The universal moduli space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors

Siqi He Gregory J. Parker Thomas Walpuski

2026-09-23

Abstract

This article equips the universal moduli spaces of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors and eigenspinors with tame Fréchet manifold structures and proves that their natural projections to the corresponding parameter spaces are uniformly Fredholm. A unified construction yields the deformation theories of harmonic spinors and eigenspinors in dimension three and of chiral harmonic spinors in dimension four, recovering earlier work of Donaldson [Don21a], Parker [Par23], and Takahashi [Tak23]. As an application, a deformation theory for non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic self-dual 2–forms in dimension four is established.

Contents

1 Introduction

More than a decade ago, Taubes and his disciples began to observe that the failure of compactness for a wide variety of "new" low-dimensional gauge theories leaves behind evidence in the form of a $\mathbb{Z}/2\mathbb{Z}$ **harmonic spinor** [Tau13a; Tau13b; Tau16; Tau17; Tau22; HW15; WZ21]. The latter have since appeared in related questions in gauge theory and calibrated geometry [Tau14; Don17; DW21; He23], and might play a role in the conjectural construction of invariants of manifolds with special holonomy [DS11; Hay17; Joy18; DW19; DW24].

In light of the above it seems pressing to understand what kind of a phenomenon the existence of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors is; in particular, whether it persists under (small) deformations of the background parameters. This question of the deformation theory of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors has been approached before in the work of Donaldson [Don21a], Parker [Par23], and Takahashi [Tak23]. The purpose of this article is to develop this deformation theory in a unified framework, building upon [BW26] and a suggestion by Donaldson [Don21a]. This framework recovers the existing deformation theories through a single universal construction and also yields a deformation theory for non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic self-dual 2–forms in dimension four.

Throughout the remainder of this article, let X be a connected closed manifold of dimension d, and S a Euclidean vector bundle of rank r over X. Denote by **Met** the tame Fréchet manifold of Riemannian metrics g on X, by **Dir** the tame Fréchet manifold of Dirac bundle structures $\mathfrak{p} = (g; \gamma, \nabla)$ on S (as in [LM89, Part II Definition 5.2] and Section 2), and by $\mathfrak{Ram}$ the set of isometry classes of ramified Euclidean line bundles $[\mathfrak{l}]$ over X. The goal is to equip the universal *set* of (non-zero) $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors (up to scaling)

$$\mathbf{PHarm} := \coprod_{(\mathfrak{p},[\mathfrak{l}])\in \mathbf{Dir}\times\mathfrak{Ram}} \mathbf{P}\ker\big(D^{\mathfrak{l}}_{\mathfrak{p}} \colon H^1\Gamma(X\backslash \mathrm{Br}(\mathfrak{l}), S\otimes \mathfrak{l}) \to L^2\Gamma(X\backslash \mathrm{Br}(\mathfrak{l}), S\otimes \mathfrak{l})\big)$$

with some geometric structure and understand the nature of the projection map $\mathrm{pr}_{\mathbf{Dir}}\colon \mathbf{PHarm} \to \mathbf{Dir}$. Here $D^{\mathfrak{l}}_{\mathbf{p}}$ denotes the Dirac operator associated with $\mathbf{p}$ twisted by $\mathfrak{l}$, and $\mathbf{P}$ denotes the real projectivisation; that is: $\mathbf{P}V := (V\backslash\{0\})/\mathbb{R}^\times$. This structure would upgrade the set $\mathbf{PHarm}$ to the **universal moduli space of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors**.

The present article does not deal with $\mathbf{PHarm}$ itself, but only with the subset

$$\mathbf{PHarm}^* \subseteq \mathbf{PHarm}$$

consisting of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors $(\mathbf{p}, [\mathfrak{l}]; [\Phi])$ which are **non-degenerate** in the following sense:

(1) the branching locus $\mathrm{Br}(\mathfrak{l}) \subseteq X$ is a submanifold, and

(2) $|\Phi|$ vanishes exactly to order $\frac{1}{2}$ at every $x \in \mathrm{Br}(\mathfrak{l})$; that is: $|\Phi| \asymp_\Phi r^{1/2}$. Here r denotes the distance to $\mathrm{Br}(\mathfrak{l})$.

The first restriction is imposed because $\mathfrak{Ram}$ is rather wild and it is far from obvious what structure to put on it, unless $d = 2$, see [TW21, §3.1]. The subset $\mathbf{Ram} \subseteq \mathfrak{Ram}$ of isometry classes of ramified Euclidean line bundles $[\mathfrak{l}]$ with smooth branching locus $\mathrm{Br}(\mathfrak{l})$, however, can be equipped with the structure of a tame Fréchet manifold in a straightforward manner. With this in mind, it is natural to try to equip $\mathbf{PHarm}^*$ with the structure of a tame Fréchet manifold. As explained above, the crucial difficulty is that $D^{\mathfrak{l}}_{\mathbf{p}}\colon H^1\Gamma(X\backslash\mathrm{Br}(\mathfrak{l}), S\otimes\mathfrak{l}) \to L^2\Gamma(X\backslash\mathrm{Br}(\mathfrak{l}), S\otimes\mathfrak{l})$ is left semi-Fredholm but has an ∞-dimensional cokernel, and the naive expectation is that this can be compensated by allowing $\mathrm{Br}(\mathfrak{l})$ to vary. Indeed, provided $|\Phi|$ vanishes exactly to order $\frac{1}{2}$ along $\mathrm{Br}(\mathfrak{l})$, this additional variation essentially corresponds to passing to a closed extension of $D^{\mathfrak{l}}_{\mathbf{p}}$. If $\mathrm{rk}\, S = 4$, then this extension is Fredholm. This observation and a suitable framework enable Nash–Moser theory to prove the following.

Theorem 1.1. *If* $\dim X = 3$ *and* $\mathrm{rk}\, S = 4$*, then* $\mathbf{PHarm}^*$ *can be given the structure of a tame Fréchet manifold such that* $\mathrm{pr}_{\mathbf{Dir}}\colon \mathbf{PHarm}^* \to \mathbf{Dir}$ *is uniformly Fredholm of index* -1.

The notion of a uniform Fredholm map between tame Fréchet manifolds is defined in Definition 4.1. Crucially, by Proposition 4.3, it is stable under transverse base change. In particular, if B is a (finite-dimensional) manifold and $\Gamma\colon B \to \mathbf{Dir}$ is a smooth map which is transverse to $\mathrm{pr}_{\mathbf{Dir}}$, then the pullback $\Gamma^*\mathbf{PHarm}^*$ is a manifold of dimension $\dim B - 1$. In this sense, the appearance of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors is a codimension one phenomenon in dimension three.

Remark 1.2. If $\dim X = 2$, then $\mathbf{Ram}$ is finite-dimensional, which drastically simplifies the issue. The analogue of Theorem 1.1 in this setting is essentially [DW24, Theorem 3.40]. ♣

Remark 1.3. Theorem 1.1 is essentially [Par23, Theorem 1.5]; see also [Tak23, Theorem 1.5]. The proof of Theorem 1.1 presented in Section 4.3 is based on a suggestion by Donaldson [Don21b] and conceptually cleaner and more direct than the argument in [Par23]. ♣

Remark 1.4. The restriction to $\mathbf{PHarm}^*$ is not unproblematic, since it prevents $\mathrm{pr}_{\mathbf{Dir}}$ from being proper; cf. [TW21, §1.4]. Therefore, it would be interesting to analyse what happens if the non-degeneracy condition is relaxed either by allowing $\mathrm{Br}(\mathfrak{l})$ to be singular or $|\Phi|$ to vanish to a higher order at certain points in $\mathrm{Br}(\mathfrak{l})$. [HMT23] contains some initial work in the direction of allowing $\mathrm{Br}(\mathfrak{l})$ to be a graph. ♣

Remark 1.5. The tame Fréchet manifold structure on $\mathbf{PHarm}^*$ is obtained by exhibiting it as a tame Fréchet submanifold of the projectivisation $\mathbf{PE}_{00}$ of a tame Fréchet space bundle $\mathbf{E}_{00}$ over $\mathbf{Dir} \times \mathbf{Ram}$. Moreover, locally, $\mathbf{PHarm}^*$ can be thickened to a submanifold $\mathbf{PHarm}^{\mathrm{t}} \subseteq \mathbf{PE}_{00}$ such that $\mathscr{V} := \mathrm{pr}_{\mathbf{Dir}}(\mathbf{PHarm}^{\mathrm{t}})$ is open and $\mathrm{pr}_{\mathscr{V}} \colon \mathbf{PHarm}^{\mathrm{t}} \to \mathscr{V}$ is a uniform submersion. This additional information can be useful to determine pullbacks of $\mathbf{PHarm}^*$ along tame smooth maps that fail to be transverse to $\mathrm{pr}_{\mathbf{Dir}}$. ♣

The argument in the proof of Theorem 1.1 is somewhat robust to the addition of mild lower-order terms. As an application one obtains the following result regarding

$$\mathbf{PEig}^* := \coprod_{(\mathfrak{p},[\mathfrak{l}],\lambda)\in\mathbf{Dir}\times\mathbf{Ram}\times\mathbb{R}} \mathbf{P}\big\{\Phi \in H^1\Gamma(X\backslash\mathrm{Br}(\mathfrak{l}), S\otimes\mathfrak{l}) : \Phi \text{ satisfies (2) and } D^{\mathfrak{l}}_{\mathfrak{p}}\Phi = \lambda\Phi\big\},$$

the **universal moduli space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ eigenspinors**.

Theorem 1.6 (cf. [Par23, Corollary 1.6]). *If* $\dim X = 3$ *and* $\operatorname{rk} S = 4$, *then* $\mathbf{PEig}^*$ *can be given the structure of a tame Fréchet manifold such that* $\mathrm{pr}_{\mathbf{Dir}} \colon \mathbf{PEig}^* \to \mathbf{Dir}$ *is uniformly Fredholm of index* 0 *and the projection map* $\lambda \colon \mathbf{PEig}^* \to \mathbb{R}$ *is smooth.*

Of course, $\mathbf{PHarm}^* = \lambda^{-1}(0) \subseteq \mathbf{PEig}^*$.

The hypothesis $\operatorname{rk} S = 4$ is essential in the above. As is explained at the end of Section 3.7, the infinitesimal deformation theory of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors at $(\mathfrak{p},[\mathfrak{l}];\Phi)$ is governed by the Dirac operator $D^{\mathfrak{l}}_{\mathfrak{p}}$ subject to a residue condition arising from the inclusion $NZ \hookrightarrow \check{S}$ of the normal bundle of Z into the bundle of possible residue values, encoding the leading order behaviour of Φ at Z. For this residue condition to be elliptic it is necessary that $2 = \operatorname{rk} NZ = \frac{1}{2}\operatorname{rk}\check{S} = \frac{1}{2}\operatorname{rk} S$.

If $\dim X \geqslant 4$, then $\operatorname{rk} S \geqslant 8$ for every Dirac bundle S; however, if X is oriented and $\dim X = 0 \pmod 4$, then every choice of $\mathfrak{p} \in \mathbf{Dir}$ orthogonally decomposes S into

$$S = S^+ \oplus S^- \quad \text{with} \quad S^\pm := \ker(\varepsilon \mp \mathbf{1}) \quad \text{and} \quad \varepsilon := \gamma(\mathrm{vol}_g);$$

moreover, for every $[\mathfrak{l}] \in \mathfrak{Ram}$, $D^{\mathfrak{l}}_{\mathfrak{p}}$ decomposes accordingly as

$$D^{\mathfrak{l}}_{\mathfrak{p}} = \begin{pmatrix} 0 & D^{\mathfrak{l},-}_{\mathfrak{p}} \\ D^{\mathfrak{l},+}_{\mathfrak{p}} & 0 \end{pmatrix}.$$

The universal set of (non-zero) chiral $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors (up to scaling)

$$\mathbf{PHarm}^\pm := \coprod_{(\mathfrak{p},[\mathfrak{l}])\in\mathbf{Dir}\times\mathfrak{Ram}} \mathbf{P}\ker\big(D^{\mathfrak{l},\pm}_{\mathfrak{p}} \colon H^1\Gamma(X\backslash\mathrm{Br}(\mathfrak{l}), S^\pm\otimes\mathfrak{l}) \to L^2\Gamma(X\backslash\mathrm{Br}(\mathfrak{l}), S^\mp\otimes\mathfrak{l})\big)$$

contains the subset

$$\mathbf{PHarm}^{\pm,*} \subseteq \mathbf{PHarm}^\pm$$

of non-degenerate chiral $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors (up to scaling). The restriction of the closed extension of $D^{\mathfrak{l}}_{\mathfrak{p}}$ considered above leads to a closed extension of $D^{\mathfrak{l},\pm}_{\mathfrak{p}}$ which is Fredholm if $\operatorname{rk} S = 8$. The proof of Theorem 1.1 adapts with minor cosmetic modifications to establish the following result.

Theorem 1.7. *If X is oriented,* $\dim X = 4$*, and* $\operatorname{rk} S = 8$*, then* $\mathbf{PHarm}^{\pm,*}$ *can be given the structure of a tame Fréchet manifold such that* $\mathrm{pr}_{\mathbf{Dir}}\colon \mathbf{PHarm}^{\pm,*} \to \mathbf{Dir}$ *is uniformly Fredholm of index*

$$\mp\frac{1}{4}\langle p_1(S), [X]\rangle \pm \sigma(X) + \frac{1}{2}\chi(Z) \mp \frac{5}{4}\langle e(NZ), [Z]\rangle - 1.$$

Remark 1.8. If X is oriented, $\dim X = 4$, and $\operatorname{rk} S = 8$, then $\varepsilon := \gamma(\mathrm{vol}_g)$ is the unique chirality operator on S up to sign. Indeed, if δ is another chirality operator, then it commutes with ε; therefore, $\delta\varepsilon = \varepsilon\delta$ is Clifford linear, self-adjoint, and parallel; hence, by Schur's Lemma $\delta = \pm\varepsilon$. In particular, there is no additional loss of generality in the above setup. ♣

The above results (and their proof) can be brought to bear on $\mathbb{Z}/2\mathbb{Z}$ harmonic 1–forms as follows.

Assume that $\dim X = 3$, X is oriented, and $S = \mathbb{R} \oplus T^*X$. For every $g \in \mathbf{Met}$ there is a Dirac bundle structure on S whose associated Dirac operator is

$$D_g := \begin{pmatrix} 0 & \mathrm{d}^* \\ \mathrm{d} & *\mathrm{d} \end{pmatrix}\colon\ \Omega^0(X) \oplus \Omega^1(X) \to \Omega^0(X) \oplus \Omega^1(X),$$

a truncation of the **Hodge–de Rham operator** $\mathrm{d} + \mathrm{d}^*\colon\ \Omega^\bullet(X) \to \Omega^\bullet(X)$. This defines a tame smooth map $\iota_{\mathrm{HdR}}\colon\ \mathbf{Met} \to \mathbf{Dir}$. By integration by parts, if $(f, \alpha) \in \Omega^0(X, \mathfrak{l}) \oplus \Omega^1(X, \mathfrak{l})$ is $\mathbb{Z}/2\mathbb{Z}$ harmonic, then $f = 0$—except in the edge case $\mathrm{Br}(\mathfrak{l}) = \varnothing$, which shall be disregarded in the remaining discussion. Therefore, the pullback

$$\mathbf{PHarm}^*_{\Omega^1} := \iota^*_{\mathrm{HdR}}\mathbf{PHarm}^*$$

is the **universal moduli space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic 1–forms**. Although ι_{HdR} is *not transverse* to $\mathrm{pr}\colon\ \mathbf{Harm}^* \to \mathbf{Dir}$, the theory underlying the proof of Theorem 1.1, see Remark 1.5, provides detailed information on $\mathbf{PHarm}^*_{\Omega^1}$ and the projection map $\mathrm{pr}_{\mathbf{Met}}\colon\ \mathbf{PHarm}^*_{\Omega^1} \to \mathbf{Met}$.

There is a finite rank vector bundle $\mathscr{H}^1 \to \mathbf{Met} \times \mathbf{Ram}$ whose fibre over $(g, [\mathfrak{l}])$ is $\mathscr{H}^1(g, \mathfrak{l})$, the space of L^2 harmonic 1–forms $\alpha \in L^2\Omega^1(X, \mathfrak{l})$. A singular version of Hodge theory [Tel83] induces an isomorphism $\mathscr{H}^1(g, \mathfrak{l}) \cong \mathrm{H}^1_{\mathrm{dR}}(\tilde{X})^-$. Here $\tilde{X} \to X$ denotes the branched double cover induced by $\mathfrak{l}$ and the superscript indicates taking the anti-invariant part with respect to the sheet-swapping involution. Evidently, there is a canonical inclusion map $\jmath\colon\ \mathbf{PHarm}^*_{\Omega^1} \hookrightarrow \mathbf{P}\mathscr{H}^1$.

Theorem 1.9. *If* $\dim X = 3$ *and X is oriented, then* $\mathbf{PHarm}^*_{\Omega^1}$ *can be given the structure of a tame Fréchet manifold such that:*

(1) $\mathrm{pr}_{\mathbf{Met}}\colon\ \mathbf{PHarm}^*_{\Omega^1} \to \mathbf{Met}$ *is a uniform submersion, and*

(2) $\jmath$ *is smooth and induces an isomorphism* $T\mathbf{PHarm}^*_{\Omega^1}/\mathbf{Met} \cong \jmath^*T\mathbf{P}\mathscr{H}^1/(\mathbf{Met} \times \mathbf{Ram})$ *of the vertical tangent bundles.*

In particular, the index of $\mathrm{pr}_{\mathbf{Met}}$ *at* $(g, [\mathfrak{l}]; [\alpha]) \in \mathbf{PHarm}^*_{\Omega^1}$ *agrees with* $\dim \mathrm{H}^1_{\mathrm{dR}}(\tilde{X})^- - 1$.

Remark 1.10. Theorem 1.9 recovers [Don21a, Theorem 1.1] in dimension three; see also [HP24] for an adaptation of [Par23] to $\mathbb{Z}/2\mathbb{Z}$ harmonic 1–forms. The proof of Theorem 1.9 presented in Section 4.6 is different from the proof in [Don21a], but, of course, related to the alternative proof of [Don21a, Theorem 1.1] mentioned in [Don21b]. ♣

Remark 1.11. The vector bundle $\mathscr{H}^1$ has a flat Gauß–Manin connection. In particular, if $\mathscr{U} \subseteq \mathbf{Met} \times \mathbf{Ram}$ is a simply-connected neighbourhood of $(g, [\mathfrak{l}])$, then it induces a trivialisation $\mathscr{H}^1|_{\mathscr{U}} \cong \mathrm{H}^1_{\mathrm{dR}}(\tilde{X})^-$. As a consequence of Theorem 1.9, every $(g, [\mathfrak{l}]; [\alpha]) \in \mathbf{PHarm}^*_{\Omega^1}$ has an open neighbourhood $\mathscr{U} \subseteq \mathbf{PHarm}^*_{\Omega^1}$ such that there is a well-defined **local period map** $\mathbf{Per}\colon \mathscr{U} \to \mathrm{PH}^1_{\mathrm{dR}}(\tilde{X})^-$ and $\mathrm{pr}_{\mathbf{Met}} \times \mathbf{Per}\colon \mathscr{U} \to \mathbf{Met} \times \mathrm{PH}^1_{\mathrm{dR}}(\tilde{X})^-$ is an open embedding. This formulation is closer to the one found in [Don21a, Theorem 1.1]. ♣

There is a variant of the preceding discussion assuming that $\dim X = 4$, X is oriented, and $S = \mathbb{R} \oplus \Lambda^+ T^*X \oplus T^*X$. For every $g \in \mathbf{Met}$ there is a Dirac bundle structure on S whose associated Dirac operator is

$$D_g := \begin{pmatrix} 0 & \mathrm{d} & \sqrt{2}\mathrm{d}^* \\ \mathrm{d}^* & 0 & 0 \\ \sqrt{2}\mathrm{d}^+ & 0 & 0 \end{pmatrix} \colon \Omega^1(X) \oplus \Omega^0(X) \oplus \Omega^+(X) \to \Omega^1(X) \oplus \Omega^0(X) \oplus \Omega^+(X).$$

This defines a tame smooth map $\iota_{\mathrm{HdR}}\colon \mathbf{Met} \to \mathbf{Dir}$. A brief computation shows that

$$\mathbf{PHarm}^*_{\Omega^1} := \iota^*_{\mathrm{HdR}} \mathbf{PHarm}^{+,*} \quad \text{and} \quad \mathbf{PHarm}^*_{\Omega^+} := \iota^*_{\mathrm{HdR}} \mathbf{PHarm}^{-,*}$$

are the **universal moduli spaces of non-degenerate** $\mathbb{Z}/2\mathbb{Z}$ **harmonic** 1**–forms** and **self-dual** 2**–forms** respectively. There are finite rank vector bundles $\mathscr{H}^1 \to \mathbf{Met} \times \mathbf{Ram}$ and $\mathscr{H}^+ \to \mathbf{Met} \times \mathbf{Ram}$ whose fibres over $(g, [\mathfrak{l}])$ are $\mathscr{H}^1(g, \mathfrak{l})$, as above, and $\mathscr{H}^+(g, \mathfrak{l})$, the space of L^2 harmonic self-dual 2–forms $\alpha \in L^2\Omega^+(X, \mathfrak{l})$. As before, $\mathscr{H}^1(g, \mathfrak{l}) \cong \mathrm{H}^1_{\mathrm{dR}}(\tilde{X})^-$; moreover, $\mathscr{H}^+(g, \mathfrak{l})$ is isomorphic to $\mathrm{H}^+_{\mathrm{dR}}(\tilde{X})^-$, a maximal subspace of $\mathrm{H}^2_{\mathrm{dR}}(\tilde{X})^-$ on which the intersection form is positive definite. There are canonical inclusions $\jmath\colon \mathbf{PHarm}^*_{\Omega^1} \hookrightarrow \mathbf{P}\mathscr{H}^1$ and $\jmath\colon \mathbf{PHarm}^*_{\Omega^+} \hookrightarrow \mathbf{P}\mathscr{H}^+$.

Theorem 1.12. *If* $\dim X = 4$ *and* X *is oriented, then:*

(1) $\mathbf{PHarm}^*_{\Omega^1}$ *can be given the structure of a tame Fréchet manifold such that:*

  (a) $\mathrm{pr}_{\mathbf{Met}}\colon \mathbf{PHarm}^*_{\Omega^1} \to \mathbf{Met}$ *is a uniform submersion, and*

  (b) $\jmath$ *is smooth and induces an isomorphism* $T\mathbf{PHarm}^*_{\Omega^1}/\mathbf{Met} \cong \jmath^* T\mathbf{P}\mathscr{H}^1/(\mathbf{Met} \times \mathbf{Ram})$.

  In particular, the index of $\mathrm{pr}_{\mathbf{Met}}$ *at* $(g, [\mathfrak{l}]; [\alpha])$ *agrees with* $\dim \mathrm{H}^1_{\mathrm{dR}}(\tilde{X})^- - 1$.

(2) $\mathbf{PHarm}^*_{\Omega^+}$ *can be given the structure of a tame Fréchet manifold such that:*

  (a) $\mathrm{pr}_{\mathbf{Met}}\colon \mathbf{PHarm}^*_{\Omega^+} \to \mathbf{Met}$ *is a uniform submersion, and*

  (b) $\jmath$ *is smooth and induces an isomorphism* $T\mathbf{PHarm}^*_{\Omega^+}/\mathbf{Met} \cong \jmath^* T\mathbf{P}\mathscr{H}^+/(\mathbf{Met} \times \mathbf{Ram})$.

  In particular, the index of $\mathrm{pr}_{\mathbf{Met}}$ *at* $(g, [\mathfrak{l}]; [\alpha])$ *agrees with* $\dim \mathrm{H}^+_{\mathrm{dR}}(\tilde{X})^- - 1$.

Remark 1.13. The assertion about $\mathbf{PHarm}^*_{\Omega^1}$ is again nothing but [Don21a, Theorem 1.1] in dimension four. However, the assertion about $\mathbf{PHarm}^*_{\Omega^+}$ is novel. In fact, for self-dual 2–forms the reduction to a scalar equation used in [Don21a] is not available. ♣

Preliminaries This article is based on the theory of tame Fréchet manifolds as developed by Hamilton [Ham82]. Readers are assumed either to be familiar with this subject or to be willing to accept it as a black box. It should be pointed out that, as a small deviation from [Ham82], instead of [Ham82, Part II Definition 1.3.2] the following definition is in effect.

Definition 1.14. A graded Fréchet space $(F, (\|-\|_k)_{k\in\mathbf{N}_0})$ is **tame** if it has the **smoothing operator property**; that is: there is a family of smoothing operators $(S_\varepsilon : \varepsilon \in (0,1))$ on F such that:

(S1) For every $\varepsilon \in (0,1)$, $k, \ell \in \mathbf{N}_0$ with $k \geqslant \ell$, and $x \in F$

$$\|S_\varepsilon x\|_k \lesssim \varepsilon^{\ell-k}\|x\|_\ell.$$

(S2) For every $\varepsilon \in (0,1)$, $k, \ell \in \mathbf{N}_0$ with $k \leqslant \ell$, and $x \in F$

$$\|(S_\varepsilon - \mathbf{1})x\|_k \lesssim \varepsilon^{\ell-k}\|x\|_\ell.$$

(S3) For every $\varepsilon \in (0,1)$, $k, \ell \in \mathbf{N}_0$, and $x \in F$

$$\varepsilon\|\partial_\varepsilon S_\varepsilon x\|_k \lesssim \varepsilon^{\ell-k}\|x\|_\ell. \qquad \bullet$$

This is a weaker condition than [Ham82, Part II Definition 1.3.2], but sufficient for the proof of the Nash–Moser inverse function theorem [Ham82, Part III Theorem 1.1.1].

Acknowledgements The authors thank Fabian Lehmann for his explanations of [DL25], Thibault Langlais for discussions concerning [Tel83] and the proof of Proposition 4.28, and both Thibault Langlais and Jacek Rzemieniecki for numerous helpful comments and suggestions on an earlier version of this article. This material is based in part upon work carried out while the authors were in residence at the Simons Laufer Mathematical Sciences Institute (previously known as MSRI) in Berkeley, California, during the Fall 2022 semester. G.P. is partially supported by an NSF Mathematical Sciences Postdoctoral Research Fellowship (Award No. 2303102).

Statement on the use of AI The authors used GPT-6 Astra to speed up the process of proving the index formula in Theorem B.1 and for the purpose of proofreading the entire article.

2 Dirac operators twisted by ramified line bundles

The technical foundation of this article is [BW26]. The latter is, of course, inspired by [BB12] and most of its results could plausibly be cobbled together from the powerful machinery developed, e.g., in [Maz91; MV14; AG16; AG23].

For the readers' convenience this section provides an *informal* overview of some of the material in [BW26]. Readers who are familiar with it can skim this section or skip it entirely. Throughout this section, the following data are assumed to be fixed:

(1) a Riemannian metric g on X;

(2) a **Dirac bundle structure** (γ, ∇) on S with respect to g; that is:

(a) a linear map $\gamma\colon TX \to \mathfrak{o}(S)$, the **Clifford multiplication**, such that for every $v \in TX$

$$\gamma(v)^2 = -|v|^2 \mathbf{1}_S,$$

and

(b) a covariant derivative $\nabla\colon \Gamma(X,S) \to \Omega^1(X,S)$, the **spin connection**, such that for every $v, w \in \mathrm{Vect}(X)$ and $\phi \in \Gamma(X,S)$

$$\nabla_v(\gamma(w)\phi) = \gamma(\nabla_v w)\phi + \gamma(w)\nabla_v\phi;$$

and

(3) a **ramified Euclidean line bundle** over X; that is:

(a) a closed subset $Z =: \mathrm{Br}(\mathfrak{l}) \subseteq X$, the **branching locus**, and

(b) a Euclidean line bundle $\mathfrak{l}$ over $X\backslash Z$

such that

(c) if $W \subseteq Z$ is closed and $\mathfrak{l}$ extends over $X\backslash W$, then $W = Z$.

Moreover, it is *assumed* that Z is a submanifold, necessarily of codimension two.

Associated with these data is a **Dirac operator**

$$D\colon \Gamma(X\backslash Z, S \otimes \mathfrak{l}) \to \Gamma(X\backslash Z, S \otimes \mathfrak{l})$$

defined by

$$D\phi := \sum_{i=1}^{d} \gamma(e_i)\nabla_{e_i}\phi.$$

Here $(e_1, \dots, e_d)$ is an arbitrary local orthonormal frame and ∇ is the twist of the spin connection by the unique orthogonal connection on $\mathfrak{l}$.

2.1 The Gelfand–Robbin quotient

To begin putting the Dirac operator into a functional analytic context, consider the **minimal extension**

$$D_{\min} = D\colon \mathrm{dom}(D_{\min}) := H^1\Gamma(X\backslash Z, S \otimes \mathfrak{l}) \to L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l}).$$

$D_{\min}$ is left semi-Fredholm but its cokernel is ∞–dimensional [BW26, Proposition 2.3 and Lemma 3.3]. Its adjoint, in the sense of unbounded operators, is the **maximal extension**

$$D_{\max} = D_{\min}^*\colon \mathrm{dom}(D_{\max}) \to L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l});$$

in fact, by [BW26, Remark 2.10],

$$\mathrm{dom}(D_{\max}) = \left\{\phi \in H^1_{\mathrm{loc}}\Gamma(X\backslash Z, S \otimes \mathfrak{l}) : \phi, D\phi \in L^2\Gamma(X\backslash Z, S \otimes \mathfrak{l})\right\}.$$

Of course, $D_{\max}$ is right semi-Fredholm but its kernel is ∞–dimensional.

The intermediate extensions of $D_{\min}$ are controlled by the **Gelfand–Robbin quotient**

$$\check{\mathbf{H}} := \frac{\mathrm{dom}(D_{\max})}{\mathrm{dom}(D_{\min})}.$$

This is a symplectic Hilbert space; its symplectic form, the **Green's** form, is defined by

$$G([\phi],[\psi]) := \langle D\phi,\psi\rangle_{L^2} - \langle \phi, D\psi\rangle_{L^2}.$$

Every closed extension of $D_{\min}$ is of the form

$$D_R := D_{\max}|_{\mathrm{dom}(D_R)} \quad \text{with} \quad \mathrm{dom}(D_R) = \{\phi \in \mathrm{dom}(D_{\max}) : [\phi] \in R\};$$

where $R \subseteq \check{\mathbf{H}}$ is a closed subspace [BW26, Proposition 2.14]; moreover,

$$D_R^* = D_{R^G};$$

that is: the adjoint of D_R is obtained by forming the symplectic complement R^G of R [BW26, Proposition 2.17]. As in [BW26, Definition 2.13], a **residue condition** is a closed subspace $R \subseteq \check{\mathbf{H}}$.

2.2 The residue map

In an abstract way, $[\phi] \in \check{\mathbf{H}}$ encodes the leading-order behaviour of $\phi \in \mathrm{dom}(D_{\max})$ near Z. This leads to the following more geometric realisation of the Gelfand–Robbin quotient.

It is instructive to begin by considering an untwisted spinor $\phi \in \Gamma(X, S)$ vanishing along Z. The Taylor expansion of ϕ at Z has a leading-order term $\phi^{(1)} \in \Gamma(Z, \mathrm{Hom}(NZ, S|_Z))$. A brief computation shows that if $D\phi = 0$, then $\phi^{(1)}$ satisfies the algebraic constraint

$$\phi^{(1)} \circ \mathrm{or} = -\gamma(\mathrm{or}) \circ \phi^{(1)}$$

for every $x \in Z$ and $\mathrm{or} \in \Lambda^2 N_x Z$. Here $\Lambda^2 N_x Z$ is identified with $\mathfrak{o}(N_x Z)$ using the Riemannian metric g so that $(u \wedge v)(w) = \langle u, w\rangle v - \langle v, w\rangle u$ as in [LM89, Part I §6 (6.3)].

If Z is cooriented, then there is a preferred $\mathrm{or} \in \Lambda^2 NZ$ of unit length which defines complex structures on $S|_Z$ and NZ. The algebraic constraint is then simply that $\phi^{(1)}$ is complex anti-linear. However, Z might not be coorientable. Nevertheless, the Riemannian metric defines an isomorphism $(\Lambda^2 NZ)^{\otimes 2} \cong \mathbb{R}$ and this produces a flat bundle

$$\mathbb{A} = \mathbb{R} \oplus i\Lambda^2 NZ$$

of normed $\mathbb{R}$–algebras over Z whose fibres are isomorphic to $\mathbb{C}$, but canonically only up to complex conjugation. The preceding discussion exhibits both NZ and $S|_Z$ as $\mathbb{A}$–modules. If $D\phi = 0$, then $\phi^{(1)}$ is $\mathbb{A}$–anti-linear.

The analogous construction for $\phi \in \mathrm{dom}(D_{\max})$ is quite a bit more involved. Its outcome is very briefly stated as follows. The Dirac bundle structure on S restricts to a Dirac bundle structure on $S|_Z$ [BW26, Definition 3.9]. As explained in [BW26, Definition 3.21 and Remark

3.22], the Euclidean line bundle $\mathfrak{l}$ defines an $\mathbb{A}$–module $NZ^{-1/2}$ together with an $\mathbb{A}$–linear orthogonal connection, and an isomorphism

$$NZ^{-1/2} \otimes_{\mathbb{A}} NZ^{-1/2} \cong \mathrm{Hom}_{\mathbb{A}}(NZ, \mathbb{A}).$$

This endows the **residue bundle**

$$\check{S} := S|_Z \otimes_{\mathbb{A}} NZ^{-1/2}$$

with a Dirac bundle structure; in particular, $\check{S}$ has a Dirac operator $D_{\check{S}}$. Moreover, the restriction of the Clifford multiplication to NZ induces a $\mathbb{A}$–anti-linear almost complex structure

$$J \in \Gamma(Z, \overline{\mathrm{End}}_{\mathbb{A}}(\check{S})).$$

This together with the Euclidean inner product induces a symplectic structure $\check{\Omega}$ on $\check{S}$. Roughly speaking, after identifying $N_xZ = \mathbb{C}$, an element of $S|_Z \otimes_{\mathbb{A}} NZ^{-1/2}$ can be viewed as a term of the form $\phi\bar{z}^{-1/2}$. This should make it plausible that the Gelfand–Robbin quotient $\check{\mathbf{H}}$ can be realised as a suitable completion of $\Gamma(Z, \check{S})$.

The norm with respect to which $\Gamma(Z, \check{S})$ has to be completed is defined in terms of the **branching locus operator**

$$A := -JD_{\check{S}}.$$

Since J is skew-adjoint, $D_{\check{S}}$ is (formally) self-adjoint, and J and $D_{\check{S}}$ anti-commute, A is (formally) self-adjoint. Denote by $\mathbf{1}_{(-\infty,0)}(A)$ and $\mathbf{1}_{[0,\infty)}(A)$ the orthogonal projections to the negative and non-negative eigenspaces of A respectively. Define the norm $\|-\|_{\check{H}} \colon \Gamma(Z, \check{S}) \to [0, \infty)$ by

$$\|\phi\|_{\check{H}} := \|\mathbf{1}_{(-\infty,0)}(A)\phi\|_{H^{1/2}} + \|\mathbf{1}_{[0,\infty)}(A)\phi\|_{H^{-1/2}},$$

and denote by $\check{H}\Gamma(Z, \check{S})$ the completion of $\Gamma(Z, \check{S})$ with respect to $\|-\|_{\check{H}}$. The symplectic form $\check{\Omega}$ on $\check{S}$ induces a symplectic form on $\check{H}\Gamma(Z, \check{S})$ which shall also be denoted by $\check{\Omega}$.

[BW26, Definition 3.40 (1)] constructs the **residue map**

$$\mathrm{res} \colon \mathrm{dom}(D_{\max}) \to \check{H}\Gamma(Z, \check{S})$$

which satisfies

$$\ker \mathrm{res} = \mathrm{dom}(D_{\min})$$

and is surjective; in fact, [BW26, Definition 3.40 (2)] constructs a right inverse, the **extension map**

$$\mathrm{ext} \colon \check{H}\Gamma(Z, \check{S}) \to \mathrm{dom}(D_{\max}).$$

As a consequence of this, res induces an isomorphism

$$\check{\mathbf{H}} \cong \check{H}\Gamma(Z, \check{S});$$

moreover, through this isomorphism the Green's form corresponds to $\check{\Omega}$. In combination with the discussion in Section 2.1, this shows that every closed extension of $D_{\min}$ corresponds to a residue condition considered as a closed subspace $R \subseteq \check{H}\Gamma(Z, \check{S})$.

Although the residue map is constructed analytically in [BW26, §3], it can be seen to arise from a suitable restriction map as follows. *Choose* a tubular neighbourhood of Z and use it to construct a blow-up $\beta\colon\ \hat{X} \longrightarrow X$ of X along Z. The restriction of β to the boundary $\partial\hat{X}$ is identified with the frame bundle

$$\pi\colon\ F := \{v \in NZ : |v| = 1\} \longrightarrow Z.$$

Set $\hat{S} := \beta^* S$. Since the inclusion $X\backslash Z \subseteq \hat{X}$ is a homotopy equivalence, $\mathfrak{l}$ extends to a Euclidean line bundle $\hat{\mathfrak{l}}$ over $\hat{X}$. [BW26, Proposition 3.23] constructs an isomorphism

$$\pi^* NZ^{-1/2} \cong \hat{\mathfrak{l}}|_{\partial\hat{X}} \otimes \pi^*\mathbb{A}.$$

In particular, this defines an inclusion

$$\pi^*\colon\ \Gamma(Z, \check{S}) \hookrightarrow \Gamma(\partial\hat{X}, \hat{S}\otimes\hat{\mathfrak{l}})$$

whose image should be understood to correspond to the fibrewise $-\frac{1}{2}$ Fourier modes. If

$$r\colon\ X\backslash Z \longrightarrow (0,\infty)$$

denotes a **regularised distance** to Z, that is: r is smooth, positive, and agrees with the distance to Z on a neighbourhood of Z, then it turns out that $\mathrm{dom}(D_{\max}) \cap r^{-1/2}\Gamma(\hat{X}, \hat{S}\otimes\hat{\mathfrak{l}})$ is dense in $\mathrm{dom}(D_{\max})$ and that for every $r^{-1/2}\phi \in \mathrm{dom}(D_{\max}) \cap r^{-1/2}\Gamma(\hat{X}, \hat{S}\otimes\hat{\mathfrak{l}})$

$$\phi|_{\partial\hat{X}} = \pi^*\,\mathrm{res}(r^{-1/2}\phi).$$

2.3 Conormal and adapted Sobolev spaces

A vector field $v \in \mathrm{Vect}(\hat{X})$ is **conormal** if $v|_{\partial\hat{X}} \in \mathrm{Vect}(\partial\hat{X})$. Denote the subspace of conormal vector fields by $\mathrm{Vect}_b(\hat{X})$. Denote by $\mathrm{DiffOp}^\bullet(S\otimes\mathfrak{l})$ the $\mathbf{N}_0$–filtered ring of differential operators acting on $S\otimes\mathfrak{l}$. The space $\mathrm{DiffOp}_b^\bullet(S\otimes\mathfrak{l}) \subseteq \mathrm{DiffOp}^\bullet(S\otimes\mathfrak{l})$ of **conormal** differential operators is the filtered subring generated by $\Gamma(\hat{X}, \mathrm{End}(\hat{S}\otimes\hat{\mathfrak{l}}))$ and differential operators of the form ∇_v with $v \in \mathrm{Vect}_b(\hat{X})$.

