MTH931 Riemannian Geometry II

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Contents

1 Riemannian metrics

Definition 1.1. Let M be a manifold. A **Riemannian metric** on M is a bilinear form $g \in \Gamma(S^2T^*M)$ on TM which is positive definite. A **Riemannian manifold** is a pair (M, g) consisting of a manifold M and a Riemannian metric g on M.

Notation 1.2. If (M, g) is a Riemannian manifold, $x \in M$, and $v, w \in T_xM$, then we set

$$\langle v, w \rangle_g := g(v, w) \quad \text{and} \quad |v|_g := \sqrt{g(v, v)}. \tag{1.3}$$

Notation 1.4. If $x^1, \dots, x^n \colon M \supset U \to \mathbf{R}$ are local coordinates, for $a, b \in \{1, \dots, n\}$, we set

$$\partial_a := \frac{\partial}{\partial x^a} \quad \text{and} \quad g_{ab} := g(\partial_a, \partial_b).$$

Definition 1.5. The **musical isomorphisms** $\cdot^\flat \colon TM \to T^*M$ and $\cdot^\sharp \colon T^*M \to TM$ are defined by

$$v^\flat := \langle v, \cdot \rangle \quad \text{and} \quad \left\langle \alpha^\sharp, \cdot \right\rangle := \alpha(\cdot).$$

Definition 1.6. Let (M, g) be Riemannian manifold. Let $f \in C^\infty(M)$. The **gradient** of f is the vector field ∇f defined by

$$\langle \nabla f, \cdot \rangle := \mathrm{d}f.$$

The **Hessian** of f is the bilinear form $\operatorname{Hess} f \in \Gamma(S^2T^*M)$ defined by

$$\operatorname{Hess} f := \frac{1}{2}\mathscr{L}_{\nabla f} g.$$

2 The Riemannian distance

Definition 2.1. The **length** of a curve $\gamma \colon [t_0, t_1] \to M$ is defined by

$$\ell(\gamma) := \int_{t_0}^{t_1} |\dot\gamma(t)| \, \mathrm{d}t. \tag{2.2}$$

Remark 2.3. The length functional ℓ is invariant under reparametrizations of γ.

Definition 2.4. A curve $\gamma \colon [t_0, t_1] \to M$ is **parametrized by arc-length** or has **unit speed** if

$$\ell(\gamma|_{[t_0,t]}) = t - t_0 \quad \text{or, equivalently,} \quad |\dot\gamma| = 1. \tag{2.5}$$

Definition 2.6. The **Riemannian distance** associated with (M, g) is the function $d\colon M\times M \to [0, \infty]$ defined by

$$d(x, y) \coloneqq \inf\{\ell(\gamma) : \gamma \in C^\infty([t_0, t_1], M) \text{ with } \gamma(t_0) = x \text{ and } \gamma(t_1) = y\}. \tag{2.7}$$

Proposition 2.8. *(M, d) is a metric space.*

3 The Riemanian volume form

Definition 3.1. Let (M, g) be an oriented Riemannian manifold. The **Riemannian volume form** is the unique positive volume form

$$\mathrm{vol}_g \quad \text{satisfying} \quad |\mathrm{vol}_g| = 1. \tag{3.2}$$

Proposition 3.3. *In local coordinates $x^1, \dots, x^n$,*

$$\mathrm{vol}_g = \sqrt{\det g}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

Definition 3.4. Let (M, g) be an oriented Riemannian manifold of dimension n. The **Hodge star operator** is the linear map $\star\colon \Lambda^\bullet T^*M \to \Lambda^{\bullet-n}T^*M$ defined by

$$\alpha \wedge \star\beta = \langle\alpha, \beta\rangle\mathrm{vol}_g.$$

Definition 3.5. Let (M, g) be an Riemannian manifold. The **divergence** of $v \in \mathrm{Vect}(M)$ is the function $\operatorname{div} v \in C^\infty(M)$ defined by

$$\operatorname{div}(v)\mathrm{vol}_g = \mathscr{L}_v\mathrm{vol}_g.$$

(Here vol_g need only be locally defined.)

Definition 3.6. Let (M, g) be Riemannian manifold. The **Laplacian** of $f \in C^\infty(M)$ is the function $\Delta f \in C^\infty(M)$ defined by

$$\Delta f = -\operatorname{div}\nabla f.$$

Proposition 3.7. *For $f \in C^\infty(M)$, in local coordinates $x^1, \dots, x^n$,*

$$\operatorname{Hess} f = \sum_{a,b=1}^{n}\left(\partial_a\partial_b f - \sum_{c=1}^{n}\Gamma_{ab}^c\partial_c f\right)\mathrm{d}x^a \otimes \mathrm{d}x^b \quad \textit{and}$$
$$\Delta f = -\sum_{a,b=1}^{n}\frac{1}{\sqrt{\det g}}\partial_a\left(\sqrt{\det g}\cdot g^{ab}\partial_b f\right).$$

Proposition 3.8. *For $f \in C^\infty(M)$,*

$$\Delta f = -\operatorname{tr}\operatorname{Hess} f.$$

4 The Levi-Civita connection

Definition 4.1. Let M be a manifold. An **affine connection** is a connection on TM. An affine connection ∇ is called **torsion-free** if for all $v, w \in \mathrm{Vect}(M)$,

$$\nabla_v w - \nabla_w v = [v, w].$$

Definition 4.2. Let (M, g) be Riemannian manifold. An affine connection ∇ is called **metric** if

$$\nabla g = 0;$$

that is: for all $v, w \in \mathrm{Vect}(M)$,

$$dg(v, w) = g(\nabla v, w) + g(v, \nabla w).$$

Theorem 4.3 (Fundamental Theorem of Riemannian Geometry). *Let (M, g) be a Riemannian manifold.*

1. *There exists a unique affine connection ∇^{LC} which is torsion-free and metric.*
2. *The affine connection ∇^{LC} satisfies **Koszul's formula**:*

$$\begin{aligned} 2\left\langle \nabla^{\mathrm{LC}}_u v, w \right\rangle &= \mathscr{L}_u \langle v, w\rangle + \mathscr{L}_v \langle w, u\rangle - \mathscr{L}_w \langle u, v\rangle \\ &\quad + \langle [u, v], w\rangle - \langle [u, w], v\rangle - \langle [v, w], u\rangle. \end{aligned} \tag{4.4}$$

3. *Suppose $x^1, \ldots, x^n$ are local coordinates on M. The **Christoffel symbols** Γ^c_{ab} defined by*

$$\nabla^{\mathrm{LC}}_{\partial_a} \partial_b = \sum_{c=1}^{n} \Gamma^c_{ab} \partial_c. \tag{4.5}$$

satisfy

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bd} - \partial_d g_{ab} + \partial_b g_{ad}). \tag{4.6}$$

Definition 4.7. We call ∇^{LC} the **Levi-Civita connection** associated with (M, g).

Remark 4.8. It is customary to drop the super-script LC.

5 The Riemann curvature tensor

Theorem 5.1. *Let (M, g) be a Riemannian manifold.*

1. *There exists a unique tensor field $R_g \in \Omega^2(M, \mathfrak{o}(TM))$ satisfying*

$$R_g(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w. \tag{5.2}$$

2. *The tensor field R_g satisfies*

$$\langle R_g(u, v)w, z \rangle = \langle R_g(w, z)u, v \rangle. \tag{5.3}$$

3. *The tensor field R_g satisfies the* ***algebraic Bianchi identity****:*

$$R_g(u, v)w + R_g(v, w)u + R_g(w, u)v = 0. \tag{5.4}$$

4. *The tensor field R_g satisfies the* ***differential Bianchi identity****:*

$$\mathrm{d}^\nabla R_g = 0. \tag{5.5}$$

Definition 5.6. We call R_g the **Riemann curvature tensor** of (M, g).

Remark 5.7. Let V be a Euclidean space of dimension n. The space of **algebraic curvature tensors** on V is

$$\mathscr{R}(V) := \ker\left(S^2\Lambda^2 V \xrightarrow{\wedge} \Lambda^4 V\right) \subset \Lambda^2 V \otimes \Lambda^2 V.$$

Since

$$\dim \mathscr{R}(V) = \frac{n^4 - n^2}{12},$$

at each point $x \in M$, the Riemann curvature tensor R_g has $(n^4 - n^2)/12$ components.

Definition 5.8. The **sectional curvature** of (M, g) is the map $\sec_g \colon \Lambda^2 TM \backslash \{0\} \to \mathbf{R}$ defined by

$$\sec_g(v \wedge w) := \frac{\langle R_g(v, w)w, v \rangle}{|v \wedge w|^2}. \tag{5.9}$$

Remark 5.10. The Riemann curvature tensor R_g can be recovered from the sectional curvature $\sec_g$ *algebraically*.

Remark 5.11. The sectional curvature *really* is a map $\mathrm{Gr}_2(TM) \to \mathbf{R}$.

Definition 5.12. The **curvature operator** is the self-adjoint map $\mathfrak{R}_g \in \Gamma(\mathrm{Sym}^2(\Lambda^2 TM))$ defined by

$$\langle \mathfrak{R}_g(u \wedge v), w \wedge z \rangle := \langle R_g(u, v)z, w \rangle.$$

6 Model spaces

Example 6.1 ($\mathbf{R}^n$). $\mathbf{R}^n$ with the Riemannian metric

$$g_0 := \sum_{a=1}^{n} \mathrm{d}x^a \otimes \mathrm{d}x^a$$

has vanishing Riemann curvature tensor: $R = 0$.

Exercise 6.2 (S^n). Consider the n–dimensional unit sphere

$$S^n := \left\{x \in \mathbf{R}^{n+1} : |x| = 1\right\}$$

with the Riemannian metric g_1 induced by g_0 on $\mathbf{R}^{n+1}$. Prove that:

1. If $u, v, w \in \mathrm{Vect}(S^n) \subset C^\infty(S^n, \mathbf{R}^{n+1})$, then at every point $x \in S^n$

$$\nabla_v w = \partial_v w + \langle v, w\rangle x.$$

2. The Riemannian curvature tensor of (S^n, g_1) is given by

$$R(u, v)w = \langle v, w\rangle u - \langle u, w\rangle v; \quad \text{that is:} \quad \sec = 1.$$

Exercise 6.3 (H^n). Consider $\mathbf{R}^{n+1}$ with the Lorentzian metric

$$g_L = -\mathrm{d}x^0 \otimes \mathrm{d}x^0 + \sum_{a=1}^{n} \mathrm{d}x^a \otimes \mathrm{d}x^a.$$

Set

$$H^n := \left\{x \in \mathbf{R}^{n+1} : g_L(x, x) = -1 \text{ and } x_0 > 0\right\}.$$

Prove that:

1. The symmetric bilinear form g_{-1} obtained by restricting g_L to H^n is positive definite; that is: a Riemannian metric.

2. The Riemannian curvature tensor of (H^n, g_{-1}) is given by

$$R(u, v)w = -\langle v, w\rangle u + \langle u, w\rangle v; \quad \text{that is:} \quad \sec = -1.$$

Definition 6.4. Let $n \in \{2, 3, \ldots\}$ and $\kappa \in \mathbf{R}$. The n–dimensional **model space** of constant sectional curvature κ is

$$(S^n_\kappa, g_\kappa) := \begin{cases} \left(S^n, \kappa^{-1/2} g_1\right) & \text{if } \kappa > 0, \\ (\mathbf{R}^n, g_0) & \text{if } \kappa = 0, \\ \left(H^n, (-\kappa)^{-1/2} g_1\right) & \text{if } \kappa < 0. \end{cases}$$

Theorem 6.5 (Riemann [Rie68], Killing [Kil91], and Hopf [Hop25]). *If (M, g) is a simply-connected Riemannian manifold of constant sectional curvature $\kappa \in \mathbf{R}$, then it is isometric to an open subset of (S^n_κ, g_κ).*

Proof sketch. The proof relies on Proposition 11.10, which can then be combined with a unique continuation argument. □

Definition 6.6. Let $n \in \{2, 3, \ldots\}$ and $\kappa \in \mathbf{R}$. The function $V^n_\kappa \colon [0, \infty) \to [0, \infty)$ is defined by

$$V^n_\kappa(r) := \mathrm{vol}(B_r(x)) \tag{6.7}$$

for $B_r(x) \subset S^n_\kappa$.

Remark 6.8. The functions V^n_κ satisfy the scaling relation

$$V^n_\kappa(r) = V^n_{r^2\kappa}(1).$$

Definition 6.9. For $\kappa \in \mathbf{R}$, set

$$\sin_\kappa(r) := \begin{cases} \sin(\sqrt{\kappa} r) & \text{if } \kappa > 0, \\ r & \text{if } \kappa = 0, \\ \sinh(\sqrt{-\kappa} r) & \text{if } \kappa < 0. \end{cases}$$

Exercise 6.10. Let $n \in \{2, 3, \ldots\}$ and $\kappa \in \mathbf{R}$. Denote by

$$\mathrm{vol}^n_\kappa$$

the Riemannian volume form of (S^n_κ, g_κ). Prove that, in geodesic polar coordinates,

$$g_\kappa = \mathrm{d}r \otimes \mathrm{d}r + \sin_\kappa(r)^2 g_{S^{n-1}} \quad \text{and} \tag{6.11}$$

$$\mathrm{vol}^n_\kappa = \sin_\kappa(r)^{n-1} \mathrm{d}r \wedge \mathrm{vol}_{S^{n-1}}. \tag{6.12}$$

Remark 6.13. It is exercise to compute that

$$\mathrm{vol}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}; \quad \text{and thus:} \quad V_{n,0}(r) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} r^n. \tag{6.14}$$

V^n_κ for $\kappa \neq 0$ can be expressed in terms of trigonometric/hyperbolic functions and Gauß' hypergeometric function ${}_2F_1$; but these formulae are unwieldy.

