\Upsilon^{\phi_t})$ has at least 3 zeros. ■

Example 4.9. Set $T^7 \coloneqq \mathbf{R}^7/\mathbf{Z}^7$. Define the torsion-free G_2–structure ϕ_0 by

$$\phi_0 \coloneqq \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - \sum_{a=1}^{3} \mathrm{d}x_a \wedge \omega_a \quad \text{with}$$

$$\omega_1 \coloneqq \mathrm{d}x_4 \wedge \mathrm{d}x_5 + \mathrm{d}x_6 \wedge \mathrm{d}x_7, \quad \omega_2 \coloneqq \mathrm{d}x_4 \wedge \mathrm{d}x_6 + \mathrm{d}x_7 \wedge \mathrm{d}x_5, \quad \omega_3 \coloneqq \mathrm{d}x_4 \wedge \mathrm{d}x_7 + \mathrm{d}x_5 \wedge \mathrm{d}x_6.$$

Define $\iota_1, \iota_2, \iota_3, \lambda \in \operatorname{Isom}(T^7)$ by

$$\begin{aligned}
\iota_1[x_1, \dots, x_7] &\coloneqq [x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7], \\
\iota_2[x_1, \dots, x_7] &\coloneqq [x_1, -x_2, -x_3, x_4, x_5, \tfrac{1}{2} - x_6, -x_7], \\
\iota_3[x_1, \dots, x_7] &\coloneqq [-x_1, x_2, -x_3, x_4, \tfrac{1}{2} - x_5, x_6, \tfrac{1}{2} - x_7], \quad \text{and} \\
\lambda[x_1, \dots, x_7] &\coloneqq [x_1, -x_2, -x_3, x_4, x_5, -x_6, -x_7].
\end{aligned}$$

$(Y_0 \coloneqq T^7/\langle \iota_1, \iota_2, \iota_3 \rangle, \phi_0)$ is the closed flat G_2–orbifold from [Joy96b, Example 3]. Its singular set has $12 = 3 \cdot 4$ components S_α ($\alpha \in A = A^1 \amalg A^2 \amalg A^3$). Here A^a groups those components arising from the fixed-point set of ι_a. The situation is analogous to that in Example 4.3 except that, for every $\alpha \in A$, S_α is isometric to T^3 and $G_\alpha \coloneqq \Lambda = \langle i, j, k \rangle < \operatorname{Im} \mathbf{H}$.

The involution λ descends to Y_0: it can be identified with an action of C_2 on Y_0 as in Remark 2.20. The induced action on A fixes the elements of A^1 and permutes those of A^2 and A^3. Completing the C_2–equivariant resolution data for $\alpha \in A^2 \amalg A^3$ presents no difficulty. For $\alpha \in A^1$, $\lambda_\alpha = R_2$ as in (2.5), and $\rho_\alpha(q) = -iqi$ as in Example 4.3. Therefore, completing the resolution data for $\alpha \in A^1$ amounts to a choice of

$$\boldsymbol{\zeta}_\alpha \in (\Delta^\circ)^{R_2} = \left\{ [\zeta, -\zeta] \in \Delta^\circ : \zeta \in \mathbf{R}i \cup (\mathbf{R}i)^\perp \right\}.$$

If $\boldsymbol{\zeta}_\alpha = [\zeta_\alpha, -\zeta_\alpha]$ with $\zeta_\alpha \in \mathbf{R}i$, then the hypotheses of Proposition 4.8 are satisfied with $\hat{\xi} = i$, $L = 1$, and Σ as in Example 4.3; indeed: C_2 acts on T^2 by $[x_2, x_3] \mapsto [-x_2, -x_3]$; hence: $\operatorname{Hom}(\pi_1(T^2), \mathbf{R})^{C_2} = 0$. This exhibits up to $12 = 4 \cdot 3$ associative submanifolds in (Y_t, ϕ_t) for every $t \in (0, T_1)$ depending on the choice of C_2–equivariant resolution data. ♠

Remark 4.10. If X is a $K3$ surface with a non-symplectic involution τ, then the fixed-point locus X^τ (typically) contains a surface of genus $g \neq 1$ [Nik83, §4]. The twisted connected sum construction [Kov03; KL11; CHNP15]—in fact: a trivial version thereof—produces closed G_2–orbifolds (Y_0, ϕ_0) from a matching pair of $K3$ surfaces $\Sigma_\pm$ equipped with non-symplectic involutions $\tau_\pm$. The singular set of Y_0 is $S^1 \times M$ with $M \coloneqq X_+^\tau \cup X_-^\tau$ and the transverse singularity model is $\mathbf{H}/C_2$. An extension of the generalised Kummer construction due to Joyce and Karigiannis [JK17] resolves Y_0 into a family $(Y_t, \phi_t)_{t \in (0, T_0)}$ of closed G_2–manifolds. It seems plausible that an extension of the techniques in the present article could produce $\mathbf{P}\colon M \to \mathscr{S}$ transverse to $\ker \delta\Upsilon$ (as in Lemma 3.6). Since (typically) $\chi(M) \neq 0$, this would produce associatives in Joyce and Karigiannis's G_2–manifolds. ♣

Remark 4.11. It is also possible to construct coassociative submanifolds in G_2–manifolds obtained from the generalised Kummer construction using similar techniques. In fact, the situation is quite a bit simpler because the deformation theory of coassociative submanifolds is always unobstructed [McL98, §4]. Details and examples will appear in the forthcoming work of Gutwein [Gut22]. ♣

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