For every $k \in \mathbf{N}_0$ the **conormal Sobolev space** $H_b^k\Gamma(X\backslash Z, S\otimes\mathfrak{l})$ is defined by

$$H_b^k\Gamma(X\backslash Z, S\otimes\mathfrak{l}) := \left\{\phi \in H_{\mathrm{loc}}^k\Gamma(X\backslash Z, S\otimes\mathfrak{l}) : \begin{array}{l} P\phi \in L^2\Gamma(X\backslash Z, S\otimes\mathfrak{l}) \text{ for} \\ \text{every } P \in \mathrm{DiffOp}_b^k(S\otimes\mathfrak{l}) \end{array}\right\}.$$

Choose a finite subset $\mathscr{P}_b^k \subseteq \mathrm{DiffOp}_b^k(S\otimes\mathfrak{l})$ which spans $\mathrm{DiffOp}_b^k(S\otimes\mathfrak{l})$ over $\Gamma(\hat{X}, \mathrm{End}(\hat{S}\otimes\hat{\mathfrak{l}}))$. Define the norm $\|-\|_{H_b^k}\colon\ H_b^k\Gamma(X\backslash Z, S\otimes\mathfrak{l}) \longrightarrow [0,\infty)$ by

$$\|\phi\|_{H_b^k}^2 := \sum_{P\in\mathscr{P}_b^k} \|P\phi\|_{L^2}^2.$$

Of course, these are the norms that appear in Melrose's b-calculus; see, e.g., [Mel91; Mel93]. They should be regarded as higher regularity analogues of $L^2\Gamma(X\backslash Z, S\otimes\mathfrak{l})$. The core of the scale of conormal Sobolev spaces $(H^k_b\Gamma(X\backslash Z, S\otimes \mathfrak{l}), (\|-\|_{H^k_b})_{k\in\mathbf{N}_0})$ is a tame Fréchet space [BW26, Proposition 4.32].

For every $k\in\mathbf{N}_0$ the **adapted Sobolev space** $H^{k+1}_a\Gamma(X\backslash Z, S\otimes\mathfrak{l})$ is defined by

$$H^{k+1}_a\Gamma(X\backslash Z, S\otimes\mathfrak{l}) := \left\{\phi\in H^{k+1}_{\mathrm{loc}}\Gamma(X\backslash Z, S\otimes\mathfrak{l}) : \begin{array}{l}\phi\in H^{k+1}_b\Gamma(X\backslash Z, S\otimes\mathfrak{l}) \text{ and}\\ D\phi\in H^k_b\Gamma(X\backslash Z, S\otimes\mathfrak{l})\end{array}\right\}.$$

Define the norm $\|-\|_{H^{k+1}_a}\colon H^{k+1}_a\Gamma(X\backslash Z, S\otimes\mathfrak{l})\to[0,\infty)$ by

$$\|\phi\|_{H^{k+1}_a} := \|\phi\|_{H^{k+1}_b} + \|D\phi\|_{H^k_b}.$$

These should be regarded as higher regularity analogues of $\mathrm{dom}(D_{\max})$. Again, the core of the scale of adapted Sobolev spaces $(H^{k+1}_a\Gamma(X\backslash Z, S\otimes\mathfrak{l}), (\|-\|_{H^{k+1}_a})_{k\in\mathbf{N}_0})$ is a tame Fréchet space [BW26, Proposition 4.37].

By [BW26, Lemma 4.12 and Lemma 4.13], for every $k\in\mathbf{N}_0$ the residue map $\mathrm{res}\colon \mathrm{dom}(D_{\max})\to \check{H}\Gamma(Z,\check{S})$ restricts to a bounded surjective linear map

$$\mathrm{res}\colon H^{k+1}_a\Gamma(X\backslash Z, S\otimes\mathfrak{l})\to H^{k+1/2}\Gamma(Z,\check{S}).$$

The extension map $\mathrm{ext}\colon \check{H}\Gamma(Z,\check{S})\to\mathrm{dom}(D_{\max})$ restricts to a right-inverse

$$\mathrm{ext}\colon H^{k+1/2}\Gamma(Z,\check{S})\to H^{k+1}_a\Gamma(X\backslash Z, S\otimes\mathfrak{l}).$$

Moreover, if $\phi\in H^{k+1}_a\Gamma(X\backslash Z, S\otimes\mathfrak{l})$ satisfies $\mathrm{res}(\phi)=0$, then

$$\nabla_v\phi,\, r^{-1}\phi\in H^k_b\Gamma(X\backslash Z, S\otimes\mathfrak{l})$$

for every $v\in\mathrm{Vect}(X)$ [BW26, Proposition 4.27]. Therefore, $\ker\mathrm{res}\subseteq H^{k+1}_a\Gamma(X\backslash Z, S\otimes\mathfrak{l})$ should be regarded as a higher regularity analogue of $\mathrm{dom}(D_{\min}) = H^1\Gamma(X\backslash Z, S\otimes\mathfrak{l})$.

2.4 Elliptic residue conditions

For every residue condition $R\subseteq\check{H}\Gamma(Z,\check{S})$ and every $k\in\mathbf{N}_0$ set

$$H^{k+1}_a\Gamma(X\backslash Z, S\otimes\mathfrak{l}; R) := H^{k+1}_a\Gamma(X\backslash Z, S\otimes\mathfrak{l})\cap\mathrm{dom}(D_R)$$

and consider the restriction of D_R to

$$D^{(k)}_R\colon H^{k+1}_a\Gamma(X\backslash Z, S\otimes\mathfrak{l}; R)\to H^k_b\Gamma(X\backslash Z, S\otimes\mathfrak{l}).$$

A residue condition $R\subseteq\check{H}\Gamma(Z,\check{S})$ is **∞–regular** if for every $k\in\mathbf{N}_0$ and every $\phi\in R$

$$\|\phi\|_{H^{k+1/2}}\lesssim_{R,k}\|\mathbf{1}_{(-\infty,0)}(A)\phi\|_{H^{k+1/2}} + \|\phi\|_{\check{H}}.$$

It is **elliptic** if both R and R^G are ∞–regular. In this case, $D^{(k)}_R$ and $D^{(k)}_{R^G}$ satisfy elliptic estimates and elliptic regularity [BW26, Theorem 4.17]; in particular, $D^{(k)}_R$ is Fredholm, and $\ker D^{(k)}_R$ and $\mathrm{coker}\, D^{(k)}_R$ are independent of $k\in\mathbf{N}_0$ [BW26, Theorem 4.17 and Proposition 4.21].

Evidently, the **APS residue condition** defined by

$$R_{\mathrm{APS}} \coloneqq \mathbf{1}_{(-\infty,0)}(A) H^{1/2}\Gamma(Z,\check{S}) \subset \check{H}\Gamma(Z,\check{S})$$

is elliptic. For a **local residue condition**, that is: a residue condition of the form

$$R_V \coloneqq \check{H}\Gamma(Z,V) \subseteq \check{H}\Gamma(Z,\check{S})$$

for a subbundle $V \subseteq \check{S}$, there is the following symbolic criterion for ellipticity [BW26, Theorem 4.40]: R_V is elliptic if for every $v \in TZ\backslash\{0\}$

$$V \oplus J\gamma(v)V = \check{S}.$$

If $\mathrm{rk}_{\mathbb{A}}\,\check{S} = 2$ and $V \subseteq \check{S}$ is an $\mathbb{A}$–linear subbundle with $\mathrm{rk}_{\mathbb{A}}\,V = 1$, then R_V satisfies this criterion because $J\gamma(v)$ is skew-adjoint and $\mathbb{A}$–anti-linear; in particular, R_V is elliptic [BW26, Example 4.44].

2.5 Once more, with chirality

Suppose that (S,γ,∇) is equipped with a **chirality operator**; that is: a self-adjoint parallel isometry $\varepsilon \in \Gamma(X,\mathrm{O}(S))$ such that

$$\gamma\varepsilon + \varepsilon\gamma = 0.$$

This induces an orthogonal decomposition of S into ∇–parallel subbundles

$$S = S^+ \oplus S^- \quad \text{with} \quad S^\pm \coloneqq \ker(\varepsilon \mp \mathbf{1}).$$

Consequently, the Dirac operator D decomposes as

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

with $D^\pm \colon \Gamma(X\backslash Z, S^\pm \otimes \mathfrak{l}) \to \Gamma(X\backslash Z, S^\mp \otimes \mathfrak{l})$ denoting the **chiral Dirac operators**.

The minimal and maximal extensions decompose analogously. Therefore, the Gelfand–Robbin quotient $\check{\mathbf{H}}$ orthogonally decomposes as

$$\check{\mathbf{H}} = \check{\mathbf{H}}^+ \oplus \check{\mathbf{H}}^- \quad \text{with} \quad \check{\mathbf{H}}^\pm \coloneqq \frac{\mathrm{dom}(D^\pm_{\max})}{\mathrm{dom}(D^\pm_{\min})};$$

moreover, $\check{\mathbf{H}}^\pm \subseteq \check{\mathbf{H}}$ are Lagrangian. A **chiral residue condition** is a closed subspace $R^\pm \subseteq \check{\mathbf{H}}^\pm$. For every chiral residue condition $R^\pm \subseteq \check{\mathbf{H}}^\pm$ there is a unique complementary chiral residue condition $R^\mp \subseteq \check{\mathbf{H}}^\mp$ such that $R \coloneqq R^+ \oplus R^- \subseteq \check{\mathbf{H}}$ is a Lagrangian residue condition. As before, the chiral residue conditions $R^\pm \subseteq \check{\mathbf{H}}^\pm$ precisely correspond to the closed extensions $D^\pm_{R^\pm}$ of $D^\pm_{\min}$.

The chirality operator also induces a parallel orthogonal decomposition $\check{S} = \check{S}^+ \oplus \check{S}^-$. With respect to it the branching locus operator A decomposes into $A^\pm \coloneqq -JD^\pm_{\check{S}} \colon \Gamma(Z,\check{S}^\pm) \to \Gamma(Z,\check{S}^\pm)$; consequently,

$$\check{H}\Gamma(Z,\check{S}) = \check{H}\Gamma(Z,\check{S}^+) \oplus \check{H}\Gamma(Z,\check{S}^-).$$

The residue map res correspondingly decomposes into $\mathrm{res}^{\pm}$, and these induce isomorphisms

$$\check{\mathbf{H}}^{\pm} \cong \check{H}\Gamma(Z, \check{S}^{\pm}).$$

Of course, the conormal and adapted Sobolev spaces also decompose according to the chirality operator. In particular, for every chiral residue condition $R^{\pm} \subseteq \check{H}\Gamma(Z, \check{S}^{\pm})$ and $k \in \mathbf{N}_0$ there are the following higher regularity versions of $D^{\pm}_{R^{\pm}}$:

$$D^{\pm,(k)}_{R^{\pm}} \colon H^{k+1}_a\Gamma(X\backslash Z, S^{\pm} \otimes \mathfrak{l}; R^{\pm}) \to H^k_b\Gamma(X\backslash Z, S^{\mp} \otimes \mathfrak{l}).$$

If the Lagrangian residue condition $R = R^+ \oplus R^-$ is elliptic, then both operators $D^{\pm,(k)}_{R^{\pm}}$ satisfy elliptic estimates and elliptic regularity; in particular, each is Fredholm.

A **local chiral residue condition** is a chiral residue condition of the form

$$R^{\pm}_{V^{\pm}} \coloneqq \check{H}\Gamma(Z, V^{\pm}) \subseteq \check{H}\Gamma(Z, \check{S}^{\pm})$$

for some subbundle $V^{\pm} \subseteq \check{S}^{\pm}$. For every subbundle $V^{\pm} \subseteq \check{S}^{\pm}$ there is a unique subbundle $V^{\mp} \subseteq \check{S}^{\mp}$ such that $V^+ \oplus V^- \subseteq \check{S} = \check{S}^+ \oplus \check{S}^-$ is Lagrangian. Of course, $V^{\mp}$ induces the complementary chiral residue condition: $R_V \coloneqq R^+_{V^+} \oplus R^-_{V^-}$ is Lagrangian. A moment's thought shows that R_V is elliptic if $V^{\pm}$ satisfies the symbolic criterion, namely for every $v \in TZ\backslash\{0\}$

$$V^{\pm} \oplus J\gamma(v)V^{\pm} = \check{S}^{\pm}.$$

This holds in particular if $\mathrm{rk}_{\mathbb{A}}\, \check{S}^{\pm} = 2$ and $V^{\pm} \subseteq \check{S}^{\pm}$ is an $\mathbb{A}$–linear subbundle with $\mathrm{rk}_{\mathbb{A}}\, V^{\pm} = 1$.

3 The universal Dirac operator and residue map

This section sets up the family version of the theory developed in [BW26] and surveyed in Section 2. After describing **Dir**, the space of Dirac bundle structures on S, and **Ram**, the moduli space of ramified line bundles (with smooth branching locus) over X, the spaces $H^{\infty}_a\Gamma(X\backslash\mathrm{Br}(\mathfrak{l}), S \otimes \mathfrak{l})$, $H^{\infty}_b\Gamma(X\backslash\mathrm{Br}(\mathfrak{l}), S \otimes \mathfrak{l})$, and $\Gamma(\mathrm{Br}(\mathfrak{l}), \check{S})$ are assembled into tame Fréchet space bundles **E**, **F**, and **G** over **Dir** × **Ram**. The Dirac operators and residue maps assemble into the universal Dirac operator $\mathbf{D} \colon \mathbf{E} \to \mathbf{F}$ and the universal residue map $\mathbf{res} \colon \mathbf{E} \to \mathbf{G}$. These bundles are equipped with partial covariant derivatives, and the corresponding infinitesimal variation of **D** is determined. Within this framework it is possible to consider residue conditions in families. Finally, it is explained what additional structure arises from the chirality operator $\varepsilon \coloneqq \gamma(\mathrm{vol}_g)$ if X is oriented and $\dim X = 0 \pmod 4$.

3.1 The space of Dirac bundles

For the purposes of this article, the following is the relevant parameter space and half of the base of the tame Fréchet space bundles **E**, **F**, and **G**.

Definition 3.1. Denote by $\mathbf{Met} \subseteq \Gamma(X, \mathrm{Hom}(S^2TX, \mathbb{R}))$ the space of Riemannian metrics on X. The **space of Dirac bundle structures** on S is the tame Fréchet submanifold

$$\mathbf{Dir} \coloneqq \left\{ (g; \gamma, \nabla) \in \mathbf{Met} \times \Gamma(X, \mathrm{Hom}(TX, \mathfrak{o}(S))) \times \mathscr{A}(S) : \begin{array}{l} (\gamma, \nabla) \text{ is a Dirac bundle structure} \\ \text{with respect to } g \end{array} \right\}.$$

In the following a triple $(g;\gamma,\nabla) \in \mathbf{Dir}$ is often abbreviated to $\mathbf{p}$ (which, of course, stands for parameter). •

The projection map $p\colon \mathbf{Dir} \to \mathbf{Met}$ exhibits $\mathbf{Dir}$ as a tame smooth Fréchet fibre bundle over $\mathbf{Met}$. Bourguignon and Gauduchon [BG92] construct an Ehresmann connection on $\mathbf{Dir}$; see also [AWW16, §3.2; MN17, Definition 2.9; AD25, §2.4, §2.5]. Moreover, the **vertical tangent bundle**

$$T\mathbf{Dir}/\mathbf{Met} \coloneqq \ker(Tp\colon T\mathbf{Dir} \to T\mathbf{Met})$$

further decomposes as follows.

Proposition 3.2. *For every $\mathbf{p} = (g;\gamma,\nabla) \in \mathbf{Dir}$ the tangent space $T_{\mathbf{p}}\mathbf{Dir}$ decomposes into the tame Fréchet subspaces*

$$T_{\mathbf{p}}\mathbf{Dir} = H_{\mathrm{BG},\mathbf{p}} \oplus V_{\mathscr{G},\mathbf{p}} \oplus V_{\nabla,\mathrm{Uh};\mathbf{p}}$$

defined as follows:

(1) *$H_{\mathrm{BG},\mathbf{p}}$ is the horizontal distribution at $\mathbf{p} \in \mathbf{Dir}$ of the **Bourguignon–Gauduchon connection** [BG92]; that is: the unique Ehresmann connection on $\mathbf{Dir} \to \mathbf{Met}$ which horizontally lifts $\dot g \in T_g\mathbf{Met}$ to $(\dot g;\dot\gamma^{\mathrm{BG}},\dot\nabla^{\mathrm{BG}}) \in T_{\mathbf{p}}\mathbf{Dir}$ defined by*

$$\dot\gamma^{\mathrm{BG}} \coloneqq \frac{1}{2}\sum_{i=1}^{d} \dot g(-,e_i)\gamma(e_i) \quad \textit{and} \quad \dot\nabla^{\mathrm{BG}} \coloneqq \frac{1}{8}\sum_{i,j=1}^{d} (\nabla_{e_i}\dot g)(e_j,-)[\gamma(e_i),\gamma(e_j)].$$

Here and throughout $(e_1,\dots,e_d)$ denotes a local g–orthonormal frame.

(2) *$V_{\mathscr{G},\mathbf{p}}$ is generated by the action of the **gauge group** $\mathscr{G}(S) \coloneqq \Gamma(X,\mathrm{O}(S))$ on $\mathbf{Dir}$; that is:*

$$V_{\mathscr{G},\mathbf{p}} \coloneqq \bigl\{(0;[\xi,\gamma],-\nabla\xi) : \xi \in \Gamma(X,\mathfrak{o}(S))\bigr\}.$$

(3) *$V_{\nabla,\mathrm{Uh};\mathbf{p}}$ consists of infinitesimal variations of the spin connection in Uhlenbeck gauge; that is:*

$$V_{\nabla,\mathrm{Uh};\mathbf{p}} \coloneqq \bigl\{(0;0,\dot\nabla) : \dot\nabla \in \Omega^1(X,\mathfrak{o}_{\mathrm{C}\ell}(S)) \text{ satisfying } \mathrm{d}_\nabla^*\dot\nabla = 0\bigr\}.$$

Here $\mathfrak{o}_{\mathrm{C}\ell}(S)$ denotes the bundle of skew-adjoint endomorphisms of S which commute with the Clifford multiplication γ.

In fact, the vertical tangent space $T_{\mathbf{p}}\mathbf{Dir}/\mathbf{Met}$ contains every infinitesimal variation of the spin connection; that is:

$$V_{\nabla,\mathbf{p}} \coloneqq \bigl\{(0;0,\dot\nabla) : \dot\nabla \in \Omega^1(X,\mathfrak{o}_{\mathrm{C}\ell}(S))\bigr\} \subseteq V_{\mathscr{G},\mathbf{p}} \oplus V_{\nabla,\mathrm{Uh};\mathbf{p}}.$$

The above turns out to be useful in the following to organise various computations.

Proof of Proposition 3.2. To prove that H_{BG} is an Ehresmann connection on $\mathbf{Dir} \to \mathbf{Met}$, it suffices to verify the formulae

$$\dot\gamma^{\mathrm{BG}}(v)\gamma(v) + \gamma(v)\dot\gamma^{\mathrm{BG}}(v) = -\dot g(v,v) \quad \text{and} \tag{3.3}$$

$$[\dot\nabla^{\mathrm{BG}}_v,\gamma(w)] = \dot\gamma^{\mathrm{BG}}(\nabla_v w) + \gamma(\dot\nabla_v w) - [\nabla_v,\dot\gamma^{\mathrm{BG}}(w)] \tag{3.4}$$

with $\dot{\nabla}$ denoting the infinitesimal deformation of the Levi–Civita connection corresponding to $\dot{g} \in T_g\mathbf{Met}$. The verification of (3.3) is straightforward from the formula in (1). By differentiating Koszul's formula,

$$2g(\dot{\nabla}_v w, u) = (\nabla_v \dot{g})(w, u) + (\nabla_w \dot{g})(u, v) - (\nabla_u \dot{g})(v, w).$$

Therefore,

$$\gamma(\dot{\nabla}_v w) = \frac{1}{2}\sum_{j=1}^{d}\big((\nabla_v \dot{g})(w, e_j) + (\nabla_w \dot{g})(v, e_j) - (\nabla_{e_j} \dot{g})(v, w)\big)\gamma(e_j).$$

A direct computation using the identity

$$[[\gamma(u), \gamma(v)], \gamma(w)] = 4(\gamma(v)g(u, w) - \gamma(u)g(v, w)) \tag{3.5}$$

yields

$$\begin{aligned}
[\dot{\nabla}_v^{\mathrm{BG}}, \gamma(w)] &= \frac{1}{2}\sum_{j=1}^{d}\big((\nabla_w \dot{g})(e_j, v) - (\nabla_{e_j} \dot{g})(v, w)\big)\gamma(e_j),\\
\dot{\gamma}^{\mathrm{BG}}(\nabla_v w) &= \frac{1}{2}\sum_{j=1}^{d}\dot{g}(\nabla_v w, e_j)\gamma(e_j), \quad \text{and}\\
[\nabla_v, \dot{\gamma}^{\mathrm{BG}}(w)] &= \frac{1}{2}\sum_{j=1}^{d}\big((\nabla_v \dot{g})(w, e_j) + \dot{g}(\nabla_v w, e_j)\big)\gamma(e_j).
\end{aligned}$$

(3.4) follows directly from the preceding identities.

Evidently, $V_{\mathscr{G},\mathfrak{p}}$ and $V_{\nabla,\mathfrak{p}}$ are contained in the vertical tangent bundle $T\mathbf{Dir}/\mathbf{Met}$. The Uhlenbeck gauge fixing condition in (3) ensures that $V_{\mathscr{G},\mathfrak{p}} \cap V_{\nabla,\mathrm{Uh};\mathfrak{p}} = 0$. To prove that $V_{\mathscr{G},\mathfrak{p}} \oplus V_{\nabla,\mathrm{Uh};\mathfrak{p}}$ is the entire vertical tangent space $T_{\mathfrak{p}}\mathbf{Dir}/\mathbf{Met}$, it suffices to show that every infinitesimal deformation $\dot{\gamma}$ of a Clifford multiplication γ is of the form $[\xi, \gamma]$ with $\xi \in \Gamma(X, \mathfrak{o}(S))$. This, however, is a consequence of the fact that Clifford algebras are separable finite-dimensional $\mathbb{R}$–algebras and, therefore, their representations are infinitesimally rigid; see, e.g., [Hoc46, Theorem 5] or [Wei94, Lemma 9.1.9 and Theorem 9.2.11]. In fact, the arguments found loc. cit. reveal that $\dot{\gamma} = [\xi, \gamma]$ holds for

$$\xi \coloneqq 2^{-d} \sum_{I \subseteq \{1,\dots,d\}} \xi_I \in \Gamma(X, \mathfrak{o}(S))$$

with

$$\xi_I \coloneqq \sum_{\ell=1}^{k}(-1)^{\ell}\gamma(e_{i_1})\cdots\gamma(e_{i_{\ell-1}})\dot{\gamma}(e_{i_\ell})\gamma(e_{i_\ell})\gamma(e_{i_{\ell-1}})\cdots\gamma(e_{i_1})$$

for $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, d\}$ with $i_1 < \dots < i_k$. Of course, this can also be verified by direct computation. ■

Remark 3.6. The $V_{\mathscr{G},\mathfrak{p}}$ and $V_{\nabla,\mathrm{Uh};\mathfrak{p}}$ need not form subbundles because there can be $\mathfrak{p} \in \mathbf{Dir}$ for which the map $\Gamma(X, \mathfrak{o}(S)) \to V_{\mathscr{G},\mathfrak{p}}$ has a non-trivial kernel. Of course, the $H_{\mathrm{BG},\mathfrak{p}}$ and $V_{\nabla,\mathfrak{p}}$ do form subbundles. ♣

Remark 3.7. Here is how to understand the Lie algebra bundle $\mathfrak{o}_{C\ell}(S)$. For every $x \in X$ and choice of orthogonal frame of T_xX, the fibre S_x is a representation of the Clifford algebra $C\ell_d$. The representation S_x decomposes into irreducible representations, the **pinor representations** [Har90, Definition 11.10], which can be read off from the classification [Har90, Theorem 11.3]:

(1) If $d = 0, 1, 2 \pmod 4$, then there is a unique pinor representation $\mathbf{P}$ and, therefore, $S_x = \mathbf{P}^m$ for some $m \in \mathbf{N}_0$. The commuting algebra $\mathbb{K} := \mathrm{End}_{C\ell}(\mathbf{P})$ of $\mathbf{P}$ is isomorphic to $\mathbb{R}$; $\mathbb{C}$; $\mathbb{H}$ depending on whether $d = 0, 6; 1, 5; 2, 4 \pmod 8$, respectively. Therefore,

$$\mathfrak{o}_{C\ell}(S_x) \cong \begin{cases} \mathfrak{o}(m) & \text{if } d = 0, 6 \pmod 8 \\ \mathfrak{u}(m) & \text{if } d = 1, 5 \pmod 8 \\ \mathfrak{sp}(m) & \text{if } d = 2, 4 \pmod 8. \end{cases}$$

(2) If $d = 3 \pmod 4$, then there are two pinor representations $\mathbf{P}^{\pm}$, distinguished by whether the volume element acts as $\pm\mathbf{1}$. Therefore, $S_x = (\mathbf{P}^+)^{m_+} \oplus (\mathbf{P}^-)^{m_-}$. The commuting algebra $\mathbb{K} := \mathrm{End}_{C\ell}(\mathbf{P}^{\pm})$ of $\mathbf{P}^{\pm}$ is isomorphic to $\mathbb{H}$, $\mathbb{R}$, depending on whether $d = 3, 7 \pmod 8$, respectively. Therefore,

$$\mathfrak{o}_{C\ell}(S_x) \cong \begin{cases} \mathfrak{sp}(m_+) \oplus \mathfrak{sp}(m_-) & \text{if } d = 3 \pmod 8 \\ \mathfrak{o}(m_+) \oplus \mathfrak{o}(m_-) & \text{if } d = 7 \pmod 8. \end{cases}$$

If X is spin and a spin structure $\mathfrak{s}$ on (X, g) is chosen, then the pinor representations give rise to pinor bundles $\mathbf{P}$ or $\mathbf{P}^+, \mathbf{P}^-$ over X and, consequently, $S = M \otimes_{\mathbb{K}} \mathbf{P}$ or $S = M^+ \otimes_{\mathbb{K}} \mathbf{P}^+ \oplus M^- \otimes_{\mathbb{K}} \mathbf{P}^-$ with M or M^+, M^- encoding the multiplicities. ♣

Remark 3.8. For every $(g; \gamma, \nabla) \in \mathbf{Dir}$ the curvature $F_\nabla \in \Omega^2(X, \mathfrak{o}(S))$ decomposes into **Riemannian spin curvature** $\mathrm{Riem}_g^S \in \Omega^2(X, \mathfrak{o}(S))$ and the **twisting curvature** $F_\nabla^{\mathrm{tw}} \in \Omega^2(X, \mathfrak{o}_{C\ell}(S))$ defined by

$$\begin{aligned} \mathrm{Riem}_g^S(u, v) &:= \frac{1}{8} \sum_{k,\ell=1}^{d} \langle \mathrm{Riem}_g(u, v) e_k, e_\ell \rangle [\gamma(e_k), \gamma(e_\ell)] \quad \text{and} \\ F_\nabla^{\mathrm{tw}} &:= F_\nabla - \mathrm{Riem}_g^S; \end{aligned}$$

see [BGV92, Proposition 3.43]. This decomposition is useful, because the contribution of Riem_g^S to various formulae often simplifies quite notably. ♣

3.2 The space of ramified Euclidean line bundles

Here is the other half of the base of the tame Fréchet space bundles $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$.

Definition 3.9 (cf. [DW24, §3.4.3]). The **moduli space of ramified Euclidean line bundles** over X is constructed as follows:

(1) Denote by $\mathbf{Sub}$ the *set* of closed codimension-two submanifolds $Z \subseteq X$. Equip $\mathbf{Sub}$ with the tame Fréchet manifold structure described in [Ham82, Part I Example 4.1.7 and Example 4.4.7, and Part II Corollary 2.3.7].

Given $Z \in \mathbf{Sub}$ and a tubular neighbourhood $\jmath\colon NZ \supseteq U \to X$ of Z, define the map $\phi_\jmath^{-1}\colon \Gamma(Z, NZ) \supseteq \Gamma(Z, U) \to \mathbf{Sub}$ by

$$\phi_\jmath^{-1}(v) := \mathrm{im}(\jmath \circ v).$$

Denote by $\phi_\jmath$ the inverse of $\phi_\jmath^{-1}\colon \Gamma(Z, U) \to \mathrm{im}\,\phi_\jmath^{-1}$. Equip $\Gamma(Z, NZ)$ with the usual structure of a tame Fréchet space; see [Ham82, Corollary II.1.3.9]. The charts constructed in [Ham82, Example I.4.1.7] are of the form $\phi_\jmath$.

(2) Denote by **Ram** the *set* of isometry classes of ramified Euclidean line bundles $(Z, \mathfrak{l})$ over X with smooth Z. Consider the map

$$\mathrm{Br}\colon \mathbf{Ram} \to \mathbf{Sub}$$

assigning to $(Z, [\mathfrak{l}])$ the branching locus Z.

(3) For every open subset $U \subseteq X$ denote by $\mathbf{Sub}_U \subseteq \mathbf{Sub}$ the open subset of those $Z \in \mathbf{Sub}$ for which:

   (a) $Z \subseteq U$, and
   (b) the restriction $\mathrm{H}^1(X\backslash Z, \{\pm1\}) \to \mathrm{H}^1(X\backslash U, \{\pm1\})$ is an isomorphism.

   The subset $\mathbf{Sub}_U$ is open and the monodromy representations assemble into an injection

$$\tau_U\colon \mathrm{Br}^{-1}(\mathbf{Sub}_U) \to \mathbf{Sub}_U \times \mathrm{H}^1(X\backslash U, \{\pm1\}).$$

   Equip **Ram** with the coarsest *topology* with respect to which the maps τ_U are continuous.

(4) With respect to the above topology $\mathrm{Br}\colon \mathbf{Ram} \to \mathbf{Sub}$ is a locally finite covering map; cf. [tDie08, proof of Theorem 3.3.2]. In particular, **Ram** inherits the structure of a tame Fréchet manifold from **Sub**. Moreover, for every $(Z, [\mathfrak{l}]) \in \mathbf{Ram}$,

$$T_{(Z,[\mathfrak{l}])}\mathbf{Ram} = T_Z\mathbf{Sub} = \Gamma(Z, NZ).$$

In the following $(Z, [\mathfrak{l}])$ is frequently abbreviated to $[\mathfrak{l}]$. This is harmless, because Z can be recovered as $\mathrm{Br}(\mathfrak{l})$. •

Remark 3.10. Here are some further comments regarding (3). Euclidean line bundles $\mathfrak{l}$ over $X\backslash Z$ are classified by their Stiefel–Whitney class $w_1(\mathfrak{l}) \in \mathrm{Hom}(\pi_1(X\backslash Z), \{\pm1\}) = \mathrm{H}^1(X\backslash Z, \{\pm1\})$. The latter cohomology group fits into the exact sequence

$$\mathrm{H}^1(X; \{\pm1\}) \xhookrightarrow{i^*} \mathrm{H}^1(X\backslash Z; \{\pm1\}) \xrightarrow{\delta} \mathrm{H}^2(X, X\backslash Z; \{\pm1\}) \xrightarrow{j^*} \mathrm{H}^2(X, \{\pm1\}).$$

Excision and the Thom isomorphism identify $\mathrm{H}^2(X, X\backslash Z; \{\pm1\}) \cong \mathrm{H}^0(Z, \{\pm1\})$. The monodromy condition means that $\delta(w_1(\mathfrak{l})) = \mathbf{-1} \in \mathrm{H}^0(Z, \{\pm1\})$. In particular, its image under the final map vanishes. But this is exactly the Poincaré dual of $[Z] \in \mathrm{H}_{d-2}(X, \{\pm1\})$. Therefore, the latter must vanish for Z to arise as the branching locus; moreover, if it vanishes, then the space of isometry classes of ramified Euclidean line bundles with branching locus Z is an $\mathrm{H}^1(X, \{\pm1\})$–torsor. ♣

Proposition 3.11 (cf. [DW24, Proposition 3.60]). **Ram** *carries a* ***twisted universal ramified Euclidean line bundle*** *in the following sense:*

(1) *Consider the* ***universal non-branching locus***

$$\mathring{\mathbf{X}} \coloneqq \{([\mathfrak{l}], x) \in \mathbf{Ram} \times X : x \notin \mathrm{Br}(\mathfrak{l})\} \subseteq \mathbf{Ram} \times X.$$

The projection map $p\colon \mathring{\mathbf{X}} \to \mathbf{Ram}$ *is a fibre bundle.*

(2) *Denote by* $\mathfrak{U}$ *the set of those open subsets* $\mathscr{U} \subseteq \mathbf{Ram}$ *for which* $p\colon \mathring{\mathbf{X}}|_{\mathscr{U}} \to \mathscr{U}$ *is trivial. For every* $\mathscr{U} \in \mathfrak{U}$ *there is a Euclidean line bundle* $\mathfrak{L}_{\mathscr{U}}$ *over* $\mathring{\mathbf{X}}|_{\mathscr{U}}$ *and for every* $[\mathfrak{l}] \in \mathscr{U}$ *an isomorphism*

$$\mathfrak{L}_{\mathscr{U}}|_{[\mathfrak{l}]\times(X\backslash\mathrm{Br}(\mathfrak{l}))} \cong \mathfrak{l}$$

unique up to $\mathrm{Aut}(\mathfrak{l}) = \{\pm 1\}$.

(3) *There is a Čech* 2*–cocycle* $\lambda \in \check{\mathrm{C}}^2(\mathfrak{U}, \underline{\{\pm 1\}})$ *together with, for every* $\mathscr{U}, \mathscr{V} \in \mathfrak{U}$*, an isomorphism* $\phi_{\mathscr{V}}^{\mathscr{U}}\colon \mathfrak{L}_{\mathscr{U}}|_{\mathring{\mathbf{X}}|_{\mathscr{U}\cap\mathscr{V}}} \cong \mathfrak{L}_{\mathscr{V}}|_{\mathring{\mathbf{X}}|_{\mathscr{U}\cap\mathscr{V}}}$ *such that for every* $\mathscr{U}, \mathscr{V}, \mathscr{W} \in \mathfrak{U}$

$$p^*\lambda_{\mathscr{U}\mathscr{V}\mathscr{W}} \coloneqq \phi_{\mathscr{U}}^{\mathscr{W}}\phi_{\mathscr{W}}^{\mathscr{V}}\phi_{\mathscr{V}}^{\mathscr{U}} \in \check{\mathrm{C}}^0(\mathring{\mathbf{X}}|_{\mathscr{U}\cap\mathscr{V}\cap\mathscr{W}}, \{\pm 1\}).$$

Here $\underline{\{\pm 1\}}$ *is the constant sheaf associated with the abelian group* $\{\pm 1\}$.

Proof. The proof is identical to that of [DW24, Proposition 3.60] and, therefore, omitted. ■

The Čech cohomology class $[\lambda] \in \check{\mathrm{H}}^2(\mathbf{Ram}, \underline{\{\pm 1\}})$ is the obstruction to gluing $\{\mathfrak{L}_{\mathscr{U}} : \mathscr{U} \in \mathfrak{U}\}$ to an untwisted universal ramified Euclidean line bundle $\mathfrak{L}$ over $\mathring{\mathbf{X}}$. Unfortunately, it does not always vanish. A pragmatic solution to this problem is to instead work with the Euclidean line bundle

$$\mathfrak{L} \coloneqq \coprod_{\mathscr{U}\in\mathfrak{U}} \mathfrak{L}_{\mathscr{U}} \quad \text{and} \quad \mathbf{Ram}^0 \coloneqq \coprod_{\mathscr{U}\in\mathfrak{U}} \mathscr{U}.$$

The significance of $\mathfrak{L}$ is that for every $(\mathscr{U}; [\mathfrak{l}]) \in \mathbf{Ram}^0$ it selects a choice of representative of $[\mathfrak{l}]$. Henceforth, this choice shall be denoted by $\mathfrak{l}$, abusing but simplifying notation. Moreover, in the upcoming discussion, $(\mathscr{U}; [\mathfrak{l}])$ is abbreviated to $[\mathfrak{l}]$ and $\mathbf{Ram}^0$ to $\mathbf{Ram}$. This abuse of notation is inconsequential, because ultimately $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors are considered up to the action of $\mathbb{R}^\times$ and $\{\pm 1\} \leqslant \mathbb{R}^\times$: forming the quotient by $\{\pm 1\}$ removes the ambiguity.

Remark 3.12. The origin of the above issue is the stabiliser $\mathrm{Aut}(\mathfrak{l}) = \{\pm 1\}$ and, in principle, one ought to work with orbifolds to deal with it. The Čech 2–cocycle λ produces a proper étale tame Fréchet Lie groupoid which presents a tame Fréchet orbifold $\underline{\mathbf{Ram}}$: the **fine moduli space of ramified Euclidean line bundles**; cf. [Moe02, §3]. The corresponding orbifold counterpart $\underline{\mathring{\mathbf{X}}}$ of $\mathring{\mathbf{X}}$ does admit a **universal ramified Euclidean line bundle** $\underline{\mathfrak{L}}$; cf. [Moe02, §5]. ♣

3.3 The extended gauge group and the Kosmann lift

The following symmetries of $\mathbf{Dir} \times \mathbf{Ram}$ are crucial for the upcoming discussion.