If $\kappa > 0$, then V^n_κ is constant equal to $\kappa^{n/2}\mathrm{vol}(S^n)$ on $[\pi/\sqrt{\kappa}, \infty)$.

For $\kappa < 0$ and $r \gg 1$,

$$\sinh(\sqrt{-\kappa} r)^{n-1} \sim \frac{e^{(n-1)\sqrt{-\kappa} r}}{2^{n-1}}.$$

Therefore,

$$V^n_\kappa(r) \sim \frac{\pi^{n/2}}{(n-1)2^{n-2}\Gamma(n/2)\sqrt{-\kappa}} e^{(n-1)\sqrt{-\kappa} r}. \tag{6.15}$$

7 Geodesics

Definition 7.1. Let $I \subset \mathbf{R}$ be an interval. A curve $\gamma\colon I \to M$ is called a **geodesic** if

$$\nabla_t \dot{\gamma} = 0. \tag{7.2}$$

Here ∇_t is the pull-back of the Levi-Civita connection to $\gamma^* TM$.

Remark 7.3. Suppose $x^1, \dots, x^n$ are local coordinates on M. Setting $\gamma^i \coloneqq x^i \circ \gamma$, (7.2) becomes

$$\ddot{\gamma}^k + \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0. \tag{7.4}$$

Theorem 7.5 (Existence and Uniqueness of Geodesics).

1. *Given $t_\star \in \mathbf{R}$, $x \in M$, and $v \in T_x M$, there exists an open interval I containing $t_\star$ and a geodesic $\gamma\colon I \to M$ satisfying $\gamma(t_\star) = x$ and $\dot{\gamma}(t_\star) = v$.*

2. *Let $\gamma\colon I \to M$ and $\delta\colon J \to M$ be two geodesics. If there is and $t_\star \in I \cap J$ such that $\gamma(t_\star) = \delta(t_\star)$ and $\dot{\gamma}(t_\star) = \dot{\delta}(t_\star)$, then γ and δ agree on $I \cap J$.*

3. *Let $\gamma\colon I \to M$ be a geodesic with $I = (t_0, t_1)$ maximal. If $t_0 \neq -\infty$, then for every compact subset $K \subset M$, there exists a $t_0^K \in (t_0, t_1)$ such that if $t \in (t_0, t_0^K)$, then $\gamma(t) \notin K$. An analogous statement holds if $t_1 \neq +\infty$.*

   In particular, if M is compact, then $I = \mathbf{R}$.

Definition 7.6. We say that (M, g) is **geodesically complete** (at $x \in M$) if every geodesic (passing through x) can be extended to $\mathbf{R}$.

8 The exponential map

Definition 8.1. Given $x \in M$ and $v \in T_x M$, denote by $\gamma_v^x\colon I_v^x \to M$ the maximal geodesic with $\gamma_v^x(0) = x$ and $\dot{\gamma}_v^x(0) = v$. Set

$$\mathscr{O}_x \coloneqq \left\{ v \in T_x M : 1 \in I_v^x \right\} \quad \text{and} \quad \mathscr{O} \coloneqq \bigcup_{x \in M} \mathscr{O}_x \subset TM \tag{8.2}$$

The **exponential map** at x is the map $\exp_x\colon \mathscr{O}_x \to M$ defined by

$$\exp_x(v) \coloneqq \gamma_v(1). \tag{8.3}$$

The **exponential map** $\exp\colon \mathscr{O} \to M$ is defined by $\exp|_{\mathscr{O}_x} \coloneqq \exp_x$.

Proposition 8.4.

1. *$\mathscr{O}$ is open and* $\exp$ *is smooth.*

2. *The derivative at* $0 \in \mathscr{O}_x$ *of the exponential map* $\exp_x$*,*

(8.5)
$$\mathrm{d}_0 \exp_x \colon\ T_0 T_x M \to T_x M,$$

   is invertible; in fact, it is the inverse of the canonical isomorphism $T_x M \cong T_0 T_x M$*.*

   In particular, $\exp_x$ *induces a diffeomorphism between a neighborhood of the origin in* $T_x M$ *and a neighborhood of* x *in* M*.*

3. *The derivative of the map* $(\pi, \exp)\colon\ \mathscr{O} \to M \times M$ *along the zero section is invertible.*

   In particular, this map induces a diffeomorphism between a neighborhood of the zero section of TM *and a neighborhood if the diagonal in* $M \times M$*.*

Lemma 8.6 (Gauß' Lemma). *Let* $x \in M$*. Denote by* $\partial_r \in \mathrm{Vect}(T_x M)$ *the radial vector field. Suppose* $B_r(0) \subset \mathscr{O}_x$ *is such that* $\exp|_{B_r(0)}$ *is a diffeomorphism onto its image. On* $B_r(0)$*,*

(8.7)
$$\Big\langle (\exp_x)_* \partial_r, (\exp_x)_* w \Big\rangle = \langle \partial_r, w \rangle.$$

Theorem 8.8 (Short geodesics are minimal). *Let* $r > 0$ *and* $x \in M$*. If* $\exp_x\colon\ B_r(0) \to M$ *is a diffeomorphism onto its image, then, for every* $v \in B_r(0)$*,* $\gamma\colon\ [0,1] \to M$ *defined by* $\gamma(t) \coloneqq \exp_x(tv)$ *is the unique minimal geodesic from* x *to* $\exp_x(v)$*; in particular:* $\exp_x(B_r(0)) = B_r(x)$*.*

Proof. We have $\ell(\gamma) = |v|$. Let $\delta\colon\ [0,1] \to M$ be a curve from x to $y = \exp_x(v)$. We will show that

(8.9)
$$\ell(\delta) \geqslant |v|.$$

We can assume that the image δ is contained $\exp_x(\bar{B}_{|v|}(0))$; otherwise, the upcoming argument shows that part of δ already has length at least $|v|$. Set

(8.10)
$$w(t) \coloneqq \exp_x^{-1}(\delta(t)).$$

By the Cauchy–Schwarz inequality and Lemma 8.6,

$$\begin{aligned}
\ell(\delta) &= \int_0^1 |\dot{\delta}(t)|\, \mathrm{d}t \\
&\geqslant \int_0^1 \Big\langle \dot{\delta}(t), (\exp_p)_* \partial_r \Big\rangle\, \mathrm{d}t \\
&= \int_0^1 \langle \dot{w}(t), \partial_r \rangle\, \mathrm{d}t = \int_0^1 \frac{\langle \dot{w}(t), w(t) \rangle}{|w(t)|}\, \mathrm{d}t = \int_0^1 \partial_t |w(t)|\, \mathrm{d}t = |v|.
\end{aligned}$$

Equality holds if and only if $\dot{w}(t)$ and ∂_r are parallel. ∎

Definition 8.11. Let $x \in M$. **Normal coordinates** of M at x are coordinates obtained by composing $\exp_x^{-1}$ with an isometry $T_xM \cong \mathbf{R}^n$.

Proposition 8.12. *Suppose $x^1, \ldots, x^n$ are normal coordinates.*

1. *The Christoffel symbols Γ_{ij}^k vanish at the origin.*
2. *We have $g_{ij}x^j = \delta_{ij}x^j$.*
3. *Setting*

$$R_{ijk\ell} \coloneqq \left\langle R(\partial_i, \partial_j)\partial_k, \partial_\ell \right\rangle, \tag{8.13}$$

we have

$$g_{ij} = \delta_{ij} + \frac{1}{3}\sum_{k,\ell=1}^{n} R_{ik\ell j}x^k x^\ell + O(|x|^3). \tag{8.14}$$

9 The energy functional

Definition 9.1. A **variation** of a curve $\gamma\colon [t_0, t_1] \to M$ is a smooth map $\boldsymbol{\gamma}\colon (-\varepsilon, \varepsilon) \times [t_0, t_1] \to M$ such that $\boldsymbol{\gamma}(0, \cdot) = \gamma$. The variation $\boldsymbol{\gamma}$ is called **proper** if $\boldsymbol{\gamma}(\cdot, t_0)$ and $\boldsymbol{\gamma}(\cdot, t_1)$ are constant. We set $\gamma_s(t) \coloneqq \boldsymbol{\gamma}(s, t)$.

Proposition 9.2 (First Variation Formula). *Given a variation $\boldsymbol{\gamma}$ of a curve $\gamma\colon [t_0, t_1] \to M$,*

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\gamma_s) = &-\int_{t_0}^{t_1} \left\langle \nabla_t \dot\gamma_0(t), \partial_s\boldsymbol{\gamma}(0,t) \right\rangle \mathrm{d}t \\ &+ \left\langle \dot\gamma_0(t_0), \partial_s\boldsymbol{\gamma}(0,t_0) \right\rangle - \left\langle \dot\gamma_0(t_1), \partial_s\boldsymbol{\gamma}(0,t_1) \right\rangle. \end{aligned} \tag{9.3}$$

Corollary 9.4. *A curve $\gamma\colon [t_0, t_1] \to M$ is a geodesic if and only if, for every proper variation $\boldsymbol{\gamma}$, 0 is a critical point of $s \mapsto E(\gamma_s)$; that is:*

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\gamma_s) = 0. \tag{9.5}$$

Proposition 9.6. *For every curve $\gamma\colon [t_0, t_1] \to M$,*

$$\ell(\gamma) \leqslant \sqrt{2E(\gamma)} \cdot \sqrt{t_1 - t_0} \tag{9.7}$$

with equality if and only if $|\dot\gamma|$ is constant.

Corollary 9.8. *Let $t_0 < t_1$ and $x, y \in M$. Set*

$$\gamma \in P_{x,y} \coloneqq \{\delta \in C^\infty([t_0, t_1], M) : \delta(t_0) = x \text{ and } \delta(t_1) = y\}. \tag{9.9}$$

If $\gamma \in P_{x,y}$ satisfies $\partial_t |\dot\gamma| = 0$ and minimizes ℓ in $P_{x,y}$, then γ also minimizes E in $P_{x,y}$; in particular: it is a geodesic.

10 The second variation formula

Lemma 10.1 (The second variation formula). *Let $\gamma\colon [t_0, t_1] \to M$ be a geodesic. If $\boldsymbol{\gamma}$ is a variation of γ, then*

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}s^2}\bigg|_{s=0} E(\gamma_s) = \int_{t_0}^{t_1} &\left|\nabla_t \partial_s \boldsymbol{\gamma}(0,t)\right|^2 - \left\langle R(\partial_s \boldsymbol{\gamma}(0,t), \dot\gamma(t))\dot\gamma(t), \partial_s \boldsymbol{\gamma}(0,t)\right\rangle \mathrm{d}t \\ &+ \left\langle \nabla_s \partial_s \boldsymbol{\gamma}(0,t), \partial_t \gamma(t_1)\right\rangle - \left\langle \nabla_s \partial_s \boldsymbol{\gamma}(0,t), \partial_t \gamma(t_0)\right\rangle. \end{aligned} \tag{10.2}$$

Remark 10.3. If $\boldsymbol{\gamma}$ is a proper variation, then (10.2) depends only on $V \coloneqq \partial_s \boldsymbol{\gamma}(0, \cdot)$.

Definition 10.4. Let $\gamma\colon [t_0, t_1] \to M$ be a geodesic. The **index form** of γ is the bilinear map $I\colon S^2\Gamma(\gamma^*TM) \to \mathbf{R}$ is defined by

$$I(v, w) = \int_{t_0}^{t_1} \langle \nabla_t v, \nabla_t w\rangle - \langle R(v, \dot\gamma(t))\dot\gamma(t), w\rangle \,\mathrm{d}t$$

Definition 10.5. Let $\gamma\colon [t_0, t_1] \to M$ be a geodesic passing through x and y. We say that x and y are **conjugate along** γ if there is a non-zero Jacobi field J along γ with $J(t_0) = 0$ and $J(t_1) = 0$.

Definition 10.6. Let $x \in M$. The **conjugate locus** of x in T_xM is the set of points $v \in \mathcal{O}_x$ such that x and $\exp_x(v)$ are conjugate along $t \mapsto \exp_x(tv)$.

Remark 10.7. The **conjugate locus** of x is the set of points $v \in \mathcal{O}_x$ such that $\mathrm{d}_v \exp_x$ is not injective.

Theorem 10.8 (Jacobi). *Let $\gamma\colon [t_0, t_1] \to M$ with $\gamma(t_0) = x$ and $y = \gamma(t_\star)$ for $t_\star \in (t_0, t_1)$. If x and y are are conjugate along $\gamma|_{[t_0,t_\star]}$, then γ is not minimal; that is: there is a proper variation $\boldsymbol{\gamma}$ of γ with*

$$\ell(\boldsymbol{\gamma}(s, \cdot)) < \ell(\gamma) \tag{10.9}$$

for all $s \neq 0$.

Sketch proof. Since x and y are conjugate along $\gamma|_{[t_0,t_\star]}$, there is a non-trivial Jacobi field J along $\gamma|_{[t_0,t_\star]}$ vanishing at t_0 and $t_\star$. Extend J to a piecewise smooth Jacobi field along γ by declaring it to vanish on $[t_\star, t_1]$.

$\nabla_t J(t_\star) \neq 0$ for otherwise J would be trivial. Choose $V \in \Gamma(\gamma^*TM)$ with

$$V(t_0) = 0, \quad V(t_1) = 0, \quad \text{and} \quad \langle \nabla_t J(t_\star), V(t_\star)\rangle < 0. \tag{10.10}$$

For $0 < \varepsilon \ll 1$, set

$$J_\varepsilon \coloneqq J + \varepsilon V.$$

Define the pricewise smooth proper variation $\boldsymbol{\gamma}_\varepsilon$ of γ by

$$\boldsymbol{\gamma}_\varepsilon(s,t) \coloneqq \exp(sJ_\varepsilon(t)). \tag{10.11}$$

By Lemma 10.1,

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} E(\gamma_s) = I(J_\varepsilon, J_\varepsilon) = 2\varepsilon I(J,V) + \varepsilon^2 I(V,V). \tag{10.12}$$

An integration by parts, shows that

$$I(J,V) = -\int_{t_0}^{t_1} \left\langle \nabla_t^2 J + R(J,\dot\gamma(t))\dot\gamma(t), V\right\rangle \mathrm{d}t + \langle \nabla_t J(t_\star), V(t_\star)\rangle. \tag{10.13}$$

The first term vanishes since J is a Jacobi field and the second term is negative. Consequently, $I(J_\varepsilon, J_\varepsilon) < 0$ provided $0 < \varepsilon \ll 1$. □

Definition 10.14. Let $x \in M$. The **cut locus** of x is the subset of those $v \in \mathscr{O}_x$ such that $\gamma(t) \coloneqq \exp_x(tv)$ is minimizing for $t \in [0,1]$ but fails to be minimizing for $t \in [0, 1+\varepsilon)$ for every $\varepsilon > 0$.

Proposition 10.15. *If v is in the cut locus if x, then*

1. *v is in the conjugate locus of x or*
2. *there is more than one minimal geodesic from x to $\exp_x(v)$.*

11 Jacobi fields

Proposition 11.1. *Let $\boldsymbol{\gamma}$ be a variation of a geodesic $\gamma\colon [t_0,t_1] \to M$. If every γ_s is a geodesic, then the vector field $J \in \Gamma(\gamma^*TM)$ defined by*

$$J \coloneqq \partial_s \boldsymbol{\gamma}(0,\cdot) \tag{11.2}$$

*satisfies the **Jacobi equation**:*

$$\nabla_t^2 J + R(J,\dot\gamma)\dot\gamma = 0. \tag{11.3}$$

Definition 11.4. Let γ be a geodesic. A vector field $J \in \Gamma(\gamma^*TM)$ is called a **Jacobi field** along γ if (11.3) holds.

Proposition 11.5. *Let $\gamma\colon [0,1] \to M$ be a geodesic. Given $J_0, \dot{J}_0 \in T_{\gamma(0)}M$, there exists a unique Jacobi field along γ with $J(0) = J_0$ and $\nabla_t J(0) = \dot{J}_0$.*

Proposition 11.6. *Let $x \in M$, $v \in \mathscr{O}_x$, and $w \in T_xM = T_vT_xM$. Denote by J the Jacobi field along $t \mapsto \exp_x(tv)$ with*

$$J(0) = 0 \quad \text{and} \quad \nabla_t J(0) = w. \tag{11.7}$$

Then

$$\mathrm{d}_v \exp_x(w) = J(1).$$

Theorem 11.8 (Hadamard). *If (M,g) is complete and $\sec_g \leqslant 0$, then $\exp_x\colon T_xM \to M$ is a covering map. In particular, the universal cover of M is diffeomorphic to $\mathbf{R}^n$.*

Sketch proof. Suppose that J is a Jacobi field along $\gamma(t) = \exp(tv)$ with $J(0) = 0$ and $\nabla_t J(0) = w$. Since

$$\partial_t |J|^2 = 2\langle \nabla_t J, J\rangle \quad \text{and} \quad \partial_t^2 |J|^2 = -2\langle R(J,\dot\gamma)\dot\gamma, J\rangle + 2|\nabla_t J|^2, \tag{11.9}$$

the function $t \mapsto |J(t)|^2$ vanishes at 0 and is strictly convex. Consequently, $J(1) \neq 0$. □

Proposition 11.10. *Let (M,g) be a Riemannian manifold of dimension n and let $x \in \mathbf{R}$. Suppose $r > 0$ is such that $B_r(x)$ lies within the cut-locus of x. Suppose that for some $\kappa \in \mathbf{R}$, and every $y \in B_r(x)$ and $v \in \partial_r^\perp \subset T_yM$,*

$$\sec(\partial_r, v) = \kappa.$$

Let $p \in S^n_\kappa$ and fix an isometry $T_xM \cong T_pS^n_\kappa$. Then the map

$$B_r(x) \xrightarrow{\exp_p^{S^n_\kappa} \circ \exp_x^{-1}} B_r^{S^n_\kappa}(p)$$

is an isometry.

Proof. Let $v \in B_r(0) \subset T_xM$ and $w \in T_vT_xM \cong T_vM$. By Lemma 8.6, we can assume that $w \perp v$. Denote by J the Jacobi field along $t \mapsto \exp(tv)$ with initial condition

$$J(0) = 0 \quad \text{and} \quad \nabla_t J(0) = w.$$

Denote by W the parallel vector field along $t \mapsto \exp(tv)$ with

$$W(0) = w.$$

By (11.3),

$$J(t) = \frac{\sin_\kappa(|v|t)}{|v|} W(t).$$

Consequently,

$$|\mathrm{d}_v \exp_x(w)|^2 = \frac{\sin_\kappa(|v|)^2}{|v|^2}|w|^2.$$

This computation depended only on the radial sectional curvatures being precisely κ. Therefore, $\exp_p^{S^n_\kappa} \circ \exp_x^{-1}$ is an isometry. □

12 Ricci curvature

Definition 12.1. Let (M, g) be a Riemannian manifold. The **Ricci curvature** of (M, g) is the tensor field $\mathrm{Ric}_g \in \Gamma(S^2T^*M)$ defined by

$$\mathrm{Ric}_g(v, w) := \mathrm{tr}(R_g(\cdot, v)w) = \sum_{a=1}^{n} \langle R(e_a, v)w, e_a\rangle. \tag{12.2}$$

Here $(e_1, \dots, e_n)$ is a orthonormal basis of T_xM.

Remark 12.3. If $\sec_g = \kappa$, then

$$\mathrm{Ric}_g = (n-1)\kappa g.$$

Remark 12.4. The map

$$\mathrm{tr}\colon \mathscr{R}(V) \to S^2V$$

is injective if $n < 3$, bijective for $n = 3$, and surjective for $n > 3$. Therefore, for $n \geqslant 3$, the Ricci curvature Ric_g has $\binom{n+1}{2}$ components. For a Riemannian 3–manifold (M, g), Ric_g determines all of R_g.

Proposition 12.5. *In normal coordinates,*

$$\frac{\mathrm{vol}_g}{\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n} = 1 - \frac{1}{6}\sum_{a,b=1}^{n} \mathrm{Ric}_{ab}x^a x^b + O(|x|^3). \tag{12.6}$$

Proof. Let $x \in M$. Set

$$\theta := \frac{\mathrm{vol}_g}{\mathrm{vol}_{T_xM}}.$$

Let $v \in T_xM$ with $|v| = 1$. Set $\gamma(t) = \exp_t(v)$. Let $e_1 = v, \dots, e_n$ be a positive orthonormal basis of T_xM. For $a = 1, \dots, n$, let $J_a(t)$ be the Jacobi field along γ with $J_a(0) = 0$ and $\nabla_t J_a(0) = e_a$. By Proposition 11.6,

$$d_{tv} \exp_x(e_a) = \frac{J_a(t)}{t}.$$

Therefore,

$$\theta(tv) = t^{-n+1}\sqrt{\det G(t)} \quad \text{with} \quad G_{ab}(t) = \langle J_a(t), J_b(t)\rangle.$$

Since J_a is a Jacobi field, its Taylor expansion is given by

$$J_a(t) = te_a - \frac{t^3}{6}R(e_a, v)v + O(t^4).$$

Hence, the Taylor expansion of $G(t)/t^2$ is

$$\frac{G_{ab}(t)}{t^2} = \delta_{ab} - \frac{t^2}{6}\left[\langle R(e_a, v)v, e_b\rangle + \langle R(e_b, v)v, e_a\rangle\right] + O(t^3).$$

Therefore,

$$\theta(tv) = t^{-n+1}\sqrt{\det G(t)} = 1 - \frac{t^2}{6}\mathrm{Ric}(v, v) + O(t^3).$$ □

13 Scalar curvature

Definition 13.1. Let (M, g) be a Riemannian manifold. The **scalar curvature** of (M, g) is the function $\mathrm{scal}_g \in C^\infty(M)$ defined by

$$\mathrm{scal}_g := \mathrm{tr}(\mathrm{Ric}_g) = \sum_{a,b=1}^{n} \left\langle R_g(e_a, e_b)e_b, e_a\right\rangle. \tag{13.2}$$

Here $(e_1, \ldots, e_n)$ is a orthonormal basis of T_xM.

Proposition 13.3. *If (M, g) is a Riemannian manifold and $x \in M$, then, for $0 < r \ll 1$,*

$$\frac{\mathrm{vol}(B_x(r))}{V_0^n(r)} = 1 - \frac{\mathrm{scal}_g(x)}{6(n+2)}r^2 + O(r^3).$$

14 Einstein Metrics

Definition 14.1. A Riemannian metric g is called a **Einstein metric** if there is a constant $\lambda \in \mathbf{R}$ such that

$$\mathrm{Ric}_g = \lambda g. \tag{14.2}$$

If g is a Einstein metric on M, then (M, g) is called a **Einstein manifold**.

Lemma 14.3 (Schur's Lemma). *Let (M, g) be a Riemannian manifold of dimension $n \geqslant 3$. If $\lambda \in C^\infty(M)$ and*

$$\mathrm{Ric}_g = \lambda g,$$

then λ is locally constant.

Corollary 14.4. *Let (M, g) be a connected Riemannian manifold of dimension $n \geqslant 3$. If*

$$\mathrm{Ric}_g^\circ := \mathrm{Ric}_g - \frac{\mathrm{scal}_g}{n} g = 0,$$

then g is an Einstein metric.

Proposition 14.5 (Contracted Bianchi identity). *If (M, g) is a Riemannian manifold, then*

$$\mathrm{dscal}_g = -2\nabla^* \mathrm{Ric}_g. \tag{14.6}$$

Proof. Let $x \in M$ and let $e_1, \dots, e_n$ be a local orthonormal frame such that $(\nabla_{e_a} e_b)(x) = 0$. At the point x, by the differential Bianchi identity (5.5),

$$\begin{aligned}
e_c \cdot \mathrm{scal} &= \sum_{a,b=1}^n \left\langle (\nabla_{e_c} R)(e_a, e_b)e_b, e_a \right\rangle \\
&= -\sum_{a,b=1}^n \left\langle (\nabla_{e_a} R)(e_b, e_c)e_b, e_a \right\rangle + \left\langle (\nabla_{e_b} R)(e_c, e_a)e_b, e_a \right\rangle \\
&= 2\sum_{a,b=1}^n \left\langle (\nabla_{e_a} R)(e_b, e_a)e_c, e_b \right\rangle \\
&= -(\nabla^* \mathrm{Ric})(e_c).
\end{aligned}$$

□

Proof of Lemma 14.3. By hypothesis, $\mathrm{dscal}_g = n\mathrm{d}\lambda$. However, by the contracted Bianchi identity (14.6), $\mathrm{dscal}_g = 2\mathrm{d}\lambda$. Therefore, $\mathrm{d}\lambda = 0$. □

15 Bochner's vanishing theorem for harmonic 1–forms

Theorem 15.1 (Bochner's vanishing theorem for harmonic 1–forms [Boc46, Theorem 1]). *Let (M, g) be a closed, connected Riemannian manifold of dimension n. If $\mathrm{Ric}_g \geqslant 0$, then the following hold:*

1. *Every harmonic 1–form α is parallel and satisfies $\mathrm{Ric}_g(\alpha^\sharp, \alpha^\sharp) = 0$. In particular, $b_1(M) \leqslant n$.*
2. *If there exists some $x \in M$ with $\mathrm{Ric}_g(x) > 0$, then every harmonic 1–form vanishes. In particular, $b_1(M) = 0$.*

Proposition 15.2 (Bochner–Weitzenböck formula for 1–forms [Boc46, Lemma 2]). *Let (M, g) be a Riemannian manifold. For every $\alpha \in \Omega^1(M)$,*

$$(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}_g(\alpha^\sharp, \cdot). \tag{15.3}$$

in particular,

$$\frac{1}{2}\Delta|\alpha|^2 = \langle(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})\alpha, \alpha\rangle - |\nabla\alpha|^2 - \mathrm{Ric}_g(\alpha^\sharp, \alpha^\sharp). \tag{15.4}$$