Definition 3.13. The **extended gauge group** of S is the tame Fréchet Lie group $\widehat{\mathscr{G}}(S)$ constructed as follows:

(1) An **extended isometry** of S is a diffeomorphism $u \in \mathrm{Diff}(S)$ for which there exists a diffeomorphism $\check{u} \in \mathrm{Diff}(X)$ such that u defines an isometry $S \cong \check{u}^* S$. These form a subgroup $\widehat{\mathscr{G}}(S) \leqslant \mathrm{Diff}(S)$.

(2) A moment's thought shows that extended isometries lift to $\mathrm{O}(r)$–equivariant diffeomorphisms of $\mathrm{Fr}(S)$, the orthogonal frame bundle of S; in fact, $\widehat{\mathscr{G}}(S) = \mathrm{Diff}(\mathrm{Fr}(S))^{\mathrm{O}(r)}$, abusing notation. By [Mol10, Proposition 2.6], $\widehat{\mathscr{G}}(S) \leqslant \mathrm{Diff}(\mathrm{Fr}(S))$ is a tame Fréchet Lie subgroup.

(3) From the above perspective, $\widehat{\mathscr{G}}(S)$ acts smoothly and tamely on $S = \mathrm{Fr}(S) \times_{\mathrm{O}(r)} \mathbb{R}^r$ and $X = \mathrm{Fr}(S) \times_{\mathrm{O}(r)} \{*\}$ from the left by

$$u \cdot [p, \phi] \coloneqq [u(p), \phi] \quad \text{and} \quad u \cdot [p, *] \coloneqq [u(p), *],$$

respectively. As a consequence, it acts smoothly and tamely on $\Gamma(X, S)$, $\mathbf{Dir}$, and $\mathbf{Ram}$ from the right by pullback.[1] •

The subset $\mathrm{Diff}_S(X) \subseteq \mathrm{Diff}(X)$ of those diffeomorphisms that lift to $\widehat{\mathscr{G}}(S)$ forms an open subgroup. In fact, $\widehat{\mathscr{G}}(S)$ fits into the exact sequence of tame Fréchet Lie groups

$$\mathscr{G}(S) \hookrightarrow \widehat{\mathscr{G}}(S) \twoheadrightarrow \mathrm{Diff}_S(X).$$

This induces a corresponding exact sequence of tame Fréchet Lie algebras

$$\Gamma(X, \mathfrak{o}(S)) \hookrightarrow \mathrm{Lie}(\widehat{\mathscr{G}}(S)) \coloneqq T_1\widehat{\mathscr{G}}(S) \twoheadrightarrow \mathrm{Vect}(X).$$

As a sequence of tame Fréchet *spaces*, this splits, but not canonically so. For example, for every orthogonal connection ∇ on S the **horizontal lift**

$$\mathrm{lift}_\nabla \colon \mathrm{Vect}(X) \to \mathrm{Lie}(\widehat{\mathscr{G}}(S)) \subseteq \mathrm{Vect}(S)$$

defines a right splitting. The right Lie algebra action of $\mathrm{lift}_\nabla(v)$ on $\mathbf{p} = (g; \gamma, \nabla) \in \mathbf{Dir}$ yields $(\mathscr{L}_v g; \dot{\gamma}, \dot{\nabla}) \in T_{\mathbf{p}}\mathbf{Dir}$ with $\dot{\gamma}(w) \coloneqq \gamma(\nabla_w v)$. For the purposes of this article it is convenient to use the following lift that fits better with the Bourguignon–Gauduchon connection defined in Proposition 3.2 (1).

Definition 3.14 (cf. [Kos72, Definition III.2.1; BG92, Proposition 17]). For every $\mathbf{p} = (g; \gamma, \nabla) \in \mathbf{Dir}$ the **Kosmann lift** is the tame linear map $\mathrm{lift}^K_{\mathbf{p}} \colon \mathrm{Vect}(X) \to \mathrm{Lie}(\widehat{\mathscr{G}}(S))$ defined by

$$\mathrm{lift}^K_{\mathbf{p}}(v) \coloneqq \mathrm{lift}_\nabla(v) + \kappa_v \quad \text{with}$$

$$\kappa_v \coloneqq -\frac{1}{8}\sum_{i,j=1}^{d} g(\nabla_{e_i} v, e_j)[\gamma(e_i), \gamma(e_j)] \in \Gamma(X, \mathfrak{o}(S)). \qquad \bullet$$

[1] The reader should be warned that $\widehat{\mathscr{G}}(S)$ does not lift to an action on $\mathbf{Ram}^0$; however, the action lifts locally. This is sufficient for Proposition 3.22.

Proposition 3.15 (cf. [Kos72, §V.1.2]). *Let* $\mathbf{p} \in \mathbf{Dir}$ *and* $v \in \mathrm{Vect}(X)$.

(1) *The right Lie algebra action of* $\mathrm{lift}_{\mathbf{p}}^K(v) \in \mathrm{Lie}(\widehat{\mathscr{G}}(S))$ *on* $\phi \in \Gamma(X, S)$ *is given by the* ***Kosmann Lie derivative****; that is:*

$$\phi \cdot \mathrm{lift}_{\mathbf{p}}^K(v) = \mathscr{L}_v^K \phi \quad \text{with} \quad \mathscr{L}_v^K := \nabla_v + \kappa_v.$$

(2) *The right Lie algebra action of* $\mathrm{lift}_{\mathbf{p}}^K(v) \in \mathrm{Lie}(\widehat{\mathscr{G}}(S))$ *on* $\mathbf{p} \in \mathbf{Dir}$ *is related to the Bourguignon–Gauduchon lift* $(\dot{g}; \dot{\gamma}^{\mathrm{BG}}, \dot{\nabla}^{\mathrm{BG}}) \in T_{\mathbf{p}}\mathbf{Dir}$ *of* $\dot{g} := \mathscr{L}_v g \in T_g\mathbf{Met}$*, defined in Proposition 3.2 (1), by*

$$\mathbf{p} \cdot \mathrm{lift}_{\mathbf{p}}^K(v) = \big(\dot{g}; \dot{\gamma}^{\mathrm{BG}}, \dot{\nabla}^{\mathrm{BG}} + F_{\nabla}^{\mathrm{tw}}(v, -)\big). \tag{3.16}$$

Remark 3.17. The following should help to make the Kosmann lift more palatable:

(1) The requirement that the right Lie algebra action of the lift $\tilde{v} \in \mathrm{Lie}(\widehat{\mathscr{G}}(S))$ of $v \in \mathrm{Vect}(X)$ reproduces $\dot{\gamma}^{\mathrm{BG}}$ determines $\tilde{v}$ up to the addition of an arbitrary $\xi_v \in \Gamma(X, \mathfrak{o}_{C\ell}(S))$.

(2) The requirement (3.16) determines $\mathrm{lift}_{\mathbf{p}}^K(v)$ uniquely up to the addition of an arbitrary $\xi_v \in \ker\big(\nabla \colon \Gamma(X, \mathfrak{o}_{C\ell}(S)) \to \Omega^1(X, \mathfrak{o}_{C\ell}(S))\big)$. The latter vanishes for typical $\mathbf{p} \in \mathbf{Dir}$. Therefore, (3.16) and a suitable continuity condition determine the Kosmann lift uniquely.

(3) For generic $\mathbf{p} \in \mathbf{Dir}$ and $v \in \mathrm{Vect}(X)$, $F_{\nabla}^{\mathrm{tw}}(v, -) \notin \mathrm{im}\big(\nabla \colon \Gamma(X, \mathfrak{o}_{C\ell}(S)) \to \Omega^1(X, \mathfrak{o}_{C\ell}(S))\big)$. As a consequence, this term cannot be eliminated from (3.16).

(4) The reader should be warned that, despite its name, the Kosmann Lie derivative does not define a Lie algebra homomorphism $\mathrm{Vect}(X) \to \mathrm{Lie}(\widehat{\mathscr{G}}(S))$; see [BG92, Proposition 18]. It does not even agree with the Lie derivative on $\Omega^\bullet(X)$ for the Dirac bundle that gives rise to the Hodge–de Rham operator $\mathrm{d} + \mathrm{d}^*$. ♣

Proof of Proposition 3.15. (1) is obvious. From this (2) is derived by the following direct computation. Set

$$(\dot{g}; \dot{\gamma}, \dot{\nabla}) := \mathbf{p} \cdot \mathrm{lift}_{\mathbf{p}}^K(v).$$

Evidently, $\dot{g} = \mathscr{L}_v g$. Since

$$(\mathscr{L}_v g)(w, z) = g(\nabla_w v, z) + g(\nabla_z v, w), \tag{3.18}$$

the formulae in Proposition 3.2 (1) concretise to

$$\dot{\gamma}^{\mathrm{BG}}(w) = \frac{1}{2}\sum_{i=1}^{d}\big(g(\nabla_w v, e_i) + g(\nabla_{e_i} v, w)\big)\gamma(e_i) \quad \text{and}$$
$$\dot{\nabla}_w^{\mathrm{BG}} = \frac{1}{8}\sum_{i,j=1}^{d}\big(g(\nabla^2_{e_i,e_j} v, w) + g(\nabla^2_{e_i,w} v, e_j)\big)[\gamma(e_i), \gamma(e_j)].$$

Therefore, using (3.5),

$$\begin{aligned}\dot{\gamma}(w) &= [\mathscr{L}_v^K, \gamma(w)] - \gamma([v,w]) = [\kappa_v, \gamma(w)] + \gamma(\nabla_w v) \\ &= -\frac{1}{8}\sum_{i,j=1}^{d} g(\nabla_{e_i} v, e_j)[[\gamma(e_i), \gamma(e_j)], \gamma(w)] + \gamma(\nabla_w v) \\ &= \frac{1}{2}\sum_{i=1}^{d}\big(g(\nabla_w v, e_i) + g(\nabla_{e_i} v, w)\big)\gamma(e_i) = \dot{\gamma}^{\mathrm{BG}}(w).\end{aligned}$$

Moreover,

$$\begin{aligned}\dot{\nabla}_w &= [\mathscr{L}_v^K, \nabla_w] - \nabla_{[v,w]} = [\nabla_v, \nabla_w] - \nabla_{[v,w]} - [\nabla_w, \kappa_v] \\ &= \mathrm{Riem}_g^S(v,w) + F_\nabla^{\mathrm{tw}}(v,w) + \frac{1}{8}\sum_{i,j=1}^{d} g(\nabla^2_{w,e_i} v, e_j)[\gamma(e_i), \gamma(e_j)] \\ &= \dot{\nabla}_w^{\mathrm{BG}} + F_\nabla^{\mathrm{tw}}(v,w) + \mathrm{R}\end{aligned}$$

with R denoting the remainder

$$\mathrm{R} := \frac{1}{8}\sum_{i,j=1}^{d}\big(g(\mathrm{Riem}_g(v,w)e_i, e_j) + g(\nabla^2_{w,e_i} v, e_j) - g(\nabla^2_{e_i,e_j} v, w) - g(\nabla^2_{e_i,w} v, e_j)\big)[\gamma(e_i), \gamma(e_j)].$$

As a consequence of the algebraic Bianchi identity,

$$\begin{aligned}\mathrm{R} &= \frac{1}{16}\sum_{i,j=1}^{d}\big(g(\mathrm{Riem}_g(v,w)e_i, e_j) + 2g(\mathrm{Riem}_g(w,e_i)v, e_j)\big)[\gamma(e_i), \gamma(e_j)] \\ &= \frac{1}{16}\sum_{i,j=1}^{d}\big(g(\mathrm{Riem}_g(v,w)e_i, e_j) + g(\mathrm{Riem}_g(v,e_j)w, e_i) + g(\mathrm{Riem}_g(v,e_i)e_j, w)\big)[\gamma(e_i), \gamma(e_j)] \\ &= 0.\end{aligned}$$

■

3.4 The universal bundles of (admissible) singular spinors and residues

The following discussion constructs **E**, **F**, and **G** as well as **D** and **res** using the method introduced by Donaldson [Don21a, §4].

Definition 3.19. Here is the first stage of the construction:

(1) The **universal bundle of admissible singular spinors** is the *set*

$$\mathbf{E} := \coprod_{(\mathfrak{p},[\mathfrak{l}])\in\mathbf{Dir}\times\mathbf{Ram}} H_a^\infty\Gamma(X\backslash\mathrm{Br}(\mathfrak{l}), S\otimes\mathfrak{l};\mathfrak{p})$$

together with the projection map $\mathbf{E}\to\mathbf{Dir}\times\mathbf{Ram}$ and the tame Fréchet space structures on the fibres, defined in Section 2.3 and [BW26, Definition 4.2 (b)]. Here and throughout the remainder of this article, additional decorations with $\mathfrak{p}$ and $\mathfrak{l}$ are used to indicate the dependence on these parameters.

(2) The **universal bundle of singular spinors** is the *set*

$$\mathbf{F} := \coprod_{(\mathfrak{p},[\mathfrak{l}])\in\mathbf{Dir}\times\mathbf{Ram}} H^\infty_b\Gamma(X\backslash\mathrm{Br}(\mathfrak{l}), S\otimes\mathfrak{l};\mathfrak{p})$$

together with the projection map $\mathbf{F}\longrightarrow\mathbf{Dir}\times\mathbf{Ram}$ and the tame Fréchet space structures on the fibres, defined in Section 2.3 and [BW26, Definition 4.2 (a)].

(3) The **universal bundle of residues** is the *set*

$$\mathbf{G} := \coprod_{(\mathfrak{p},[\mathfrak{l}])\in\mathbf{Dir}\times\mathbf{Ram}} \Gamma(\mathrm{Br}(\mathfrak{l}), \check{S}_{\mathfrak{p}})$$

together with the projection map $\mathbf{G}\longrightarrow\mathbf{Dir}\times\mathbf{Ram}$ and the usual tame Fréchet space structures on the fibres.

(4) The **universal Dirac operator** $\mathbf{D}\colon\ \mathbf{E}\longrightarrow\mathbf{F}$ is defined by

$$\mathbf{D}(\mathfrak{p},[\mathfrak{l}];\phi) := (\mathfrak{p},[\mathfrak{l}];D^{\mathfrak{l}}_{\mathfrak{p}}\phi).$$

(5) The **universal residue map res**$\colon\ \mathbf{E}\longrightarrow\mathbf{G}$ is defined by

$$\mathbf{res}(\mathfrak{p},[\mathfrak{l}];\phi) := (\mathfrak{p},[\mathfrak{l}];\mathrm{res}\,\phi).$$

It remains to endow $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ with the structure of tame Fréchet space bundles such that $\mathbf{D}$ and $\mathbf{res}$ become maps of tame Fréchet space bundles. •

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "\mathbf{D}"] \arrow[d, "\mathbf{res}"'] & \mathbf{F} \\
\mathbf{G} &
\end{tikzcd}$$

Figure 1: The universal Dirac operator and residue map.

The above task is not entirely trivial. Even if $(Z_0,[\mathfrak{l}_0])\in\mathbf{Ram}$ is held fixed and $\mathfrak{p}_0,\mathfrak{p}\in\mathbf{Dir}$ are close, $H^\infty_a(X\backslash Z_0, S\otimes\mathfrak{l}_0;\mathfrak{p}_0)$ and $H^\infty_a(X\backslash Z_0, S\otimes\mathfrak{l}_0;\mathfrak{p})$ need not agree, even as subspaces of $\Gamma(X\backslash Z_0, S\otimes\mathfrak{l}_0)$, unless a much stronger relation between $\mathfrak{p}_0$ and $\mathfrak{p}$ is assumed.

Definition 3.20. A set of **normal data** for a codimension two submanifold $Z_0\subseteq X$ consists of:

(N1) a rank 2 subbundle $N\subseteq TX|_{Z_0}$ complementary to TZ_0,

(N2) a Euclidean metric h on N, and

(N3) an $I\in\Gamma(Z_0,\mathrm{Hom}(\Lambda^2N,\mathrm{End}(S|_{Z_0})))$ such that $I(\mathrm{or})\in\mathrm{End}(S_x)$ is an orthogonal complex structure for every $x\in Z_0$ and $\mathrm{or}\in\Lambda^2N_x$ with $|\mathrm{or}|=1$.

Every $\mathbf{p} = (g; \gamma, \nabla) \in \mathbf{Dir}$ defines normal data for a codimension two submanifold $Z \subseteq X$ by declaring

$$N \coloneqq TZ^{\perp}, \quad h \coloneqq g|_N, \quad \text{and} \quad I \coloneqq \gamma|_{\Lambda^2 N}.$$

The **slice** associated with a ramified Euclidean line bundle $(Z_0, [\mathfrak{l}_0]) \in \mathbf{Ram}$ and normal data (N, h, I) for Z_0 is the tame Fréchet submanifold

$$\mathbf{Sl} \coloneqq \left\{ (\mathbf{p}, [\mathfrak{l}_0]) \in \mathbf{Dir} \times \mathbf{Ram} : \mathbf{p} \text{ induces the normal data } (N, h, I) \text{ for } Z_0 \right\} \subseteq \mathbf{Dir} \times \mathbf{Ram}.$$

These slices foliate $\mathbf{Dir} \times \mathbf{Ram}$ in the sense that every $(\mathbf{p}, [\mathfrak{l}]) \in \mathbf{Dir} \times \mathbf{Ram}$ is contained in some slice. •

Proposition 3.21. *Let* $\mathbf{Sl} \subseteq \mathbf{Dir} \times \mathbf{Ram}$ *be a slice and* $(\mathbf{p}_0, [\mathfrak{l}_0]) \in \mathbf{Sl}$.

(1) *For every* $(\mathbf{p}, [\mathfrak{l}]) \in \mathbf{Sl}$ *the identity maps induce tame isomorphisms*

$$\begin{aligned} H_a^\infty\Gamma(X\backslash \mathrm{Br}(\mathfrak{l}_0), S \otimes \mathfrak{l}_0; \mathbf{p}_0) &\cong H_a^\infty\Gamma(X\backslash \mathrm{Br}(\mathfrak{l}), S \otimes \mathfrak{l}; \mathbf{p}) = \mathbf{E}_{\mathbf{p},\mathfrak{l}}, \\ H_b^\infty\Gamma(X\backslash \mathrm{Br}(\mathfrak{l}_0), S \otimes \mathfrak{l}_0; \mathbf{p}_0) &\cong H_b^\infty\Gamma(X\backslash \mathrm{Br}(\mathfrak{l}), S \otimes \mathfrak{l}; \mathbf{p}) = \mathbf{F}_{\mathbf{p},\mathfrak{l}}, \quad \textit{and} \\ \Gamma(\mathrm{Br}(\mathfrak{l}_0), \check{S}_{\mathbf{p}_0}) &\cong \Gamma(\mathrm{Br}(\mathfrak{l}), \check{S}_{\mathbf{p}}) = \mathbf{G}_{\mathbf{p},\mathfrak{l}}. \end{aligned}$$

(2) *The maps*

$$\begin{aligned} \mathbf{D}|_{\mathbf{Sl}} \colon \mathbf{Sl} \times H_a^\infty\Gamma(X\backslash \mathrm{Br}(\mathfrak{l}_0), S \otimes \mathfrak{l}_0; \mathbf{p}_0) &\to H_b^\infty\Gamma(X\backslash \mathrm{Br}(\mathfrak{l}_0), S \otimes \mathfrak{l}_0; \mathbf{p}_0) \quad \textit{and} \\ \mathbf{res}|_{\mathbf{Sl}} \colon \mathbf{Sl} \times H_a^\infty\Gamma(X\backslash \mathrm{Br}(\mathfrak{l}_0), S \otimes \mathfrak{l}_0; \mathbf{p}_0) &\to \Gamma(\mathrm{Br}(\mathfrak{l}_0), \check{S}_{\mathbf{p}_0}) \end{aligned}$$

induced by $\mathbf{D}$ *and* $\mathbf{res}$ *are tame smooth.*

Proof. By [BW26, Proposition 3.14], $D_0 \coloneqq D_{\mathbf{p}_0}^{\mathfrak{l}_0}$ and $D_{\mathbf{p}}^{\mathfrak{l}_0}$ differ from (cut-offs of) the corresponding model operators $\mathring{D}_{\mathbf{p}_0}^{\mathfrak{l}_0}$ and $\mathring{D}_{\mathbf{p}}^{\mathfrak{l}_0}$ [BW26, Definition 3.12] by operators in $\mathrm{DiffOp}_b^1(S \otimes \mathfrak{l}_0)$. Direct inspection of the model operators and a glance at [BW26, Proof of Lemma 4.29] show that

$$D_{\mathbf{p}}^{\mathfrak{l}_0} = D_{\mathbf{p}_0}^{\mathfrak{l}_0} + B_{\mathbf{p},\mathbf{p}_0}$$

with $B_{\mathbf{p},\mathbf{p}_0} \in \mathrm{DiffOp}_b^1(S \otimes \mathfrak{l}_0)$. This immediately implies (1). Inspection of $B_{\mathbf{p},\mathbf{p}_0}$ establishes (2) as in [Ham82, Part II Theorem 3.3.3]. ∎

This provides the desired tame Fréchet bundle structures for the restrictions $\mathbf{E}|_{\mathbf{Sl}}$, $\mathbf{F}|_{\mathbf{Sl}}$, and $\mathbf{G}|_{\mathbf{Sl}}$ to slices: in fact, it canonically trivialises them. These tame Fréchet bundle structures can be extended with the help of the following method due to Donaldson [Don21a].

Proposition 3.22 (cf. [Don21a, Proposition 4.2]). *Let* $\mathbf{Sl}$ *be a slice. For every* $(\mathbf{p}_0, [\mathfrak{l}_0]) \in \mathbf{Sl}$ *there is a* ***local uniformiser****; that is: an open neighbourhood* $\mathscr{U} \subseteq \mathbf{Dir} \times \mathbf{Ram}$ *of* $(\mathbf{p}_0, [\mathfrak{l}_0])$ *and a tame smooth map*

$$u \colon \mathscr{U} \to \widehat{\mathscr{G}}(S)$$

such that for every $(\mathbf{p}, [\mathfrak{l}]) \in \mathscr{U}$

$$u(\mathbf{p}, [\mathfrak{l}])^*(\mathbf{p}, [\mathfrak{l}]) \in \mathbf{Sl}.$$

Moreover, given a compact subset $K \subseteq X\backslash \mathrm{Br}(\mathfrak{l}_0)$*, the local uniformiser can be chosen such that* $u(\mathbf{p}, [\mathfrak{l}])|_K = \mathbf{1}_S$ *for every* $(\mathbf{p}, [\mathfrak{l}]) \in \mathscr{U}$.

Proof. [Don21a, Proof of Proposition 4.2] explains the construction of a tame smooth map $\check{u}\colon \mathscr{U} \to \mathrm{Diff}_0(X)$ which admits a lift $u\colon \mathscr{U} \to \widehat{\mathscr{G}}(S)$ such that, for every $(\mathbf{p}, [\mathfrak{l}]) \in \mathscr{U}$, $u(\mathbf{p}, [\mathfrak{l}])|_K = \mathbf{1}_S$ and $u(\mathbf{p}, [\mathfrak{l}])^*\mathbf{p}$ defines the same normal data as $\mathbf{p}_0$—except possibly for the almost complex structure I, because in [Don21a] there is no analogue of (N3). After possibly shrinking $\mathscr{U}$, this defect can be repaired by observing the following:

(1) For every I as in (N3) that is sufficiently close to I_0, there is a $u_0 \in \mathscr{G}(S|_{Z_0})$ close to $\mathbf{1}_{S|_{Z_0}}$ with $u_0^*I = I_0$;

(2) Since u_0 is close to $\mathbf{1}_{S|_{Z_0}}$, it can be extended to $\overline{u}_0 \in \mathscr{G}(S)$.

Evidently, this repair can be carried out in a way that produces the desired tame smooth map $u\colon \mathscr{U} \to \widehat{\mathscr{G}}(S)$. ■

The following two observations are verified by direct inspection.

Proposition 3.23. *Let $(\mathbf{p}, [\mathfrak{l}]) \in \mathbf{Dir} \times \mathbf{Ram}$. Every extended gauge transformation $u \in \widehat{\mathscr{G}}(S)$ induces tame isomorphisms*

$$
\begin{aligned}
u\colon H^\infty_a\Gamma(X\backslash\mathrm{Br}(\mathfrak{l}), S\otimes\mathfrak{l};\mathbf{p}) &\to H^\infty_a\Gamma(X\backslash\mathrm{Br}(\check{u}^*\mathfrak{l}), S\otimes\check{u}^*\mathfrak{l};u^*\mathbf{p}),\\
u\colon H^\infty_b\Gamma(X\backslash\mathrm{Br}(\mathfrak{l}), S\otimes\mathfrak{l};\mathbf{p}) &\to H^\infty_b\Gamma(X\backslash\mathrm{Br}(\check{u}^*\mathfrak{l}), S\otimes\check{u}^*\mathfrak{l};u^*\mathbf{p}), \quad \textit{and}\\
u\colon \Gamma(\mathrm{Br}(\mathfrak{l}), \check{S}_{\mathbf{p}}) &\to \Gamma(\mathrm{Br}(\check{u}^*\mathfrak{l}), \check{S}_{u^*\mathbf{p}}).
\end{aligned}
$$

In particular, these lift the action of $\widehat{\mathscr{G}}(S)$ on the right of **Dir** × **Ram** *to* **E**, **F**, *and* **G**. ■

Proposition 3.24. *Let* **Sl** *be a slice and $(\mathbf{p}_0, [\mathfrak{l}_0]) \in \mathbf{Sl}$.*

(1) *The subgroup*

$$\mathrm{Stab}_{\widehat{\mathscr{G}}(S)}(\mathbf{Sl}) := \left\{u \in \widehat{\mathscr{G}}(S) : u^*\mathbf{Sl} = \mathbf{Sl}\right\} \subset \widehat{\mathscr{G}}(S)$$

is a tame Fréchet Lie subgroup.

(2) *The right actions of $\mathrm{Stab}_{\widehat{\mathscr{G}}(S)}(\mathbf{Sl})$ on $H^\infty_a(X\backslash Z_0, S\otimes\mathfrak{l}_0;\mathbf{p}_0)$, $H^\infty_b(X\backslash Z_0, S\otimes\mathfrak{l}_0;\mathbf{p}_0)$, and $\Gamma(Z_0, \check{S}_{\mathbf{p}_0})$ are tame smooth.* ■

Definition 3.25. The following completes the construction begun in Definition 3.19. Endow **E**, **F**, and **G** with the unique tame smooth Fréchet bundle structure in which for every $(\mathbf{p}_0, [\mathfrak{l}_0]) \in$ **Dir** × **Ram** and for every local uniformiser $u\colon \mathscr{U} \to \widehat{\mathscr{G}}(S)$ as in Proposition 3.22 the bijections $\sigma\colon \mathbf{E}|_{\mathscr{U}} \to \mathscr{U}\times H^\infty_a(X\backslash\mathrm{Br}(\mathfrak{l}_0), S\otimes\mathfrak{l}_0;\mathbf{p}_0)$, $\tau\colon \mathbf{F}|_{\mathscr{U}} \to \mathscr{U}\times H^\infty_b(X\backslash\mathrm{Br}(\mathfrak{l}_0), S\otimes\mathfrak{l}_0;\mathbf{p}_0)$, and $\upsilon\colon \mathbf{G}|_{\mathscr{U}} \to \mathscr{U}\times\Gamma(\mathrm{Br}(\mathfrak{l}_0), \check{S}_{\mathbf{p}_0})$ defined by

$$\sigma(\mathbf{p}, [\mathfrak{l}];\phi) := (\mathbf{p}, [\mathfrak{l}]; u(\mathbf{p}, [\mathfrak{l}])\phi), \quad \tau(\mathbf{p}, [\mathfrak{l}];\phi) := (\mathbf{p}, [\mathfrak{l}]; u(\mathbf{p}, [\mathfrak{l}])\phi), \quad \text{and}$$
$$\upsilon(\mathbf{p}, [\mathfrak{l}];\rho) := (\mathbf{p}, [\mathfrak{l}]; u(\mathbf{p}, [\mathfrak{l}])\rho)$$

are local trivialisations. By Proposition 3.23, these are compatible with the tame Fréchet space structures on the fibres. By Proposition 3.24, they form an atlas. •

Given the above construction, the following is an immediate consequence of Proposition 3.21.

Corollary 3.26. *The universal Dirac operator* $\mathbf{D}\colon \mathbf{E} \to \mathbf{F}$ *and the universal residue map* $\mathbf{res}\colon \mathbf{E} \to \mathbf{G}$ *are maps of tame Fréchet bundles.* ■

Remark 3.27. Here are some observations regarding the above discussion:

(1) The construction of the tame Fréchet space bundle structures is quite robust, insofar as it depends only on Proposition 3.21, Proposition 3.23, and Proposition 3.24. In particular, the method shall be used later in this article to construct certain tame Fréchet subbundles of $\mathbf{E}$.

(2) The construction of $\mathbf{F}$ could have been carried out with a less delicate definition of slice, because $H_b^\infty\Gamma(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p})$ does not depend on the normal structure.

(3) For $k \in \mathbf{N}_0$ the right actions of $\mathrm{Stab}_{\widehat{\mathscr{G}}(S)}(\mathbf{Sl})$ on the Hilbert spaces $H_a^k(X\backslash Z_0, S \otimes \mathfrak{l}_0; \mathbf{p}_0)$, $H_b^k(X\backslash Z_0, S \otimes \mathfrak{l}_0; \mathbf{p}_0)$, and $H^k(Z_0, \check{S}_{\mathbf{p}_0})$ fail to be smooth (for the same reason that the translation action $\mathbb{R} \circlearrowright H^1(\mathbb{R})$ fails to be smooth). Therefore, $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ are not (in an obvious way) inverse limits of Hilbert space bundles—at least not in the usual sense, but see [TW21, §4.4].

(4) By construction, $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ are $\widehat{\mathscr{G}}(S)$–equivariant tame Fréchet space bundles over $\mathbf{Dir} \times \mathbf{Ram}$, and $\mathbf{D}$ and $\mathbf{res}$ are $\widehat{\mathscr{G}}(S)$–equivariant (up to the caveat pointed out in (*a*)). ♣

3.5 Covariant derivatives of the universal Dirac operator

This section explains how to differentiate the universal Dirac operator $\mathbf{D}$. Of course, this requires the choice of (partial) connections.

Definition 3.28. The **slice distribution** $\mathscr{F}^{\mathbf{Sl}} \subseteq T(\mathbf{Dir} \times \mathbf{Ram})$ is the distribution induced by the slices foliating $\mathbf{Dir} \times \mathbf{Ram}$; that is: for every $(\mathbf{p}, [\mathfrak{l}]) \in \mathbf{Dir} \times \mathbf{Ram}$

$$\mathscr{F}^{\mathbf{Sl}}_{\mathbf{p},[\mathfrak{l}]} := T_{\mathbf{p},[\mathfrak{l}]}\mathbf{Sl}_{\mathbf{p},[\mathfrak{l}]}$$

with $\mathbf{Sl}_{\mathbf{p},[\mathfrak{l}]}$ denoting the unique slice through $(\mathbf{p}, [\mathfrak{l}])$. The canonical trivialisations of $\mathbf{E}|_{\mathbf{Sl}}$, $\mathbf{F}|_{\mathbf{Sl}}$, and $\mathbf{G}|_{\mathbf{Sl}}$ defined by Proposition 3.21 define the **slice partial covariant derivatives** $\nabla^{\mathbf{Sl}}$ with respect to $\mathscr{F}^{\mathbf{Sl}}$ on $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$. •

To compute $\nabla^{\mathbf{Sl}}\mathbf{D}$, it is helpful to decompose $\mathscr{F}^{\mathbf{Sl}}$ according to Proposition 3.2.

Proposition 3.29. *For every* $(Z_0, [\mathfrak{l}_0]) \in \mathbf{Ram}$*, normal data* (N, h, I) *for* Z_0*, and* $(\mathbf{p}, [\mathfrak{l}]) \in \mathbf{Sl}$ *the tangent space* $T_{\mathbf{p},[\mathfrak{l}]}\mathbf{Sl}$ *decomposes into three tame Fréchet subspaces*

$$T_{\mathbf{p},[\mathfrak{l}]}\mathbf{Sl} = H^{\mathbf{Sl}}_{\mathrm{BG};\mathbf{p},[\mathfrak{l}]} \oplus V^{\mathbf{Sl}}_{\mathscr{G};\mathbf{p},[\mathfrak{l}]} \oplus V^{\mathbf{Sl}}_{\nabla,\mathrm{Uh};\mathbf{p},[\mathfrak{l}]}$$

defined by

$$\begin{aligned}
H^{\mathbf{Sl}}_{\mathrm{BG};\mathbf{p},[\mathfrak{l}]} &:= \left\{(\dot{g}; \dot{\gamma}^{\mathrm{BG}}, \dot{\nabla}^{\mathrm{BG}}; 0) \in H_{\mathrm{BG};\mathbf{p},[\mathfrak{l}]} : \dot{g}|_Z \in \Gamma(Z, \mathrm{Hom}(S^2TZ, \mathbb{R}))\right\}, \\
V^{\mathbf{Sl}}_{\mathscr{G};\mathbf{p},[\mathfrak{l}]} &:= \left\{(0; [\xi, \gamma], -\nabla\xi; 0) \in V_{\mathscr{G};\mathbf{p},[\mathfrak{l}]} : [\xi|_Z, \gamma|_{\Lambda^2 N}] = 0\right\}, \quad \textit{and} \\
V^{\mathbf{Sl}}_{\nabla,\mathrm{Uh};\mathbf{p},[\mathfrak{l}]} &:= V_{\nabla,\mathrm{Uh};\mathbf{p},[\mathfrak{l}]};
\end{aligned}$$

in particular, $V_{\nabla;\mathbf{p}} \oplus 0 \subseteq T_{\mathbf{p},[\mathfrak{l}]}\mathbf{Sl}$.

Proof. Direct inspection reveals that $(\dot g; \dot\gamma, \dot\nabla; 0) \in T_{\mathbf{p}}\mathbf{Dir} \oplus T_{[\mathfrak{l}]}\mathbf{Ram}$ lies in $\mathscr{F}^{\mathrm{Sl}}$ if and only if:

(1) $\dot g|_Z \in \Gamma(Z, \mathrm{Hom}(S^2TZ, \mathbb{R}))$ and

(2) for every $x \in Z_0$ and every orthonormal basis (e_1, e_2) of N_x

$$\dot\gamma(e_1)\gamma(e_2) + \gamma(e_1)\dot\gamma(e_2) = 0.$$

This together with Proposition 3.2 implies the assertion. ∎

Proposition 3.30. *For every* $(\mathbf{p}, [\mathfrak{l}]) \in \mathbf{Dir} \times \mathbf{Ram}$ *the following hold:*

(1) *For every* $\dot{\mathbf{p}}_{\mathrm{BG}} = (\dot g; \dot\gamma^{\mathrm{BG}}, \dot\nabla^{\mathrm{BG}}; 0) \in H^{\mathrm{Sl}}_{\mathrm{BG};\mathbf{p},[\mathfrak{l}]}$

$$\nabla^{\mathrm{Sl}}_{\dot{\mathbf{p}}_{\mathrm{BG}}}\mathbf{D}(\mathbf{p}, \mathfrak{l}; \phi) = -\frac{1}{2}\sum_{i,j=1}^{d} \dot g(e_i, e_j)\gamma(e_i)\nabla_{e_j}\phi + \frac{1}{4}\gamma\big((\nabla^*\dot g)^\sharp + \nabla(\mathrm{tr}_g\,\dot g)\big)\phi.$$

(2) *For every* $\dot{\mathbf{p}}_{\mathscr{G}} = (0; [\xi, \gamma], -\nabla\xi; 0) \in V^{\mathrm{Sl}}_{\mathscr{G};\mathbf{p},[\mathfrak{l}]}$

$$\nabla^{\mathrm{Sl}}_{\dot{\mathbf{p}}_{\mathscr{G}}}\mathbf{D}(\mathbf{p}, \mathfrak{l}; \phi) = \sum_{i=1}^{d}\big([\xi, \gamma(e_i)]\nabla_{e_i}\phi - \gamma(e_i)(\nabla_{e_i}\xi)\phi\big).$$

(3) *For every* $\dot{\mathbf{p}}_{\nabla} = (0; 0, \dot\nabla; 0) \in V^{\mathrm{Sl}}_{\nabla;\mathbf{p},[\mathfrak{l}]}$

$$\nabla^{\mathrm{Sl}}_{\dot{\mathbf{p}}_{\nabla}}\mathbf{D}(\mathbf{p}, \mathfrak{l}; \phi) = \sum_{i=1}^{d}\gamma(e_i)\dot\nabla_{e_i}\phi.$$

Moreover, $\nabla^{\mathrm{Sl}}\mathbf{res} = 0$.