Proof. Let $x \in M$ and let $e_1, \ldots, e_n$ be a local orthonormal frame such that $(\nabla_{e_a} e_b)(x) = 0$. At the point x,

$$\begin{aligned}
(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})\alpha &= -\sum_{a,b=1}^n e^a \wedge i(e_b)\nabla_{e_a}\nabla_{e_b}\alpha + i(e_a)e^b \wedge \nabla_{e_a}\nabla_{e_b}\alpha \\
&= -\sum_{a=1}^n \nabla_{e_a}\nabla_{e_a}\alpha - \sum_{a,b=1}^n e^a i(e_b)[\nabla_{e_a}, \nabla_{e_b}]\alpha \\
&= \nabla^*\nabla\alpha + \sum_{a,b=1}^n \alpha(R(e_a, e_b)e_b)e^a \\
&= \nabla^*\nabla\alpha + \sum_{a,b,c=1}^n \alpha(e_c)\langle R(e_a, e_b)e_b, e_c\rangle e^a.
\end{aligned}$$

This proves (15.3). The identity (15.4) follows from

$$\Delta|\alpha|^2 = 2\langle\nabla^*\nabla\alpha, \alpha\rangle - 2|\nabla\alpha|^2. \qquad \square$$

Proof of Theorem 15.1. Let α be a harmonic 1–form. By the Bochner–Weitzenböck formula (15.4),

$$\begin{aligned}
0 &= -\frac{1}{2}\int_M \Delta|\alpha|^2 \\
&= \int_M |\nabla\alpha|^2 + \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp).
\end{aligned}$$

Both terms under the integral are non-negative. Therefore, they vanish. This proves (1). If $\mathrm{Ric}_g(x) > 0$, then α must vanish in order for the second term to vanish. This proves (2). $\square$

Remark 15.5. If $\mathrm{Ric}_g \geqslant 0$ and M is non-compact, then there often (always?) are many non-parallel harmonic 1–forms.

16 Bochner's vanshing theorem for Killing fields

Definition 16.1. Let (M, g) be a Riemannian manifold. A vector field $v \in \mathrm{Vect}(M)$ is called a **Killing field** if

$$\mathscr{L}_v g = 0. \tag{16.2}$$

The space of Killing fields is denoted by

$$\mathfrak{iso}(M, g) := \{v \in \mathrm{Vect}(M) : \mathscr{L}_v g = 0\}.$$

Remark 16.3. If $u, v, w \in \mathrm{Vect}(M)$, then

$$(\mathscr{L}_u g)(v, w) = \langle \nabla_v u, w \rangle + \langle \nabla_w u, v \rangle \tag{16.4}$$

Exercise 16.5 (Bochner). Let v be Killing field and let $\alpha \in \Omega^1(M)$ be a harmonic 1–form. Prove that the function $\alpha(v)$ is constant.

Theorem 16.6 (Bochner's vanishing theorem for Killing fields [Boc46, Theorem 2]). *Let (M, g) be a closed, connected Riemannian manifold of dimension n. If $\mathrm{Ric}_g \leqslant 0$, then the following hold:*

1. *Every Killing field v is parallel and satisfies $\mathrm{Ric}_g(v, v) = 0$. In particular, $\dim \mathfrak{iso}(M, g) \leqslant n$.*
2. *If there exists some $x \in M$ with $\mathrm{Ric}_g(x) < 0$, then every Killing field vanishes. In particular, $\mathrm{Iso}(M, g)$ is finite.*

This follows from the following proposition and the argument from the proof of Theorem 15.1.

Proposition 16.7 (Bochner–Weitzenböck formula for vector fields [Boc46, Lemma 2]). *Let (M, g) be a Riemannian manifold. For every $v \in \mathrm{Vect}(M)$,*

$$\nabla^* \mathscr{L}_v g - \mathrm{d}\,\mathrm{div}\, v = \langle \nabla^* \nabla v, \cdot \rangle - \mathrm{Ric}_g(v, \cdot) \tag{16.8}$$

in particular,

$$\frac{1}{2}\Delta |v|^2 = (\nabla^* \mathscr{L}_v g)(v) - \mathscr{L}_v\, \mathrm{div}\, v + \mathrm{Ric}_g(v, v) - |\nabla v|^2. \tag{16.9}$$

Proof. Let $x \in M$ and let $e_1, \dots, e_n$ be a local orthonormal frame such that $(\nabla_{e_a} e_b)(x) = 0$. At the point x,

$$\begin{aligned}
(\nabla^*\mathscr{L}_v g)(w) &= -\sum_{a=1}^n (\nabla_{e_a}\mathscr{L}_v g)(e_a, w) \\
&= -\sum_{a=1}^n \nabla_{e_a}[(\mathscr{L}_v g)(e_a, w)] - (\mathscr{L}_v g)(e_a, \nabla_{e_a} w) \\
&= -\sum_{a=1}^n \nabla_{e_a}(\langle \nabla_{e_a} v, w\rangle + \langle \nabla_w v, e_a\rangle) - \langle \nabla_{e_a} v, \nabla_{e_a} w\rangle - \langle e_a, \nabla_{\nabla_{e_a} w} v\rangle \\
&= \langle \nabla^*\nabla v, w\rangle - \sum_{a=1}^n \nabla_{e_a}\langle \nabla_w v, e_a\rangle - \langle e_a, \nabla_{\nabla_{e_a} w} v\rangle;
\end{aligned}$$

and, furthermore,

$$\begin{aligned}
-\sum_{a=1}^n \nabla_{e_a}\langle \nabla_w v, e_a\rangle - \langle \nabla_{\nabla_{e_a} v} v, e_a\rangle &= -\sum_{a=1}^n \langle \nabla_{e_a}\nabla_w v, e_a\rangle - \langle \nabla_{\nabla_{e_a} w} v, e_a\rangle \\
&= -\mathrm{Ric}(v, w) + \sum_{a=1}^n \langle \nabla_w \nabla_{e_a} v, e_a\rangle \\
&= -\mathrm{Ric}(v, w) + \mathscr{L}_w \operatorname{div} w. \qquad \square
\end{aligned}$$

Corollary 16.10. *If (M, g) is a compact Riemannian manifold, then $v \in \mathrm{Vect}(M)$ is a Killing field if and only if*

$$\nabla^*\nabla v - \mathrm{Ric}_g(v, \cdot)^\flat = 0.$$

Application to Riemann surfaces Theorem 16.6 can be used to proof Hurwitz' automorphism theorem.

Theorem 16.11 (Uniformization Theorem). *Let (Σ, j) be a closed Riemann surface.*

1. *If $\chi(\Sigma) = 2$, then $(\Sigma, j) \cong \mathbf{C}P^1$.*
2. *If $\chi(\Sigma) = 0$, then $(\Sigma, j) \cong \mathbf{C}/\Lambda$ with $\Lambda \subset \mathbf{C}$ a co-compact lattice.*
3. *If $\chi(\Sigma) < 0$, then $(\Sigma, j) \cong H^2/\Gamma$ with $\Gamma \subset \mathrm{PSL}_2(\mathbf{R})$ a discrete subgroup acting freely on $\mathbf{H}^2$.*

Theorem 16.12 (Metric Uniformization Theorem). *Let (Σ, j) be a closed Riemann surface. In the conformal class determined by j there is a unique Riemannian metric g satisfying*

$$\mathrm{Ric}_g = \lambda g \quad \textit{with} \quad \lambda = \begin{cases} 1 & \textit{if } \chi(\Sigma) = 2, \\ 0 & \textit{if } \chi(\Sigma) = 0, \\ -1 & \textit{if } \chi(\Sigma) < 0. \end{cases}$$

Theorem 16.13 (Hurwitz' Automorphism Theorem [Hur93, p. 424]). *If (Σ, j) is a closed Riemann surface with $\chi(\Sigma) < 0$, then*

$$\# \operatorname{Aut}(\Sigma, j) \leqslant -42\chi(\Sigma). \tag{16.14}$$

Equality holds in (16.14) if and only if Σ is a branched cover of $\mathbf{C}P^1$ with ramification indicies 2, 3, and 7.

Remark 16.15. See de Saint-Gervais [dSai16] for an account of the history of the Uniformization Theorem.

Proof of Theorem 16.13. By Theorem 16.12, for the Einstein metric g,

$$\operatorname{Aut}(\Sigma, j) = \operatorname{Iso}(\Sigma, g).$$

Therefore, by Theorem 16.6, $\operatorname{Aut}(\Sigma, j)$ is finite.

The inequality is proved by a somewhat tedious—but nevertheless enlightening—case distinction. Set $\Gamma := \operatorname{Aut}(\Sigma, j)$. Consider the quotient map

$$\pi\colon\ \Sigma \to S := \Sigma/\Gamma.$$

Since $\operatorname{Aut}(\Sigma, j)$ acts holomorphically, π is locally given by $z \mapsto z^n$. Therefore, S is a Riemann surface and π is a branched covering map.

A point $z \in \Sigma$ is called a **ramification point** of the stabilizer Γ_z is non-trivial. In this case, $\Gamma_z = \mathbf{Z}/n\mathbf{Z}$ and we call $e_z := \# G_z$ the **ramification index** of z. A point $w \in S$ is called a **branch point** if it is the image of a ramification point. Since Γ acts transitively on $\pi^{-1}(w)$, every $z \in \pi^{-1}(w)$ is a ramification point and they all have the same ramification index. The **ramification index** w is the ramification index of any of its preimages and denoted by e_w. By the preceding discussion,

$$\#\pi^{-1}(w) = \frac{\#\Gamma}{e_w}.$$

Denote by $z_1, \ldots, z_m$ the ramification points and by $w_1, \ldots, w_k$ the branch points of π. By the Riemann–Hurwitz formula,

$$\begin{aligned} -\chi(\Sigma) &= -\#\Gamma \cdot \chi(S) + \sum_{a=1}^{m} (e_{z_a} - 1) \\ &= \#\Gamma \cdot \underbrace{\left[-\chi(S) + \sum_{a=1}^{k} \left(1 - \frac{1}{e_{w_a}}\right) \right]}_{=:A}. \end{aligned}$$

Therefore,

$$\#\Gamma = -\frac{\chi(\Sigma)}{A}.$$

Since $\chi(\Sigma) < 0$, $A > 0$. The following case distinction shows that $A \geqslant \frac{1}{42}$:

- If $\chi(S) < 0$, then $A \geqslant 2$.
- If $\chi(S) = 0$, then $k \geqslant 1$; therefore: $A \geqslant \frac{1}{2}$.
- It remains to analyze the case $\chi(S) = 2$. In this case $k \geqslant 3$.
  - If $k \geqslant 5$, then $A \geqslant \frac{1}{2}$.
  - If $k = 4$, then at least one of the ramification indices is bigger than two; therefore: $A \geqslant 1/6$.
  - A further case distinction in the case $k = 3$ shows that $A \geqslant 1/42$ with equality achieved if the ramification indices are 2, 3, and 7.

This finishes the proof. □

Remark 16.16. It should be stressed that they crucial part of the above proof is establishing that $\mathrm{Aut}(\Sigma, j)$ is finite. The remainder, although much longer, is really just bookkeeping.

17 Myers' Theorem

Theorem 17.1 (Myers [Mye41]). *Let (M, g) be a connected, complete Riemannian manifold. Let $\kappa > 0$. If*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g,$$

then

$$\mathrm{diam}(M, g) \leqslant \pi/\sqrt{\kappa}.$$

In particular, $\pi_1(M)$ is finite.

Proof. If $\pi_1(M)$ is not finite, then the universal cover $\tilde{M}$ has infinite diameter but also satisfies the lower Ricci bound: a contradiction to the asserted diameter bound.

The diameter bound follows once we prove that if $\gamma\colon [0, T] \to M$ is a minimal geodesic parametrized by arc-length, then

$$T \leqslant \pi/\sqrt{\kappa}.$$

To see this, let $e_1 = \dot{\gamma}, e_2, \ldots, e_n$ be a parallel orthonormal frame along γ. For $a = 2, \ldots, n$, set

$$V_a := \sin\left(\frac{\pi}{T}t\right) e_a \tag{17.2}$$

and let $\boldsymbol{\gamma}_a$ be a proper variation of γ with $\partial_s \boldsymbol{\gamma}_a(0, \cdot) = V_a$. By the second variation formula for the energy functional,

$$
\begin{aligned}
\sum_{a=2}^{n} \frac{\mathrm{d}^2}{\mathrm{d}s^2}\bigg|_{s=0} E\left(\gamma_{a,s}\right) &= \sum_{a=2}^{n} \int_0^T |\nabla_t V_a(t)|^2 - \langle R(V_a(t), \dot\gamma(t))\dot\gamma(t), V_a(t)\rangle \,\mathrm{d}t \\
&= (n-1)\left(\frac{\pi}{T}\right)^2 \int_0^T \cos\left(\frac{\pi}{T}t\right)^2 \mathrm{d}t - \int_0^T \sin\left(\frac{\pi}{T}t\right)^2 \mathrm{Ric}(e_1(t), e_1(t))\,\mathrm{d}t \\
&\leqslant (n-1)\left(\frac{\pi}{T}\right)^2 \int_0^T \cos\left(\frac{\pi}{T}t\right)^2 \mathrm{d}t - (n-1)\kappa \int_0^T \sin\left(\frac{\pi}{T}t\right)^2 \,\mathrm{d}t \\
&= \left[\left(\frac{\pi}{T}\right)^2 - \kappa\right] \frac{(n-1)T}{2}.
\end{aligned}
$$

If $T > \pi/\sqrt{\kappa}$, then this is negative; hence, one of the $\boldsymbol{\gamma}_a$ is an energy decreasing (hence: length decreasing) variation; therefore, γ is cannot be minimal. □

Remark 17.3. The diameter bound in Theorem 17.1 is sharp since S^n has $\mathrm{Ric} = g$.

Remark 17.4. The conclusion of Theorem 17.1 is much stronger than that of Theorem 15.1; however, so is its hypothesis. In fact, $\mathrm{Ric}_g \geqslant (n-1)\kappa g$ with $\kappa > 0$ is a much stronger condition than $\mathrm{Ric}_g > 0$ which in turn is much stronger than $\mathrm{Ric}_g \geqslant 0$.

18 Laplacian comparison theorem

Definition 18.1. Let (M, g) be a Riemannian manifold of dimension n and $x \in M$. The **distance function** associated with x is the function $r\colon M \to [0, \infty)$ defined by

$$r(y) \coloneqq d(x, y).$$

Remark 18.2. Within of the cut-locus of x in M, the distance function r associated with x is smooth and, by Gauß' lemma, satisfies

$$|\nabla r| = 1.$$

Theorem 18.3 (Laplacian Comparison Theorem). *Let (M, g) be a Riemannian manifold and let $x \in M$. Let $\kappa \in \mathbf{R}$. If*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g,$$

then, within the cut-locus of x in M,

$$-\Delta r \leqslant (n-1)\frac{\sin_\kappa'(r)}{\sin_\kappa(r)}. \tag{18.4}$$

Moreover, at a point y within the cut-locus of x in M, equality holds in (18.4) if and only if all radial sectional curvatures are equal to κ; that is: for all $v \in \partial_r^\perp \subset T_yM$,

$$\sec_g(\partial_r, v) = \kappa.$$

This might seems to be a "technical" result, but we will see that it has far-reaching consequences. As a first indication, we give a second proof of Myers' theorem.

Proof of Theorem 17.1 using Theorem 18.3. Suppose $\kappa > 0$ and $\mathrm{Ric}_g \geqslant (n-1)\kappa g$. If $\mathrm{diam}(M, g) > \pi/\sqrt{k}$, then there is a minimal geodesic $\gamma\colon [0, T] \to M$ parametrized by arc-length with $T > \pi/\sqrt{k}$. Set $x := \gamma(0)$. The geodesic is contained in within of the cut-locus of x in M and $r \circ \gamma(t) = t$. This contradicts (18.4) because the function $\sin_\kappa'(t)/\sin_\kappa(t) = \sqrt{\kappa}\cot(\sqrt{\kappa}t)$ diverges to $-\infty$ as t tends $\pi/\sqrt{k}$. ◻

The following propositions prepare the proof of Theorem 18.3.

Proposition 18.5 (Bochner–Weitzenböck formula for gradients). *Let (M, g) be a Riemannian manifold. For $f \in C^\infty(M)$,*

$$\frac{1}{2}\Delta|\nabla f|^2 = \langle \nabla\Delta f, \nabla f\rangle - |\mathrm{Hess}\, f|^2 - \mathrm{Ric}(\nabla f, \nabla f). \tag{18.6}$$

Proof. By the Bochner–Weitzenböck formula (15.4) for $\alpha = \mathrm{d}f$,

$$\begin{aligned}\frac{1}{2}\Delta|\nabla f|^2 &= \langle \mathrm{d}\mathrm{d}^*\mathrm{d}f, \nabla f\rangle - |\nabla \mathrm{d}f|^2 - \mathrm{Ric}(\mathrm{d}f^\sharp, \mathrm{d}f^\sharp)\\ &= \langle \nabla\Delta f, \nabla f\rangle - |\mathrm{Hess}\, f|^2 - \mathrm{Ric}(\nabla f, \nabla f).\end{aligned}$$ ◻

Proposition 18.7. *In the situation of Theorem 18.3,*

$$-\,\partial_r\Delta r + \frac{(\Delta r)^2}{n-1} + (n-1)\kappa \leqslant 0. \tag{18.8}$$

Equality holds in (18.8) if and only if

$$\mathrm{Hess}\, r = -\frac{\Delta r}{n-1}(g - \mathrm{d}r \otimes \mathrm{d}r) \quad \textit{and} \quad \mathrm{Ric}_g = (n-1)\kappa g. \tag{18.9}$$

Proof. By the Cauchy–Schwarz inequality, if $A \in \mathbf{R}^{m\times m}$ is symmetric, then

$$\frac{(\mathrm{tr}\, A)^2}{m} \leqslant |A|^2 \tag{18.10}$$

with equality if and only if $A = \frac{\mathrm{tr}\, A}{m}\mathbf{1}$. Therefore and since $\mathrm{Hess}\, r(\partial_r, \cdot) = 0$,

$$\frac{(\Delta r)^2}{n-1} \leqslant |\mathrm{Hess}\, r|^2$$

with equality if and only if the first part of (18.9) holds.

By the Bochner–Weitzenböck formula for gradients (18.6),

$$0 = -\partial_r\Delta r + |\mathrm{Hess}\, r|^2 + \mathrm{Ric}(\partial_r, \partial_r).$$

Consequently, (18.8) holds with equality if and only if (18.9). ◻

Proposition 18.11 (Riccati Comparison Principle). *Let $\kappa \in \mathbf{R}$. If $f\colon (0,T) \to \mathbf{R}$ satisfies*

$$f' + f^2 + \kappa \leqslant 0 \quad \textit{and} \quad f(t) = \frac{1}{t} + O(1),$$

then

$$f(t) \leqslant \frac{\sin_\kappa'(t)}{\sin_\kappa(t)}.$$

Proof. The function $f_\kappa = \sin_\kappa'/\sin_\kappa$ satisfies the **Riccati equation**

$$f_\kappa' + f_\kappa^2 + \kappa = 0 \quad \text{and} \quad f_\kappa(t) = \frac{1}{t} + O(1),$$

Choose a smooth function $G\colon (0,T) \to \mathbf{R}$ such that

$$G' = f + f_\kappa \quad \text{and} \quad G(t) = 2\log t + O(1).$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^G(f - f_\kappa)\right] = e^G\left(f' - f_\kappa' + f^2 - f_\kappa^2\right) \leqslant 0,$$

the function $e^G(f - f_\kappa)$ is decreasing. This implies the assertion because

$$\lim_{t\to 0} e^{G(t)}(f(t) - f_\kappa(t)) = 0. \qquad \square$$

Proof of Theorem 18.3. Let $\gamma\colon [0,T] \to M$ be a geodesic emerging from x and parametrized by arc-length. Set

$$f(t) \coloneqq -\frac{\Delta r}{n-1} \circ \gamma(t).$$

By (18.8),

$$f' + f^2 + \kappa \leqslant 0.$$

Since

$$\Delta r = \frac{n-1}{r} + O(1),$$

(18.4) follows from Proposition 18.11.

If equality holds in (18.4) at y, then, by Proposition 18.7,

$$\operatorname{Hess} r = \frac{\sin_\kappa'(r)}{\sin_\kappa(r)}(g - \mathrm{d}r \otimes \mathrm{d}r).$$

Let $e_1 = \partial_r, e_2, \dots, e_n$ be a local orthonormal frame defined near y such that at y, for all $a, b = 2, \dots, n$,

$$\operatorname{Hess} r(e_a, e_b) = \frac{\sin_\kappa'(r)}{\sin_\kappa(r)}\delta_{ab} \quad \text{and} \quad [\partial_r, e_a] = 0.$$

Since

$$\operatorname{Hess} r(e_a, e_b) = \left\langle \nabla_{e_a} \partial_r, e_b \right\rangle,$$

the former means that

$$\nabla_{e_a} \partial_r = \frac{\sin_\kappa'(r)}{\sin_\kappa(r)} e_a = f_\kappa e_a.$$

Therefore,

$$\begin{aligned}
\sec_g(\partial_r \wedge e_a) &= -\left\langle \nabla_{\partial_r} \nabla_{e_a} \partial_r, e_a \right\rangle \\
&= -\left\langle \nabla_{\partial_r} f_\kappa e_a, e_a \right\rangle \\
&= -\left\langle (f_\kappa' + f_\kappa^2) e_a, e_a \right\rangle \\
&= \kappa. \qquad \square
\end{aligned}$$

Remark 18.12. There is also is a proof of Theorem 18.3 using Jacobi fields.

19 The Lichnerowicz–Obata Theorem

Theorem 19.1 (Lichnerowicz [Lic58] and Obata [Oba62, Theorems 1 and 2]). *Let $\kappa > 0$. Let (M, g) be a closed Riemannian manifold of dimension n with $\operatorname{Ric}_g \geqslant (n-1)\kappa g$. If λ is a non-zero eigenvalue of the Laplacian, then*

$$\lambda \geqslant n\kappa. \tag{19.2}$$

Equality is achieved in (19.2) if and only if (M, g) is isometric to (S^n_κ, g_κ).

The analysis of the case $\lambda = n\kappa$ requires the following result.

Theorem 19.3 (Obata [Oba62, Theorem A]). *Let (M, g) be a complete Riemannian manifold of dimension n. Let $\kappa > 0$. There exists a non-zero function $f \in C^\infty(M)$ satisfying*

$$\operatorname{Hess} f = -\kappa f g \tag{19.4}$$

if and only if (M, g) is isometric to (S^n_κ, g_κ).

Remark 19.5. Is straight-forward to verify that the coordinate functions $x_1, \dots, x_{n+1}$ on $S^n \subset \mathbf{R}^{n+1}$ have satisfy (19.4).

Proof of Theorem 19.1 assuming Theorem 19.3. Suppose $f \in C^\infty(M)$ is an eigenfunction of the Laplacian with eigenvalue $\lambda \neq 0$. By (18.6),

$$\frac{1}{2}\Delta|\nabla f|^2 = \lambda|\nabla f|^2 - |\operatorname{Hess} f|^2 - \operatorname{Ric}(\nabla f, \nabla f).$$

Using the lower-bound on Ric_g and (18.10),

$$\frac{1}{2}\Delta|\nabla f|^2 \leqslant (\lambda-(n-1)\kappa)|\nabla f|^2 - \frac{\lambda^2 f^2}{n}.$$

By integrating both sides and by integration by parts,

$$\begin{aligned} 0 &\leqslant (\lambda-(n-1)\kappa)\int_M |\nabla f|^2 - \frac{\lambda^2}{n}\int_M f^2 \\ &= \frac{(n-1)}{n}\lambda(\lambda-n\kappa)\int_M f^2. \end{aligned}$$

Since $\lambda \neq 0$, (19.2) holds.

It follows from the above, that equality holds in (19.2) if and only if

$$\mathrm{Hess}\, f = -\kappa f g \quad \text{and} \quad \mathrm{Ric}(\nabla f, \nabla f) = \kappa|\nabla f|^2.$$

The result thus follows from Theorem 19.3. ◻

Proof of Theorem 19.3. Without loss of generality, we restrict to the case $\kappa = 1$ and assume that the maximum of f is equal to 1. Let $x_\star$ be a point at which f achieves its maximum.

Proposition 19.6. *For every geodesic $\gamma\colon [0,T] \to M$ parametrized by arc-length with $\gamma(0) = x_\star$,*

$$f \circ \gamma = \cos.$$

In particular, with $r = r_{x_\star}$,

$$f = \cos(r).$$

Proof. The function $F\colon [0,T] \to \mathbf{R}$ defined by

$$F \coloneqq f \circ \gamma.$$

satisfies

$$F'' + F = 0.$$

Hence, there are constants $A, B \in \mathbf{R}$ such that

$$F(t) = A\cos(t) + B\sin(t).$$

Since f achieves its maximum at x and $f(x) = 1$, the coefficients are $A = 1$ and $B = 0$. ◻

For $y \in B_\pi(x_\star)$, if $\gamma\colon [0, r(y)] \to M$ is a minimizing geodesic parametrized by arc-length from $x_\star$ to y, then

$$\nabla f(y) = -\sin(r(y))\dot\gamma(r(y)).$$

Therefore and since $\sin(r) \neq 0$ for $r \in (0, \pi)$, γ is uniquely determined by y. Thus, $\exp_x \colon B_\pi(0) \to B_\pi(x_\star)$ is a diffeomorphism.

We have

$$\nabla f = -\sin(r)\partial_r.$$

Thus, for $v, w \perp \partial_r$,

$$\begin{aligned}
-\sin(r)\operatorname{Hess} r(v, w) &= -\sin(r)\langle \nabla_v \partial_r, w\rangle \\
&= \langle \nabla_v \nabla f, w\rangle \\
&= \operatorname{Hess} f(v, w) \\
&= -\cos(r)g(v, w).
\end{aligned}$$

Therefore,

$$\operatorname{Hess} r = \cot(r)(g - \mathrm{d}r \otimes \mathrm{d}r).$$

It follows as in the proof of Theorem 18.3, that the sectional curvature on $B_\pi(x_\star)$ is equal to 1. One can now use Theorem 17.1, to argue that the sectional curvature is equal to 1 on all of M and then appeal to Theorem 6.5.