Proof. The variation $\dot\sharp$ of the isomorphism $\sharp\colon T^*X \to TX$ induced by $\dot g \in T_g\mathbf{Met}$ satisfies

$$\dot\sharp\alpha = -\sum_{i=1}^{d}\dot g(\alpha^\sharp, e_i)e_i.$$

Therefore, for every $(\mathbf{p}; [\mathfrak{l}]) \in \mathbf{Dir} \times \mathbf{Ram}$ and $\dot{\mathbf{p}} = (\dot g; \dot\gamma, \dot\nabla; 0) \in \mathscr{F}^{\mathrm{Sl}}_{\mathbf{p},[\mathfrak{l}]}$,

$$\nabla^{\mathrm{Sl}}_{\dot{\mathbf{p}}}\mathbf{D}(\mathbf{p}, \mathfrak{l}; \phi) = \sum_{i=1}^{d}\big(\dot\gamma(e_i)\nabla_{e_i}\phi + \gamma(e_i)\dot\nabla_{e_i}\phi\big) - \sum_{i,j=1}^{d}\dot g(e_i, e_j)\gamma(e_i)\nabla_{e_j}\phi. \tag{3.31}$$

For every $\dot{\mathbf{p}} = \dot{\mathbf{p}}_{\mathrm{BG}} = (\dot g; \dot\gamma^{\mathrm{BG}}, \dot\nabla^{\mathrm{BG}}; 0)$ as in Proposition 3.2 (1) this concretises to

$$\begin{aligned}\nabla^{\mathrm{Sl}}_{\dot{\mathbf{p}}_{\mathrm{BG}}}\mathbf{D}(\mathbf{p}, \mathfrak{l}; \phi) &= -\frac{1}{2}\sum_{i,j=1}^{d}\dot g(e_i, e_j)\gamma(e_i)\nabla_{e_j}\phi + \frac{1}{8}\sum_{i,j,k=1}^{d}(\nabla_{e_i}\dot g)(e_j, e_k)\gamma(e_k)[\gamma(e_i), \gamma(e_j)]\phi \\ &= -\frac{1}{2}\sum_{i,j=1}^{d}\dot g(e_i, e_j)\gamma(e_i)\nabla_{e_j}\phi + \frac{1}{4}\gamma\big((\nabla^*\dot g)^\sharp + \nabla(\mathrm{tr}_g\,\dot g)\big)\phi.\end{aligned}$$

The final step uses the observation

$$\gamma(e_k)[\gamma(e_i),\gamma(e_j)] + \gamma(e_j)[\gamma(e_i),\gamma(e_k)] = -2\delta_{ij}\gamma(e_k) + 4\delta_{jk}\gamma(e_i) - 2\delta_{ki}\gamma(e_j).$$

This proves (1). The remaining formulae (2) and (3) are obvious. ■

Remark 3.32. Direct inspection of the above formulae and a glance at Proposition 3.30 reveal that

$$\begin{aligned}&\nabla^{\mathrm{Sl}}_{\dot{\mathfrak{p}}_{\mathrm{BG}}}\mathbf{D}(\mathfrak{p},\mathfrak{l};-),\nabla^{\mathrm{Sl}}_{\dot{\mathfrak{p}}_{\nabla}}\mathbf{D}(\mathfrak{p},\mathfrak{l};-) \in \mathrm{DiffOp}^1_b(S\otimes\mathfrak{l}) \quad \text{and}\\ &\nabla^{\mathrm{Sl}}_{\dot{\mathfrak{p}}_{\mathscr{G}}}\mathbf{D}(\mathfrak{p},\mathfrak{l};-) \in \mathrm{DiffOp}^0_b(S\otimes\mathfrak{l})D^{\mathfrak{l}}_{\mathfrak{p}} + \mathrm{DiffOp}^1_b(S\otimes\mathfrak{l}).\end{aligned}$$ ♣

Remark 3.33. The differential operator $\nabla^{\mathrm{Sl}}_{\dot{\mathfrak{p}}_{\mathrm{BG}}}\mathbf{D}(\mathfrak{p},\mathfrak{l};-)$ is not formally self-adjoint. In fact, a brief computation shows that its formal adjoint satisfies

$$\nabla^{\mathrm{Sl}}_{\dot{\mathfrak{p}}_{\mathrm{BG}}}\mathbf{D}(\mathfrak{p},\mathfrak{l};-)^* = \nabla^{\mathrm{Sl}}_{\dot{\mathfrak{p}}_{\mathrm{BG}}}\mathbf{D}(\mathfrak{p},\mathfrak{l};-) - \frac{1}{2}\gamma(\nabla(\mathrm{tr}_g\,\dot{g})).$$

This can be rectified by using the **volume-renormalised slice partial covariant derivative**

$$\nabla^{\mathrm{Sl,vol}}_{\dot{\mathfrak{p}}} := \nabla^{\mathrm{Sl}}_{\dot{\mathfrak{p}}} + \tfrac{1}{4}\,\mathrm{tr}_g\,\dot{g}$$

instead of ∇^{Sl}; cf. [Mai97, §2.2; AD25, §2.5]. For the purposes of this article, the above is not an issue and it suffices to work with ∇^{Sl}. However, it would be one in setting up a Brill–Noether theory of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors; cf. [DW23, §1.A]. ♣

The right action of $\widehat{\mathscr{G}}(S)$ on $\mathbf{Dir}\times\mathbf{Ram}$ induces a right action

$$T(\mathbf{Dir}\times\mathbf{Ram}) \circlearrowleft T\widehat{\mathscr{G}}(S) = \widehat{\mathscr{G}}(S)\ltimes \mathrm{Lie}\,\widehat{\mathscr{G}}(S).$$

This can be used to construct (local) partial covariant derivatives with respect to

$$\mathscr{F}^{\mathbf{Ram}} := \mathrm{pr}_2^* T\mathbf{Ram} \subseteq T(\mathbf{Dir}\times\mathbf{Ram})$$

from ∇^{Sl} and the following infinitesimal version of Proposition 3.22.

Proposition 3.34. *For every $(\mathfrak{p}_0,[\mathfrak{l}_0]) \in \mathbf{Dir}\times\mathbf{Ram}$ there is a **local infinitesimal uniformiser**; that is: an open neighbourhood $(\mathfrak{p}_0,[\mathfrak{l}_0]) \in \mathscr{U} \subseteq \mathbf{Dir}\times\mathbf{Ram}$ and a tame smooth map*

$$\begin{aligned}\mathscr{F}^{\mathbf{Ram}}|_{\mathscr{U}} &\to \mathrm{Vect}(X)\\ (\mathfrak{p},[\mathfrak{l}];v) &\mapsto \bar{v} = \bar{v}(\mathfrak{p},[\mathfrak{l}];v)\end{aligned}$$

such that:

(1) *for every $(\mathfrak{p},[\mathfrak{l}];v) \in \mathscr{F}^{\mathbf{Ram}}|_{\mathscr{U}}$*

$$(\mathfrak{p},[\mathfrak{l}];v)\cdot \mathrm{lift}^K_{\mathfrak{p}}(\bar{v}) \in \mathscr{F}^{\mathrm{Sl}}_{\mathfrak{p},[\mathfrak{l}]},$$

(2) *$\bar{v}(\mathfrak{p},[\mathfrak{l}];-)$ is $\mathbb{R}$-linear, and*

(3) *for every* $(\mathbf{p},[\mathfrak{l}];v)\in\mathscr{F}^{\mathbf{Ram}}|_{\mathscr{U}}$

$$\|\bar{v}(\mathbf{p},[\mathfrak{l}];v)\|_{H^{k+1}}\lesssim_k\|v\|_{H^k}.$$

Moreover, given a compact subset $K\subseteq X\backslash\mathrm{Br}(\mathfrak{l}_0)$*, the local infinitesimal uniformiser can be chosen such that* $\bar{v}(\mathbf{p},[\mathfrak{l}];v)|_K=0$ *for every* $(\mathbf{p},[\mathfrak{l}];v)\in\mathscr{F}^{\mathbf{Ram}}|_{\mathscr{U}}$*.*

The proof of Proposition 3.34 (3) uses the following well-known observation.

Lemma 3.35. *Let* $d,k\in\mathbf{N}_0$*. There is a* 1*–**jet extension map***

$$\mathrm{ext}^{(1)}\colon\ \mathcal{S}(\mathbb{R}^d)\oplus\mathcal{S}(\mathbb{R}^d)^k\to\mathcal{S}(\mathbb{R}^d\oplus\mathbb{R}^k)$$

such that for every $(f,g_1,\dots,g_k)\in\mathcal{S}(\mathbb{R}^d)^{k+1}$ *its extension* $F:=\mathrm{ext}^{(1)}(f,g_1,\dots,g_k)\in\mathcal{S}(\mathbb{R}^d\oplus\mathbb{R}^k)$ *satisfies*

$$F(-,0)=f\quad\text{and}\quad\frac{\partial F}{\partial y_i}(-,0)=g_i,\tag{3.36}$$

and for every $s\in\mathbb{R}$

$$\|F\|_{H^s}\lesssim_s\|f\|_{H^{s-k/2}}+\sum_{i=1}^k\|g_i\|_{H^{s-k/2-1}}.\tag{3.37}$$

Proof. Choose a $\chi\in\mathcal{S}(\mathbb{R}^k)$ with $\chi(0)=1$ and $\nabla^\ell\chi(0)=0$ for every $\ell\in\mathbf{N}$. For $(f,g_1,\dots,g_k)\in\mathcal{S}(\mathbb{R}^d)^{k+1}$ define $F:=\mathrm{ext}^{(1)}(f,g_1,\dots,g_k)$ by

$$\begin{aligned}F(x,y):=&\int_{\mathbb{R}^d}e^{2\pi i\langle\xi,x\rangle}\chi(\langle\xi\rangle y)\hat{f}(\xi)\,\mathrm{d}\xi\\&+\sum_{i=1}^k y_i\int_{\mathbb{R}^d}e^{2\pi i\langle\xi,x\rangle}\chi(\langle\xi\rangle y)\hat{g}_i(\xi)\,\mathrm{d}\xi.\end{aligned}$$

(3.36) is an immediate consequence of the Fourier inversion theorem.

By direct computation, with the substitution $\tilde{\eta}=\eta/\langle\xi\rangle$,

$$\begin{aligned}\left\|\int_{\mathbb{R}^d}e^{2\pi i\langle\xi,x\rangle}\chi(\langle\xi\rangle y)\hat{f}(\xi)\,\mathrm{d}\xi\right\|_{H^s}^2&=\int_{\mathbb{R}^d}|\hat{f}(\xi)|^2\int_{\mathbb{R}^k}\langle(\xi,\eta)\rangle^{2s}|\mathscr{F}_y(\chi(\langle\xi\rangle y))(\eta)|^2\,\mathrm{d}\eta\,\mathrm{d}\xi\\&=\int_{\mathbb{R}^d}\langle\xi\rangle^{2s-k}|\hat{f}(\xi)|^2\underbrace{\int_{\mathbb{R}^k}\langle\tilde{\eta}\rangle^{2s}|\hat{\chi}(\tilde{\eta})|^2\,\mathrm{d}\tilde{\eta}}_{=:c(s)}\,\mathrm{d}\xi\\&=c(s)\|f\|_{H^{s-k/2}}^2,\end{aligned}$$

and

$$\begin{aligned}\left\|y_i\int_{\mathbb{R}^d}e^{2\pi i\langle\xi,x\rangle}\chi(\langle\xi\rangle y)\widehat{g}_i(\xi)\,\mathrm{d}\xi\right\|_{H^s}^2&=\int_{\mathbb{R}^d}|\widehat{g}_i(\xi)|^2\int_{\mathbb{R}^k}\langle(\xi,\eta)\rangle^{2s}|\mathscr{F}_y(y_i\chi(\langle\xi\rangle y))(\eta)|^2\,\mathrm{d}\eta\,\mathrm{d}\xi\\&=\int_{\mathbb{R}^d}\langle\xi\rangle^{2s-k-2}|\widehat{g}_i(\xi)|^2\underbrace{\int_{\mathbb{R}^k}\langle\eta\rangle^{2s}|\mathscr{F}_y(y_i\chi(y))(\eta)|^2\,\mathrm{d}\eta}_{=:c_i(s)}\,\mathrm{d}\xi\\&=c_i(s)\|g_i\|_{H^{s-k/2-1}}^2.\end{aligned}$$

Here $\mathscr{F}_y$ denotes the Fourier transform in the variable $y \in \mathbb{R}^k$. Since $c(s), c_i(s) < \infty$, the above prove (3.37). ∎

Proof of Proposition 3.34. Let $(\mathbf{p}; Z, [\mathfrak{l}]; v) \in \mathscr{F}^{\mathbf{Ram}}|_{\mathscr{U}}$ and $\bar{v} \in \mathrm{Vect}(X)$. By Proposition 3.15 (2),

$$(\mathbf{p}, [\mathfrak{l}]; v) \cdot \mathrm{lift}_{\mathbf{p}}^K(\bar{v}) = \big(\mathscr{L}_{\bar{v}}g; \dot{\gamma}^{\mathrm{BG}}, \dot{\nabla}^{\mathrm{BG}} + F_\nabla^{\mathrm{tw}}(\bar{v}, -); v - \bar{v}|_Z^N\big)$$

with $(\mathscr{L}_{\bar{v}}g; \dot{\gamma}^{\mathrm{BG}}, \dot{\nabla}^{\mathrm{BG}}) \in T_{\mathbf{p}}\mathbf{Dir}$ denoting the Bourguignon–Gauduchon lift of $\mathscr{L}_{\bar{v}}g \in T_g\mathbf{Met}$. Here $-^N$ denotes the orthogonal projection onto NZ. By inspection of Proposition 3.29, condition (1) translates into

$$\bar{v}|_Z^N = v \quad \text{and} \quad (\mathscr{L}_{\bar{v}}g)|_Z \in \Gamma(Z, \mathrm{Hom}(S^2TZ, \mathbb{R})).$$

By (3.18), the second condition means that for every $x \in Z$, $t \in T_xZ$, $n, m \in N_xZ$

$$g(\nabla_n\bar{v}, t) = -g(n, \nabla_t\bar{v}) \quad \text{and} \quad g(\nabla_n\bar{v}, m) + g(n, \nabla_m\bar{v}) = 0.$$

These are algebraic conditions on the 1–jet $J^1\bar{v}|_Z$ of $\bar{v}$ at Z. Evidently, it is possible to algebraically solve for $J^1\bar{v} \in \Gamma(Z, J^1TX)$ and to extend it to a vector field $\bar{v} \in \mathrm{Vect}(X)$. In fact, using Lemma 3.35, this can be done locally in $\mathbf{Dir} \times \mathbf{Ram}$ in such a way that $\bar{v}$ depends tame smoothly on $(\mathbf{p}, [\mathfrak{l}]; v)$ and satisfies (2) and (3). ∎

Definition 3.38. Given a local infinitesimal uniformiser $\mathscr{F}^{\mathbf{Ram}}|_{\mathscr{U}} \to \mathrm{Vect}(X)$, $(\mathbf{p}, [\mathfrak{l}]; v) \mapsto \bar{v} = \bar{v}(\mathbf{p}, [\mathfrak{l}], v)$ as in Proposition 3.34, define the partial covariant derivatives $\nabla^{\mathbf{Ram}}$ with respect to $\mathscr{F}^{\mathbf{Ram}}$ on $\mathbf{E}|_{\mathscr{U}}$, $\mathbf{F}|_{\mathscr{U}}$, and $\mathbf{G}|_{\mathscr{U}}$ by

$$\nabla_v^{\mathbf{Ram}} := \nabla^{\mathbf{Sl}}_{(\mathbf{p},[\mathfrak{l}],v)\cdot \mathrm{lift}_{\mathbf{p}}^K(\bar{v})}. \qquad \bullet$$

Unsurprisingly, $\nabla_v^{\mathbf{Ram}}\mathbf{D}$ can be expressed in terms of the Kosmann Lie derivative defined in Proposition 3.15. This is crucial for the discussion in Section 4.

Proposition 3.39 (cf. [Kos72, §V.1.2]). *In the situation of Definition 3.38, for every* $(\mathbf{p}; [\mathfrak{l}]; \phi) \in \mathbf{E}|_{\mathscr{U}}$ *and* $v \in T_{[\mathfrak{l}]}\mathbf{Ram}$

$$\nabla_v^{\mathbf{Ram}}\mathbf{D}(\mathbf{p}, [\mathfrak{l}]; \phi) = [\mathscr{L}_{\bar{v}}^K, D_{\mathbf{p}}^{\mathfrak{l}}]\phi.$$

Proof. By Proposition 3.15,

$$\begin{aligned}[\mathscr{L}_{\bar{v}}^K, \gamma(e_i)] &= \dot{\gamma}^{\mathrm{BG}}(e_i) + \gamma([\bar{v}, e_i]) \quad \text{and} \\ [\mathscr{L}_{\bar{v}}^K, \nabla_{e_i}] &= \dot{\nabla}_{e_i}^{\mathrm{BG}} + F_\nabla^{\mathrm{tw}}(\bar{v}, e_i) + \nabla_{[\bar{v}, e_i]}.\end{aligned}$$

Therefore,

$$[\mathscr{L}_{\bar{v}}^K, D] = \sum_{i=1}^d \dot{\gamma}^{\mathrm{BG}}(e_i)\nabla_{e_i} + \gamma(e_i)\dot{\nabla}_{e_i}^{\mathrm{BG}} + \gamma(e_i)F_\nabla^{\mathrm{tw}}(\bar{v}, e_i) - \gamma(\nabla_{e_i}\bar{v})\nabla_{e_i} - \gamma(e_i)\nabla_{\nabla_{e_i}\bar{v}}.$$

A glance at (3.31) reveals that this is precisely $\nabla_v^{\mathbf{Ram}}\mathbf{D}(\mathbf{p}, [\mathfrak{l}]; -)$. ∎

Remark 3.40. The formula in Proposition 3.39 has to be treated with some care, because

$$\mathscr{L}_{\bar{v}}^K D_{\mathbf{p}}^{\mathfrak{l}} \quad \text{and} \quad D_{\mathbf{p}}^{\mathfrak{l}} \mathscr{L}_{\bar{v}}^K$$

do not map $\mathbf{E}_{\mathbf{p},\mathfrak{l}}$ to $\mathbf{F}_{\mathbf{p},\mathfrak{l}}$, but their difference does. In fact, as a consequence of Remark 3.32,

$$\nabla_v^{\mathbf{Ram}}\mathbf{D}(\mathbf{p}, \mathfrak{l}; -) \in \mathrm{DiffOp}_b^1(S \otimes \mathfrak{l}). \qquad ♣$$

3.6 Families of residue conditions

This subsection explains how to impose residue conditions in families.

Definition 3.41. A **family of residue conditions** $(B, f, \mathbf{R})$ consists of:

(1) a tame Fréchet manifold B,

(2) a tame smooth map $f\colon B \to \mathbf{Dir} \times \mathbf{Ram}$, and

(3) a tame Fréchet space subbundle $\mathbf{R} \subseteq f^*\mathbf{G}$. •

Here is an example of a family of (spectral) residue conditions.

Example 3.42. Let $\lambda \in \mathbb{R}$. Denote by $\mathscr{U}_\lambda \subseteq \mathbf{Dir} \times \mathbf{Ram}$ the open subset of those $(\mathfrak{p}, [\mathfrak{I}]) \in \mathbf{Dir} \times \mathbf{Ram}$ for which the branching locus operator $A_{\mathfrak{p},[\mathfrak{I}]}$ does not have λ as an eigenvalue; that is: $\lambda \notin \operatorname{spec}(A_{\mathfrak{p},[\mathfrak{I}]})$. There is a tame Fréchet space subbundle

$$\mathbf{APS}_\lambda \subseteq \mathbf{G}|_{\mathscr{U}_\lambda}$$

such that, for every $(\mathfrak{p}, [\mathfrak{I}]) \in \mathscr{U}_\lambda$,

$$\mathbf{APS}_{\lambda;\mathfrak{p},[\mathfrak{I}]} = \mathbf{1}_{(-\infty,\lambda)}(A_{\mathfrak{p},[\mathfrak{I}]})\Gamma(\mathrm{Br}(\mathfrak{I}), \check{S}_{\mathfrak{p}}).$$

This defines the **family of APS residue conditions** with threshold $\lambda \in \mathbb{R}$. ♠

Proposition 3.43. *For every family of residue conditions* $(B, f, \mathbf{R})$

$$\mathbf{E}_{\mathbf{R}} := (f^*\mathbf{res})^{-1}(\mathbf{R}) \subseteq f^*\mathbf{E}$$

is a tame Fréchet space subbundle.

$$\begin{tikzcd}
\mathbf{E}_{\mathbf{R}} \arrow[r, "\mathbf{D}_{\mathbf{R}}"] \arrow[d, "\mathbf{res}"] \arrow[dr, hook] & f^*\mathbf{F} \arrow[dr, equal] & \\
\mathbf{R} \arrow[dr, hook] & f^*\mathbf{E} \arrow[r, "f^*\mathbf{D}"] \arrow[d, "f^*\mathbf{res}"] & f^*\mathbf{F} \\
& f^*\mathbf{G} &
\end{tikzcd}$$

Figure 2: The construction of $\mathbf{D}_{\mathbf{R}}$.

Definition 3.44. In the situation of Proposition 3.43, the restriction of $f^*\mathbf{D}$ to $\mathbf{E}_{\mathbf{R}}$ is denoted by

$$\mathbf{D}_{\mathbf{R}}\colon \mathbf{E}_{\mathbf{R}} \to f^*\mathbf{F}. \quad \bullet$$

The proof of Proposition 3.43 relies on the following.

Proposition 3.45. *The **universal bundle of** $\mathbb{Z}/2\mathbb{Z}$ **spinors***

$$\mathbf{E}_0 \coloneqq \ker \mathbf{res} \subseteq \mathbf{E}$$

is a tame Fréchet space subbundle. Moreover, the exact sequence

$$\mathbf{E}_0 \hookrightarrow \mathbf{E} \xrightarrow{\mathbf{res}} \mathbf{G}$$

of tame Fréchet space bundles locally splits; that is: for every $(\mathbf{p}, [\mathfrak{l}]) \in \mathbf{Dir} \times \mathbf{Ram}$ *there are an open neighbourhood* $(\mathbf{p}, [\mathfrak{l}]) \in \mathscr{U} \subseteq \mathbf{Dir} \times \mathbf{Ram}$ *and a map of tame Fréchet space bundles* $\mathbf{ext}\colon \mathbf{G}|_{\mathscr{U}} \to \mathbf{E}|_{\mathscr{U}}$ *satisfying* $\mathbf{res} \circ \mathbf{ext} = \mathbf{1}$.

Proof. That $\mathbf{E}_0 \subseteq \mathbf{E}$ is a tame Fréchet space subbundle is evident by inspection of the trivialisations constructed in Definition 3.25. The splitting arises from the extension map mentioned in Section 2.2 and constructed in [BW26, Definition 3.40 (2)]. ■

Remark 3.46. Here are some observations regarding $\mathbf{E}_0$:

(1) The construction of $\mathbf{E}_0$ could be carried out directly with a less delicate definition of slice, because $\ker \operatorname{res} = H^\infty_a\Gamma(X\backslash Z, S \otimes \mathfrak{l}; 0; \mathbf{p})$ does not depend on the normal structure; cf. Remark 3.27 (2).

(2) For every $v \in \operatorname{Vect}(X)$ and $(\mathbf{p}, [\mathfrak{l}]; \Phi) \in \mathbf{E}_0$, $(\mathbf{p}, [\mathfrak{l}]; \nabla_v\Phi) \in \mathbf{F}$ and the corresponding map

$$\nabla\colon \operatorname{Vect}(X) \times \mathbf{E}_0 \to \mathbf{F}$$

is tame smooth [BW26, Proposition 4.27]. ♣

Proof of Proposition 3.43. Using Proposition 3.45, $\mathbf{E}_\mathbf{R}$ can locally be presented as

$$\mathbf{E}_\mathbf{R}|_{f^{-1}(\mathscr{U})} = f^*\mathbf{E}_0 \oplus (f^*\mathbf{ext})(\mathbf{R}).$$ ■

The arguments in Section 4 use families of residue conditions of the following kind.

Definition 3.47. Here is how to consider local residue conditions in families:

(1) (a) The **universal branching locus** is the tame Fréchet submanifold

$$\mathbf{Br} \coloneqq \left\{(\mathbf{p}, [\mathfrak{l}]; x) \in \mathbf{Dir} \times \mathbf{Ram} \times X : x \in \operatorname{Br}(\mathfrak{l})\right\} \subseteq \mathbf{Dir} \times \mathbf{Ram} \times X.$$

The projection map $p\colon \mathbf{Br} \to \mathbf{Dir} \times \mathbf{Ram}$ is a fibre bundle.

(b) The normal bundle $\mathbf{N} \coloneqq N\mathbf{Br}$ shall be identified with

$$\mathbf{N} = \coprod_{(\mathbf{p},[\mathfrak{l}]) \in \mathbf{Dir}\times\mathbf{Ram}} N\operatorname{Br}(\mathfrak{l}).$$

It inherits a Euclidean metric from the metrics on X which are part of $\mathbf{p} \in \mathbf{Dir}$. This induces an isomorphism $(\Lambda^2\mathbf{N})^{\otimes 2} \cong \mathbb{R}$ and endows

$$\mathbb{A} \coloneqq \mathbb{R} \oplus i\Lambda^2\mathbf{N}$$

with the structure of a bundle of algebras over $\mathbf{Br}$, whose fibres are isomorphic to $\mathbb{C}$, but canonically only up to complex conjugation.

(c) The construction from [BW26, Definition 3.21 and Remark 3.22] explained in Section 2.2, parametrised by $(\mathbf{p}, [\mathfrak{l}]) \in \mathbf{Dir} \times \mathbf{Ram}$, assembles into an $\mathbb{A}$–module $\mathbf{N}^{-1/2}$.

(d) Clifford multiplication endows the restriction of $\mathrm{pr}_X^* S$ to $\mathbf{Br}$ with the structure of an $\mathbb{A}$–module. The **universal residue bundle** is the rank r Euclidean bundle over $\mathbf{Br}$ defined by

$$\check{\mathbf{S}} \coloneqq \mathrm{pr}_X^* S \otimes_{\mathbb{A}} \mathbf{N}^{-1/2} = \coprod_{(\mathbf{p},[\mathfrak{l}])\in\mathbf{Dir}\times\mathbf{Ram}} \check{S}_{\mathbf{p}}.$$

(2) A **family of local residue conditions** $(\mathbf{B}, f, \mathbf{V})$ consists of:

(a) a tame Fréchet manifold $\mathbf{B}$,

(b) a tame smooth map $f\colon \mathbf{B} \to \mathbf{Dir} \times \mathbf{Ram}$, and

(c) a subbundle $\mathbf{V} \subseteq (p^* f)^* \check{\mathbf{S}}$.

Here $f^* p\colon f^*\mathbf{Br} \to \mathbf{B}$ is the pullback of the universal branching locus and $p^* f\colon f^*\mathbf{Br} \to \mathbf{Br}$ is the projection map; see Figure 3.

(3) A family of local residue conditions $(\mathbf{B}, f, \mathbf{V})$ defines a family of residue conditions $(\mathbf{B}, f, \mathbf{R})$ such that for every $b \in \mathbf{B}$

$$\mathbf{R}_b \coloneqq \Gamma(Z_b, \mathbf{V}|_{Z_b}) \subseteq \Gamma(Z_b, \check{S}_{\mathbf{p}_b}) \quad \text{with} \quad (\mathbf{p}_b; Z_b, [\mathfrak{l}_b]) \coloneqq f(b). \qquad \bullet$$

Remark 3.48. Here are some comments regarding Definition 3.47:

(1) $\check{\mathbf{S}}$ is related to the universal bundle of residues $\mathbf{G}$, constructed in Section 3.4, by

$$\mathbf{G} = p_* \check{\mathbf{S}};$$

in the sense that, for every $(\mathbf{p}, [\mathfrak{l}]) \in \mathbf{Dir} \times \mathbf{Ram}$, $\mathbf{G}_{\mathbf{p},[\mathfrak{l}]} = \Gamma(p^{-1}(\mathbf{p}, [\mathfrak{l}]), \check{\mathbf{S}})$.

(2) Identifying $f^*\mathbf{G} = f^* p_* \check{\mathbf{S}} = (f^* p)_* (p^* f)^* \check{\mathbf{S}}$, the tame Fréchet space subbundle $\mathbf{R} \subseteq f^*\mathbf{G}$ is

$$\mathbf{R} \coloneqq (f^* p)_* \mathbf{V} \subseteq f^* \mathbf{G}. \qquad \clubsuit$$

For the purposes of this article, it is very desirable for $\mathbf{D}_{\mathbf{R}}$ to have the following property.

Definition 3.49. Let E and F be tame Fréchet space bundles over a tame Fréchet manifold B. A map of tame Fréchet space bundles $L\colon E \to F$ is **uniformly Fredholm** if for every $b \in B$ there exist an open neighbourhood $b \in \mathscr{U} \subseteq B$, finite-rank vector bundles Def and Ob over $\mathscr{U}$, and maps of tame Fréchet space bundles $\pi\colon E|_{\mathscr{U}} \twoheadrightarrow \mathrm{Def}$ and $\iota\colon \mathrm{Ob} \hookrightarrow F|_{\mathscr{U}}$ such that the **thickening**

$$\begin{pmatrix} L & \iota \\ \pi & 0 \end{pmatrix} \colon E|_{\mathscr{U}} \oplus \mathrm{Ob} \to F|_{\mathscr{U}} \oplus \mathrm{Def}$$

is an isomorphism of tame Fréchet space bundles. $\bullet$

$$\begin{tikzcd}
\mathbf{V} \arrow[r, hook] & (p^*f)^*\check{\mathrm{S}} \arrow[r] \arrow[d] & \check{\mathrm{S}} \arrow[d] \\
& f^*\mathbf{Br} \arrow[r, "p^*f"] \arrow[d, "f^*p"'] & \mathbf{Br} \arrow[d, "p"] \\
& \mathbf{B} \arrow[r, "f"'] & \mathbf{Dir} \times \mathbf{Ram}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{R} := (f^*p)_*\mathbf{V} \arrow[r, hook] & f^*\mathbf{G} \arrow[r] \arrow[d] & \mathbf{G} = p_*\check{\mathrm{S}} \arrow[d] \\
& \mathbf{B} \arrow[r, "f"'] & \mathbf{Dir} \times \mathbf{Ram}
\end{tikzcd}$$

(a) (b)

Figure 3: Families of local residue conditions.

Proposition 3.50. *Let $(\mathbf{B}, f, \mathbf{V})$ be a family of local residue conditions such that for every $b \in \mathbf{B}$, $x \in Z_b$, and $v \in T_xZ_b\backslash\{0\}$*

$$\mathbf{V}_{(b,x)} \oplus J_{\mathfrak{p}_b}\gamma(v)\mathbf{V}_{(b,x)} = \check{S}_{\mathfrak{p}_b,x}.$$

Denote by $(\mathbf{B}, f, \mathbf{R})$ the corresponding family of residue conditions. If $\mathfrak{a}\colon f^*\mathbf{F} \to f^*\mathbf{F}$ is a map of tame Fréchet bundles of order zero, then

$$\mathbf{D}_{\mathbf{R}} + \mathfrak{a}\colon \mathbf{E}_{\mathbf{R}} \to f^*\mathbf{F}$$

is uniformly Fredholm. Moreover, for every $b_0 \in \mathbf{B}$ the maps $\pi\colon \mathbf{E}_{\mathbf{R}}|_{\mathscr{U}} \to \mathrm{Def}$ and $\iota\colon \mathrm{Ob} \hookrightarrow f^*\mathbf{F}|_{\mathscr{U}}$ in Definition 3.49 can be chosen such that there is a compact subset $K \subseteq X$ such that for every $b \in \mathscr{U}$ the following hold:

1. *$K \subseteq X\backslash Z_b$;*
2. *if $\phi \in \mathbf{E}_{\mathbf{R},b}$ has $\mathrm{supp}(\phi) \cap K = \varnothing$, then $\pi(\phi) = 0$; and*
3. *$\mathrm{supp}(\iota(o)) \subseteq K$ for every $o \in \mathrm{Ob}_b$.*

Proof. By [BW26, Proposition 4.41], for every $b \in \mathbf{B}$ the residue condition $\mathbf{R}_b$ is elliptic; in particular: for every $k \in \mathbf{N}_0$ the operator

$$D^{(k)}_{\mathbf{R}_b} + \mathfrak{a}^{(k)}_b := D^{\mathfrak{l}_b,(k)}_{\mathfrak{p}_b,\mathbf{R}_b} + \mathfrak{a}^{(k)}_b\colon H^{k+1}_a\Gamma(X\backslash Z_b, S \otimes \mathfrak{l}_b; \mathbf{R}_b; \mathfrak{p}_b) \to H^k_b\Gamma(X\backslash Z_b, S \otimes \mathfrak{l}_b; \mathfrak{p}_b)$$

is Fredholm, and $\ker(D^{(k)}_{\mathbf{R}_b} + \mathfrak{a}^{(k)}_b)$ and $\mathrm{coker}(D^{(k)}_{\mathbf{R}_b} + \mathfrak{a}^{(k)}_b)$ are independent of $k \in \mathbf{N}_0$.

Let $b_0 \in \mathbf{B}$. Choose the following:

(I) a compact subset $K \subseteq X\backslash Z_{b_0}$ with non-empty interior,

(II) an open neighbourhood $b_0 \in \mathscr{U} \subseteq \mathbf{B}$ such that $K \subseteq X\backslash Z_b$ for every $b \in \mathscr{U}$,

(III) finite rank Euclidean vector bundles Def and Ob over $\mathscr{U}$ and maps of tame Fréchet space bundles $\pi\colon \mathbf{E}_{\mathbf{R}}|_{\mathscr{U}} \twoheadrightarrow \mathrm{Def}$ and $\iota\colon \mathrm{Ob} \hookrightarrow f^*\mathbf{F}|_{\mathscr{U}}$ such that for every $b \in \mathscr{U}$:

(a) for every $\phi \in \mathbf{E}_{\mathbf{R},b}$

$$\|\pi(\phi)\| \lesssim \|\phi\|_{H^1_a},$$

(b) (i) if $\operatorname{supp}(\phi) \cap K = \varnothing$, then $\pi(\phi) = 0$,
(ii) $\operatorname{supp}(\iota(o)) \subseteq K$ for every $b \in \mathscr{U}$ and $o \in \mathrm{Ob}_b$,

(c) for every $b \in \mathscr{U}$ and $k \in \mathbf{N}_0$

$$L_b^{(k)} := \begin{pmatrix} D^{(k)}_{\mathbf{R}_b} + \mathfrak{a}^{(k)}_b & \iota_b \\ \pi_b & 0 \end{pmatrix}$$

is invertible.

(III.b) can be arranged with the help of Proposition 3.22. To see that (III.c) can be achieved, observe the following. If $L_b^{(0)}$ is invertible, then $L_b^{(k)}$ is invertible for every $k \in \mathbf{N}_0$ by elliptic regularity [BW26, Theorem 4.17]. Since $b \mapsto L_b^{(0)}$ is a continuous family of bounded operators between Banach spaces (in a suitable trivialisation), it suffices to ensure that $L_{b_0}^{(0)}$ is invertible. The latter holds provided the induced maps $\ker(D^{(0)}_{\mathbf{R}_{b_0}} + \mathfrak{a}^{(0)}_{b_0}) \to \mathrm{Def}_{b_0}$ and $\mathrm{Ob}_{b_0} \to \operatorname{coker}(D^{(0)}_{\mathbf{R}_{b_0}} + \mathfrak{a}^{(0)}_{b_0})$ are isomorphisms. Because the kernels of both $D_{\mathbf{R}_{b_0}} + \mathfrak{a}_{b_0}$ and its adjoint have the unique continuation property, this can easily be arranged while satisfying the desired support properties for a suitable $K \subseteq X \backslash Z_{b_0}$.

The argument in [Ham82, Part II Proofs of Theorem 3.3.3 and Theorem 3.3.5] shows that inverses obtained from (III.c) assemble into an inverse of the map of tame Fréchet bundles

$$\begin{pmatrix} D_{\mathbf{R}} + \mathfrak{a} & \iota \\ \pi & 0 \end{pmatrix} : \mathbf{E}_{\mathbf{R}}|_{\mathscr{U}} \oplus \mathrm{Ob} \to f^*\mathbf{F} \oplus \mathrm{Def}. \qquad \blacksquare$$

3.7 The universal subbundle of admissible $\mathbb{Z}/2\mathbb{Z}$ spinors

Naively, one might hope to use $\mathbf{E}_0$ as the ambient space for the construction of tame Fréchet manifold structures on the universal moduli spaces of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors introduced in Section 1. However, for the arguments in Section 4.3 it is crucial to resolve the issue raised in Remark 3.40 by working with a smaller ambient tame Fréchet space bundle $\mathbf{E}_{00}$, the **universal bundle of admissible $\mathbb{Z}/2\mathbb{Z}$ spinors**. The following describes the fibres of $\mathbf{E}_{00}$.

Proposition 3.51. *For every $\mathbf{p} \in \mathbf{Dir}$ and $(Z, [\mathfrak{l}]) \in \mathbf{Ram}$ the graded Fréchet space $(K^\infty(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p}), (\|-\|_{K^k})_{k\in\mathbf{N}_0})$ defined by*

$$K^\infty(X\backslash Z, S\otimes\mathfrak{l};\mathbf{p}) := H^\infty_a(X\backslash Z, S\otimes\mathfrak{l};0;\mathbf{p}) \cap (D^{\mathfrak{l}}_{\mathbf{p}})^{-1} w_{\log}^{-1} H^\infty_a(X\backslash Z, S\otimes\mathfrak{l};0;\mathbf{p}), \quad \textit{and}$$
$$\|\phi\|_{K^k} := \|\phi\|_{H^{k+2}_b} + \|\nabla\phi\|_{H^{k+1}_b} + \|w_{\log} D^{\mathfrak{l}}_{\mathbf{p}}\phi\|_{H^{k+1}_b} + \|\nabla w_{\log} D^{\mathfrak{l}}_{\mathbf{p}}\phi\|_{H^k_b}$$

is tame; that is, it has the smoothing operator property. Here $w_{\log}$ denotes multiplication by $\langle\log(r)\rangle = \sqrt{1+\log(r)^2}$ and r denotes the distance to Z.

The proof uses the following observation.