One can also argue directly. First of all $B_\pi(x_\star)$ is isometric to $S^n \backslash \{p\}$ by Proposition 11.10. There must be a unique point $x_\dagger$ at distance π from $x_\star$. To see this, note that: f achieves its minimum -1 on $\partial B_\pi(x_\star)$. Every $y \in \partial B_\pi(x_\star)$ is a non-degenerate critical point; hence, $\partial B_\pi(x_\star)$ is discrete. On the other hand, $\partial B_\pi(x_\star)$ is connected and thus consists of precisely one point $x_\dagger$. The above argument also shows that, $B_\pi(x_\dagger)$ is isometric to $S^n \backslash \{q\}$ with q antipodal to p.

M is covered by $B_\pi(x_\star)$ and $B_\pi(x_\dagger)$; and their intersection is precisely

$$B_\pi(x_\star)\backslash\{x_\star\} = B_\pi(x_\dagger)\backslash\{x_\star\}.$$

The isometries $B_\pi(x_\star) \to S^n\backslash\{p\}$ and $B_\pi(x_\dagger) \to S^n\backslash\{q\}$ glue to a global isometry $M \to S^n$. □

Remark 19.7. A more detailed proof can be found in Berger, Gauduchon, and Mazet [BGM71, Théorème d'Obata D.1.6].

20 Bishop–Gromov volume comparison

Theorem 20.1 (Bishop–Gromov's Relative Volume Comparison Theorem [BC64, Section 11.10 Corollary 3; Gro81a, Section 2.1]). *Let (M, g) be a complete Riemannian manifold of dimension n and let $x \in M$. Let $\kappa \in \mathbf{R}$ and $0 < r \leqslant R$. If*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g$$

on $B_R(x)$, then

$$\frac{\mathrm{vol}(B_R(x))}{\mathrm{vol}(B_r(x))} \leqslant \frac{V_\kappa^n(R)}{V_\kappa^n(r)}. \tag{20.2}$$

Moreover, equality holds in (20.2) if and only if all radial sectional curvatures are equal to κ on $B_R(x)$.

Remark 20.3. The conclusion of Theorem 20.1 is equivalent to the function $\theta\colon (0, R] \to (0, \infty)$ defined by

$$\theta(r) := \frac{\mathrm{vol}(B_R(r))}{V_\kappa^n(r)} \tag{20.4}$$

being non-increasing.

Theorem 20.5 (Bishop's Absolute Volume Comparison Theorem [Bis63; BC64, Section 11.10 Corollary 4]). *Let (M, g) be a complete Riemannian manifold of dimension n and let $x \in M$. Let $\kappa \in \mathbf{R}$. If*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g,$$

then, for all $r > 0$,

$$\mathrm{vol}(B_r(x)) \leqslant V_\kappa^n(r).$$

Proof of Theorem 20.5 assuming Theorem 20.1. This is a consequence of

$$\lim_{r \to 0} \frac{\mathrm{vol}(B_r(x))}{V_\kappa^n(r)} = 1. \qquad \square$$

The following proposition prepares the proof of Theorem 20.1.

Definition 20.6. For $n \in \{2, 3, \ldots\}$ and $\kappa \in \mathbf{R}$, set

$$v_\kappa^n(r) := \begin{cases} 0 & \text{if } \kappa > 0 \text{ and } r \geqslant \pi/\sqrt{\kappa} \text{ and} \\ \sin_\kappa(r)^{n-1} & \text{otherwise.} \end{cases}$$

Proposition 20.7. *Let (M, g) be a Riemannian manifold of dimension n and let $x \in M$. Within of the cut-locus of x in T_xM, define v by*

$$\exp_x^* \mathrm{vol}_g = v\mathrm{d}r \wedge \mathrm{vol}_{S^{n-1}}.$$

Let $\kappa \in \mathbf{R}$. If

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g,$$

then

$$\partial_r\left(\frac{v}{v_\kappa^n}\right) \leqslant 0. \tag{20.8}$$

Moreover, equality holds in (20.8) if and only if all radial sectional curvatures are equal to κ within of the cut-locus of x in M.

Proof. For $f \in C^\infty(M)$,

$$\mathscr{L}_{\nabla f}\mathrm{vol}_g = \mathrm{d}i(\nabla f)\mathrm{vol}_g = (\mathrm{div}\, \nabla f)\mathrm{vol}_g = -\Delta f \mathrm{vol}_g.$$

Therefore,

$$\partial_r v = -v\Delta r.$$

Hence, by Theorem 18.3,

$$\frac{\partial_r v}{v} \leqslant (n-1)\frac{\sin_\kappa'(r)}{\sin_\kappa(r)}.$$

Since

$$\frac{\partial_r v_\kappa^n}{v_\kappa^n} = (n-1)\frac{\sin_\kappa'(r)}{\sin_\kappa(r)},$$

it follows that

$$\partial_r\left(\frac{v}{v_\kappa^n}\right) = \frac{\partial_r v}{v_\kappa^n} - \frac{v}{v_\kappa^n}\frac{\partial_r v_\kappa^n}{v_\kappa^n} \leqslant 0.$$

Equality holds in (20.8) if and only if equality holds in (18.4). By Theorem 18.3, the latter holds if and only if all radial sectional curvatures are equal to κ within the cut-locus of x in M. ◻

Proof of Theorem 20.1. By Remark 20.3, it suffices to prove that the function $\theta\colon (0, R] \to (0, \infty)$ defined by (20.4) is non-increasing.

Let $v\colon T_xM \to [0, \infty)$ be such that, within the cut-locus of x in T_xM,

$$\exp_x^* \mathrm{vol}_g = v\mathrm{d}r \wedge \mathrm{vol}_{S^n};$$

and $v = 0$ on and beyond the cut-locus of x in T_xM. For $r > 0$,

$$\mathrm{vol}(B_r(x)) = \int_{B_r(0)} v\mathrm{d}r \wedge \mathrm{vol}_{S^{n-1}} \quad \text{and} \quad V_\kappa^n(r) = \int_{B_r(0)} v_\kappa^n \mathrm{d}r \wedge \mathrm{vol}_{S^{n-1}}.$$

Therefore,

$$
\begin{aligned}
&V_\kappa^n(r)^2 \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{vol}(B_r(x))}{V_\kappa^n(r)}\right) \\
&\quad = \mathrm{vol}(\partial B_r(x)) \cdot V_\kappa^n(r) - \mathrm{vol}(B_r(x)) \cdot (V_\kappa^n)'(r) \\
&\quad = \int_0^r \left(\int_{S^{n-1}} v(r\hat{x}) \cdot \int_{S^{n-1}} v_\kappa^n(s) - \int_{S^{n-1}} v(s\hat{x}) \cdot \int_{S^{n-1}} v_\kappa^n(r)\right) \mathrm{d}s \\
&\quad = \mathrm{vol}(S^{n-1}) \int_0^r \int_{S^{n-1}} \left(v(r\hat{x}) v_\kappa^n(s) - v(s\hat{x}) v_\kappa^n(r)\right) \mathrm{d}s.
\end{aligned}
$$

The integrand is non-positive if and only if, for $0 < s \leqslant r$,

$$
\frac{v(r\hat{x})}{v_\kappa^n(r)} \leqslant \frac{v(s\hat{x})}{v_\kappa^n(s)}
$$

which follows from (20.8). This proves that θ is non-increasing.

Equality holds in (20.2) if and only if equality holds in (20.8) on $B_R(x)$. By Proposition 20.7, the latter holds if and only if and only if all radial sectional curvatures are equal to κ on $B_R(x)$. □

A minimal modification of the proof of Theorem 20.1 establishes the following variant.

Definition 20.9. Let $\Gamma \subset S^{n-1}$ be measurable and $0 \leqslant r \leqslant R$. Set

$$
A_{r,R}^\Gamma \coloneqq \{\rho\hat{x} \in \mathbf{R}^n : \hat{x} \in \Gamma \text{ and } \rho \in [r, R]\}
$$

Let (M, g) a Riemannian manifold, $x \in M$, and $0 \leqslant r \leqslant R$. The **annular sector** associated with Γ centered at x and with radii r and R is

$$
A_{r,R}^\Gamma(x) = A_{r,R}^{\Gamma,M}(x) \coloneqq \exp_x(A_{r,R}^\Gamma).
$$

For $n \in \{2, 3, \ldots\}$ and $\kappa \in \mathbf{R}$, set

$$
V_\kappa^n(\Gamma, r, R) \coloneqq \mathrm{vol}(A_{r,R}^{\Gamma, S_\kappa^n}(x)).
$$

Theorem 20.10 (Relative Volume Comparison Theorem for annular sectors [Zhu97, Theorem 3.1]). *Let (M, g) be a complete Riemannian manifold of dimension n and let $x \in M$. Let $\kappa \in \mathbf{R}$. Suppose that*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g.$$

If $0 \leqslant r \leqslant R$ and $0 \leqslant s \leqslant S$ with $r \leqslant s$ and $R \leqslant S$, then

$$\frac{\mathrm{vol}(A^{\Gamma}_{s,S}(x))}{\mathrm{vol}(A^{\Gamma}_{r,R}(x))} \leqslant \frac{V^n_\kappa(\Gamma, s, S)}{V^n_\kappa(\Gamma, r, R)}. \tag{20.11}$$

Moreover, equality holds in (20.11) if and only if all all radial sectional curvatures are equal to κ on $A^{\Gamma}_{r,S}(x)$.

21 Volume growth

Exercise 21.1. Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric}_g \geqslant 0$. Prove that, for $r \geqslant 1$,

$$\mathrm{vol}(B_r(x)) \leqslant \mathrm{vol}(B_1(x)) r^n.$$

Exercise 21.2 (maximal volume growth rigidity). Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric}_g \geqslant 0$. Prove that if

$$\lim_{r\to\infty} \frac{\mathrm{vol}(B_r(x))}{r^n} \geqslant V^n_0(1) = \frac{\pi^{n/2}}{\Gamma(n/2+1)},$$

then (M, g) is isometric to $\mathbf{R}^n$.

Remark 21.3. The preceding exercise shows, in particular, that asymptotically Euclidean manifold cannot be Ricci flat without being flat. There are, however, Ricci flat manifolds which are asymptotic to $\mathbf{R}^n/\Gamma$ for Γ acting freely outside the origin.

Theorem 21.4 (Yau [Yau76, Theorem 7]). *If (M, g) is a connected, complete, non-compact Riemannian manifold of dimension n and with $\mathrm{Ric}_g \geqslant 0$, then, for $x \in M$ and $r \geqslant 1$,*

$$\mathrm{vol}(B_r(x)) \gtrsim_n \mathrm{vol}(B_1(x)) r.$$

Definition 21.5. Let (M, g) be a Riemannian manifold. A **geodesic ray** is a geodesic $\gamma \colon [0, \infty) \to M$ satisfying

$$d(\gamma(s), \gamma(t)) = |t - s| \tag{21.6}$$

for all $s, t \in [0, \infty)$. A **geodesic line** is a geodesic $\gamma \colon \mathbf{R} \to M$ satisfying (21.6) for all $s, t \in \mathbf{R}$.

Exercise 21.7. Let (M, g) be a connected, complete, non-compact Riemannian manifold Prove that every for every $x \in M$ there is a geodesic ray γ with $\gamma(0) = x$.

Remark 21.8. The following proof is not due Yau's original proof. If you know how this proof is due to, let me know.

Proof of Theorem 21.4. There is a geodesic ray $\gamma\colon [0,\infty) \to M$ with $\gamma(0) = x$. By Theorem 20.10, for $t \geqslant 2$,

$$\begin{aligned}\frac{\mathrm{vol}(B_{t+1}(\gamma(t))\backslash B_{t-1}(\gamma(t))}{\mathrm{vol}(B_{t-1}(\gamma(t)))} &\leqslant \frac{(t+1)^n - (t-1)^n}{(t-1)^n} \\ &= \left(\frac{2}{t-1} + 1\right)^n - 1 \\ &\lesssim_n \frac{1}{t}.\end{aligned}$$

Therefore, for $t \geqslant 2$,

$$\begin{aligned}t\mathrm{vol}(B_1(x)) &\leqslant t\mathrm{vol}(B_{t+1}(\gamma(t))\backslash B_{t-1}(\gamma(t))) \\ &\lesssim_n \mathrm{vol}(B_{t-1}(\gamma(t))) \\ &\lesssim \mathrm{vol}(B_{2t}(x)).\end{aligned}$$

This proves the assertion for $r \geqslant 4$. Since the assertion holds trivially for $r \in [1,4]$, the proof is complete. ◻

Example 21.9. The flat metric on $T^{n-k} \times \mathbf{R}^k$ has polynomial volume growth of order k.

Remark 21.10. For more interesting examples of complete, Ricci-flat manifolds with linear volume growth see Hein [Hei12], Biquard and Minerbe [BM11], and Haskins, Hein, and Nordström [HHN15]

22 S.Y. Cheng's maximal diameter sphere theorem

Theorem 22.1 (S.Y. Cheng's maximal diameter sphere theorem [Che75]). *Let (M, g) be a complete Riemannian manifold. Let $\kappa > 0$. If*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g \quad \textit{and} \quad \mathrm{diam}(M, g) = \pi/\sqrt{\kappa},$$

then (M, g) is isometric to (S^n_κ, g_κ).

Remark 22.2. The analogous result for $\sec_g$ is due to Topogonov.

Remark 22.3. The following proof is due to Shiohama [Shi83, Section 2].

Proof of Theorem 22.1. Without loss of generality $\kappa = 1$. Let $x, y \in M$ with

$$d(x, y) = \pi$$

The cut-locus of x lies in the complement of $B_\pi(x)$. The balls $B_{\pi/2}(x)$ and $B_{\pi/2}(y)$ do not intersect. By hypothesis,

$$\mathrm{vol}(B_\pi(x)) = \mathrm{vol}(\bar{B}_\pi(y)) = \mathrm{vol}(M).$$

Therefore and by Theorem 20.1,

$$\begin{aligned} 2\mathrm{vol}(M) &= \mathrm{vol}(B_\pi(x)) + \mathrm{vol}(B_\pi(y)) \\ &\leqslant \frac{V_1^n(\pi)}{V_1^n(\pi/2)}\left(\mathrm{vol}(B_{\pi/2}(x)) + \mathrm{vol}(B_{\pi/2}(y))\right) \\ &\leqslant 2\left(\mathrm{vol}(B_{\pi/2}(x)) + \mathrm{vol}(B_{\pi/2}(y))\right). \end{aligned}$$

By Theorem 20.1, all radial sectional curvatures are equal to 1 in $B_\pi(x)$. Therefore, for every $p \in S^n$, the composition

$$B_\pi(x) \xrightarrow{\exp_p^{S^n} \circ \exp_x^{-1}} B_\pi^{S^{n-1}}(p)$$

is an isometry; see Proposition 11.10. Hence, for every $z \in B_\pi(x)\backslash\{x\}$ there is a point $w \in M$ with $d(z,w) = \pi$. Therefore, x was arbitrary and it follows that

$$\sec_g = 1.$$

The assertion thus follows from Theorem 6.5. (Alternatively, one can argue directly that the isometry $B_\pi(x) \cong B_\pi^{S^{n-1}}(p)$ extends to an isometry $M \cong S^n$.) □

Remark 22.4. Theorem 22.1 can be used to give analyze the equality case in Theorem 19.1; see Xia [Xia13, Theorem 1.6]. In the proof of Theorem 19.1, assuming $\kappa = 1$, if f is an eigenfunction with eigenvalue n, then

$$|\nabla f|^2 + f^2$$

is constant. Without loss of generality,

$$|\nabla f|^2 + f^2 = 1; \quad \text{that is:} \quad \frac{|\nabla f|}{\sqrt{1-f^2}} = 1.$$

Let $x_\star$ be a point where f achieves its maximum and let $x_\dagger$ be a point where f achieves its minimum. By the above, $f(x_\star) = 1$ and $f(x_\dagger) = -1$. If γ be a minimizing geodesic parametrized by arc-length, then by the coarea formula,

$$\begin{aligned} \ell(\gamma) &= \int_0^{d(x,y)} |\dot{\gamma}(t)|\mathrm{d}t \\ &= \int_\gamma \frac{|\nabla f|}{\sqrt{1-f^2}} \circ \gamma(t)\mathrm{d}t \\ &\geqslant \int_{-1}^{1} \frac{1}{\sqrt{1-s^2}}\mathrm{d}s \\ &= \pi. \end{aligned}$$

Thus, $\mathrm{diam}(M,g) \geqslant \pi$.

23 The growth of groups

Definition 23.1. Let G be a group. Suppose G is finitely generated and S is a finite generating set. The **word length** with respect to S is the map $\ell_S\colon\ G \to \mathbf{N}_0$ defined by

$$\ell_S(g) \coloneqq \min\bigl\{m : g = g_1 \cdots g_m \text{ with } g_a \in S \cup S^{-1}\bigr\}.$$

For $r \in \mathbf{N}$, set

$$B_S(r) \coloneqq \{g \in G : \ell_s(g) \leqslant r\} \quad \text{and} \quad V_S(r) \coloneqq \#B_r^S.$$

Proposition 23.2. *Let G be a finitely generated group and let $S, T \subset G$ both be finite generating sets. With*

$$c \coloneqq \max\left(\{\ell_S(g) : g \in T\} \cup \{\ell_T(g) : g \in S\}\right)$$

the inequalities

$$\frac{1}{c}\ell_T \leqslant \ell_S \leqslant c\ell_T.$$

hold. □

Definition 23.3. Let G be a finitely generated group. G is said to have **polynomial growth** of rate $\nu \geqslant 0$ if, for some (hence: every) finite generating set S,

$$V_S(r) \asymp r^\nu.$$

G is said to have **exponential growth** if, for some (hence: every) finite generating set S,

$$\lim_{r\to\infty} V_S(r)^{1/r} > 1.$$

Exercise 23.4. Try to work out the growth rates for a few of your favorite groups. (If you have no favorite group, try: the free abelian group $\mathbf{Z}^n$, the free group F_n, surfaces groups $\pi_1(\Sigma)$, the lamp-lighter group, ...)

The above can also be phrased in terms of Cayley graph equipped with the counting measure and the obvious metric.

Definition 23.5. Let G be a group and let S be a generating set. The **Cayley graph** associated with S is the colored, directe graph whose vertices are the elements of G with (g, h) an directed edge colored by $s \in S$ if and only if $h = gs$.

Finally, we have to mention the celebrated theorem of Gromov.

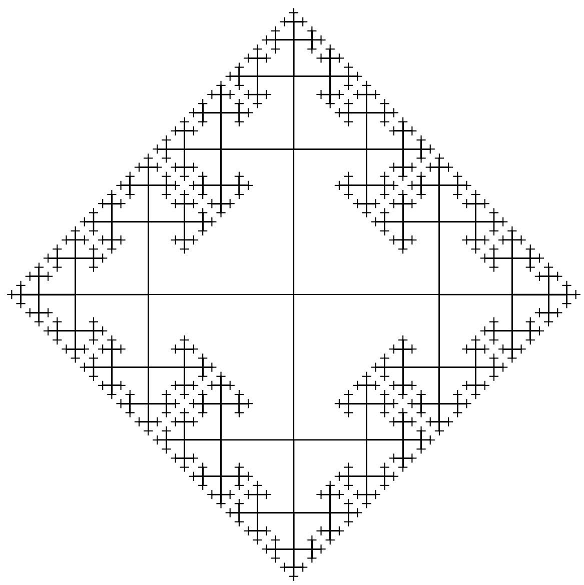

Figure 1: $B_7(e)$ in the Cayley graph of F_2.

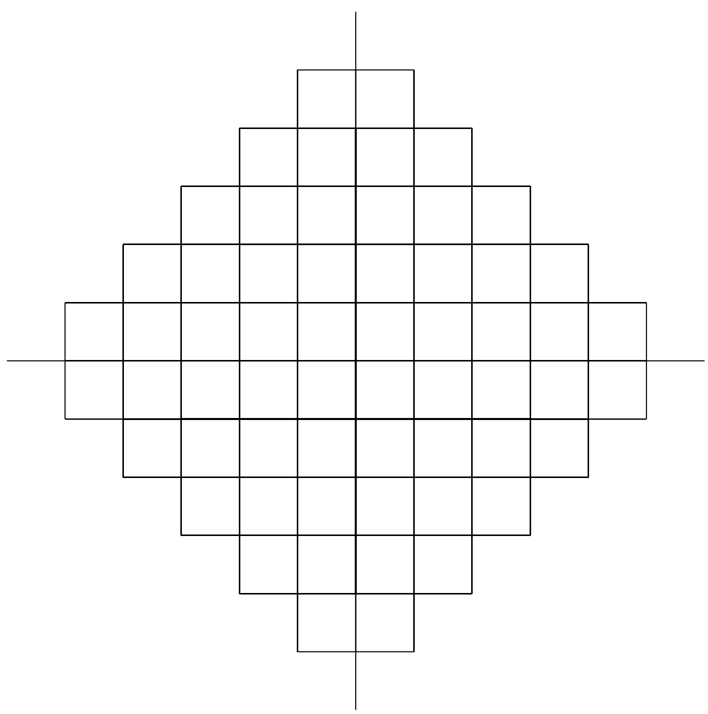

Figure 2: $B_7(e)$ in the Cayley graph of $\mathbf{Z}^2$.

Definition 23.6. Let G be a group. The **lower central series** of G is the sequence of groups $(G_a)_{a\in\mathbf{N}_0}$ defined by

$$G_0 \coloneqq G \quad \text{and} \quad G_{a+1} \coloneqq [G, G_a].$$

The group G is called **nilpotent** if its lower central series terminates in the trivial group after finitely many steps. The group G is called **virtually nilpotent** if it has a finite index subgroup which is nilpotent.

Example 23.7. $\mathbf{Z}^n$ is nilpotent.

Example 23.8. The **discrete Heisenberg group**

$$H_3(\mathbf{Z}) \coloneqq \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbf{Z} \right\}$$

is nilpotent.

Theorem 23.9 (Bass [Bas72, Theorem 2] and Guivarc'h [Gui73, Théorème II.4]). *If G is nilpotent, then it has polynomial volume growth of rate*

$$\sum_{a\geqslant 0} a \cdot \operatorname{rk}(G_a/G_{a+1}).$$

Exercise 23.10. Show that $H_3(\mathbf{Z})$ has polynomial growth of rate 4; that is: although $H_3(\mathbf{Z})$ appears to be 3–dimensional, it behaves 4–dimensional.

Theorem 23.11 (Gromov's theorem on groups of polynomial growth [Gro81b]). *A finitely generated group has polynomial growth if and only if it is virtually nilpotent.*

Remark 23.12. An elementary, but long, proof can be found on Terry Tao's blog.

24 Non-negative Ricci curvature and $\pi_1(M)$

Theorem 24.1 (Milnor [Mil68, Theorem 1]). *If (M, g) is a complete Riemannian manifold of dimension n with $\operatorname{Ric}_g \geqslant 0$, then every finitely generated subgroup $G < \pi_1(M)$ has polynomial growth of rate at most n.*

Conjecture 24.2 (Milnor's finite generation conjecture [Mil68]). *If (M, g) is a complete Riemannian manifold of dimension n with $\operatorname{Ric}_g \geqslant 0$, then $\pi_1(M)$ is finitely generated.*

Remark 24.3. Li [Li86, Theorem 2] and Anderson [And90b, Corollary 1.5] proved Conjecture 24.2 assuming M has maximal volume growth. Sormani [Sor00, Theorem 1] proved Conjecture 24.2 assuming small linear diameter growth. Liu [Liu13, Corollary 1] proved Conjecture 24.2 in dimension three using minimal surface techniques; see also, Pan [Pan18, Theorem 1.1] for a proof using Cheeger–Colding theory.

Proof of Theorem 24.1. Denote by $\pi\colon \tilde{M} \to M$ the universal cover. Let $\tilde{x}_0 \in \pi^{-1}(x_0)$. The fundamental group $\pi_1(M, x_0)$ acts on $\tilde{M}$ as the deck transformation group $\mathrm{Deck}(\tilde{M})$. Let $S \subset \pi_1(M, x_0) \cong \mathrm{Deck}(M)$ be finite subset and denote by G the group generated by S. Set

$$D := \max\{d(\tilde{x}_0, g\tilde{x}_0) : g \in S\}.$$

For $r \in \mathbf{N}$, $\bar{B}_{Dr}(\tilde{x}_0)$ contains at least $V_S(r)$ distinct points of the form $g\tilde{x}_0$ with $g \in \mathrm{Deck}(\tilde{M})$.

Set

$$\delta := \inf\{d(\tilde{x}_0, g\tilde{x}_0) : g \in \mathrm{Deck}(\tilde{M})\}.$$

Since $\mathrm{Deck}(M)$ acts discretely, $\delta > 0$. The ball $B_{Dr+\delta}(\tilde{x}_0)$ contains at least $V_S(r)$ disjoint subsets of the form $B_\delta(g\tilde{x}_0)$. Therefore and by Theorem 20.1,

$$\begin{aligned} V_S(r) &\leqslant \frac{\mathrm{vol}(B_{Dr+\delta}(\tilde{x}_0))}{\mathrm{vol}(B_\delta(\tilde{x}_0))} \\ &\leqslant \frac{(Dr+\delta)^n}{\delta^n} \\ &\leqslant \left(\frac{D}{\delta^n} + 1\right) r^n. \end{aligned}$$

□

Theorem 24.4 (Milnor [Mil68, Theorem 2]). *If (M, g) is a complete Riemannian manifold of dimension n with $\sec_g < 0$, then $\pi_1(M)$ has exponential growth.*

Theorem 24.5 (Anderson [And90c, Theorem 2.3]). *Given $n \in \mathbf{N}$, $\kappa \in \mathbf{R}$, $D, V > 0$, there are only finitely many isomorphism types of groups which appear as $\pi_1(M)$ for connected, closed Riemannian manifolds (M, g) of dimension n with*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa, \quad \mathrm{vol}(M, g) \geqslant V, \quad \textit{and} \quad \mathrm{diam}(M, g) \leqslant D.$$

The proof relies on the following lemma.