Proposition 3.52. *For every* $\mathbf{p} \in \mathbf{Dir}$, $(Z, [\mathfrak{l}]) \in \mathbf{Ram}$, $\phi \in K^\infty(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p})$, *and* $k \in \mathbf{N}_0$

$$\|w_{\log}\phi\|_{H_b^{k+2}} + \|r^{-1}w_{\log}\phi\|_{H_b^{k+1}} + \|w_{\log}\nabla\phi\|_{H_b^{k+1}} \lesssim_k \|\phi\|_{K^k}.$$

Proof. By [BW26, Lemma 3.3 and Proposition 4.27],

$$\|r^{-1}\phi\|_{H_b^{k+1}} \lesssim_k \|\phi\|_{H_b^{k+1}} + \|\nabla\phi\|_{H_b^{k+1}} \lesssim_k \|\phi\|_{K^k}. \tag{3.53}$$

Therefore, since $[D_{\mathbf{p}}^{\mathfrak{l}}, \log(r)] = r^{-1}\gamma(\partial_r)$,

$$\|w_{\log}\phi\|_{H_b^{k+1}} + \|D_{\mathbf{p}}^{\mathfrak{l}}w_{\log}\phi\|_{H_b^{k+1}} \lesssim_k \|\phi\|_{H_b^{k+1}} + \|\nabla\phi\|_{H_b^{k+1}} + \|w_{\log}D_{\mathbf{p}}^{\mathfrak{l}}\phi\|_{H_b^{k+1}} \leqslant \|\phi\|_{K^k}.$$

As a consequence, $w_{\log}\phi \in H_a^{k+1}(X\backslash Z, S \otimes \mathfrak{l})$. Since $\phi \in L^\infty(X\backslash Z, S \otimes \mathfrak{l})$, $\operatorname{res}(w_{\log}\phi) = 0$. By [BW26, Theorem 4.11],

$$\|w_{\log}\phi\|_{H_b^{k+2}} \lesssim_k \|w_{\log}\phi\|_{H_b^{k+1}} + \|D_{\mathbf{p}}^{\mathfrak{l}}w_{\log}\phi\|_{H_b^{k+1}}$$

and $w_{\log}\phi \in H_a^{k+2}(X\backslash Z, S \otimes \mathfrak{l}; 0; \mathbf{p})$. Therefore, by [BW26, Lemma 3.3 and Proposition 4.27],

$$\|r^{-1}w_{\log}\phi\|_{H_b^{k+1}} + \|w_{\log}\nabla\phi\|_{H_b^{k+1}} \lesssim_k \|r^{-1}\phi\|_{H_b^{k+1}} + \|\nabla w_{\log}\phi\|_{H_b^{k+1}} \lesssim \|\phi\|_{K^k}.$$ ■

Proof of Proposition 3.51. For every $\phi \in \Gamma(X\backslash Z, S \otimes \mathfrak{l})$ with $\operatorname{supp}\phi \subseteq r^{-1}([0, 1/2])$ set

$$\|\phi\|_{\mathring{K}^k} := \|\phi\|_{H_b^{k+2}} + \|\mathring{\nabla}\phi\|_{H_b^{k+1}} + \|\log(r)\mathring{D}_{\mathbf{p}}^{\mathfrak{l}}\phi\|_{H_b^{k+1}} + \|\mathring{\nabla}\log(r)\mathring{D}_{\mathbf{p}}^{\mathfrak{l}}\phi\|_{H_b^k}.$$

Here $\mathring{\nabla}$ and $\mathring{D}_{\mathbf{p}}^{\mathfrak{l}}$ are the models for ∇ and $D_{\mathbf{p}}^{\mathfrak{l}}$ constructed in [BW26, §3.2]. Since $B := D_{\mathbf{p}}^{\mathfrak{l}} - \mathring{D}_{\mathbf{p}}^{\mathfrak{l}} \in \operatorname{DiffOp}_b^1(S \otimes \mathfrak{l})$, $[B, \log(r)] = r^{-1}\sigma_B(\mathrm{d}r)$ and its conormal derivatives are bounded. Therefore, for $\phi \in K^\infty(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p})$ with $\operatorname{supp}\phi \subseteq r^{-1}([0, 1/2])$, by Proposition 3.52

$$\|\log(r)B\phi\|_{H_b^{k+1}} \lesssim_{B,k} \|\log(r)\phi\|_{H_b^{k+2}} \lesssim_{B,k} \|\phi\|_{K^k}$$

and

$$\begin{aligned}\|\nabla\log(r)B\phi\|_{H_b^k} &\lesssim_k \|r^{-1}B\phi\|_{H_b^k} + \|\log(r)\nabla B\phi\|_{H_b^k}\\ &\lesssim_{B,k} \|r^{-1}\phi\|_{H_b^{k+1}} + \|\log(r)\nabla\phi\|_{H_b^{k+1}} \lesssim_k \|\phi\|_{K^k}.\end{aligned}$$

The above (and similar estimates) imply that

$$\|\phi\|_{\mathring{K}^k} \lesssim_k \|\phi\|_{K^k} \quad \text{and} \quad \|\phi\|_{K^k} \lesssim_k \|\phi\|_{\mathring{K}^k}.$$

Appendix A constructs a family of smoothing operators for $(\|-\|_{\mathring{K}^k} : k \in \mathbf{N}_0)$. These combined with the usual smoothing operators away from Z construct the required smoothing operators as, for example, in [Par23, Proof of Lemma B.1; BW26, Proof of Proposition 4.32]. ■

Proposition 3.54. *For every* $\mathbf{p} \in \mathbf{Dir}$, $(Z, [\mathfrak{l}]) \in \mathbf{Ram}$, $\Phi \in K^\infty(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p})$, *and* $\bar{v} \in \mathrm{Vect}(X)$

$$\mathscr{L}_{\bar{v}}^K D_{\mathbf{p}}^{\mathfrak{l}} \Phi \in H_b^\infty(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p});$$

in fact, for every $k, \ell \in \mathbf{N}_0$ *with* $\ell > d/2$, *and* $\varepsilon > 0$, *there is a constant* $c_k(\varepsilon) < \infty$ *such that*

$$\|\mathscr{L}_{\bar{v}}^K D_{\mathbf{p}}^{\mathfrak{l}} \Phi\|_{H_b^k} \leqslant \varepsilon \|\bar{v}\|_{H^{k+1}} \|\Phi\|_{K^\ell} + c_k(\varepsilon) \|\bar{v}\|_{H^1} \|\Phi\|_{K^{k+\ell}}.$$

Proof. Evidently,

$$\begin{aligned}\|\mathscr{L}_{\bar{v}}^K D_{\mathbf{p}}^{\mathfrak{l}} \Phi\|_{H_b^k} &\lesssim_k \|\nabla \bar{v} \otimes D_{\mathbf{p}}^{\mathfrak{l}} \Phi\|_{H_b^k} + \|\bar{v} \otimes \nabla D_{\mathbf{p}}^{\mathfrak{l}} \Phi\|_{H_b^k} \\ &= \|\nabla \bar{v} \otimes D_{\mathbf{p}}^{\mathfrak{l}} \Phi\|_{H_b^k} + \|r^{-1} w_{\log}^{-1} \bar{v} \otimes r w_{\log} \nabla D_{\mathbf{p}}^{\mathfrak{l}} \Phi\|_{H_b^k}.\end{aligned}$$

The summands on the right-hand side are estimated individually.

By [BW26, Proposition 4.26], $H_a^\ell \Gamma(X\backslash Z, S \otimes \mathfrak{l}; 0, \mathbf{p}) \subseteq L^\infty \Gamma(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p})$. Moreover, by definition of $\|-\|_{K^k}$ and (3.53),

$$\|D_{\mathbf{p}}^{\mathfrak{l}} \Phi\|_{H_a^{k+1}} \lesssim_k \|\Phi\|_{K^k}.$$

Therefore,

$$\|\nabla \bar{v} \otimes D_{\mathbf{p}}^{\mathfrak{l}} \Phi\|_{H_b^k} \lesssim_k \sum_{m=0}^{k} \|\nabla \bar{v}\|_{H^{k-m}} \|D\Phi\|_{H_a^{m+\ell}} \lesssim_k \sum_{m=0}^{k} \|\bar{v}\|_{H^{k-m+1}} \|\Phi\|_{K^{m+\ell-1}}.$$

By (the proof of) the borderline L^2 Hardy inequality in dimension two,

$$\|r^{-1} w_{\log}^{-1} \bar{v}\|_{H_b^k} \lesssim_k \|\bar{v}\|_{H^{k+1}};$$

see, e.g., [DS13, Proposition 3.1]. Moreover, by [BW26, Proposition 4.28], $H_b^\ell \Gamma(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p}) \subseteq r^{-1} L^\infty \Gamma(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p})$. Finally, by definition of $\|-\|_{K^k}$ and (3.53),

$$\|w_{\log} \nabla D_{\mathbf{p}}^{\mathfrak{l}} \Phi\|_{H_b^k} \lesssim_k \|\Phi\|_{K^k}.$$

Therefore,

$$\|r^{-1} w_{\log}^{-1} \bar{v} \otimes r w_{\log} \nabla D_{\mathbf{p}}^{\mathfrak{l}} \Phi\|_{H_b^k} \lesssim_k \sum_{m=0}^{k} \|\bar{v}\|_{H^{k-m+1}} \|w_{\log} \nabla D_{\mathbf{p}}^{\mathfrak{l}} \Phi\|_{H_b^{m+\ell}} \lesssim_k \sum_{m=0}^{k} \|\bar{v}\|_{H^{k-m+1}} \|\Phi\|_{K^{m+\ell}}.$$

Combining the above estimates,

$$\|\mathscr{L}_{\bar{v}}^K D_{\mathbf{p}}^{\mathfrak{l}} \Phi\|_{H_b^k} \lesssim_k \sum_{m=0}^{k} \|\bar{v}\|_{H^{k-m+1}} \|\Phi\|_{K^{m+\ell}}.$$

This implies the asserted estimate using interpolation as in [Ham79, §4.2; DL25, Lemma A.4]. ∎

Proposition 3.55 (cf. [BW26, Remark 4.45]). *Let* $(\mathbf{p},[\mathfrak{l}]) \in \mathbf{Dir} \times \mathbf{Ram}$ *and set* $Z := \mathrm{Br}(\mathfrak{l})$. *Choose a local infinitesimal uniformiser as in Proposition 3.34. For every* $\Phi \in K^\infty(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p})$ *and* $v \in T_{[\mathfrak{l}]}\mathbf{Ram} = \Gamma(Z, NZ)$

$$\mathscr{L}_{\tilde{v}}^K \Phi \in H_a^\infty \Gamma(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p});$$

moreover, for every $\Phi \in K^\infty(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p})$ *there is a unique* $\mathbf{L}_\Phi \in \Gamma(Z, \mathrm{Hom}_{\mathbb{A}}(\overline{NZ}, \check{S}_{\mathbf{p}}))$ *such that for every* $v \in T_{[\mathfrak{l}]}\mathbf{Ram}$

$$\mathrm{res}(\mathscr{L}_{\tilde{v}}^K \Phi) = \mathbf{L}_\Phi(v).$$

Proof. By Remark 3.40 and Proposition 3.52, $\nabla_v^{\mathbf{Ram}} \mathbf{D}(\mathbf{p}, \mathfrak{l}; \Phi) \in H_b^\infty \Gamma(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p})$. Therefore, by Proposition 3.39 and Proposition 3.54, $\mathscr{L}_{\tilde{v}}^K \Phi \in H_a^\infty \Gamma(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p})$.

A moment's thought reveals that $v \mapsto \mathrm{res}(\mathscr{L}_{\tilde{v}}^K \Phi)$ is independent of the choice of infinitesimal uniformiser. Moreover, it is $C^\infty(Z)$–linear and, therefore, defines $\mathbf{L}_\Phi \in \Gamma(Z, \mathrm{Hom}(NZ, \check{S}_{\mathbf{p}}))$. Direct inspection of $D_{\mathbf{p}}^{\mathfrak{l}}$, or rather the model operator $\mathring{D}_{\mathbf{p}}^{\mathfrak{l}}$, shows that $\mathbf{L}_\Phi$ is $\mathbb{A}$–anti-linear. ■

By the method explained in Section 3.4, the above fibres assemble into the following tame Fréchet space bundle.

Definition 3.56. Consider the tame Fréchet space bundle

$$\mathbf{F}_{00} := w_{\log}^{-1} \mathbf{E}_0.$$

The **universal bundle of admissible** $\mathbb{Z}/2\mathbb{Z}$ **spinors** is the tame Fréchet subbundle

$$\mathbf{E}_{00} := \mathbf{E}_0 \cap \mathbf{D}^{-1}\mathbf{F}_{00} \subseteq \mathbf{E}_0;$$

indeed, the preceding discussion and direct inspection show that the atlas constructed in Definition 3.25 induces an atlas for $\mathbf{E}_{00}$; cf. Remark 3.27 (1). Of course, the tame Fréchet space structure on each fibre of $\mathbf{E}_{00}$ is the one defined in Proposition 3.51. Denote by

$$\mathbf{E}_{00}^* \subseteq \mathbf{E}_{00}$$

the open subset of **non-degenerate** admissible $\mathbb{Z}/2\mathbb{Z}$ spinors; that is: the subset consisting of those admissible $\mathbb{Z}/2\mathbb{Z}$ spinors $(\mathbf{p},[\mathfrak{l}];\Phi) \in \mathbf{E}_{00}$ for which $\mathbf{L}_\Phi$ is injective. •

The crucial observation is that every non-degenerate admissible $\mathbb{Z}/2\mathbb{Z}$ harmonic spinor defines a residue condition.

Proposition 3.57. *Assume that* $\mathrm{rk}_{\mathbb{A}} \check{S} = 2$. *Denote by* $f \colon \mathbf{E}_{00}^* \subseteq \mathbf{E}_{00} \to \mathbf{Dir} \times \mathbf{Ram}$ *the projection map and by*

$$\mathbf{V} \subseteq (p^* f)^* \check{\mathbf{S}}$$

the subbundle such that for every $(\mathbf{p},[\mathfrak{l}];\Phi) \in \mathbf{E}_{00}^*$

$$\mathbf{V}_{\mathbf{p},[\mathfrak{l}];\Phi} = \mathrm{im}\, \mathbf{L}_\Phi.$$

$(\mathbf{E}_{00}^*, f, \mathbf{V})$ *is a family of local residue conditions that satisfies the hypothesis of Proposition 3.50.*

Proof. Since $\mathbf{L}_\Phi$ is $\mathbb{A}$–anti-linear and $\dim \mathrm{im}\, \mathbf{L}_\Phi = \frac{1}{2} \mathrm{rk}\, \check{S}$, this follows from [BW26, Example 4.44]. ■

3.8 Chiral adaptations

Throughout this subsection, assume that X is oriented and $\dim X = 0 \pmod 4$. For every $\mathbf{p} \in \mathbf{Dir}$ and $[\mathfrak{l}] \in \mathbf{Ram}$ the volume form induces a chirality operator $\varepsilon := \gamma(\mathrm{vol}_g)$. This decomposes $S = S^+ \oplus S^-$ and splits the entire discussion so far into positive and negative chiral parts. Most of this is entirely formal.

Definition 3.58. Here is the decomposition of the objects constructed in Section 3.4:

(1) Decompose the tame Fréchet space bundles $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ into tame Fréchet subbundles

$$\mathbf{E} = \mathbf{E}^+ \oplus \mathbf{E}^-, \quad \mathbf{F} = \mathbf{F}^+ \oplus \mathbf{F}^-, \quad \text{and} \quad \mathbf{G} = \mathbf{G}^+ \oplus \mathbf{G}^- \tag{3.59}$$

with fibres over $(\mathbf{p}, [\mathfrak{l}]) \in \mathbf{Dir} \times \mathbf{Ram}$ given by

$$\begin{aligned} \mathbf{E}^\pm_{\mathbf{p},\mathfrak{l}} &= H^\infty_a\Gamma(X\backslash\mathrm{Br}(\mathfrak{l}), S^\pm \otimes \mathfrak{l}; \mathbf{p}), \\ \mathbf{F}^\pm_{\mathbf{p},\mathfrak{l}} &= H^\infty_b\Gamma(X\backslash\mathrm{Br}(\mathfrak{l}), S^\pm \otimes \mathfrak{l}; \mathbf{p}), \quad \text{and} \\ \mathbf{G}^\pm_{\mathbf{p},\mathfrak{l}} &= \Gamma(\mathrm{Br}(\mathfrak{l}), \check{S}^\pm_{\mathbf{p}}). \end{aligned}$$

(2) Decompose the universal Dirac operator $\mathbf{D}\colon \mathbf{E} \to \mathbf{F}$ into the **universal chiral Dirac operators** $\mathbf{D}^\pm\colon \mathbf{E}^\pm \to \mathbf{F}^\mp$ such that

$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{D}^- \\ \mathbf{D}^+ & 0 \end{pmatrix}.$$

(3) Decompose the universal residue map $\mathbf{res}\colon \mathbf{E} \to \mathbf{G}$ into the **universal chiral residue maps** $\mathbf{res}^\pm\colon \mathbf{E}^\pm \to \mathbf{G}^\pm$. •

$$\begin{array}{ccc} \mathbf{E}^\pm & \xrightarrow{\mathbf{D}^\pm} & \mathbf{F}^\mp \\ {\scriptstyle \mathbf{res}^\pm}\downarrow & & \\ \mathbf{G}^\pm & & \end{array}$$

Figure 4: The universal chiral Dirac operator and residue map.

The partial covariant derivatives constructed in Section 3.5 respect the above decomposition mostly, but not entirely.

Proposition 3.60.

(1) *The restriction of the slice partial covariant derivatives ∇^{Sl} to $H^{\mathrm{Sl}}_{\mathrm{BG}}$ and V_∇ preserves the splittings (3.59).*

(2) *In the situation of Definition 3.38, the partial covariant derivatives ∇^{Ram} preserve the splittings (3.59); moreover, for every $(\mathbf{p}, [\mathfrak{l}]; \phi) \in \mathbf{E}^\pm|_{\mathscr{U}}$ and $v \in T_{[\mathfrak{l}]}\mathbf{Ram}$*

$$\nabla^{\mathrm{Ram}}_v \mathbf{D}^\pm(\mathbf{p}, [\mathfrak{l}]; \phi) = [\mathscr{L}^K_{\tilde{v}}, D^{\mathfrak{l}}_{\mathbf{p}}]\phi.$$

Proof. (1) follows by direct computation. By direct inspection, κ_v defined in Definition 3.14 commutes with ε. This implies (2). ∎

Definition 3.61. (Local) chiral residue conditions can be imposed in families as follows:

(1) A **family of chiral residue conditions** $(B, f, \mathbf{R}^\pm)$ consists of:

  (a) a tame Fréchet manifold B,
  (b) a tame smooth map $f\colon B \to \mathbf{Dir} \times \mathbf{Ram}$, and
  (c) a tame Fréchet space subbundle $\mathbf{R}^\pm \subseteq f^*\mathbf{G}^\pm$.

Set

$$\mathbf{E}^\pm_{\mathbf{R}^\pm} := (f^*\mathbf{res}^\pm)^{-1}(\mathbf{R}^\pm)$$

and denote the restriction of $f^*\mathbf{D}$ to $\mathbf{E}^\pm_{\mathbf{R}^\pm}$ by

$$\mathbf{D}^\pm_{\mathbf{R}^\pm}\colon \mathbf{E}^\pm_{\mathbf{R}^\pm} \to f^*\mathbf{F}^\mp.$$

(2) (a) Decompose the universal residue bundle $\check{\mathbf{S}}$ as

$$\check{\mathbf{S}} = \check{\mathbf{S}}^+ \oplus \check{\mathbf{S}}^-$$

into the **universal chiral residue bundles** $\check{\mathbf{S}}^\pm$ over $\mathbf{Br}$.

  (b) A **family of local chiral residue conditions** $(\mathbf{B}, f, \mathbf{V}^\pm)$ consists of:
    (i) a tame Fréchet manifold $\mathbf{B}$,
    (ii) a tame smooth map $f\colon \mathbf{B} \to \mathbf{Dir} \times \mathbf{Ram}$, and
    (iii) a subbundle $\mathbf{V}^\pm \subseteq (p^*f)^*\check{\mathbf{S}}^\pm$.

  Here $f^*p\colon f^*\mathbf{Br} \to \mathbf{B}$ is the pullback of the universal branching locus and $p^*f\colon f^*\mathbf{Br} \to \mathbf{Br}$ is the projection map.

  (c) A family of local chiral residue conditions $(\mathbf{B}, f, \mathbf{V}^\pm)$ defines a family of chiral residue conditions $(\mathbf{B}, f, \mathbf{R}^\pm)$ such that for every $b \in \mathbf{B}$

$$\mathbf{R}^\pm_b := \Gamma(Z_b, \mathbf{V}^\pm|_{Z_b}) \subseteq \Gamma(Z_b, \check{S}^\pm_{\mathfrak{p}_b}) \quad \text{with} \quad (\mathfrak{p}_b; Z_b, [\mathfrak{l}_b]) := f(b). \qquad \bullet$$

It is immediate from the discussion in Section 2.5 that the following variant of Proposition 3.50 holds.

Proposition 3.62. *Let $(\mathbf{B}, f, \mathbf{V}^\pm)$ be a family of local chiral residue conditions such that for every $b \in \mathbf{B}$, $x \in Z_b$, and $v \in T_x Z_b \backslash \{0\}$*

$$\mathbf{V}^\pm_{(b,x)} \oplus J_{\mathfrak{p}_b}\gamma(v)\mathbf{V}^\mp_{(b,x)} = \check{S}^\pm_{\mathfrak{p}_b,x}.$$

Denote by $(\mathbf{B}, f, \mathbf{R}^\pm)$ the corresponding family of chiral residue conditions. If $\mathfrak{a}\colon f^*\mathbf{F}^\pm \to f^*\mathbf{F}^\mp$ is a map of tame Fréchet bundles of order zero, then

$$\mathbf{D}^\pm_{\mathbf{R}^\pm} + \mathfrak{a}\colon \mathbf{E}^\pm_{\mathbf{R}^\pm} \to f^*\mathbf{F}^\mp$$

is uniformly Fredholm. Moreover, for every $b_0 \in \mathbf{B}$ the maps $\pi\colon \mathbf{E}^\pm_{\mathbf{R}^\pm}|_{\mathscr{U}} \to \mathrm{Def}$ and $\iota\colon \mathrm{Ob} \hookrightarrow f^*\mathbf{F}^\mp|_{\mathscr{U}}$ in Definition 3.49 can be chosen such that there is a compact subset $K \subseteq X$ such that for every $b \in \mathscr{U}$ the following hold:

(1) $K \subseteq X \backslash Z_b$;

(2) *if* $\phi \in \mathbf{E}^{\pm}_{\mathbf{R}^{\pm},b}$ *has* $\operatorname{supp}(\phi) \cap K = \varnothing$, *then* $\pi(\phi) = 0$; *and*

(3) $\operatorname{supp}(\iota(o)) \subseteq K$ *for every* $o \in \mathrm{Ob}_b$. ■

Finally, it remains to decompose the universal bundle of admissible $\mathbb{Z}/2\mathbb{Z}$ spinors and the resulting family of local residue conditions.

Definition 3.63. The tame Fréchet space bundles $\mathbf{E}_{00}$ and $\mathbf{F}_{00}$ defined in Definition 3.56 decompose into

$$\mathbf{E}_{00} = \mathbf{E}^{+}_{00} \oplus \mathbf{E}^{-}_{00} \quad \text{and} \quad \mathbf{F}_{00} = \mathbf{F}^{+}_{00} \oplus \mathbf{F}^{-}_{00}$$

with $\mathbf{E}^{\pm}_{00}$ denoting the **universal bundle of admissible chiral $\mathbb{Z}/2\mathbb{Z}$ spinors**. The latter contains the open subset

$$\mathbf{E}^{\pm,*}_{00} \subseteq \mathbf{E}^{\pm}_{00}$$

of **non-degenerate** admissible chiral $\mathbb{Z}/2\mathbb{Z}$ spinors. For every $(\mathbf{p}, [\mathfrak{l}]; \Phi) \in \mathbf{E}^{\pm,*}_{00}$, the map $\mathbf{L}_{\Phi}$ constructed in Proposition 3.55 corestricts to

$$\mathbf{L}^{\pm}_{\Phi} \in \Gamma(Z, \operatorname{Hom}_{\mathbb{A}}(\overline{NZ}, \check{S}^{\pm}_{\mathbf{p}})).$$

The images define a family of local chiral residue conditions

$$(\mathbf{E}^{\pm,*}_{00}, f, \mathbf{V}^{\pm}).$$

If $\operatorname{rk}_{\mathbb{A}} \check{S}^{\pm} = 2$, this family satisfies the hypothesis of Proposition 3.62. •

4 The constructions of the tame Fréchet manifold structures

Using the framework laid out in Section 3, this section constructs tame Fréchet manifold structures on $\mathbf{PHarm}^*$, $\mathbf{PEig}^*$, and $\mathbf{PHarm}^{\pm,*}$, thereby establishing Theorem 1.1, Theorem 1.6, and Theorem 1.7. The final two subsections derive Theorem 1.9 and Theorem 1.12.

4.1 Uniformly Fredholm maps

The results to be proved in this section refer to the following notion.

Definition 4.1. Let $\mathbf{X}, \mathbf{Y}$ be tame Fréchet manifolds. A tame smooth map $f\colon \mathbf{X} \to \mathbf{Y}$ is **uniformly Fredholm** if for every $x \in \mathbf{X}$ there are a tame Fréchet space $\mathbf{F}$, a chart $\phi\colon U \to \tilde{U} \subseteq \mathbf{F} \times \mathbb{R}^k$ of $\mathbf{X}$ with $x \in U$, a chart $\psi\colon V \to \tilde{V} \subseteq \mathbf{F} \times \mathbb{R}^\ell$ of $\mathbf{Y}$ with $f(x) \in V$, and a smooth map $F\colon \tilde{U} \to \mathbb{R}^\ell$ such that

$$\psi \circ f \circ \phi^{-1}(y, z) = (y, F(y, z)). \quad \bullet$$

In this case, $k - \ell$ is independent of the choice of charts and defines the **index** of f as a locally constant map

$$\operatorname{index} f\colon \mathbf{X} \to \mathbb{Z}.$$

Remark 4.2. A uniformly Fredholm map for which one can always choose $\ell = 0$ is nothing but a **uniform submersion** in the sense of [Glö16, p. 2].[2] Variations of the above notion are discussed in [Die19, §2.2.3; DR22, §3.3]. ♣

The significance of uniformly Fredholm maps arises from the fact that they are compatible with transverse base change.

Proposition 4.3. *Let* $\mathbf{X}$, $\mathbf{Y}$, *and* $\mathbf{Z}$ *be tame Fréchet manifolds,* $f\colon \mathbf{X} \to \mathbf{Y}$ *uniformly Fredholm and* $g\colon \mathbf{Z} \to \mathbf{Y}$ *tame smooth. If* f *and* g *are* ***transverse****; that is: for every* $(x,z) \in \mathbf{X} \times \mathbf{Z}$ *with* $f(x) = g(z) =: y$ *the map* $T_x f + T_z g\colon T_x\mathbf{X} \oplus T_z\mathbf{Z} \to T_y\mathbf{Y}$ *is surjective; then:*

(1) *the fibre product*
$$g^*\mathbf{X} := \big\{(x,z) \in \mathbf{X} \times \mathbf{Z} : f(x) = g(z)\big\}$$
is a tame Fréchet submanifold, and

(2) *the map* $g^*f\colon g^*\mathbf{X} \to \mathbf{Z}$ *is uniformly Fredholm and* $\operatorname{index} g^*f = (\operatorname{index} f) \circ f^*g$.

Proof. It suffices to consider the situation in which $\mathbf{F}$ is a tame Fréchet space, $\mathbf{X}$ is an open subset of $\mathbf{F} \times \mathbb{R}^k$, $\mathbf{Y}$ is an open subset of $\mathbf{F} \times \mathbb{R}^\ell$, and $f(x_0, x_1) = (x_0, F(x_0, x_1))$. By direct inspection,
$$g^*\mathbf{X} = \big\{(z, x_0, x_1) \in \mathbf{Z} \times \mathbf{F} \times \mathbb{R}^k : x_0 = \mathrm{pr}_1 \circ g(z), F(\mathrm{pr}_1 \circ g(z), x_1) = \mathrm{pr}_2 \circ g(z)\big\}.$$
By the transversality assumption, at every $(z, x_0, x_1) \in g^*\mathbf{X}$, the derivative of the map $(z, x_1) \mapsto F(\mathrm{pr}_1 \circ g(z), x_1) - \mathrm{pr}_2 \circ g(z)$ is surjective. Therefore, the assertion follows from the version of the regular value theorem stated in [Ham82, Part III Theorem 2.3.1]. ■

The hypothesis in Proposition 4.3 is generic according to the following parametric transversality result.

Proposition 4.4. *Let* $\mathbf{X}$, $\mathbf{Y}$, *and* $\mathbf{P}$ *be second-countable tame Fréchet manifolds,* Z *a second-countable (finite-dimensional) manifold, and* $f\colon \mathbf{X} \to \mathbf{Y}$ *uniformly Fredholm. If* $G\colon \mathbf{P} \times Z \to \mathbf{Y}$ *is transverse to* f*, in particular, if it is a* ***naive submersion****; that is:* $T_{(p,z)}G\colon T_p\mathbf{P} \oplus T_z Z \to T_{G(p,z)}\mathbf{Y}$ *is surjective for every* $(p,z) \in \mathbf{P} \times Z$*, then the subset*
$$\mathbf{P}^{\pitchfork} := \big\{p \in \mathbf{P} : g_p := G(p, -)\colon Z \to \mathbf{Y} \text{ is transverse to } f\big\}$$
is comeager.

The proof relies on the following observation.

Theorem 4.5 (Smale [Sma65]). *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be second-countable tame Fréchet manifolds. If* $f\colon \mathbf{X} \to \mathbf{Y}$ *is uniformly Fredholm, then its set of regular values*
$$\mathrm{RegVal}(f) := \{y \in \mathbf{Y} : T_x f \text{ is surjective for every } x \in f^{-1}(y)\}$$
is comeager.

Proof. As pointed out by [DR22, Remark 3.4], the proof in [Sma65] carries over to the present situation. ■

[2] Glöckner does not use the adjective *uniform*.

Proof of Proposition 4.4. By Proposition 4.3, $G^*\mathbf{X}$ is a second-countable tame Fréchet submanifold, and $G^*f\colon G^*\mathbf{X} \to \mathbf{P} \times Z$ is uniformly Fredholm. Let $(p,z,x) \in G^*\mathbf{X}$ and set $y := f(x)$. The Snake Lemma applied to

$$\begin{tikzcd}
T_zZ \oplus T_x\mathbf{X} \arrow[r, hook] \arrow[d, "T_zg_p - T_xf"'] & T_p\mathbf{P} \oplus T_zZ \oplus T_x\mathbf{X} \arrow[r, two heads] \arrow[d, "T_{(p,z)}G - T_xf"] & T_p\mathbf{P} \\
T_y\mathbf{Y} \arrow[r, equal] & T_y\mathbf{Y} &
\end{tikzcd}$$

yields an exact sequence

$$T_{(p,z,x)}G^*\mathbf{X} \xrightarrow{T_{(p,z,x)}(\mathrm{pr}_\mathbf{P} \circ G^*f)} T_p\mathbf{P} \twoheadrightarrow \operatorname{coker}(T_zg_p - T_xf).$$

Therefore, $p \in \mathbf{P}^{\pitchfork}$ if and only if it is a regular value of $\mathrm{pr}_\mathbf{P} \circ G^*f\colon G^*\mathbf{X} \to \mathbf{P}$. Since Z is finite-dimensional, $\mathrm{pr}_\mathbf{P} \circ G^*f$ is uniformly Fredholm. Therefore, the assertion follows from Theorem 4.5. ■

Remark 4.6. There is a version of Theorem 4.5 for uniformly left semi-Fredholm maps [Qui70]. Using this the hypothesis that Z is finite-dimensional can be dropped. ♣

4.2 The Nash–Moser submersion theorem

The proofs of Theorem 1.1, Theorem 1.6, and Theorem 1.7 use the following Nash–Moser version of the submersion theorem.

Theorem 4.7. *Let $\mathbf{X}$ and $\mathbf{B}$ be tame Fréchet manifolds, $\mathbf{V}$ a tame Fréchet space bundle over $\mathbf{X}$, $\mathrm{pr}_\mathbf{B}\colon \mathbf{X} \to \mathbf{B}$ a uniform submersion, $s \in \Gamma(\mathbf{X},\mathbf{V})$ a tame smooth section, and ∇ a partial connection on $\mathbf{V}$ with respect to the vertical tangent bundle $T\mathbf{X}/\mathbf{B} := \ker T\mathrm{pr}_\mathbf{B} \subseteq T\mathbf{X}$. If there is a finite rank vector bundle* Def *over $\mathbf{X}$ and a map of tame Fréchet space bundles $\pi\colon T\mathbf{X}/\mathbf{B} \twoheadrightarrow \mathrm{Def}$ such that the thickening*

$$\begin{pmatrix} \nabla s \\ \pi \end{pmatrix} \colon T\mathbf{X}/\mathbf{B} \to \mathbf{V} \oplus \mathrm{Def} \tag{4.8}$$

is an isomorphism of tame Fréchet space bundles, then:

(1) *the zero locus*

$$\mathbf{Z} := s^{-1}(0) \subseteq \mathbf{X}$$

is a tame Fréchet submanifold,

(2) *the projection $\mathrm{pr}_\mathbf{B}\colon \mathbf{Z} \to \mathbf{B}$ is a uniform submersion, and*

(3) $T\mathbf{Z}/\mathbf{B} = \ker T(\mathrm{pr}_\mathbf{B}|_\mathbf{Z}) \cong \mathrm{Def}|_\mathbf{Z}$.

Proof. Since the statement is local, it suffices to consider the situation in which $\mathbf{X} = \mathbf{U} \times \mathbf{B}$ is the product of open neighbourhoods of the origin in two tame Fréchet spaces, and $\mathbf{V}$ is a product bundle, whose fibre we continue to denote by $\mathbf{V}$. In this case,

$$s\colon \mathbf{U} \times \mathbf{B} \to \mathbf{V}$$

is a tame smooth map, and we may further assume that $(0,0) \in s^{-1}(0)$.

The assumptions imply that $\mathrm{Def} \cong \ker(\nabla s)$; let Def_0 denote the fibre over $(0,0)$, and let $\pi_0\colon T\mathbf{U} \to \mathrm{Def}_0$ be a projection to this fibre. For an open neighbourhood of the origin $\mathbf{K} \subseteq \mathrm{Def}_0$, consider the family of augmented maps

$$\bar{s}\colon\ \mathbf{X} \times \mathbf{K} \to \mathbf{V} \oplus \mathrm{Def}_0$$

defined by $\bar{s}(x,k) := (s(x), \pi_0(x) - k)$, where $\pi_0(x)$ is understood by viewing $\mathbf{U} \subseteq T\mathbf{U}$ as a subset. The covariant derivative

$$\nabla\bar{s} := (\nabla s, \pi_0)\colon\ T\mathbf{X}/\mathbf{B} \to \mathbf{V} \oplus \mathrm{Def}_0$$

is an isomorphism; indeed, viewing $\mathrm{Def} \subseteq T\mathbf{U}$ as $\ker(\nabla s)$ as above, $\pi_0\colon\ \mathrm{Def} \to \mathrm{Def}_0$ is a map of finite-rank vector bundles that is the identity over the origin, hence we may assume it is an isomorphism after shrinking $\mathbf{U}$, $\mathbf{B}$, and $\mathbf{K}$.

Let $G\colon\ \mathbf{V} \to T\mathbf{U}$ denote the restriction of the inverse of (4.8) to the first component in the codomain, which is a tame smooth family of right inverses of ∇s. Then

$$\overline{G}\colon\ \mathbf{V} \oplus \mathrm{Def}_0 \to T\mathbf{X}/\mathbf{B} \quad \text{with} \quad \overline{G}_{(u,b)}(v,q) := G_{(u,b)}v + \pi_0^{-1}\big(q - \pi_0 G_{(u,b)}v\big)$$

is a tame smooth family of inverses for $\nabla\bar{s}$. Moreover,

$$(T\bar{s})_{(u,b)}\overline{G}_{(u,b)}(v,q) = (v,q) - \big(\Gamma_{(u,b)}\big(\overline{G}_{(u,b)}(v,q)\big)s, 0\big).$$

Thus $\overline{G}$ is an inverse of $T\bar{s}$ up to an $\bar{s}$–quadratic error term. [Ham82, Part III Theorem 3.3.4] then applies to show that there is a tame smooth map $\Phi\colon\ \mathbf{B} \times \mathbf{K} \to \mathbf{X}$ whose value at (b,k) is the unique solution to

$$s(\Phi(b,k)) = 0 \quad \text{and} \quad \pi_0\big(\mathrm{pr}_{\mathbf{U}}\Phi(b,k)\big) = k$$

in $\mathbf{X}$. By construction, $(\mathrm{pr}_{\mathbf{B}}, \pi_0 \circ \mathrm{pr}_{\mathbf{U}}) \circ \Phi = \mathrm{id}$, hence Φ provides a chart on $\mathbf{Z} = s^{-1}(0)$ which is already of the form in Definition 4.1. This shows (1) and (2). The same composition in the second factor alone shows that $T\Phi(b,-)\colon\ \mathrm{Def}_0 \to T\mathbf{Z}/\mathbf{B}$ is an isomorphism, which shows (3). ■

4.3 The universal space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors

This subsection proves Theorem 1.1; that is: assuming that $\dim X = 3$ and $\mathrm{rk}\, S = 4$, the space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors

$$\mathbf{Harm}^* := \big\{(\mathfrak{p}, [\mathfrak{l}]; \Phi) \in \mathbf{E}_0 : D^{\mathfrak{l}}_{\mathfrak{p}}\Phi = 0 \text{ and } \Phi \text{ is non-degenerate}\big\}$$

shall be equipped with the structure of a tame Fréchet manifold such that the projection map $\mathrm{pr}_{\mathbf{Dir}}\colon\ \mathbf{Harm}^* \to \mathbf{Dir}$ is uniformly Fredholm of index zero. Since

$$\mathbf{PHarm}^* = \mathbf{Harm}^*/\mathbb{R}^\times,$$

this immediately implies Theorem 1.1. Of course, throughout the following four subsubsections it is assumed that $\dim X = 3$ and $\mathrm{rk}\, S = 4$.

4.3.1 Local thickenings of the universal space of $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors

The following is the essential step in the proof of Theorem 1.1.

Proposition 4.9. *For every* $(\mathbf{p}_0, [\mathfrak{l}_0]; \Phi_0) \in \mathbf{Harm}^*$ *there are an open subset* $\mathscr{U} \subseteq \mathbf{E}^*_{00}$ *with* $(\mathbf{p}_0, [\mathfrak{l}_0]; \Phi_0) \in \mathscr{U}$ *and a finite rank subbundle* $\mathrm{Ob} \subseteq f^*\mathbf{F}_{00}|_{\mathscr{U}}$ *such that:*

(1) $\mathscr{V} := \mathrm{pr}_{\mathbf{Dir}}(\mathscr{U}) \subseteq \mathbf{Dir}$ *is open,*

(2) *the local thickening*

$$\mathbf{Harm}^{\mathbf{t}} := \left\{(\mathbf{p}, [\mathfrak{l}]; \Phi) \in \mathscr{U} : D^{\mathfrak{l}}_{\mathbf{p}}\Phi \in \mathrm{Ob}\right\} \subseteq \mathscr{U}$$

is a tame Fréchet manifold, and

(3) *the projection map* $\mathrm{pr}_{\mathscr{V}} \colon \mathbf{Harm}^{\mathbf{t}} \to \mathscr{V}$ *is a uniform submersion.*

Proof. The proof consists of five steps. Denote by $(\mathbf{E}^*_{00}, f, \mathbf{R})$ the residue condition from Proposition 3.57. Let $(\mathbf{p}_0, [\mathfrak{l}_0]; \Phi_0) \in \mathbf{Harm}^*$.

Step 1. *Choice of* $\mathscr{U}$ *and* $\mathrm{Ob} \subseteq \mathbf{F}_{00}|_{\mathscr{U}}$.

By Proposition 3.50 and Proposition 3.57, there are an open neighbourhood $(\mathbf{p}_0, [\mathfrak{l}_0]; \Phi_0) \in \mathscr{U} \subseteq \mathbf{E}^*_{00}$, finite rank vector bundles Def and Ob over $\mathscr{U}$, and maps of tame Fréchet space bundles $\pi \colon \mathbf{E}_{\mathbf{R}}|_{\mathscr{U}} \twoheadrightarrow \mathrm{Def}$ and $\iota \colon \mathrm{Ob} \hookrightarrow f^*\mathbf{F}|_{\mathscr{U}}$ such that

$$\begin{pmatrix} \mathbf{D}_{\mathbf{R}} & \iota \\ \pi & 0 \end{pmatrix}$$

is invertible. Moreover, Proposition 3.50 guarantees that ι can be chosen so that it factors through $\jmath \colon \mathrm{Ob} \hookrightarrow f^*\mathbf{F}_{00}|_{\mathscr{U}}$; in fact, the $\iota(o)$ are always supported away from Z. Henceforth, Ob shall be regarded as a subbundle of $f^*\mathbf{F}_{00}|_{\mathscr{U}}$.

Since $\mathrm{pr}_{\mathbf{Dir}} \colon \mathbf{E}^*_{00} \to \mathbf{Dir}$ is a submersion, $\mathscr{V} := \mathrm{pr}_{\mathbf{Dir}}(\mathscr{U}) \subseteq \mathbf{Dir}$ is open; therefore, (1) holds. The task at hand is to prove that, after possibly shrinking $\mathscr{U}$, (2) and (3) hold with the help of Theorem 4.7.

Step 2. *Setup for Theorem 4.7.*

Consider the tame Fréchet space bundle

$$\mathbf{Q} := f^*\mathbf{F}_{00}/\mathrm{Ob}$$

over $\mathscr{U}$. Define the section $s \in \Gamma(\mathscr{U}, \mathbf{Q})$ by

$$s(\mathbf{p}, [\mathfrak{l}]; \Phi) := D^{\mathfrak{l}}_{\mathbf{p}}\Phi \pmod{\mathrm{Ob}_{\mathbf{p},[\mathfrak{l}];\Phi}}.$$

Evidently, $\mathbf{Harm}^{\mathbf{t}} = s^{-1}(0)$.

The vertical tangent bundle of $\mathscr{U}$ fits into the short exact sequence

$$f^*\mathbf{E}_{00} \hookrightarrow T\mathscr{U}/\mathbf{Dir} \twoheadrightarrow f^*\mathscr{F}^{\mathrm{Ram}}.$$

Choose a local infinitesimal uniformiser and define ∇^{Ram} as in Definition 3.38. This splits the above short exact sequence and defines a partial covariant derivative on $\mathbf{Q}$ such that, for every $(\mathbf{p},[\mathfrak{l}];\Phi)\in\mathscr{U}$

$$
\begin{aligned}
(\nabla s)_{(\mathbf{p},[\mathfrak{l}];\Phi)}\colon\ & \mathbf{E}_{00;\mathbf{p},[\mathfrak{l}]}\oplus T_{[\mathfrak{l}]}\mathbf{Ram}\to \mathbf{Q}_{\mathbf{p},[\mathfrak{l}];\Phi} \\
& (\phi,v)\mapsto D_{\mathbf{p}}^{\mathfrak{l}}\phi+(\nabla_v^{\mathrm{Ram}}\mathbf{D})(\mathbf{p},[\mathfrak{l}];\Phi)\pmod{\mathrm{Ob}_{\mathbf{p},[\mathfrak{l}];\Phi}}.
\end{aligned}
\tag{4.10}
$$

It can be arranged that the subsets $\operatorname{supp}(\nabla_v^{\mathrm{Ram}}\mathbf{D})(\mathbf{p},[\mathfrak{l}];\Phi)$ are always disjoint from the compact subset in which the elements of Ob are supported.

The task at hand is to prove that, after possibly shrinking $\mathscr{U}\ni(\mathbf{p}_0,[\mathfrak{l}_0];\Phi_0)$, the thickening

$$
\begin{pmatrix}\nabla s\\ \pi\end{pmatrix}
$$

is an isomorphism of tame Fréchet space bundles and thus the hypotheses of Theorem 4.7 are satisfied. The latter directly implies (2) and (3).

Step 3. *Reduction to the invertibility of a lift.*

It is evident from (4.10) that ∇s lifts to

$$
\widetilde{\nabla s}\colon f^*(\mathbf{E}_0\oplus\mathscr{F}^{\mathrm{Ram}})\to\tilde{\mathbf{Q}}\coloneqq f^*\mathbf{F}|_{\mathscr{U}}/\mathrm{Ob}
$$

such that the following diagram commutes

$$
\begin{tikzcd}
f^*(\mathbf{E}_{00}\oplus\mathscr{F}^{\mathrm{Ram}}) \arrow[r, "(\nabla s,\pi)"] \arrow[d, hook] & \mathbf{Q}\oplus\mathrm{Def} \arrow[d, hook] \\
f^*(\mathbf{E}_0\oplus\mathscr{F}^{\mathrm{Ram}}) \arrow[r, "(\widetilde{\nabla s},\pi)"'] & \tilde{\mathbf{Q}}\oplus\mathrm{Def}.
\end{tikzcd}
$$

By Proposition 3.52 and Remark 3.40, for every $(\mathbf{p},[\mathfrak{l}];\Phi)\in\mathbf{E}_{00}$, $(\nabla_v\mathbf{D})(\mathbf{p},[\mathfrak{l}];\Phi)\in\mathbf{F}_{00;\mathbf{p},[\mathfrak{l}]}$. Therefore, if $(\phi,v)\in\mathbf{E}_{0;\mathbf{p},[\mathfrak{l}]}\oplus\mathscr{F}^{\mathrm{Ram}}_{\mathbf{p},[\mathfrak{l}]}$ and $\psi\in\mathbf{F}_{00;\mathbf{p},[\mathfrak{l}]}$ satisfy

$$
D_{\mathbf{p}}^{\mathfrak{l}}\phi+\nabla_v^{\mathrm{Ram}}\mathbf{D}(\mathbf{p},[\mathfrak{l}];\Phi)=\psi\pmod{\mathrm{Ob}_{\mathbf{p},[\mathfrak{l}];\Phi}},
$$

then $D_{\mathbf{p}}^{\mathfrak{l}}\phi\in\mathbf{F}_{00;\mathbf{p},[\mathfrak{l}]}$. As a consequence, if $(\widetilde{\nabla s},\pi)$ is an isomorphism of tame Fréchet space bundles, then $(\nabla s,\pi)$ is an isomorphism of tame Fréchet space bundles.

The next two steps verify that $(\widetilde{\nabla s},\pi)$ is an isomorphism of tame Fréchet space bundles, after possibly shrinking $\mathscr{U}$.

Step 4. *A slight change of perspective.*

Define $\Theta_{\mathrm{ext}},\Theta_K\colon f^*(\mathbf{E}_0\oplus\mathscr{F}^{\mathrm{Ram}})\to\mathbf{E}_{\mathbf{R}}$ by

$$
\begin{aligned}
\Theta_{\mathrm{ext}}(\mathbf{p},[\mathfrak{l}],\Phi;\phi,v)&\coloneqq(\mathbf{p},[\mathfrak{l}];\Phi;\phi-\mathrm{ext}(\mathbf{L}_\Phi v))\quad\text{and}\\
\Theta_K(\mathbf{p},[\mathfrak{l}],\Phi;\phi,v)&\coloneqq(\mathbf{p},[\mathfrak{l}];\Phi;\phi-\mathscr{L}_{\tilde v}^K\Phi).
\end{aligned}
$$

By the discussion in Section 3.6, Θ_{ext} is an isomorphism of tame Fréchet space bundles with inverse

$$\Theta_{\mathrm{ext}}^{-1}(\mathbf{p},[\mathfrak{l}],\Phi;\psi) := (\mathbf{p},[\mathfrak{l}],\Phi;\psi - \mathrm{ext}(\mathrm{res}(\psi)), -\mathbf{L}_\Phi^{-1}\,\mathrm{res}(\psi)).$$

By direct computation,

$$\Theta_{\mathrm{ext}}^{-1} \circ \Theta_K(\mathbf{p},[\mathfrak{l}],\Phi;\phi,v) = (\mathbf{p},[\mathfrak{l}],\Phi;\phi - \Psi_\Phi(v),v)$$

with

$$\Psi_\Phi(v) := \mathscr{L}_{\bar{v}}^K\Phi - \mathrm{ext}(\mathbf{L}_\Phi v).$$

By construction, $\mathrm{res}(\Psi_\Phi(v)) = \mathrm{res}(\mathscr{L}_{\bar{v}}^K\Phi) - \mathrm{res}(\mathrm{ext}(\mathbf{L}_\Phi v)) = 0$. Evidently, $\Theta_{\mathrm{ext}}^{-1} \circ \Theta_K$ is an isomorphism of tame Fréchet space bundles; hence, so is Θ_K.

From Proposition 3.39 and (4.10) it follows that

$$\widetilde{\nabla s} \circ \Theta_K^{-1} = \mathbf{D}_{\mathbf{R}} + \mathbf{P} \pmod{\mathrm{Ob}}$$

with the perturbation $\mathbf{P}$ defined by

$$\mathbf{P}(\mathbf{p},[\mathfrak{l}],\Phi;\psi) := (\mathbf{p},[\mathfrak{l}],\Phi;\mathscr{L}_{\bar{v}_\Phi(\psi)}^K D_{\mathbf{p}}^{\mathfrak{l}}\Phi) \quad \text{and} \quad \bar{v}_\Phi(\psi) := \bar{v}(\mathbf{p},[\mathfrak{l}],-\mathbf{L}_\Phi^{-1}\,\mathrm{res}(\psi)).$$

Step 5. *Proof of invertibility.*

By construction, the thickening

$$\overline{\mathbf{D}}_{\mathbf{R}} := \begin{pmatrix} \mathbf{D}_{\mathbf{R}} \\ \pi \end{pmatrix} \pmod{\mathrm{Ob}}$$

is an isomorphism of tame Fréchet space bundles and its inverse $\overline{\mathbf{D}}_{\mathbf{R}}^{-1}$ satisfies tame estimates of the form

$$\|\overline{\mathbf{D}}_{\mathbf{R}}^{-1}(\psi,\delta)\|_{H_a^{k+1}} \lesssim \|\psi\|_{H_b^k} + \|\delta\|.$$

By Proposition 3.54, $\mathbf{P}$ is a small perturbation in the sense of [DL25, Appendix A]. Therefore, as a consequence of [DL25, Proposition A.5], $\overline{\mathbf{D}}_{\mathbf{R}} + \mathbf{P}$ is invertible. ■

4.3.2 Transversality of the obstruction map

Proposition 4.11. *In the situation of Proposition 4.9 the* ***obstruction map*** $\mathrm{ob}\colon \mathbf{Harm}^{\mathfrak{t}} \to \mathrm{Ob}$ *defined by*

$$\mathrm{ob}(\mathbf{p},[\mathfrak{l}];\Phi) := D_{\mathbf{p}}^{\mathfrak{l}}\Phi$$

is transverse to the zero section.

Proof of Proposition 4.11. Let $(\mathbf{p},[\mathfrak{l}];\Phi) \in \mathrm{ob}^{-1}(0) \subseteq \mathbf{Harm}^*$. By Proposition 3.30, for every $\dot{\nabla} \in \Omega^1(X,\mathfrak{o}_{C\ell}(S))$, $\dot{\mathbf{p}}_\nabla = (0;0;\dot{\nabla};0) \in T_{(\mathbf{p},[\mathfrak{l}])}\mathbf{Sl}$ and

$$\nabla_{\dot{\mathbf{p}}_\nabla}^{\mathbf{Sl}} \mathbf{D}(\mathbf{p},[\mathfrak{l}];\Phi) = \sum_{i=1}^{3} \gamma(e_i)\dot{\nabla}_{e_i}\Phi.$$

There is a compact subset $K \subseteq X\backslash Z$ with non-empty interior on which Φ does not vanish. By Remark 3.7 and Lemma 4.12, stated after this proof, for every $\psi \in \Gamma(X\backslash Z, S \otimes \mathfrak{l})$ with $\mathrm{supp}(\psi) \subseteq K$ there is a $\dot{\nabla}$ such that

$$\sum_{i=1}^{3} \gamma(e_i)\dot{\nabla}_{e_i}\Phi = \psi.$$

Since $\mathrm{pr}_{\mathbf{Dir}} \colon \mathbf{Harm}^{\mathbf{t}} \to \mathbf{Dir}$ is a submersion, there is a lift $\Omega^1(X, \mathfrak{o}_{C\ell}(S)) \to T_{(\mathbf{p},[\mathfrak{l}];\Phi)}\mathbf{Harm}^{\mathbf{t}}$. From the above it is clear that the composition

$$\Omega^1(X, \mathfrak{o}_{C\ell}(S)) \hookrightarrow T_{(\mathbf{p},[\mathfrak{l}];\Phi)}\mathbf{Harm}^{\mathbf{t}} \xrightarrow{T_{(\mathbf{p},[\mathfrak{l}];\Phi)}\mathrm{ob}} T_{(\mathbf{p},[\mathfrak{l}];\Phi;0)}\mathrm{Ob} \twoheadrightarrow \mathrm{Ob}_{(\mathbf{p},[\mathfrak{l}];\Phi)}$$

is surjective. Therefore, ob is transverse to the zero section. ■

Lemma 4.12. *For every $\phi \in \mathbb{H}\backslash\{0\}$ and $\psi \in \mathbb{H}$ there are $\xi, \eta, \zeta \in \mathrm{Im}\,\mathbb{H}$ such that*

$$i\phi\xi + j\phi\eta + k\phi\zeta = \psi.$$

Proof. By direct inspection, it is possible to choose $\tilde{\xi}, \tilde{\eta}, \tilde{\zeta} \in \mathrm{Im}\,\mathbb{H}$ such that $i\tilde{\xi} + j\tilde{\eta} + k\tilde{\zeta} = \psi\phi^{-1}$. Set $\xi := \phi^{-1}\tilde{\xi}\phi$, $\eta := \phi^{-1}\tilde{\eta}\phi$, $\zeta := \phi^{-1}\tilde{\zeta}\phi$. ■

Proof of Theorem 1.1. It is an immediate consequence of Proposition 4.9, Proposition 4.11, and [Ham82, Part III, Theorem 2.3.1] that $\mathbf{Harm}^* \subseteq \mathbf{E}^*_{00}$ is a tame Fréchet submanifold and that $\mathrm{pr}_{\mathbf{Dir}} \colon \mathbf{Harm}^* \to \mathbf{Dir}$ is uniformly Fredholm.

By [BW26, Corollary 2.30], the family of residue conditions introduced in Proposition 3.57 is Lagrangian. As a consequence $\mathrm{rk}\,\mathrm{Def} = \mathrm{rk}\,\mathrm{Ob}$ and, therefore, $\mathrm{pr}_{\mathbf{Dir}}$ has index zero. This directly implies Theorem 1.1. ■

Remark 4.13. The proof of Theorem 1.1 identifies the vertical tangent spaces of $\mathbf{Harm}^*$: indeed, if $(\mathbf{p}, [\mathfrak{l}]; \Phi) \in \mathbf{Harm}^*$, then identifying $\mathbf{E}_{0;\mathbf{p},[\mathfrak{l}]} \oplus T_{[\mathfrak{l}]}\mathbf{Ram} = \mathbf{E}_{R;\mathbf{p},[\mathfrak{l}];\Phi}$

$$\ker T_{(\mathbf{p},[\mathfrak{l}];\Phi)}\mathrm{pr}_{\mathbf{Dir}} = \ker D_{R_\Phi}. \quad ♣$$

Of course, the corresponding vertical tangent space of $\mathbf{PHarm}^*$ is $\ker D_{R_\Phi}/\mathbb{R}\langle\Phi\rangle$.

4.3.3 A transversality criterion

Here is how to verify whether or not a smooth map $\Gamma \colon B \to \mathbf{Dir}$ is transverse to $\mathrm{pr}_{\mathbf{Dir}} \colon \mathbf{PHarm}^* \to \mathbf{Dir}$.

Definition 4.14. Let $(\mathbf{p}, [\mathfrak{l}]) \in \mathbf{Dir} \times \mathbf{Ram}$ and $\phi, \psi \in \Gamma(X\backslash\mathrm{Br}(\mathfrak{l}), S \otimes \mathfrak{l})$.

(1) The **stress-energy tensor** $T_{\phi,\psi} \in \Gamma(X\backslash Z, \mathrm{Hom}(S^2TX, \mathbb{R}))$ is defined by

$$T_{\phi,\psi}(v, w) := -\frac{1}{8}(\langle\gamma(v)\nabla_w\phi, \psi\rangle + \langle\gamma(w)\nabla_v\phi, \psi\rangle + \langle\gamma(v)\nabla_w\psi, \phi\rangle + \langle\gamma(w)\nabla_v\psi, \phi\rangle).$$

(2) Define $\mu(\phi,\psi) \in \Omega^1(X\backslash Z, \mathfrak{o}_{C\ell}(S))$ by

$$\mu(\phi,\psi) \coloneqq \frac{1}{2}\bar{\gamma}^*(\phi\langle\psi,-\rangle + \psi\langle\phi,-\rangle)$$

with $\bar{\gamma}^*$ denoting the adjoint of the linear map $\bar{\gamma}\colon T^*X \otimes \mathfrak{o}_{C\ell}(S) \to \mathrm{Sym}(S)$ defined by $\bar{\gamma}(\alpha\otimes\xi) \coloneqq \gamma(\alpha^\sharp)\xi$. •

Proposition 4.15. *Let B be a manifold and $\Gamma\colon B \to$ **Dir** a smooth map. The following are equivalent:*

(1) *Γ is transverse to $\mathrm{pr}_{\mathbf{Dir}}\colon$ $\mathbf{PHarm}^* \to \mathbf{Dir}$.*

(2) *For every $b \in B$ and $(\mathbf{p}, [\mathfrak{l}]; \Phi) \in \mathbf{Harm}^*$ with $\Gamma(b) = \mathbf{p}$ the map*

$$\delta_\Phi\colon\ T_bB \to (\ker D_{R_\Phi})^*$$

defined by

$$\delta_\Phi(v)\kappa \coloneqq \big\langle T_{\Phi,\kappa}, T_{\mathbf{p}}\mathrm{pr}_{\mathbf{Met}} \circ T_b\Gamma(v)\big\rangle_{L^2} + \big\langle \mu(\Phi,\kappa), \mathrm{pr}_{V_\nabla} \circ T_b\Gamma(v)\big\rangle_{L^2}$$

is surjective. Here R_Φ denotes the residue condition induced by Φ.

Proof. Let $b \in B$ and $(\mathbf{p}, [\mathfrak{l}]; \Phi) \in \mathbf{Harm}^*$ with $\Gamma(b) = \mathbf{p}$. In the construction of $\mathbf{Harm}^{\mathbf{t}}$, Ob can be chosen such that the L^2 orthogonal projection Π onto $\ker D_{R_\Phi}$ induces an isomorphism $\mathrm{Ob}_{(\mathbf{p},[\mathfrak{l}];\Phi)} \cong \ker D_{R_\Phi}$. In particular, as a consequence of the construction of $\mathbf{Harm}^*$, Γ and $\mathrm{pr}_{\mathbf{Dir}}$ are transverse at b and $(\mathbf{p}, [\mathfrak{l}]; \Phi)$ precisely if the map

$$\begin{aligned} T_bB &\to \ker D_{R_\Phi} \\ v &\mapsto \Pi\nabla_{T_b\Gamma(v)}\mathbf{D}(\mathbf{p}, [\mathfrak{l}]; \Phi) \end{aligned}$$

is surjective. By direct inspection this is equivalent to δ_Φ being surjective; cf. [Mai97, Proof of Proposition 4.1] or [AD25, Proof of Lemma 3.2]. Evidently, the $V_{\mathscr{G}}$ component of $T_b\Gamma(v)$ does not contribute. ■

Remark 4.16. The (dual of the) condition (2) is a **Petri condition**; cf. [Wen23, §5; DW23, Definition 1.1.11]. Here are some comments on the usefulness of the contributions to δ_Φ to verify this condition:

(1) The map $\mu(\Phi,-)\colon\ \Gamma(X\backslash Z, S\otimes\mathfrak{l}) \to \Omega^1(X\backslash Z, \mathfrak{o}_{C\ell}(S))$ factors through a linear map $S\otimes\mathfrak{l} \to \mathrm{Hom}(TX, \mathfrak{o}_{C\ell}(S))|_{X\backslash Z}$ which is injective on the dense open subset $X\backslash\Phi^{-1}(0)$ by Lemma 4.12. Of course, this is what enabled the proof of Proposition 4.11 and makes the second contribution to δ_Φ quite useful.

(2) The map $T_{\Phi,-}\colon\ \Gamma(X\backslash Z, S\otimes\mathfrak{l}) \to \Gamma(X\backslash Z, \mathrm{Hom}(S^2TX,\mathbb{R}))$ factors through $J^1(S\otimes\mathfrak{l}) \to \mathrm{Hom}(S^2TX,\mathbb{R})|_{X\backslash Z}$. If $T_{\Phi,\Phi} = 0$, then g is conformally flat by [Mai97, Proof of Theorem 1.2]. Therefore, the first contribution to δ_Φ might not always be useful. However, it is quite possible that for generic metrics it is. Maybe, the methods developed by [GK26] can be brought to bear on this.

The above issue also is what stands in the way of establishing a version of Theorem 1.1 for $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors on Dirac bundles arising from a (topological) spin structure on X. ♣

4.3.4 The universal zero locus

Here is an observation to the effect that generic non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors do not vanish outside of the branching locus. This observation essentially already appears in [HP24, Proof of Corollary 1.12] and is repeated here for the readers' convenience.

Proposition 4.17. *The **universal zero locus** of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors*

$$\mathbf{Z}_{\mathbf{PHarm}^*} \coloneqq \left\{(\mathbf{p}, [\mathfrak{l}]; [\Phi], x) \in \mathbf{PHarm}^* \times X : x \notin \mathrm{Br}(\mathfrak{l}), \Phi(x) = 0\right\}$$

is a tame Fréchet submanifold of $\mathbf{PHarm}^* \times X$ *and* $\mathrm{pr}_{\mathbf{PHarm}^*} \colon \mathbf{Z}_{\mathbf{PHarm}^*} \to \mathbf{PHarm}^*$ *has index* -1.

Proof. There is a rank 4 vector bundle $\mathscr{S}$ over $\mathring{\mathbf{X}}_{\mathbf{PHarm}^*} \coloneqq \{(\mathbf{p}, [\mathfrak{l}]; [\Phi], x) \in \mathbf{PHarm}^* \times X : x \notin \mathrm{Br}(\mathfrak{l})\}$ whose fibre over $(\mathbf{p}, [\mathfrak{l}]; [\Phi], x)$ is $\mathrm{Hom}(\mathbb{R}\langle\Phi\rangle, S_x \otimes \mathfrak{l}_x)$. Evaluation defines a section $\mathrm{ev} \in \Gamma(\mathring{\mathbf{X}}_{\mathbf{PHarm}^*}, \mathscr{S})$. If ev is transverse to the zero section, then, by [Ham82, Part III, Theorem 2.3.1], $\mathbf{Z}_{\mathbf{PHarm}^*}$ is a tame Fréchet submanifold and $\operatorname{index} \mathrm{pr}_{\mathbf{PHarm}^*} = \dim X - \operatorname{rk} \mathscr{S} = -1$.

To prove that ev is transverse to the zero section, it suffices to prove that for every $(\mathbf{p}, [\mathfrak{l}]; [\Phi], x) \in \mathbf{Z}_{\mathbf{PHarm}^*}$ and $\psi \in (S \otimes \mathfrak{l})_x$ there are $\phi \in H^\infty_a\Gamma(X\backslash Z, S \otimes \mathfrak{l}; R_\Phi; \mathbf{p})$ and $\dot{\mathbf{p}} \in T_{\mathbf{p}}\mathbf{Dir}$ such that

$$\phi(x) = \psi \quad \text{and} \quad D^{\mathfrak{l}}_{\mathbf{p}}\phi + \nabla_{\dot{\mathbf{p}}}\mathbf{D}(\mathbf{p}, [\mathfrak{l}]; \Phi) = 0.$$

To prove this, it suffices to prove that

$$\begin{aligned} \ker \mathrm{ev}_x \oplus T_{\mathbf{p}}\mathbf{Dir} &\to H^\infty_b(X\backslash Z, S \otimes \mathfrak{l}; \mathbf{p}) \\ (\phi, \dot{\mathbf{p}}) &\mapsto D^{\mathfrak{l}}_{\mathbf{p}}\phi + \nabla_{\dot{\mathbf{p}}}\mathbf{D}(\mathbf{p}, [\mathfrak{l}]; \Phi) \end{aligned}$$

is surjective. Here $\mathrm{ev}_x \colon H^\infty_a\Gamma(X\backslash Z, S \otimes \mathfrak{l}; R_\Phi; \mathbf{p}) \to S_x \otimes \mathfrak{l}_x$ denotes the evaluation map. The restriction of $D^{\mathfrak{l}}_{\mathbf{p}}$ to $\ker \mathrm{ev}_x$ is Fredholm and $\operatorname{coker} D^{\mathfrak{l}}_{\mathbf{p}}|_{\ker \mathrm{ev}_x} \cong (\ker D_{R_\Phi} \oplus D^{\mathfrak{l}}_{\mathbf{p}}\widetilde{S_x \otimes \mathfrak{l}_x})^*$ for every choice of lift $\widetilde{S_x \otimes \mathfrak{l}_x}$ of $S_x \otimes \mathfrak{l}_x$ along ev_x. With this in mind, the surjectivity of the above map follows by the argument employed in the proof of Proposition 4.11. ■

The above might be somewhat interesting in view of [DW21; Yan25a; Yan25b].

4.4 The universal space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ eigenspinors

Assuming that $\dim X = 3$ and $\operatorname{rk} S = 4$, consider the universal space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ eigenspinors

$$\mathbf{Eig}^* \coloneqq \left\{(\mathbf{p}, [\mathfrak{l}]; \Phi, \lambda) \in \mathbf{E}^*_{00} \times \mathbb{R} : D^{\mathfrak{l}}_{\mathbf{p}}\Phi = \lambda\Phi\right\}.$$

To prove that $\mathbf{Eig}^*$ is a submanifold and $\mathrm{pr}_{\mathbf{Dir}} \colon \mathbf{Eig}^* \to \mathbf{Dir}$ is uniformly Fredholm of index 1, one proceeds as in Section 4.3.

Proposition 4.18. *For every* $(\mathbf{p}_0, [\mathfrak{l}_0]; \Phi_0, \lambda_0) \in \mathbf{Eig}^*$ *there are an open subset* $\mathscr{U} \subseteq \mathbf{E}^*_{00} \times \mathbb{R}$ *with* $(\mathbf{p}_0, [\mathfrak{l}_0]; \Phi_0, \lambda_0) \in \mathscr{U}$ *and a finite rank subbundle* $\mathrm{Ob} \subseteq \mathrm{pr}_1^* f^* \mathbf{F}_{00}|_{\mathscr{U}}$ *such that:*

(1) $\mathscr{V} \coloneqq \mathrm{pr}_{\mathbf{Dir}}(\mathscr{U}) \subseteq \mathbf{Dir}$ *is open,*

(2) *the local thickening*

$$\mathbf{Eig}^{\mathrm{t}} := \left\{(\mathbf{p}, [\mathrm{I}]; \Phi, \lambda) \in \mathscr{U} : (D^{\mathrm{I}}_{\mathbf{p}} - \lambda)\Phi \in \mathrm{Ob}\right\} \subseteq \mathscr{U}$$

is a tame Fréchet manifold, and

(3) *the projection map* $\mathrm{pr}_{\mathscr{V}}\colon \mathbf{Eig}^{\mathrm{t}} \to \mathscr{V}$ *is a uniform submersion.*

Proof. The proof of Proposition 4.9 carries over with very minor modifications. In particular, Ob is constructed using Proposition 3.50 applied to $\mathbf{D} - \lambda$ and s is defined by

$$s(\mathbf{p}, [\mathrm{I}]; \Phi, \lambda) := (D^{\mathrm{I}}_{\mathbf{p}} - \lambda)\Phi \pmod{\mathrm{Ob}_{\mathbf{p},[\mathrm{I}];\Phi,\lambda}}. \qquad \blacksquare$$

Proposition 4.19. *In the situation of Proposition 4.18 the* ***obstruction map*** $\mathrm{ob}\colon \mathbf{Eig}^{\mathrm{t}} \to \mathrm{Ob}$ *defined by*

$$\mathrm{ob}(\mathbf{p}, [\mathrm{I}]; \Phi, \lambda) := (D^{\mathrm{I}}_{\mathbf{p}} - \lambda)\Phi$$

is transverse to the zero section. ■

Proof of Theorem 1.6. As a consequence of Proposition 4.18, Proposition 4.19, and [Ham82, Part III, Theorem 2.3.1], it follows that $\mathbf{Eig}^* \subseteq \mathbf{E}^*_{00} \times \mathbb{R}$ is a tame Fréchet manifold and that $\mathrm{pr}_{\mathbf{Dir}}\colon \mathbf{Eig}^* \to \mathbf{Dir}$ is uniformly Fredholm of index 1. This implies the assertion. ■

Here is the analogue of the transversality criterion Proposition 4.15.

Proposition 4.20. *Let B be a manifold and $\Gamma\colon B \to \mathbf{Dir}$ a smooth map. The following are equivalent:*

(1) Γ *is transverse to* $\mathrm{pr}_{\mathbf{Dir}}\colon \mathbf{PEig}^* \to \mathbf{Dir}$.

(2) *For every $b \in B$ and $(\mathbf{p}, [\mathrm{I}]; [\Phi], \lambda) \in \mathbf{PEig}^*$ with $\Gamma(b) = \mathbf{p}$ the map*

$$\delta_{\Phi,\lambda}\colon T_bB \times \mathbb{R} \to \left(\ker(D_{R_\Phi} - \lambda)\right)^*$$

defined by

$$\delta_{\Phi,\lambda}(v,t)\kappa := \left\langle T_{\Phi,\kappa}, T_{\mathbf{p}}\mathrm{pr}_{\mathbf{Met}} \circ T_b\Gamma(v)\right\rangle_{L^2} + \left\langle \mu(\Phi,\kappa), \mathrm{pr}_{V_\nabla} \circ T_b\Gamma(v)\right\rangle_{L^2} - t\langle \Phi, \kappa\rangle_{L^2}$$

is surjective. Here R_Φ denotes the residue condition induced by Φ. ■

Remark 4.21. It should be observed that $\mathrm{tr}_g\, T_{\Phi,\kappa} = -\frac{1}{2}\lambda\langle\Phi,\kappa\rangle$ and, therefore, $T_{\Phi,-}$ cannot vanish if $\lambda \neq 0$. ♣

Finally, as in Proposition 4.17, generic non-degenerate $\mathbb{Z}/2\mathbb{Z}$ eigenspinors do not vanish outside of the branching locus.

Proposition 4.22. *The* ***universal zero locus*** *of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ eigenspinors*

$$\mathbf{Z}_{\mathbf{PEig}^*} := \left\{(\mathbf{p}, [\mathrm{I}]; [\Phi], \lambda, x) \in \mathbf{PEig}^* \times X : x \notin \mathrm{Br}(\mathrm{I}), \Phi(x) = 0\right\}$$

is a tame Fréchet submanifold of $\mathbf{PEig}^* \times X$ and $\mathrm{pr}_{\mathbf{PEig}^*}\colon \mathbf{Z}_{\mathbf{PEig}^*} \to \mathbf{PEig}^*$ has index -1. ■

Finally, it should be pointed out that it is straightforward to verify that zero is a regular value of the projection map $\lambda\colon \mathbf{PEig}^* \to \mathbb{R}$.

4.5 The universal space of non-degenerate chiral $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors

Proof of Theorem 1.7. The proof of Theorem 1.1 carries over with cosmetic modifications, using Proposition 3.62 instead of Proposition 3.50. The index formula is a consequence of Theorem B.1. ■

Here is the analogue of the transversality criterion Proposition 4.15.

Proposition 4.23. *Let B be a manifold and $\Gamma\colon B \to \mathbf{Dir}$ a smooth map. The following are equivalent:*

(1) *Γ is transverse to $\mathrm{pr}_{\mathbf{Dir}}\colon \mathbf{PHarm}^{\pm,*} \to \mathbf{Dir}$.*

(2) *For every $b \in B$ and $(\mathbf{p}, [I]; \Phi) \in \mathbf{Harm}^{\pm,*}$ with $\Gamma(b) = \mathbf{p}$ the map*

$$\delta_\Phi\colon T_bB \to \left(\ker D^{\mp}_{R^{\mp}_\Phi}\right)^*$$

defined by

$$\delta_\Phi(v)\kappa := \left\langle T_{\Phi,\kappa}, T_{\mathbf{p}}\mathrm{pr}_{\mathbf{Met}} \circ T_b\Gamma(v)\right\rangle_{L^2} + \left\langle \mu(\Phi,\kappa), \mathrm{pr}_{V_\nabla} \circ T_b\Gamma(v)\right\rangle_{L^2}$$

is surjective. Here $R^{\pm}_\Phi$ denotes the chiral residue condition induced by Φ. ■

Remark 4.24. The reader should beware that the codomain of δ_Φ is the dual of the kernel of the *adjoint* of $D^{\pm}_{R^{\pm}_\Phi}$. In the situation considered in Proposition 4.15, the relevant operator is self-adjoint and this distinction does not arise. ♣

4.6 The universal space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic 1–forms

This subsection proves Theorem 1.9. Throughout this subsection, $\dim X = 3$, X is oriented, and $S = \underline{\mathbb{R}} \oplus T^*X$. The map $\iota_{\mathrm{HdR}}\colon \mathbf{Met} \to \mathbf{Dir}$, alluded to before the statement of Theorem 1.9, assigns to every Riemannian metric $g \in \mathbf{Met}$ the Dirac bundle structure on S with respect to which the Clifford multiplication is given by

$$\gamma(v)(f,\alpha) := (-i_v\alpha, v^\flat f + *(v^\flat \wedge \alpha))$$

and the spin connection is induced by the Levi-Civita connection. The pullback

$$\mathbf{Harm}^*_{\Omega^1} := \iota^*_{\mathrm{HdR}}\mathbf{Harm}^*$$

is the **universal space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic 1–forms** on X. Indeed, if $(g, [I]; f, \alpha) \in \mathbf{Harm}^*_{\Omega^1}$, then

$$\mathrm{d}^*\mathrm{d}f = \mathrm{d}^*(\mathrm{d}f + *\mathrm{d}\alpha) = 0 \tag{4.25}$$

and, therefore, $\mathrm{d}f = 0$ by integration by parts and using that (f,α) has vanishing residue. Since $\mathrm{Br}(I) \neq \varnothing$, an assumption that is tacitly made, $f = 0$. Therefore $([I],\alpha)$ is a $\mathbb{Z}/2\mathbb{Z}$ harmonic 1–form with respect to g. Conversely and evidently, if $([I],\alpha)$ is a non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic 1–form with respect to g, then $(g,[I];0,\alpha) \in \mathbf{Harm}^*_{\Omega^1}$.

Here is some preparation regarding the finite rank vector bundle $\mathscr{H}^1 \to \mathbf{Met} \times \mathbf{Ram}$ mentioned in Theorem 1.9.

Proposition 4.26. *For every* $[\mathfrak{l}] \in \mathbf{Ram}$ *the double cover* $\mathring{\pi}\colon\ \tilde{X}^{\circ} := \{v \in \mathfrak{l} : |v| = 1\} \to X\backslash Z$ *extends to a smooth branched double cover* $\pi\colon\ \tilde{X} \to X$.

Proof. The Euclidean line bundle $\mathfrak{l}$ determines a double cover $\widetilde{SNZ}$ of the unit sphere bundle SNZ. The latter is the boundary of the unit disc bundle DNZ. The question is whether $\widetilde{SNZ}$ also is the boundary of a disc bundle $\widetilde{DNZ}$ that is a branched double cover of DNZ. The obstruction to the existence of $\widetilde{DNZ}$ is $w_2(NZ) \in \mathrm{H}^2(Z, \{\pm 1\})$. The latter, however, is the restriction $\mathrm{PD}([Z])|_Z$ and vanishes by Remark 3.10. ■

Definition 4.27 (cf. [BW26, Example 3.48]). For every $(g, [\mathfrak{l}]) \in \mathbf{Met} \times \mathbf{Ram}$ the residue bundle $\check{S}$ over $Z := \mathrm{Br}(\mathfrak{l})$ contains the $\mathbb{A}$–linear subbundle

$$\check{S}_D := N^*Z \otimes_{\mathbb{A}} NZ^{-1/2}.$$

These form a family of local residue conditions $(\mathbf{Met} \times \mathbf{Ram}, \iota_{\mathrm{HdR}} \times \mathrm{id}_{\mathbf{Ram}}, \check{\mathbf{S}}_D)$. Denote the associated family of residue conditions by $(\mathbf{Met} \times \mathbf{Ram}, \iota_{\mathrm{HdR}} \times \mathrm{id}_{\mathbf{Ram}}, \mathbf{R}_D)$. •

This is an elliptic Lagrangian family of local residue conditions. As a consequence of Teleman's Lipschitz Hodge theory [Tel83], the associated kernels form a vector bundle.

Proposition 4.28. *For every* $(g, [\mathfrak{l}]) \in \mathbf{Met} \times \mathbf{Ram}$ *the following hold:*

(1) *The kernel of* $D^{\mathfrak{l}}_{g,R_D}$ *consists precisely of the* L^2 *harmonic* $\mathfrak{l}$*–twisted* 1*–forms; that is:*

$$\ker D^{\mathfrak{l}}_{g,R_D} = \mathscr{H}^1(g, [\mathfrak{l}]) := \{\alpha \in L^2\Omega^1(X\backslash Z, \mathfrak{l}) : \mathrm{d}\alpha = \mathrm{d}^*\alpha = 0\}$$

(2) *Pulling back along* $\pi\colon\ \tilde{X} \to X$ *induces an isomorphism*

$$\mathscr{H}^1(g, [\mathfrak{l}]) \cong \mathrm{H}^1_{\mathrm{dR}}(\tilde{X})^-.$$

Here $\mathrm{H}^1_{\mathrm{dR}}(\tilde{X})^-$ *denotes the subspace of* $\mathrm{H}^1_{\mathrm{dR}}(\tilde{X})$ *which is anti-invariant under the sheet-swapping involution* τ.

Proof. If $(f, \alpha) \in \ker D^{\mathfrak{l}}_{g,R_D}$, then $f = 0$ by the argument following (4.25). This proves that $\ker D^{\mathfrak{l}}_{g,R_D} \subseteq \mathscr{H}^1(g, [\mathfrak{l}])$. Conversely, if $\alpha \in \mathscr{H}^1(g, [\mathfrak{l}])$, then by [Don21a, (4.1)] it has an expansion of the form

$$\alpha = \mathrm{Re}(Az^{-1/2}\mathrm{d}z) + \text{higher order terms}.$$

Here z is a local complex normal coordinate and, globally, $A \in \Gamma(Z, NZ^{-1/2})$. Therefore, $(0, \alpha) \in H^\infty_a(X\backslash \mathrm{Br}(\mathfrak{l}), S \otimes \mathfrak{l})$ and $\mathrm{res}(0, \alpha) \in R_D$. This proves (1).

The proof of (2) is a variation of the proof of [Wan93, Proposition 5]. The metric π^*g degenerates along the ramification locus $\pi^{-1}(\mathrm{Br}(\mathfrak{l}))$. This is a consequence of the fact that near the ramification locus π is locally modelled on $(x, z) \mapsto (x, z^2)$. There is a Lipschitz map $\sigma\colon\ \tilde{X} \to X$ which is: (1) modelled on $(x, z) \mapsto (x, z^2/|z|)$ near the ramification locus, (2) invariant under τ, (3) homotopic to π, and (4) such that $\tilde{g} := \sigma^* g$ is a Lipschitz Riemannian metric in the sense of [Tel83, §3]. A moment's thought shows that $\mathscr{H}^1(g, [\mathfrak{l}])$ pulls back under σ precisely to the space of τ–anti-invariant L^2 harmonic forms with respect to $\tilde{g}$ in the sense of [Tel83, (4.4)]. Therefore, the assertion follows from the version of the Hodge theorem stated in [Tel83, Theorem 4.1]. ■

Corollary 4.29. *The vector spaces $\mathscr{H}^1(g,[\mathfrak{l}])$ form a finite rank subbundle of* $(\iota_{\mathrm{HdR}} \times \mathrm{id}_{\mathbf{Ram}})^*\mathbf{E} \to \mathbf{Met} \times \mathbf{Ram}$. ■

It turns out that the local residue conditions defined in Proposition 3.57 and Definition 4.27 are related as follows.

Proposition 4.30. *If* $(g,[\mathfrak{l}];\alpha) \in \mathbf{Harm}^*_{\Omega^1}$, *then*

$$\operatorname{im} \mathbf{L}_\alpha = \check{S}_D.$$

Proof. As explained in [Don21a, (4.1)], α has an expansion of the form

$$\alpha = \mathrm{Re}(Bz^{1/2}\mathrm{d}z) + \text{higher order terms}.$$

Here z is a local complex normal coordinate and, globally, $B \in \Gamma(Z, \mathrm{Hom}_{\mathbb{A}}(\overline{NZ}, NZ^* \otimes_{\mathbb{A}} NZ^{-1/2}))$. A moment's thought shows that up to a constant factor B is nothing but $\mathbf{L}_\alpha$. As a consequence, $\operatorname{im} \mathbf{L}_\alpha \subseteq \check{S}_D$. Since $\operatorname{rk} \mathbf{L}_\alpha = \operatorname{rk} \check{S}_D$, this inclusion is an identity. ■

Proof of Theorem 1.9. Since ι_{HdR} is not transverse to $\mathrm{pr}_{\mathbf{Dir}} \colon \mathbf{Harm}^* \to \mathbf{Dir}$, *a priori* $\mathbf{Harm}^*_{\Omega^1}$ might not be a tame Fréchet submanifold of $\iota^*_{\mathrm{HdR}}\mathbf{E}_{00}$.

Let $(g_0,[\mathfrak{l}_0];\alpha) \in \mathbf{Harm}^*_{\Omega^1}$. Consider a local thickening $\mathbf{Harm}^{\mathbf{t}}$ containing $(\iota_{\mathrm{HdR}}(g_0),[\mathfrak{l}_0];\alpha)$ as in Proposition 4.9. Since $\mathrm{pr}_{\mathscr{V}} \colon \mathbf{Harm}^{\mathbf{t}} \to \mathscr{V}$ is a uniform submersion, $\iota^*_{\mathrm{HdR}}\mathbf{Harm}^{\mathbf{t}} \subseteq \iota^*_{\mathrm{HdR}}\mathbf{E}^*_{00}$ is a tame Fréchet submanifold and the projection map $\iota^*_{\mathrm{HdR}}\mathbf{Harm}^{\mathbf{t}} \to \iota^{-1}_{\mathrm{HdR}}(\mathscr{V})$ is a uniform submersion. Evidently, $\iota^*_{\mathrm{HdR}}\mathbf{Harm}^{\mathbf{t}} \supseteq \mathbf{Harm}^*_{\Omega^1} \cap \iota^*_{\mathrm{HdR}}\mathscr{U}$.

By the preceding propositions the thickening $\mathbf{Harm}^{\mathbf{t}}$ can be chosen such that for every $(g,[\mathfrak{l}];\alpha) \in \iota^*_{\mathrm{HdR}}\mathbf{Harm}^{\mathbf{t}}$ the L^2 orthogonal projection $\mathrm{Ob}_{g,[\mathfrak{l}];\alpha} \to \mathscr{H}^1(g,[\mathfrak{l}])$ is injective. By construction, if $(g,[\mathfrak{l}];\alpha) \in \iota^*_{\mathrm{HdR}}\mathbf{Harm}^{\mathbf{t}}$, then

$$\mathrm{d}^*\alpha = 0 \quad \text{and} \quad *\,\mathrm{d}\alpha \in \mathrm{Ob}_{g,[\mathfrak{l}];\alpha}.$$

However, the L^2 orthogonal projection of $*\mathrm{d}\alpha$ onto $\mathscr{H}^1(g,[\mathfrak{l}])$ vanishes by integration by parts, since α has vanishing residue. As a consequence the obstruction map $\iota^*_{\mathrm{HdR}}\mathrm{ob}$ vanishes. Therefore, $\iota^*_{\mathrm{HdR}}\mathbf{Harm}^{\mathbf{t}}$ is an open neighbourhood of $(g_0,[\mathfrak{l}_0];\alpha)$ in $\mathbf{Harm}^*_{\Omega^1}$. This proves (1).

In light of Remark 4.13, (2) holds as a consequence of Proposition 4.28 and Proposition 4.30. ■

4.7 The universal space of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic self-dual 2–forms

This subsection proves Theorem 1.12. Throughout this subsection, $\dim X = 4$, X is oriented, and $S = T^*X \oplus \underline{\mathbb{R}} \oplus \Lambda^+T^*X$. The map $\iota_{\mathrm{HdR}} \colon \mathbf{Met} \to \mathbf{Dir}$, alluded to before the statement of Theorem 1.12, assigns to every Riemannian metric $g \in \mathbf{Met}$ the Dirac bundle structure on S with respect to which the Clifford multiplication is given by

$$\gamma(v)(\alpha; f, \beta) := (v^\flat f - \sqrt{2} i_v\beta; -i_v\alpha, \sqrt{2}(v^\flat \wedge \alpha)^+)$$

and the spin connection is induced by the Levi-Civita connection. The pullbacks

$$\mathbf{PHarm}^*_{\Omega^1} := \iota^*_{\mathrm{HdR}}\mathbf{PHarm}^{+,*} \quad \text{and} \quad \mathbf{PHarm}^*_{\Omega^+} := \iota^*_{\mathrm{HdR}}\mathbf{PHarm}^{-,*}$$

are the **universal moduli spaces of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic 1–forms** and **self-dual 2–forms** respectively.

Definition 4.31 (cf. [BW26, Example 3.48]). For every $(g,[\mathfrak{l}]) \in \mathbf{Met} \times \mathbf{Ram}$ the chiral residue bundle $\check{S}^{\pm}$ over $Z := \mathrm{Br}(\mathfrak{l})$ contains the $\mathbb{A}$−linear subbundle

$$\check{S}^{+}_{D} := N^{*}Z \otimes_{\mathbb{A}} NZ^{-1/2} \quad \text{or} \quad \check{S}^{-}_{D} := (T^{*}Z \wedge N^{*}Z)^{+} \otimes_{\mathbb{A}} NZ^{-1/2}$$

respectively. These form a family of local chiral residue conditions $(\mathbf{Met}\times\mathbf{Ram}, \iota_{\mathrm{HdR}} \times \mathrm{id}_{\mathbf{Ram}}, \check{\mathbf{S}}^{\pm}_{D})$. Denote the associated family of chiral residue conditions by $(\mathbf{Met}\times\mathbf{Ram}, \iota_{\mathrm{HdR}} \times \mathrm{id}_{\mathbf{Ram}}, \mathbf{R}^{\pm}_{D})$. •

$\check{S}^{+}_{D} \oplus \check{S}^{-}_{D}$ is an elliptic Lagrangian family of local residue conditions and the associated kernels form vector bundles.

Proposition 4.32. *For every $(g,[\mathfrak{l}]) \in \mathbf{Met}\times\mathbf{Ram}$ the following hold:*

(1) *The kernel of $D^{\mathfrak{l}}_{g,R^{\pm}_{D}}$ consists precisely of the L^2 harmonic $\mathfrak{l}$−twisted 1−forms and self-dual 2−forms respectively; that is:*

$$\ker D^{\mathfrak{l}}_{g,R^{+}_{D}} = \mathscr{H}^{1}(g,\mathfrak{l}) := \left\{\alpha \in L^2\Omega^1(X\backslash Z,\mathfrak{l}) : \mathrm{d}\alpha = \mathrm{d}^{*}\alpha = 0\right\} \quad \textit{and}$$
$$\ker D^{\mathfrak{l}}_{g,R^{-}_{D}} = \mathscr{H}^{+}(g,\mathfrak{l}) := \left\{\beta \in L^2\Omega^{+}(X\backslash Z,\mathfrak{l}) : \mathrm{d}\beta = \mathrm{d}^{*}\beta = 0\right\}.$$

(2) *Pulling back along $\pi\colon \tilde{X} \to X$ induces isomorphisms*

$$\mathscr{H}^{1}(g,\mathfrak{l}) \cong \mathrm{H}^{1}_{\mathrm{dR}}(\tilde{X})^{-} \quad \textit{and} \quad \mathscr{H}^{+}(g,\mathfrak{l}) \cong \mathrm{H}^{+}_{\mathrm{dR}}(\tilde{X})^{-}.$$

Here $\mathrm{H}^{\bullet}_{\mathrm{dR}}(\tilde{X})^{-}$ denotes the subspace of $\mathrm{H}^{\bullet}_{\mathrm{dR}}(\tilde{X})$ which is anti-invariant under the sheet-swapping involution τ and $\mathrm{H}^{+}_{\mathrm{dR}}(\tilde{X})^{-} \subseteq \mathrm{H}^{2}_{\mathrm{dR}}(\tilde{X})^{-}$ denotes a *maximal subspace on which the intersection form is positive-definite.*

Proof. The proof is analogous to that of Proposition 4.28 using the fact that a $\beta \in \mathscr{H}^{+}(g,\mathfrak{l})$ has an expansion of the form

$$\beta = \mathrm{Re}(Az^{-1/2}\mathrm{d}w \wedge \mathrm{d}z) + \text{higher order terms}.$$

Here w is a local complex coordinate on Z and z is a local complex normal coordinate. ■

Corollary 4.33. *The vector spaces $\mathscr{H}^{1}(g,\mathfrak{l})$ and $\mathscr{H}^{+}(g,\mathfrak{l})$ form finite rank subbundles of $(\iota_{\mathrm{HdR}} \times \mathrm{id}_{\mathbf{Ram}})^{*}\mathrm{E}^{\pm} \to \mathbf{Met}\times\mathbf{Ram}$ respectively.* ■

Proposition 4.34. *If $(g,[\mathfrak{l}];\alpha) \in \mathbf{Harm}^{*}_{\Omega^1}$ (or $(g,[\mathfrak{l}];\beta) \in \mathbf{Harm}^{*}_{\Omega^+}$), then*

$$\mathrm{im}\,\mathbf{L}_{\alpha} = \check{S}^{+}_{D} \quad (\textit{or} \quad \mathrm{im}\,\mathbf{L}_{\beta} = \check{S}^{-}_{D}).$$

Proof. The proof is analogous to that of Proposition 4.30 using that

$$\beta = \mathrm{Re}(Bz^{1/2}\mathrm{d}w \wedge \mathrm{d}z) + \text{higher order terms}.$$

Here w is a local complex coordinate on Z and z is a local complex normal coordinate and, globally, $B \in \Gamma(Z, \mathrm{Hom}_{\mathbb{A}}(\overline{NZ}, (T^{*}Z \wedge NZ^{*})^{+} \otimes_{\mathbb{A}} NZ^{-1/2}))$. ■

Proof of Theorem 1.12. The proof is analogous to that of Theorem 1.9 using the above ingredients and cosmetic modifications to track the chirality. ∎

Remark 4.35. If $(g, [\mathfrak{l}]; \alpha) \in \mathbf{Harm}^*_{\Omega^1}$, then the index of $D^{\mathfrak{l}}_{g,R^+_\alpha}$ can be determined as follows. If X is a closed oriented 4–manifold, then

$$b^1(X) - b^0(X) - b^+(X) = -\frac{1}{2}(\chi(X) + \sigma(X)).$$

By [GS99, Lemma 7.1.7],

$$\chi(\tilde{X}) = 2\chi(X) - \chi(Z) \quad \text{and} \quad \sigma(\tilde{X}) = 2\sigma(X) - \frac{1}{2}[Z] \cdot [Z].$$

If Z is not orientable, then $[Z] \cdot [Z]$ needs to be interpreted accordingly. The orientation local systems $\mathfrak{o}$ of TZ and NZ agree. Therefore, there is a well-defined Euler class $\mathfrak{e}(NZ) \in \mathrm{H}^2(Z, \mathfrak{o})$ and $[Z] \in \mathrm{H}_2(Z, \mathfrak{o})$ and both can be paired.

As a consequence of Proposition 4.32, Proposition 4.34, and the above,

$$\begin{aligned} \operatorname{index} D^{\mathfrak{l},+}_{g,R^+_\alpha} &= \dim \mathscr{H}^1(g, \mathfrak{l}) - \dim \mathscr{H}^+(g, \mathfrak{l}) \\ &= -\frac{1}{2}(\chi(\tilde{X}) + \sigma(\tilde{X})) + \frac{1}{2}(\chi(X) + \sigma(X)) \\ &= -\frac{1}{2}(\chi(X) + \sigma(X)) + \frac{1}{2}\chi(Z) + \frac{1}{4}[Z] \cdot [Z]. \end{aligned}$$

In fact, since α is non-degenerate, $[Z] \cdot [Z] = 0$. ♣

Finally, here is a straightforward construction of non-degenerate $\mathbb{Z}/2\mathbb{Z}$ harmonic self-dual 2–forms on Kähler surfaces.

Example 4.36. Let X be a Kähler surface and $q \in \mathrm{H}^0(X, K_X^{\otimes 2})$ transverse to zero. The latter defines a holomorphic branched double cover $\tilde{X} := \{(x, v) \in K_X : v^2 = q(x)\} \to X$. This induces a ramified Euclidean line bundle $\mathfrak{l}$ with $Z := \mathrm{Br}(\mathfrak{l}) = q^{-1}(0)$ and $\operatorname{Re} \sqrt{q} \in \Omega^+(X \backslash Z, \mathfrak{l})$ is a $\mathbb{Z}/2\mathbb{Z}$ harmonic self-dual 2–form.

It is not difficult to find X and q as above. By Bertini's theorem, if $|2K_X|$ is base-point free, then a general $q \in \mathrm{H}^0(X, K_X^{\otimes 2})$ is transverse to zero. A concrete example is a smooth hypersurface $X_d \subseteq \mathbb{C}P^3$ of degree $d \geqslant 5$ and a general section of $K_{X_d}^{\otimes 2} = \mathscr{O}_{X_d}(2d - 8)$. ♠

By Theorem 1.12, the $\mathbb{Z}/2\mathbb{Z}$ harmonic self-dual 2–forms obtained above persist under small deformations of the metric on X.

A Smoothing operators

The following appendix supplies the missing ingredient in the proof of Proposition 3.51; that is: it constructs a family of smoothing operators for the scale of norms $(\|-\|_{\mathring{K}^k} : k \in \mathbf{N}_0)$ defined by

$$\|\phi\|_{\mathring{K}^k} := \|\phi\|_{H_b^{k+2}} + \|\mathring{\nabla}\phi\|_{H_b^{k+1}} + \|\log(r)\mathring{D}\phi\|_{H_b^{k+1}} + \|\mathring{\nabla} \log(r)\mathring{D}\phi\|_{H_b^k}.$$

Proposition A.1. *There is a family of smoothing operators* $(S_\varepsilon : \varepsilon \in (0,1))$ *on* $L^2\Gamma(NZ\backslash Z, \mathring{S} \otimes \mathring{\mathfrak{l}})$ *such that for every* $\varepsilon \in (0,1)$*,* $\phi \in L^2\Gamma(NZ\backslash Z, \mathring{S} \otimes \mathring{\mathfrak{l}})$*, and* $k, \ell \in \mathbf{N}_0$ *with* $k \geqslant \ell$*,*

$$\|S_\varepsilon \phi\|_{\mathring{K}^k} \lesssim_{k,\ell} \varepsilon^{-k+\ell}\|\phi\|_{\mathring{K}^\ell}, \tag{A.2}$$

$$\|(S_\varepsilon - \mathbf{1})\phi\|_{\mathring{K}^\ell} \lesssim_{k,\ell} \varepsilon^{k-\ell}\|\phi\|_{\mathring{K}^k}, \tag{A.3}$$

$$\varepsilon\|\partial_\varepsilon S_\varepsilon \phi\|_{\mathring{K}^k} \lesssim_{k,\ell} \varepsilon^{-k+\ell}\|\phi\|_{\mathring{K}^\ell}, \quad \text{and} \tag{A.4}$$

$$\varepsilon\|\partial_\varepsilon S_\varepsilon \phi\|_{\mathring{K}^\ell} \lesssim_{k,\ell} \varepsilon^{k-\ell}\|\phi\|_{\mathring{K}^k}. \tag{A.5}$$

As explained in [BW26, §3.4], Fubini's theorem and spectral theory imply that the Hilbert space $L^2\Gamma(NZ\backslash Z, \mathring{S} \otimes \mathring{\mathfrak{l}})$ decomposes as

$$L^2\Gamma(NZ\backslash Z, \mathring{S} \otimes \mathring{\mathfrak{l}}) = L^2((0,\infty), r\mathrm{d}r, L^2(F, \underline{S} \otimes \underline{\mathfrak{l}})) = \bigoplus_{(\lambda,\mu)\in\sigma} L^2((0,\infty), r\mathrm{d}r; E_{\lambda,\mu}).$$

Here $\sigma \subseteq \mathbb{R}^2$ is a discrete subset and the $E_{\lambda,\mu}$ are finite-dimensional as in [BW26, Proposition 3.24]. Moreover, by [BW26, Proposition 4.8], with respect to this decomposition

$$\|\phi\|^2_{H^k_b} \asymp_k \sum_{(\lambda,\mu)\in\sigma} \sum_{\ell=0}^{k} \langle(\lambda,\mu)\rangle^{2\ell} \|(r\partial_r)^{k-\ell}\phi_{\lambda,\mu}\|^2_{r^{-1/2}L^2}$$

and

$$\|\mathring{\nabla}\phi\|^2_{H^k_b} \asymp_k \sum_{(\lambda,\mu)\in\sigma} \Big(\langle\mu\rangle^2\|\phi_{\lambda,\mu}\|^2_{H^k_b} + \big\|\tfrac{\lambda}{r}\phi_{\lambda,\mu}\big\|^2_{H^k_b} + \|\partial_r\phi_{\lambda,\mu}\|^2_{H^k_b}\Big).$$

The model Dirac operator does not quite preserve $E_{\lambda,\mu}$ but rather the direct sum $E_{\lambda,\mu} \oplus E_{-(\lambda+1),-\mu}$, on which it is given by

$$\mathring{D}^{\lambda,\mu} := \begin{pmatrix} \mu & J(\partial_r + \frac{\lambda+1}{r}) \\ J(\partial_r - \frac{\lambda}{r}) & -\mu \end{pmatrix}.$$

(The case $(\lambda,\mu) = (-1/2, 0)$ requires a bit more care as explained in [BW26, Definition 3.25], but this is inconsequential for the remaining discussion.)

The proof of Proposition 3.52, applied to the model operator, gives

$$\|\log(r)\mathring{\nabla}\phi\|_{H^{k+1}_b} \lesssim_k \|\phi\|_{\mathring{K}^k}.$$

Together with [BW26, Proposition 4.8] and the above formula for $\mathring{D}^{\lambda,\mu}$, this yields

$$\begin{aligned}\|\phi\|^2_{\mathring{K}^k} \asymp_k \sum_{(\lambda,\mu)\in\sigma} \Big(&\|\phi_{\lambda,\mu}\|^2_{H^{k+2}_b} + \|\partial_r\phi_{\lambda,\mu}\|^2_{H^{k+1}_b} + \|\tfrac{\lambda}{r}\phi_{\lambda,\mu}\|^2_{H^{k+1}_b} + \|\log(r)(\partial_r - \tfrac{\lambda}{r})\phi_{\lambda,\mu}\|^2_{H^{k+1}_b} \\ &+ \|\partial_r \log(r)(\partial_r - \tfrac{\lambda}{r})\phi_{\lambda,\mu}\|^2_{H^k_b} + \|\tfrac{\lambda}{r}\log(r)(\partial_r - \tfrac{\lambda}{r})\phi_{\lambda,\mu}\|^2_{H^k_b}\Big).\end{aligned}$$

Defining (S_ε) as the composition of a spectral cutoff at ε^{-1} in λ and μ, as in [Par23, Appendix B; BW26, §4.5], and the following smoothing operators in the radial direction establishes Proposition A.1.

Proposition A.6. *On $L^2((0,\infty), r\mathrm{d}r; \mathbb{C})$ define*

$$\|\phi\|_{H_b^k} := \sum_{\ell=0}^{k} \|(r\partial_r)^\ell \phi\|_{r^{-1/2}L^2}$$

and for every $k \in \mathbf{N}_0$ and $\lambda \in \mathbb{R}$ set

$$\begin{aligned}\|\phi\|_{\mathring{K}_\lambda^k} &:= \|\phi\|_{H_b^{k+2}} + \|\partial_r\phi\|_{H_b^{k+1}} + \|\tfrac{\lambda}{r}\phi\|_{H_b^{k+1}} + \|\log(r)(\partial_r - \tfrac{\lambda}{r})\phi\|_{H_b^{k+1}} \\ &\quad + \|\partial_r \log(r)(\partial_r - \tfrac{\lambda}{r})\phi\|_{H_b^k} + \|\tfrac{\lambda}{r}\log(r)(\partial_r - \tfrac{\lambda}{r})\phi\|_{H_b^k}.\end{aligned}$$

There is a family of smoothing operators $(S_\varepsilon : \varepsilon \in (0,1))$ satisfying the analogues of (A.2), (A.3), (A.4), and (A.5).

The family (S_ε) in Proposition A.6 is constructed by **Mellin convolution** with a family of smoothing kernels σ_ε; that is:

$$(S_\varepsilon\phi)(r) := (\sigma_\varepsilon *_{\mathcal{M}} \phi)(r) := \int_0^\infty \sigma_\varepsilon(r/s)\phi(s)\,\frac{\mathrm{d}s}{s}.$$

The Mellin convolution $*_{\mathcal{M}}$ on $(0,\infty)$ is related to the (usual) convolution $*$ on $\mathbb{R}$ via the isometry $T\colon L^2((0,\infty), r\mathrm{d}r; \mathbb{C}) \to L^2(\mathbb{R};\mathbb{C})$ defined by

$$T(f)(x) := e^x f(e^x);$$

indeed

$$(f *_{\mathcal{M}} g)(r) = T^{-1}(Tf * Tg)(r).$$

Denote by $\mathcal{F}\colon L^2(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R};\mathbb{C})$ the Fourier transform. The **Mellin transform**

$$\mathcal{M}\colon L^2((0,\infty), r\mathrm{d}r; \mathbb{C}) \to L^2(\mathbb{R};\mathbb{C})$$

is defined by

$$\mathcal{M} := \mathcal{F} \circ T.$$

In particular,

$$\mathcal{M}(f *_{\mathcal{M}} g) = \mathcal{M}(f)\mathcal{M}(g).$$

Combining the identities $T(r\partial_r)T^{-1} = \partial_x - 1$ and $\mathcal{F}\partial_x\mathcal{F}^{-1} = 2\pi i\xi$ shows that for every $\phi \in C^\infty((0,\infty), \mathbb{C})$ the norms are expressed in terms of the Mellin transform by

$$\|\phi\|_{H_b^k} = \sum_{\ell=0}^{k} \|(r\partial_r)^\ell\phi\|_{r^{-1/2}L^2} \asymp_k \|\langle\xi\rangle^k \mathcal{M}(\phi)\|_{L^2}. \tag{A.7}$$

The following is the key to Proposition A.6.

Lemma A.8. *Let $\chi \in C_c^\infty(\mathbb{R}, [0,1])$ and $\rho \in C_c^\infty(\mathbb{R}, [0,1])$ such that*

$$\chi|_{[-1,1]} = 1, \quad \mathrm{supp}(\chi) \subseteq [-2,2], \quad \rho(0) = 1, \quad \textit{and} \quad \mathrm{supp}(\rho) \subseteq [-1,1].$$

*Define the **family of smoothing kernels** $(\sigma_\varepsilon : \varepsilon \in (0,1))$ by*

$$\sigma_\varepsilon(r) := \mathcal{M}^{-1}(\chi_\varepsilon)(r)\rho(\log(r))$$

with $\chi_\varepsilon(\xi) := \chi(\varepsilon\xi)$. The following hold:

(1) *For every* $\varepsilon \in (0,1)$, $k \in \mathbf{N}_0$, *and* $\xi \in \mathbb{R}$

$$|\mathscr{M}(\sigma_\varepsilon)(\xi)| \lesssim_{\rho,k} \varepsilon^{-k}\langle\xi\rangle^{-k}, \tag{A.9}$$

$$|\mathscr{M}(\sigma_\varepsilon)(\xi) - 1| \lesssim_{\rho,k} \varepsilon^{k}\langle\xi\rangle^{k}, \tag{A.10}$$

$$|\varepsilon\partial_\varepsilon\mathscr{M}(\sigma_\varepsilon)(\xi)| \lesssim_{\chi,\rho,k} \varepsilon^{-k}\langle\xi\rangle^{-k}, \quad \textit{and} \tag{A.11}$$

$$|\varepsilon\partial_\varepsilon\mathscr{M}(\sigma_\varepsilon)(\xi)| \lesssim_{\chi,\rho,k} \varepsilon^{k}\langle\xi\rangle^{k}. \tag{A.12}$$

(2) *For every* $\varepsilon \in (0,1]$, $k \in \mathbb{Z}$, *and* $\xi \in \mathbb{R}$

$$|\varepsilon^{-1}\mathscr{M}(\log(r)\sigma_\varepsilon)(\xi)| \lesssim_{\chi,\rho,k} \varepsilon^{k}\langle\xi\rangle^{k} \quad \textit{and} \quad |\partial_\varepsilon\mathscr{M}(\log(r)\sigma_\varepsilon)(\xi)| \lesssim_{\chi,\rho,k} \varepsilon^{k}\langle\xi\rangle^{k}.$$

Proof. (1) By direct computation, using $\int_{\mathbb{R}} \hat\rho = \rho(0) = 1$,

$$\mathscr{M}(\sigma_\varepsilon) = \chi_\varepsilon * \hat\rho, \quad \mathscr{M}(\sigma_\varepsilon) - 1 = (\chi_\varepsilon - 1) * \hat\rho, \quad \text{and} \quad \partial_\varepsilon\mathscr{M}(\sigma_\varepsilon) = \partial_\varepsilon\chi_\varepsilon * \hat\rho.$$

Since $\rho \in C_c^\infty(\mathbb{R}) \subseteq \mathscr{S}(\mathbb{R})$, $\hat\rho \in \mathscr{S}(\mathbb{R})$ and, therefore,

$$|\hat\rho(\xi)| \lesssim_{\rho,k} \langle\xi\rangle^{-k}.$$

By Young's inequality and since $\|\chi_\varepsilon\|_{L^\infty} \leqslant 1$, $\|\varepsilon\partial_\varepsilon\chi_\varepsilon\|_{L^\infty} \lesssim_\chi 1$, and $\|\hat\rho\|_{L^1} \lesssim_\rho 1$,

$$\|\mathscr{M}(\sigma_\varepsilon)\|_{L^\infty} \lesssim_\rho 1, \quad \|\mathscr{M}(\sigma_\varepsilon) - 1\|_{L^\infty} \lesssim_\rho 1 \quad \text{and} \quad \|\varepsilon\partial_\varepsilon\mathscr{M}(\sigma_\varepsilon)\|_{L^\infty} \lesssim_{\chi,\rho} 1.$$

This implies (A.9) and (A.11) assuming $\varepsilon\langle\xi\rangle \leqslant 4$, and (A.10) and (A.12) assuming $\varepsilon\langle\xi\rangle \geqslant 1/2$.

Since $|\langle\xi\rangle - \langle\eta\rangle| \leqslant |\xi - \eta|$, if $\varepsilon\langle\xi\rangle \geqslant 4$ and $|\xi - \eta| \leqslant 2\varepsilon^{-1}$, then

$$\langle\eta\rangle \geqslant \langle\xi\rangle - |\xi - \eta| \geqslant \frac{1}{2}\langle\xi\rangle.$$

Therefore and since $\operatorname{supp}(\chi) \subseteq [-2,2]$ and $\varepsilon \leqslant 1$, if $\varepsilon\langle\xi\rangle \geqslant 4$, then

$$|\mathscr{M}(\sigma_\varepsilon)(\xi)| \leqslant \int_{|\xi-\eta|\leqslant 2\varepsilon^{-1}} |\hat\rho(\eta)|\,\mathrm{d}\eta \lesssim_{\rho,k} \varepsilon^{-1}\langle\xi\rangle^{-k-1} \lesssim_k \varepsilon^{-k}\langle\xi\rangle^{-k}.$$

Combined with the above observation, this proves (A.9) unconditionally. The same argument also proves (A.11) unconditionally.

Since $|\xi - \eta| \leqslant |\xi| + |\eta| \leqslant \langle\xi\rangle + \langle\eta\rangle$, if $\varepsilon\langle\xi\rangle \leqslant 1/2$ and $|\xi - \eta| \geqslant \varepsilon^{-1}$, then

$$\langle\eta\rangle \geqslant |\xi - \eta| - \langle\xi\rangle \geqslant \frac{1}{2}\varepsilon^{-1}.$$

Therefore and since $\operatorname{supp}(\chi - 1) \cap [-1,1] = \varnothing$ and $\langle\xi\rangle \geqslant 1$, if $\varepsilon\langle\xi\rangle \leqslant 1/2$, then

$$|\mathscr{M}(\sigma_\varepsilon)(\xi) - 1| \leqslant \int_{|\xi-\eta|\geqslant \varepsilon^{-1}} |\hat\rho(\eta)|\,\mathrm{d}\eta \lesssim_{\rho,k} \int_{|\xi-\eta|\geqslant \varepsilon^{-1}} \langle\eta\rangle^{-k-2}\,\mathrm{d}\eta \lesssim_k \varepsilon^k \lesssim_k \varepsilon^k\langle\xi\rangle^k.$$

Combined with the above observation, this proves (A.10) unconditionally. The same argument also proves (A.12) unconditionally.

(2) By direct computation,

$$\mathscr{M}(\log(r)\sigma_\varepsilon) = \frac{i}{2\pi}\chi_\varepsilon' * \hat\rho.$$

Since $\|\chi_\varepsilon'\|_{L^\infty} \lesssim_\chi \varepsilon$ and $\|\varepsilon\partial_\varepsilon\chi_\varepsilon'\|_{L^\infty} \lesssim_\chi \varepsilon$, and $\operatorname{supp}(\chi_\varepsilon') \subseteq [-2\varepsilon^{-1}, 2\varepsilon^{-1}]$ and $\operatorname{supp}(\chi_\varepsilon') \cap [-\varepsilon^{-1}, \varepsilon^{-1}] = \varnothing$, the above arguments prove the asserted estimates. ∎

Proof of Proposition A.6. Choose χ and ρ as in Lemma A.8 and construct σ_ε accordingly. Moreover, for every $\nu \in \mathbb{R}$ set $\sigma_\varepsilon^{(\nu)} := r^{-\nu}\sigma_\varepsilon$ and $\delta_\varepsilon^{(\nu)} := \log(r)\sigma_\varepsilon^{(\nu)}$. Observe that $\sigma_\varepsilon^{(\nu)}$ arises from χ and $\rho^{(\nu)}$ defined by $\rho^{(\nu)}(x) := e^{-\nu x}\rho(x)$. In particular, Lemma A.8 applies to $\sigma_\varepsilon^{(\nu)}$ mutatis mutandis.

By (A.7) and Lemma A.8 (1), $(S_\varepsilon := \sigma_\varepsilon *_{\mathscr{M}} - : \varepsilon \in (0,1))$ satisfies the analogues of (A.2), (A.3), (A.4), and (A.5) with respect to $\|-\|_{H_b^k}$.

Therefore, it remains to analyse the behaviour of the five differential operators appearing in $\|-\|_{\mathring{K}_\lambda^k}$ with respect to S_ε. By direct computation,

$$\begin{aligned} r\partial_r(\sigma *_{\mathscr{M}} \phi) &= \sigma *_{\mathscr{M}} r\partial_r\phi, \\ r^\nu(\sigma *_{\mathscr{M}} \phi) &= (r^\nu\sigma) *_{\mathscr{M}} (r^\nu\phi), \quad \text{and} \\ \log(r)(\sigma *_{\mathscr{M}} \phi) &= \sigma *_{\mathscr{M}} \log(r)\phi + \log(r)\sigma *_{\mathscr{M}} \phi. \end{aligned}$$

Consequently,

$$\begin{aligned} \partial_r S_\varepsilon\phi &= \sigma_\varepsilon^{(1)} *_{\mathscr{M}} \partial_r\phi, \\ \tfrac{\lambda}{r} S_\varepsilon\phi &= \sigma_\varepsilon^{(1)} *_{\mathscr{M}} \tfrac{\lambda}{r}\phi, \\ \log(r)(\partial_r - \tfrac{\lambda}{r})S_\varepsilon\phi &= \sigma_\varepsilon^{(1)} *_{\mathscr{M}} \log(r)(\partial_r - \tfrac{\lambda}{r})\phi + \delta_\varepsilon^{(1)} *_{\mathscr{M}} (\partial_r - \tfrac{\lambda}{r})\phi, \\ \partial_r \log(r)(\partial_r - \tfrac{\lambda}{r})S_\varepsilon\phi &= \sigma_\varepsilon^{(2)} *_{\mathscr{M}} \partial_r \log(r)(\partial_r - \tfrac{\lambda}{r})\phi + \delta_\varepsilon^{(2)} *_{\mathscr{M}} \partial_r(\partial_r - \tfrac{\lambda}{r})\phi, \\ \tfrac{\lambda}{r} \log(r)(\partial_r - \tfrac{\lambda}{r})S_\varepsilon\phi &= \sigma_\varepsilon^{(2)} *_{\mathscr{M}} \tfrac{\lambda}{r} \log(r)(\partial_r - \tfrac{\lambda}{r})\phi + \delta_\varepsilon^{(2)} *_{\mathscr{M}} \tfrac{\lambda}{r}(\partial_r - \tfrac{\lambda}{r})\phi. \end{aligned}$$

This combined with

$$\begin{aligned} \|(\partial_r - \tfrac{\lambda}{r})\phi\|_{H_b^k} &\lesssim \|\log(r)(\partial_r - \tfrac{\lambda}{r})\phi\|_{H_b^k}, \\ \|\partial_r(\partial_r - \tfrac{\lambda}{r})\phi\|_{H_b^k} &\lesssim \|\partial_r \log(r)(\partial_r - \tfrac{\lambda}{r})\phi\|_{H_b^k} + \|\tfrac{\lambda}{r}(\partial_r - \tfrac{\lambda}{r})\phi\|_{H_b^k}, \\ \|\tfrac{\lambda}{r}(\partial_r - \tfrac{\lambda}{r})\phi\|_{H_b^k} &\lesssim \|\tfrac{\lambda}{r} \log(r)(\partial_r - \tfrac{\lambda}{r})\phi\|_{H_b^k}, \end{aligned}$$

and Lemma A.8 proves the assertion. ■

B The index formula

Throughout this appendix, suppose that X is oriented, $\dim X = 4$, and $\operatorname{rk} S = 8$. Moreover, let $\mathbf{p} = (g;\gamma,\nabla) \in \mathbf{Dir}$ and $(Z, [\mathfrak{l}]) \in \mathbf{Ram}$.

Theorem B.1. *If $V^\pm \subseteq \check{S}^\pm$ is an $\mathbb{A}$−linear subbundle with $\operatorname{rk}_{\mathbb{A}} V^\pm = 1$, then*

$$\operatorname{index} D^{\mathfrak{l},\pm}_{R^\pm_{V^\pm}} = \mp\frac{1}{4}\langle p_1(S), [X]\rangle \pm \sigma(X) + \frac{1}{2}\chi(Z) \mp \frac{1}{4}\langle e(NZ), [Z]\rangle \pm \langle e(V^\pm), [Z]\rangle.$$

Remark B.2. In the situation of Theorem B.1,

$$\frac{1}{2}\chi(Z) \pm \frac{1}{4}\langle e(NZ), [Z]\rangle$$

is an integer. Indeed, by the formulae in Remark 4.35,

$$\frac{1}{2}\chi(Z)+\frac{1}{4}\langle e(NZ),[Z]\rangle=\chi(X)+\sigma(X)-\frac{1}{2}(\chi(\tilde{X})+\sigma(\tilde{X}))$$

and $\chi(X)+\sigma(X)$ is even for every closed oriented 4–manifold X. ♣

The proof relies on the index computation in Remark 4.35 and the following two ingredients: a relative index formula and a consequence of excision.

Proposition B.3. *If $V^+, W^+ \subseteq \check{S}^+$ are $\mathbb{A}$–linear subbundles with $\mathrm{rk}_{\mathbb{A}} V^+ = \mathrm{rk}_{\mathbb{A}} W^+ = 1$, then*

$$\operatorname{index} D^+_{R^+_{V^+}} - \operatorname{index} D^+_{R^+_{W^+}} = \langle e(V^+),[Z]\rangle - \langle e(W^+),[Z]\rangle.$$

Proposition B.4. *There is a $\mathbb{A}$–linear subbundle $V^+_D \subseteq \check{S}^+$ with $\mathrm{rk}_{\mathbb{A}} V^+_D = 1$ satisfying*

$$e(V^+_D)=e(\check{S}^+_D) \quad \textit{and} \quad \operatorname{index} D^{\mathrm{I},+}_{R^+_{V^+_D}} - \operatorname{index} D^{\mathrm{I},+}_{\mathrm{HdR},R^+_{\check{S}^+_D}} = \operatorname{index} D^+ - \operatorname{index} D^+_{\mathrm{HdR}}.$$

*Here D_{HdR} denotes the Dirac operator associated with the Dirac bundle structure $\iota_{\mathrm{HdR}}(g) \in$ **Dir**.*

Proof of Theorem B.1 assuming Proposition B.3 and Proposition B.4. It suffices to prove the formula in positive chirality, because the complementary $V^\pm \subseteq \check{S}^\pm$, related by $V^- = (JV^+)^\perp$, satisfy

$$e(V^-)=e(V^+)+e(TZ),$$

because J is $\mathbb{A}$–anti-linear and $\overline{(V^+)^\perp} \cong V^+ \otimes_{\mathbb{A}} TZ$ via Clifford multiplication.

By Proposition B.3, Proposition B.4, and Remark 4.35,

$$\begin{aligned}
\operatorname{index} D^{\mathrm{I},+}_{R^+_{V^+}} &= \operatorname{index} D^{\mathrm{I},+}_{R^+_{V^+_D}} + \langle e(V^+),[Z]\rangle - \langle e(\check{S}^+_D),[Z]\rangle \\
&= \operatorname{index} D^+ + \operatorname{index} D^{\mathrm{I},+}_{\mathrm{HdR},R^+_{\check{S}^+_D}} - \operatorname{index} D^+_{\mathrm{HdR}} - \langle e(\check{S}^+_D),[Z]\rangle + \langle e(V^+),[Z]\rangle \\
&= -\frac{1}{4}\langle p_1(S),[X]\rangle + \sigma(X) \\
&\quad + \frac{1}{2}\chi(Z) + \frac{1}{4}\langle e(NZ),[Z]\rangle - \langle e(\check{S}^+_D),[Z]\rangle + \langle e(V^+),[Z]\rangle.
\end{aligned}$$

Since $\check{S}^+_D = N^*Z \otimes_{\mathbb{A}} NZ^{-1/2}$, and $N^*Z \cong NZ$ and $NZ^{-1/2} \otimes_{\mathbf{A}} NZ^{-1/2} \cong \overline{NZ}$ as $\mathbf{A}$–bundles,

$$2e(\check{S}^+_D)=e(NZ).$$

It remains to prove that

$$\operatorname{index} D^+ = -\frac{1}{4}\langle p_1(S),[X]\rangle + \sigma(X).$$

As in [AHS78, Proof of Theorem 6.1], for the purpose of determining the index formula, there is no loss in assuming that X is spin. Denote by $\$$ the *real* spinor bundle. There is some quaternionic line bundle L such that $S = \slashed{S} \otimes_{\mathbb{H}} L$. By the Atiyah–Singer index formula [AS68b, Theorem 5.3; LM89, Part III Theorem 13.10],

$$\operatorname{index} D^+ = -\langle \hat{A}(X)\operatorname{ch}(L), [X]\rangle = \frac{1}{4}\sigma(X) + \langle c_2(L), [X]\rangle.$$

The sign is a consequence of the real and complex chirality operators differing by a factor of -1. It remains to determine $\langle c_2(L), [X]\rangle$. Since $p_1(S^\pm) = -c_2(\slashed{S}^\pm \otimes_{\mathbb{C}} L) = -2(c_2(\slashed{S}^\pm) + c_2(L))$ and $c_2(\slashed{S}^\pm) = -\frac{1}{4}p_1(TX) \pm \frac{1}{2}e(TX)$,

$$\begin{aligned}\langle c_2(L), [X]\rangle &= -\frac{1}{2}\langle p_1(S^\pm), [X]\rangle - \langle c_2(\slashed{S}^\pm), [X]\rangle \\ &= -\frac{1}{2}\langle p_1(S^\pm), [X]\rangle + \frac{1}{4}\langle p_1(TX), [X]\rangle \mp \frac{1}{2}\langle e(TX), [X]\rangle \\ &= -\frac{1}{2}\langle p_1(S^\pm), [X]\rangle + \frac{3}{4}\sigma(X) \mp \frac{1}{2}\langle e(TX), [X]\rangle.\end{aligned}$$

The above combine to the stated contribution. ■

Proof of Proposition B.4. Denote by S_{HdR} the Clifford module underlying D_{HdR}. The bundle $\operatorname{Hom}_{C\ell}(S, S_{\mathrm{HdR}})$ has rank four and every non-zero homomorphism is an isomorphism, in fact, an isometry after normalisation. A simple transversality argument shows that S and S_{HdR} are isomorphic as Clifford modules outside a finite subset avoiding Z. Choosing an isomorphism identifies V_D^+. Up to a lower order perturbation that does not affect the index, this also identifies $D^{\mathrm{I},+}_{R^+_{V_D^+}}$ and $D^{\mathrm{I},+}_{\mathrm{HdR},R^+_{\check{S}_D^+}}$ outside the same finite subset. The assertion now follows by the usual excision argument [AS68a, §8]. ■

The remainder of this appendix is devoted to the proof of Proposition B.3. The following localises the proof of this relative index formula to Z.

Proposition B.5. *Denote by $R^+_{\mathrm{APS}} \coloneqq \mathbf{1}_{(-\infty,0)}(A^+)H^{1/2}\Gamma(Z, \check{S}^+)$ the positive APS residue condition. If $V^+ \subseteq \check{S}^+$ is an $\mathbb{A}$–linear subbundle with $\operatorname{rk}_{\mathbb{A}} V^+ = 1$, then*

$$\operatorname{index} D^{\mathrm{I},+}_{R^+_{V^+}} - \operatorname{index} D^{\mathrm{I},+}_{R^+_{\mathrm{APS}}} = \operatorname{index}(\mathbf{1}_{(-\infty,0)}(A^+)\colon R^+_{V^+} \to R^+_{\mathrm{APS}}).$$

Proof. By the chiral version of [BW26, Proposition 2.24 and Lemma 2.25], for every positive Fredholm residue condition $R^+ \subseteq \check{\mathbf{H}}^+$,

$$\operatorname{index} D^{\mathrm{I},+}_{R^+} = \dim\ker D^{\mathrm{I},+}_{\min} - \dim\operatorname{coker} D^{\mathrm{I},+}_{\max} + \operatorname{index}\delta_{R^+}$$

with

$$\delta_{R^+} \coloneqq \mathrm{pr}_{\check{\mathbf{H}}^+/\Lambda^+}|_{R^+}\colon R^+ \to \check{\mathbf{H}}^+/\Lambda^+$$

denoting the restriction of the projection map.

The difference

$$\delta_{R^+_{V^+}} - \delta_{R^+_{\mathrm{APS}}}\mathbf{1}_{(-\infty,0)}(A^+)|_{R^+_{V^+}} = \mathrm{pr}_{\check{\mathbf{H}}^+/\Lambda^+}\mathbf{1}_{[0,\infty)}(A^+)|_{R^+_{V^+}}$$

is compact because the right-hand side factors through the compact inclusion

$$\mathbf{1}_{[0,\infty)}(A^+)H^{1/2}\Gamma(Z,\check{S}^+) \hookrightarrow \mathbf{1}_{[0,\infty)}(A^+)H^{-1/2}\Gamma(Z,\check{S}^+).$$

Therefore,

$$\operatorname{index} D^{\mathrm{I},+}_{R^+_{V^+}} - \operatorname{index} D^{\mathrm{I},+}_{R^+_{\mathrm{APS}}} = \operatorname{index} \delta_{R^+_{V^+}} - \operatorname{index} \delta_{R^+_{\mathrm{APS}}} = \operatorname{index} \mathbf{1}_{(-\infty,0)}(A^+)|_{R^+_{V^+}}. \qquad \blacksquare$$

The above reduces the proof of Proposition B.3 to the following relative index formula.

Proposition B.6. *If $V^+, W^+ \subseteq \check{S}^+$ are $\mathbb{A}$–linear subbundles with $\operatorname{rk}_{\mathbb{A}} V^+ = \operatorname{rk}_{\mathbb{A}} W^+ = 1$, then*

$$\begin{aligned}\operatorname{index}\big(\mathbf{1}_{(-\infty,0)}(A^+)\colon\ R^+_{V^+} \to R^+_{\mathrm{APS}}\big) - \operatorname{index}\big(\mathbf{1}_{(-\infty,0)}(A^+)\colon\ R^+_{W^+} \to R^+_{\mathrm{APS}}\big)&\\ = \langle e(V^+),[Z]\rangle - \langle e(W^+),[Z]\rangle.&\end{aligned}$$

Proof. The proof consists of four steps.

Step 1. *Reduction to orientable Z.*

Denote the orthogonal projection onto W^+ by pr_{W^+}. As explained in [BW26, Proof of Theorem 4.40],

$$\mathrm{pr}_{W^+} - \mathbf{1}_{[0,\infty)}(A^+)$$

is an elliptic pseudo-differential operator. Since

$$\mathrm{pr}_{W^+} - \mathbf{1}_{[0,\infty)}(A^+) = \mathbf{1}_{(-\infty,0)}(A^+)\mathrm{pr}_{W^+} - \mathbf{1}_{[0,\infty)}(A^+)(\mathbf{1} - \mathrm{pr}_{W^+}),$$

a pseudo-differential parametrix for $\mathrm{pr}_{W^+} - \mathbf{1}_{[0,\infty)}(A^+)$ induces a pseudo-differential parametrix

$$G_{W^+}\colon\ R^+_{\mathrm{APS}} \to R^+_{W^+} = H^{1/2}\Gamma(Z,W^+)$$

of order zero. Therefore,

$$G_{W^+}\mathbf{1}_{(-\infty,0)}(A^+)\colon\ H^{1/2}\Gamma(Z,V^+) \to H^{1/2}\Gamma(Z,W^+)$$

is an elliptic pseudo-differential operator with

$$\begin{aligned}&\operatorname{index}(G_{W^+}\mathbf{1}_{(-\infty,0)}(A^+)\colon\ H^{1/2}\Gamma(Z,V^+) \to H^{1/2}\Gamma(Z,W^+))\\ &\qquad = \operatorname{index}(\mathbf{1}_{(-\infty,0)}(A^+)\colon\ R^+_{V^+} \to R^+_{\mathrm{APS}}) - \operatorname{index}(\mathbf{1}_{(-\infty,0)}(A^+)\colon\ R^+_{W^+} \to R^+_{\mathrm{APS}}).\end{aligned}$$

Let $\pi\colon\ \tilde{Z} \to Z$ be the orientation cover. The above setup pulls back along π. By [AS68b, Theorem 2.12], the index doubles after pulling back. Moreover

$$\langle e(\pi^*V^+) - e(\pi^*W^+), [\tilde{Z}]\rangle = 2\langle e(V^+) - e(W^+), [Z]\rangle.$$

Therefore, it suffices to verify the asserted index formula for oriented Z.

Henceforth, Z is assumed to be oriented. As a consequence $\mathbb{A} = \mathbb{C}$ and I is a complex structure on $\check{S}$. The remaining steps prove that, in this situation,

$$\text{(B.7)}\qquad \operatorname{index}\big(\mathbf{1}_{(-\infty,0)}(A^+)\colon\ R^+_{V^+} \to R^+_{\mathrm{APS}}\big) = \frac{1}{2}\dim\ker A^+ + \langle e(V^+),[Z]\rangle + \frac{1}{2}\chi(Z),$$

and analogously for W^+.

Step 2. *Variations on the APS residue condition.*

Given a self-adjoint first-order elliptic operator T on $\check{S}^+$ with symbol $\sigma_T(\xi) = \sigma_{A^+}(\xi) = -J\gamma(\xi)$ and $TI = -IT$, set

$$R^+_{\mathrm{APS},T} \coloneqq \mathbf{1}_{(-\infty,0)}(T)H^{1/2}\Gamma(Z,\check{S}^+).$$

Of course, $T = A^+$ recovers the usual positive APS residue condition. Since $TI = -IT$,

$$\mathbf{1}_{(0,\infty)}(T) = -I\mathbf{1}_{(-\infty,0)}(T)I \tag{B.8}$$

and, in particular,

$$H^{1/2}\Gamma(Z,\check{S}^+) = R^+_{\mathrm{APS},T} \oplus \ker T \oplus IR^+_{\mathrm{APS},T}.$$

If T_1, T_2 are two such operators, then

$$\operatorname{index}\bigl(\mathbf{1}_{(-\infty,0)}(T_2)\colon R^+_{\mathrm{APS},T_1} \to R^+_{\mathrm{APS},T_2}\bigr) = \frac{1}{2}(\dim\ker T_2 - \dim\ker T_1). \tag{B.9}$$

This holds for the following reason. Since T_1 and T_2 have the same principal symbol,

$$\mathbf{1}_{(-\infty,0)}(T_1) - \mathbf{1}_{(-\infty,0)}(T_2) \quad \text{and} \quad \mathbf{1}_{(0,\infty)}(T_1) - \mathbf{1}_{(0,\infty)}(T_2)$$

are of order -1 and, therefore, are compact on $H^{1/2}\Gamma(Z,\check{S}^+)$. Therefore,

$$\mathbf{1}_{(-\infty,0)}(T_2)\mathbf{1}_{(-\infty,0)}(T_1) + \mathbf{1}_{\{0\}}(T_2)\mathbf{1}_{\{0\}}(T_1) + \mathbf{1}_{(0,\infty)}(T_2)\mathbf{1}_{(0,\infty)}(T_1)$$

is a compact perturbation of the identity and thus has index zero. This combined with (B.8) implies that

$$\begin{aligned}
0 &= \operatorname{index}(\mathbf{1}_{(-\infty,0)}(T_2)\colon R^+_{\mathrm{APS},T_1} \to R^+_{\mathrm{APS},T_2}) + \dim\ker T_1 - \dim\ker T_2 \\
&\quad + \operatorname{index}(I\mathbf{1}_{(-\infty,0)}(T_2)I^{-1}\colon IR^+_{\mathrm{APS},T_1} \to IR^+_{\mathrm{APS},T_2}) \\
&= 2\operatorname{index}(\mathbf{1}_{(-\infty,0)}(T_2)\colon R^+_{\mathrm{APS},T_1} \to R^+_{\mathrm{APS},T_2}) + \dim\ker T_1 - \dim\ker T_2.
\end{aligned}$$

Step 3. *A variation of the APS residue condition adapted to V^+.*

Decompose A^+ with respect to the decomposition $\check{S}^+ = V^+ \oplus V^{+\perp}$ as

$$A^+ = \begin{pmatrix} * & B^* \\ B & * \end{pmatrix}.$$

A moment's thought shows that the operator

$$T \coloneqq \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix}$$

is of the type considered above.

The composition

$$R^+_{V^+} \xrightarrow{\mathbf{1}_{(-\infty,0)}(T)} R^+_{\mathrm{APS},T} \xrightarrow{\mathbf{1}_{(-\infty,0)}(A^+)} R^+_{\mathrm{APS}}$$

differs from the direct projection onto R^+_{APS} by

$$(\mathbf{1}_{(-\infty,0)}(A^+) - \mathbf{1}_{(-\infty,0)}(T))(\mathbf{1}_{(-\infty,0)}(T) - \mathbf{1})|_{R^+_{V^+}}.$$

The latter is compact because the first factor is of order -1. Therefore, by (B.9),

$$\begin{aligned}
\operatorname{index}\bigl(\mathbf{1}_{(-\infty,0)}(A^+)\colon R^+_{V^+} \to R^+_{\mathrm{APS}}\bigr) &= \operatorname{index}\bigl(\mathbf{1}_{(-\infty,0)}(A^+)\colon R^+_{\mathrm{APS},T} \to R^+_{\mathrm{APS}}\bigr)\\
&\quad + \operatorname{index}\bigl(\mathbf{1}_{(-\infty,0)}(T)\colon R^+_{V^+} \to R^+_{\mathrm{APS},T}\bigr)\\
&= \frac{1}{2}\bigl(\dim\ker A^+ - \dim\ker B - \dim\ker B^*\bigr)\\
&\quad + \operatorname{index}\bigl(\mathbf{1}_{(-\infty,0)}(T)\colon R^+_{V^+} \to R^+_{\mathrm{APS},T}\bigr).
\end{aligned}$$

To compute the index of $\mathbf{1}_{(-\infty,0)}(T)\colon R^+_{V^+} \to R^+_{\mathrm{APS},T}$, consider the polar decomposition $B = U|B|$ with U vanishing on $\ker B$. Since

$$\mathbf{1}_{(0,\infty)}(T) - \mathbf{1}_{(-\infty,0)}(T) = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix},$$

U is a pseudo-differential operator of order zero. Moreover,

$$R^+_{\mathrm{APS},T} = \bigl\{(a, -Ua) : a \in H^{1/2}\Gamma(Z, V^+) \cap (\ker B)^\perp\bigr\}.$$

In particular, with π denoting the orthogonal projection onto $\ker B$, for every $b \in H^{1/2}\Gamma(Z, V^+)$

$$\mathbf{1}_{(-\infty,0)}(T)(b, 0) = \frac{1}{2}(b - \pi b, -U(b - \pi b)).$$

Consequently,

$$\mathbf{1}_{(-\infty,0)}(T)\colon R^+_{V^+} \to R^+_{\mathrm{APS},T}$$

is surjective and its kernel is $\ker B$. The above, therefore, simplifies to

$$\operatorname{index}\bigl(\mathbf{1}_{(-\infty,0)}(A^+)\colon R^+_{V^+} \to R^+_{\mathrm{APS}}\bigr) = \frac{1}{2}\bigl(\dim\ker A^+ + \operatorname{index} B\bigr).$$

Step 4. *Computation of* $\operatorname{index} B$*.*

The symbol of B is $\sigma_B(\xi) = -J\gamma(\xi)\colon V^+ \to (V^+)^\perp$. The Clifford multiplication defines a complex anti-linear isomorphism $(V^+)^\perp \cong V^+ \otimes_{\mathbb{C}} \Lambda^{0,1}T^*Z$. Treating this as an identification exhibits B as a complex linear differential operator $\Gamma(Z, V^+) \to \Omega^{0,1}(Z, V^+)$ with the symbol of a Cauchy–Riemann operator $\bar\partial_{V^+}$ (up to a constant factor). Therefore, by Riemann–Roch,

$$\operatorname{index} B = 2\operatorname{index}_{\mathbb{C}} \bar\partial_{V^+} = 2\langle e(V^+), [Z]\rangle + \chi(Z).$$

This combined with the previous step yields the desired (B.7). ■

Proof of Proposition B.3. This is an immediate consequence of Proposition B.5 and Proposition B.6. ■

References

[AG16] Pierre Albin and Jesse Gell-Redman. *The index of Dirac operators on incomplete edge spaces. SIGMA Symmetry Integrability Geom. Methods Appl.* 12 (2016), Paper No. 089, 45. DOI: 10.3842/SIGMA.2016.089. MR: 3544861. zbMATH: 6633211 (cit. on p. 8)

[AG23] Pierre Albin and Jesse Gell-Redman. *The index formula for families of Dirac type operators on pseudomanifolds. J. Differential Geom.* 125.2 (2023), pp. 207–343. DOI: 10.4310/jdg/1696432923. MR: 4649388. zbMATH: 7783633 (cit. on p. 8)

[AD25] Bernd Ammann and Mattias Dahl. *The space of Dirac-minimal metrics is connected in dimensions* 2 *and* 4. *SIGMA, Symmetry Integrability Geom. Methods Appl.* 21 (2025), paper 102, 18. DOI: 10.3842/SIGMA.2025.102. MR: 4998264. zbMATH: 8142854 (cit. on pp. 15, 29, 52)

[AWW16] Bernd Ammann, Hartmut Weiss, and Frederik Witt. *A spinorial energy functional: critical points and gradient flow. Math. Ann.* 365.3-4 (2016), pp. 1559–1602. DOI: 10.1007/s00208-015-1315-8. zbMATH: 6618541 (cit. on p. 15)

[AHS78] Michael F. Atiyah, Nigel J. Hitchin, and Isadore M. Singer. *Self-duality in four-dimensional Riemannian geometry. Proc. R. Soc. Lond. A* 362 (1978), pp. 425–461. DOI: 10.1098/rspa.1978.0143. MR: 506229. zbMATH: 3605562 (cit. on p. 67)

[AS68a] Michael F. Atiyah and Isadore M. Singer. *The index of elliptic operators. I. Ann. of Math. (2)* 87 (1968), pp. 484–530. DOI: 10.2307/1970715. MR: 236950 (cit. on p. 67)

[AS68b] Michael F. Atiyah and Isadore M. Singer. *The index of elliptic operators. III. Ann. of Math. (2)* 87.3 (1968), pp. 546–604. DOI: 10.2307/1970717. MR: 236952. zbMATH: 3262980 (cit. on pp. 67, 68)

[BB12] Christian Bär and Werner Ballmann. *Boundary value problems for elliptic differential operators of first order. Surveys in differential geometry.* 17. International Press, 2012, pp. 1–78. DOI: 10.4310/SDG.2012.v17.n1.a1. arXiv: 1101.1196. MR: 3076058. zbMATH: 6322496 (cit. on p. 8)

[BW26] Gorapada Bera and Thomas Walpuski. *Dirac operators twisted by ramified Euclidean line bundles. J. Reine Angew. Math.* (2026). arXiv: 2503.01392. zbMATH: 902679470. . to appear (cit. on pp. 2, 8–14, 23, 25, 34, 35, 37–41, 51, 57, 59, 61, 62, 67, 68)

[BGV92] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators.* Grundlehren der Mathematischen Wissenschaften 298. Springer-Verlag, 1992. DOI: 10.1007/978-3-642-58088-8. MR: 1215720. zbMATH: 52134 (cit. on p. 18)

[BG92] Jean-Pierre Bourguignon and Paul Gauduchon. *Spineurs, opérateurs de Dirac et variations de métriques. Communications in Mathematical Physics* 144.3 (1992), pp. 581–599. DOI: 10.1007/BF02099184. MR: 1158762. zbMATH: 54761 (cit. on pp. 15, 21, 22)

[DS13] Anda Degeratu and Mark Stern. *Witten spinors on nonspin manifolds. Comm. Math. Phys.* 324.2 (2013), pp. 301–350. DOI: 10.1007/s00220-013-1804-0. MR: 3117513. zbMATH: 6226654 (cit. on p. 40)

[Die19] Tobias Diez. *Normal Form of Equivariant Maps and Singular Symplectic Reduction in Infinite Dimensions with Applications to Gauge Field Theory*. PhD thesis. Universität Leipzig, 2019. arXiv: 1909.00744. zbMATH: 900385099 (cit. on p. 45)

[DR22] Tobias Diez and Gerd Rudolph. *Normal form of equivariant maps in infinite dimensions*. *Ann. Global Anal. Geom.* 61.1 (2022), pp. 159–213. DOI: 10.1007/s10455-021-09777-2. MR: 4367904. zbMATH: 7478462 (cit. on pp. 45, 46)

[DW19] Aleksander Doan and Thomas Walpuski. *On counting associative submanifolds and Seiberg–Witten monopoles*. *Pure and Applied Mathematics Quarterly* 15.4 (2019), pp. 1047–1133. DOI: 10.4310/PAMQ.2019.v15.n4.a4. arXiv: 1712.08383. zbMATH: 7182264. (cit. on p. 2)

[DW21] Aleksander Doan and Thomas Walpuski. *On the existence of harmonic $\mathbb{Z}_2$ spinors*. *Journal of Differential Geometry* 117.3 (2021), pp. 395–449. DOI: 10.4310/jdg/1615487003. arXiv: 1710.06781. MR: 4255067. zbMATH: 7367821. (cit. on pp. 2, 54)

[DW23] Aleksander Doan and Thomas Walpuski. *Equivariant Brill–Noether theory for elliptic operators and super-rigidity of J–holomorphic maps*. *Forum of Mathematics Sigma* 11 (2023). DOI: 10.1017/fms.2022.104. arXiv: 2006.01352. MR: 4537309. (cit. on pp. 30, 53)

[DW24] Aleksander Doan and Thomas Walpuski. *Chambered invariants of real Cauchy–Riemann operators*. 2024. arXiv: 2410.21057. zbMATH: 902593006. (cit. on pp. 2, 3, 18–20)

[DL25] Simon Donaldson and Fabian Lehmann. *Calabi–Yau threefolds with boundary*. *Pure Appl. Math. Q.* 21.3 (2025), pp. 1119–1170. DOI: 10.4310/pamq.250116012309. MR: 4852029. zbMATH: 7969235 (cit. on pp. 8, 40, 50)

[Don17] Simon K. Donaldson. *Adiabatic limits of co-associative Kovalev–Lefschetz fibrations*. *Algebra, Geometry, and Physics in the 21st Century*. Springer, 2017, pp. 1–29. DOI: 10.1007/978-3-319-59939-7_1. arXiv: 1603.08391. zbMATH: 6854365 (cit. on p. 2)

[Don21a] Simon K. Donaldson. *Deformations of multivalued harmonic functions*. *The Quarterly Journal of Mathematics* 72.1-2 (2021), pp. 199–235. DOI: 10.1093/qmath/haab018. MR: 4271385. zbMATH: 7389919 (cit. on pp. 1, 2, 6, 7, 23, 25, 26, 57, 58)

[Don21b] Simon K. Donaldson. *Deformations of singular sets and Nash–Moser theory II*. Talk delivered at the meeting "Donaldson–Thomas invariants and Resurgence" of the Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics. 2021. (cit. on pp. 3, 6)

[DS11] Simon K. Donaldson and E. P. Segal. *Gauge theory in higher dimensions, II*. *Surveys in differential geometry*. Vol. 16. Geometry of special holonomy and related topics. International Press, 2011, pp. 1–41. arXiv: 0902.3239. MR: 2893675. zbMATH: 6085990 (cit. on p. 2)

[Glö16] Helge Glöckner. *Fundamentals of submersions and immersions between infinite-dimensional manifolds.* 2016. DOI: 10.48550/arXiv.1502.05795. arXiv: 1502.05795. zbMATH: 900240202 (cit. on pp. 44, 45)

[GS99] Robert E. Gompf and András I. Stipsicz. *4–manifolds and Kirby calculus.* Vol. 20. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1999. DOI: 10.1090/gsm/020. MR: 1707327. zbMATH: 1356915 (cit. on p. 60)

[GK26] Josef Greilhuber and Willi Kepplinger. *On Arnold's Transversality Conjecture for the Laplace–Beltrami Operator.* 2026. arXiv: 2312.16939. zbMATH: 902392331 (cit. on p. 53)

[Ham79] Richard S. Hamilton. *Deformation of complex structures on manifolds with boundary. II. Families of noncoercive boundary value problems. J. Differential Geometry* 14.3 (1979), pp. 409–473. DOI: 10.4310/jdg/1214435106. MR: 594711. zbMATH: 3808012 (cit. on p. 40)

[Ham82] Richard S. Hamilton. *The inverse function theorem of Nash and Moser. Bull. Amer. Math. Soc. (N.S.)* 7.1 (1982), pp. 65–222. DOI: 10.1090/S0273-0979-1982-15004-2. MR: 656198. zbMATH: 3787692 (cit. on pp. 7, 18, 25, 37, 45, 47, 51, 53, 54)

[Har90] F. Reese Harvey. *Spinors and calibrations.* Vol. 9. Perspectives in Mathematics. Boston, MA: Academic Press Inc., 1990. zbMATH: 46293 (cit. on p. 17)

[Hay17] Andriy Haydys. *G_2 instantons and the Seiberg–Witten monopoles. Gromov–Witten theory, gauge theory and dualities, Kioloa, Australia, January 6–16, 2016.* Id/No 3. Canberra: The Australian National University, Centre for Mathematics and its Applications, 2017. arXiv: 1703.06329. zbMATH: 7255724 (cit. on p. 2)

[HMT23] Andriy Haydys, Rafe Mazzeo, and Ryosuke Takahashi. *An index theorem for $\mathbb{Z}/2$–harmonic spinors branching along a graph.* 2023. arXiv: 2310.15295 (cit. on p. 4)

[HW15] Andriy Haydys and Thomas Walpuski. *A compactness theorem for the Seiberg–Witten equation with multiple spinors in dimension three. Geom. Funct. Anal.* 25.6 (2015), pp. 1799–1821. DOI: 10.1007/s00039-015-0346-3. arXiv: 1406.5683. MR: 3432158. zbMATH: 6526260. (cit. on p. 2)

[He23] Siqi He. *The branched deformations of the special Lagrangian submanifolds. Geom. Funct. Anal.* 33.5 (2023), pp. 1266–1321. DOI: 10.1007/s00039-023-00645-8. arXiv: 2202.12282. MR: 4646409. zbMATH: 7746809 (cit. on p. 2)

[HP24] Siqi He and Gregory J. Parker. *$\mathbb{Z}_2$-Harmonic Spinors and 1–forms on Connected sums and Torus sums of 3–manifolds.* 2024. arXiv: 2407.10922. (cit. on pp. 6, 53)

[Hoc46] Gerhard Hochschild. *On the cohomology theory for associative algebras. Ann. Math. (2)* 47 (1946), pp. 568–579. DOI: 10.2307/1969093. MR: 16762. zbMATH: 3104825 (cit. on p. 17)

[Joy18] Dominic D. Joyce. *Conjectures on counting associative* 3*–folds in* G_2*–manifolds. Modern geometry: a celebration of the work of Simon Donaldson.* Proceedings of Symposia in Pure Mathematics 99. American Mathematical Society, 2018, pp. 97–160. DOI: 10.1090/pspum/099/01739. arXiv: 1610.09836. MR: 3838881. zbMATH: 7278411 (cit. on p. 2)

[Kos72] Yvette Kosmann. *Dérivées de Lie des spineurs. (Lie derivatives of spinors). Ann. Mat. Pura Appl. (4)* 91 (1972), pp. 317–395. DOI: 10.1007/BF02428822. MR: 312413. zbMATH: 3366098 (cit. on pp. 21, 32)

[LM89] H. Blaine Lawson Jr. and Marie-Louise Michelsohn. *Spin geometry.* Princeton Mathematical Series 38. Princeton University Press, 1989. MR: 1031992. zbMATH: 44927 (cit. on pp. 2, 10, 67)

[Mai97] Stephan Maier. *Generic metrics and connections on* Spin*– and* Spin^c*–manifolds. Communications in Mathematical Physics* 188.2 (1997), pp. 407–437. DOI: 10.1007/s002200050171. MR: 1471821. zbMATH: 1121181 (cit. on pp. 29, 52, 53)

[Maz91] Rafe Mazzeo. *Elliptic theory of differential edge operators. I. Communications in Partial Differential Equations* 16.10 (1991), pp. 1615–1664. DOI: 10.1080/03605309108820815. MR: 1133743. zbMATH: 28820 (cit. on p. 8)

[MV14] Rafe Mazzeo and Boris Vertman. *Elliptic theory of differential edge operators, II: Boundary value problems. Indiana Univ. Math. J.* 63.6 (2014), pp. 1911–1955. DOI: 10.1512/iumj.2014.63.5435. MR: 3298726. zbMATH: 6405945 (cit. on p. 8)

[Mel91] Richard B. Melrose. *Pseudodifferential operators, corners and singular limits. Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990).* Math. Soc. Japan, Tokyo, 1991, pp. 217–234. MR: 1159214. zbMATH: 66480 (cit. on p. 12)

[Mel93] Richard B. Melrose. *The Atiyah–Patodi–Singer index theorem.* Vol. 4. Research Notes in Mathematics. A. K. Peters, 1993. DOI: 10.1016/0377-0257(93)80040-i. MR: 1348401. zbMATH: 496294 (cit. on p. 12)

[Moe02] Ieke Moerdijk. *Orbifolds as groupoids: an introduction. Orbifolds in mathematics and physics (Madison, WI, 2001).* Vol. 310. Contemp. Math. Providence, RI: American Mathematical Society (AMS), 2002, pp. 205–222. DOI: 10.1090/conm/310/05405. MR: 1950948. zbMATH: 1942887 (cit. on p. 20)

[Mol10] Mathieu Molitor. *The group of unimodular automorphisms of a principal bundle and the Euler–Yang–Mills equations. Differ. Geom. Appl.* 28.5 (2010), pp. 543–564. DOI: 10.1016/j.difgeo.2010.04.005. MR: 2670086. zbMATH: 5772557 (cit. on p. 21)

[MN17] Olaf Müller and Nikolai Nowaczyk. *A universal spinor bundle and the Einstein–Dirac–Maxwell equation as a variational theory. Lett. Math. Phys.* 107.5 (2017), pp. 933–961. DOI: 10.1007/s11005-016-0929-4. MR: 3633031. zbMATH: 6722082 (cit. on p. 15)

[Par23] Gregory J. Parker. *Deformations of* $\mathbb{Z}_2$*–Harmonic Spinors on* 3*–Manifolds. Geom. Funct. Anal.* (2023). arXiv: 2301.06245. to appear (cit. on pp. 1–4, 6, 39, 62)

[Qui70] Frank Quinn. *Transversal approximation on Banach manifolds. Global Analysis (Proc. Sympos. Pure Math., Vols. XIV, XV, XVI, Berkeley, Calif., 1968).* Vol. XIV-XVI. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1970, pp. 213–222. MR: 264713. zbMATH: 3328095 (cit. on p. 46)

[Sma65] Stephen Smale. *An infinite dimensional version of Sard's theorem. American Journal of Mathematics* 87 (1965), pp. 861–866. DOI: 10.2307/2373250. MR: 0185604. zbMATH: 3231415 (cit. on pp. 45, 46)

[Tak23] Ryosuke Takahashi. *The moduli space of S^1–type zero loci for $\mathbb{Z}/2\mathbb{Z}$–harmonic spinors in dimension* 3. *Comm. Anal. Geom.* 31.1 (2023), pp. 119–242. DOI: 10.4310/cag.2023.v31.n1.a5. arXiv: 1503.00767. zbMATH: 7742707 (cit. on pp. 1–3)

[Tau13a] Clifford H. Taubes. $\mathbf{PSL}(2;\mathbb{C})$ *connections on* 3*–manifolds with L^2 bounds on curvature. Cambridge Journal of Mathematics* 1.2 (2013), pp. 239–397. DOI: 10.4310/CJM.2013.v1.n2.a2. arXiv: 1205.0514. MR: 3272050. zbMATH: 6292859 (cit. on p. 2)

[Tau13b] Clifford H. Taubes. *Compactness theorems for* $\mathrm{SL}(2;\mathbb{C})$ *generalizations of the* 4*–dimensional anti-self dual equations.* 2013. arXiv: 1307.6447 (cit. on p. 2)

[Tau14] Clifford H. Taubes. *The zero loci of $\mathbb{Z}/2$ harmonic spinors in dimension* 2, 3 *and* 4. 2014. arXiv: 1407.6206 (cit. on p. 2)

[Tau16] Clifford H. Taubes. *On the behavior of sequences of solutions to* $\mathrm{U}(1)$ *Seiberg–Witten systems in dimension* 4. 2016. arXiv: 1610.07163 (cit. on p. 2)

[Tau17] Clifford H. Taubes. *The behavior of sequences of solutions to the Vafa–Witten equations.* 2017. arXiv: 1702.04610. zbMATH: 900307151 (cit. on p. 2)

[Tau22] Clifford H. Taubes. *Sequences of Nahm pole solutions to the* $\mathrm{SU}(2)$ *Kapustin–Witten equations.* 2022. arXiv: 1805.02773 (cit. on p. 2)

[TW21] Clifford H. Taubes and Yingying Wu. *Topological aspects of $\mathbb{Z}/2\mathbb{Z}$ eigenfunctions for the Laplacian on S^2.* 2021. arXiv: 2108.05017 (cit. on pp. 3, 4, 27)

[Tel83] Nicolae Teleman. *The index of signature operators on Lipschitz manifolds. Inst. Hautes Études Sci. Publ. Math.* 58 (1983), pp. 39–78. MR: 720931. zbMATH: 3842838. (cit. on pp. 5, 8, 57)

[tDie08] Tammo tom Dieck. *Algebraic topology.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008. DOI: 10.4171/048. MR: 2456045. zbMATH: 5318519 (cit. on p. 19)

[WZ21] Thomas Walpuski and Boyu Zhang. *On the compactness problem for a family of generalized Seiberg–Witten equations in dimension 3. Duke Mathematical Journal* 170.17 (2021), pp. 3891–3934. DOI: 10.1215/00127094-2021-0005. arXiv: 1904.03749. zbMATH: 7467778. (cit. on p. 2)

[Wan93] Shuguang Wang. *Moduli spaces over manifolds with involutions. Math. Ann.* 296.1 (1993), pp. 119–138. DOI: 10.1007/BF01445098. MR: 1213375. zbMATH: 537290 (cit. on p. 57)

[Wei94] Charles A. Weibel. *An introduction to homological algebra.* Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994. DOI: 10.1017/CBO9781139644136. MR: 1269324. zbMATH: 595200 (cit. on p. 17)

[Wen23] Chris Wendl. *Transversality and super-rigidity for multiply covered holomorphic curves. Ann. of Math. (2)* 198.1 (2023), pp. 93–230. DOI: 10.4007/annals.2023.198.1.2. arXiv: 1609.09867. MR: 4593732. zbMATH: 7690462 (cit. on p. 53)

[Yan25a] Dashen Yan. *A Construction of non-degenerate* $\mathbb{Z}_2$*−harmonic functions on* $\mathbb{R}^n$. 2025. arXiv: 2503.19286. (cit. on p. 54)

[Yan25b] Dashen Yan. *On nondegenerate* $\mathbb{Z}_2$*−harmonic* 1*−forms with shrinking branching sets.* 2025. arXiv: 2510.07678. (cit. on p. 54)