Lemma 24.6 (Gromov [Gro81c; Gro07, Proposition 3.22]). *Let (M, g) be a closed Riemannian manifold. Given $x_0 \in M$, there are loops $\gamma_1, \ldots, \gamma_m$ based at x_0 with*

$$\ell(\gamma_a) \leqslant 3\,\mathrm{diam}(M, g)$$

and a presentation

$$\pi_1(M, x_0) = \langle [\gamma_1], \ldots, [\gamma_m] | R\rangle$$

with every relation in R of the form

$$[\gamma_a][\gamma_b] = [\gamma_c]. \tag{24.7}$$

Remark 24.8. The lemma makes no assertion about the number of generators.

Proof. Choose a triangulation K of M such that x_0 is one of the vertices, and every edge e_{ab} has length at most one-half of the injectivity radius of M. For every vertex v_a in K, denote by δ_a the minimal geodesic from x_0 to v_a. The loop

$$\gamma_{ab} := \delta_a e_{ab} \delta_b^{-1}$$

has length at most $3 \operatorname{diam} M$.

Every loop based x_0 in the 1–skeleton of K is homotopic to a product of the γ_{ab}. Since every loop based at x_0 is homotopic to a loop in the 1–skeleton of K, the γ_{ab} generate $\pi(M, x_0)$.

If v_a, v_b, v_c form a 2–simplex in K, then $\gamma_{ab}\gamma_{bc} \simeq \gamma_{ac}$. Every homotopy between two loops based at x_0 in the 1–skeleton of K is homotopic to a homotopy lying in the 2–skeleton. Since homotopies in the 2–skeleton correspond to a collection of 2–simplices, every relation is generated by relations of the form $\gamma_{ab}\gamma_{bc} = \gamma_{ac}$. □

Proof of Theorem 24.5. It suffices to estimate the number of loops γ_a in Lemma 24.6, because, if there are m generators, then there can be at most 2^{m^3} relations of the form (24.7).

Let (M, g) be a connected, closed Riemannian manifold of dimension n with $\operatorname{Ric}_g \geqslant (n-1)\kappa$, $\operatorname{vol}(M) \geqslant V$, and $\operatorname{diam}(M) \leqslant D$. Denote by $\pi\colon \tilde{M} \to M$ the universal cover of M. Let $\tilde{x}_0 \in \tilde{M}$ and set $x_0 := \pi(\tilde{x}_0)$. The $[\gamma_a]$ acts as deck transformations: $\pi_1(M, x_0) \cong \operatorname{Deck}(\tilde{M})$. Set

$$K := \left\{x \in \tilde{M} : d(x, \tilde{x}_0) \leqslant d(\gamma \cdot x, \tilde{x}_0) \text{ for all } \gamma \in \pi_1(M, x_0)\right\}$$

The sets $[\gamma_a] \cdot K$ all have volume equal to $\operatorname{vol}(M)$ and are all contained in $B_{6D}(\tilde{x}_0)$. Furthermore, they intersect in sets of measure zero. Therefore,

$$\begin{aligned} m &\leqslant \frac{\operatorname{vol}(B_{6D}(\tilde{x}_0))}{\operatorname{vol}(K)} \\ &\leqslant \frac{V_\kappa^n(6D)}{V}. \end{aligned}$$ □

Theorem 24.9 (Anderson [And90c, Theorem 2.1]). *Let $n \in \mathbf{N}$, $\kappa \in \mathbf{R}$, $D, V > 0$. Set*

$$N = N(n, \kappa, D, V) := \frac{V_\kappa^n(2D)}{V}.$$

Suppose (M, g) is a closed Riemannian manifold of dimension n with

$$\operatorname{Ric}_g \geqslant (n-1)\kappa, \quad \operatorname{vol}(M, g) \geqslant V, \quad \textit{and} \quad \operatorname{diam}(M, g) \leqslant D.$$

If γ is a loop in M for which $[\gamma]$ has order at least N in $\pi_1(M)$, then

$$\ell(\gamma) \geqslant \frac{DV}{V_\kappa^n(2D)}.$$

Exercise 24.10. Prove Theorem 24.9.

25 The Maximum Principle

Definition 25.1. Let (M,g) a Riemannian manifold. A function $f \in C^\infty(M)$ is called **subharmonic** if $\Delta f \leqslant 0$ and **superharmonic** if $\Delta f \geqslant 0$.

Theorem 25.2 (E.Hopf's Maximum Principle [Hop27]). *Let (M,g) a connected Riemannian manifold. Let $f \in C^\infty(M)$ be subharmonic. If f has a local maximum at $x \in M$, then f is constant on a neighborhood of x. In particular, f has a global maximum if and only if it is constant.*

Definition 25.3 (Calabi [Cal58]). Let (M,g) is a Riemannian manifold. Let $f \in C^0(M)$ and $g \in C^0(M)$. A **lower barrier** (resp. **upper barrier**) for f at $x \in M$ is a smooth function f_x defined in a neighborhood of x satisfying

$$f_x(x) = f(x) \quad \text{and} \quad f_x \leqslant f \quad (\text{resp. } f_x \geqslant f).$$

We say that

$$\Delta f \leqslant g \quad (\text{resp. } \Delta f \geqslant g)$$

in the barrier sense if, for every $x \in M$ and every $\varepsilon > 0$, there exists a lower barrier (resp. upper barrier) $f_{x,\varepsilon}$ for f at x such that

$$\Delta f_{x,\varepsilon} \leqslant g + \varepsilon \quad (\text{resp. } \Delta f_{x,\varepsilon} \geqslant g - \varepsilon).$$

The function f is called **subharmonic in the barrier sense** (resp.**superharmonic in the barrier sense**) if $\Delta f \leqslant 0$ (resp. $\Delta f \geqslant 0$).

Theorem 25.4 (Calabi's Laplacian Comparison Theorem [Cal58, Theorem 3]). *Let (M,g) be a complete Riemannian manifold and let $x \in M$. Let $\kappa \in \mathbf{R}$. If*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g,$$

then

$$-\Delta r \leqslant (n-1)\frac{\sin_\kappa'(r)}{\sin_\kappa(r)} \tag{25.5}$$

holds in the barrier sense on all of M.

Proof. Let $y \in M$ and denote by $\gamma\colon [0,T] \to M$ a minimal geodesic γ with $\gamma(0) = x$ and $\gamma(T) = y$ parametrized by arc-length. For $\varepsilon > 0$, define $r_\varepsilon\colon M \to [0,\infty)$ by

$$r_\varepsilon(z) \coloneqq \varepsilon + d(\gamma(\varepsilon), z).$$

By the triangle inequality, r_ε is an upper barrier for r. The point y lies within the cut-locus of $\gamma(\varepsilon)$ for $\varepsilon > 0$. (This is an easy exercise.) Thus, by Theorem 18.3,

$$\begin{aligned} -\Delta r_\varepsilon(y) &\leqslant (n-1)\frac{\sin_\kappa'(r_\varepsilon(y)-\varepsilon)}{\sin_\kappa(r_\varepsilon(y)-\varepsilon)} \\ &= (n-1)\frac{\sin_\kappa'(r(y)-\varepsilon)}{\sin_\kappa(r(y)-\varepsilon)}. \end{aligned}$$

Since $\sin_\kappa'/\sin_\kappa$ is decreasing, it follows that (25.5) holds in the barrier sense. □

Theorem 25.6 (Calabi's Maximum Principle [Cal58, Theorem 1]). *Let (M, g) a Riemannian manifold. Let $f \in C^0(M)$ be subharmonic in the barrier sense. If f has a local maximum at $x \in M$, then f is constant on a neighborhood of x. In particular, f has a global maximum if and only if it is constant.*

Proof. If f_x is a lower barrier for f at x, then x is also a local maximum for f_x. Therefore,

$$\Delta f_x(x) \geqslant 0.$$

Suppose that f achieves a local maximum at x, but f is not constant. Then there is a $0 < r \ll 1$ such that

$$f(x) = \sup\{f(y) : y \in B_r(x)\} \quad \text{and} \quad \Gamma := \{y \in \partial B_r(x) : f(y) = f(x)\} \neq \partial B_r(x).$$

As we will see shortly, there is a smooth function $g\colon \bar{B}_r(x) \to \mathbf{R}$ satisfying

$$g(x) = 0, \quad g|_\Gamma \leqslant -1/2, \quad \text{and} \quad \Delta g \leqslant -1.$$

For $0 < \delta \ll 1$,

$$\begin{aligned} f(x) &= f(x) + \delta g(x) \\ &> \sup\{f(y) + \delta g(y) : y \in \partial B_r(x)\}. \end{aligned}$$

Therefore, the function $f + \delta g$ has achieves a local maximum at some $y \in B_r(x)$. This, however, contradicts the observation in the first paragraph because $\Delta(f + \delta g) \leqslant -\delta$ in the barrier sense.

To construct g, we proceed as follows. Since $\Gamma \neq \partial B_r(x)$, we can choose a function χ satisfying

$$\chi(x) = 0, \quad \chi|_\Gamma < 0, \quad \text{and} \quad |\nabla\chi| > 0.$$

For $\Lambda \gg 1$, the function

$$g := e^{\Lambda\chi} - 1,$$

has the required properties, because

$$\Delta g = \Lambda e^{\Lambda\chi}\left(\Delta\chi - \Lambda|\nabla\chi|^2\right). \qquad □$$

Theorem 25.7. *Let (M, g) be a Riemannian manifold. If $f \in C^0(M)$ is subharmonic and superharmonic in the barrier sense, then it is smooth and harmonic.*

Proof. For $x \in M$ and $0 \leqslant r \ll 1$, standard elliptic theory constructs a continuous function $h\colon \bar{B}_r(x) \to \mathbf{R}$ which is smooth and harmonic on $B_r(x)$ and satisfies the Dirichlet boundary condition $h|_{\partial B_r(x)} = f|_{\partial B_r(x)}$. By the maximum principle applied to $f - h$ and $h - f$, $f = h$. □

26 Busemann functions

Proposition 26.1. *Let (M, g) be a Riemannian manifold. Given a geodesic ray γ in M, there exists a function $b_\gamma \colon M \to \mathbf{R}$ such that, for all $x \in M$,*

$$b_\gamma(x) := \lim_{t\to\infty} d(x, \gamma(t)) - t. \tag{26.2}$$

The function b_γ is Lipschitz with $\mathrm{Lip}(b_\gamma) \leqslant 1$.

Definition 26.3. In the situation of Proposition 26.1, b_γ is called the **Busemann function** associated with (M, g).

Remark 26.4. Morally, a Busemann function is a renormalized distance function associated to ∞.

Example 26.5. Consider $(\mathbf{R}^n, g_0)$. The Busemann function b_γ associated with geodesic ray $\gamma(t) = x_0 + tv$ is given by

$$b_\gamma(x) = \langle v, x_0 - x\rangle.$$

Exercise 26.6. Compute the Busemann functions on $\mathbf{H}^2$. What are the level sets (so called **horospheres**) of these functions?

Proof of Proposition 26.1. Define $b_\gamma^t \colon M \to \mathbf{R}$ by

$$b_\gamma^t(x) := d(x, \gamma(t)) - t.$$

By the triangle inequality:

1. $b_\gamma^t(x)$ is non-increasing in t,
2. $|b_\gamma^t(x)| \leqslant d(x, \gamma(0))$, and
3. $|b_\gamma^t(x) - b_\gamma^t(y)| \leqslant d(x, y)$.

The first two show that the limit in (26.2) exists. The last implies $\mathrm{Lip}(b_\gamma) = 1$. □

Proposition 26.7. *If (M, g) is a Riemannian manifold with $\mathrm{Ric}_g \geqslant 0$ and γ is a geodesic ray, then*

$$\Delta b_\gamma \geqslant 0$$

in the barrier sense.

Proof. Let $x \in M$. For $s \geqslant 0$, denote by $\delta_s \colon [0, T_s] \to M$ the geodesic parametrized by arc-length from x to $\gamma(s)$. There is a sequence $(s_n)_{n\in\mathbf{N}}$ converging to infinity, such that $(\delta_{s_n})_{n\in\mathbf{N}}$ converges to a geodesic ray δ with $\delta(0) = x$. This geodesic ray is called an **asymptote** from x to γ.

The function $b_\gamma(x) + b_\delta^t$ is smooth in a neighborhood of x and agrees with $b_\gamma(x)$ at x. By the triangle inequality, for $s \geqslant 0$ and $0 \leqslant t \leqslant T_s$,

$$\begin{aligned} b_\gamma^s(y) - b_\gamma^s(x) &= d(y, \gamma(s)) - d(\gamma(s), x) \\ &= d(y, \gamma(s)) - d(\gamma(s), \delta_s(t)) - d(\delta_s(t), x) \\ &\leqslant d(y, \delta_s(t)) - d(\delta_s(t), x) \\ &= d(y, \delta_s(t)) - t. \end{aligned}$$

Setting $s = s_n$ and taking the limit $n \to \infty$,

$$b_\gamma(y) \leqslant b_\gamma(x) + b_\delta^t(y). \tag{26.8}$$

Therefore, $b_\gamma(x) + b_\delta^t$ is an upper barrier for b_γ at x.

By Theorem 18.3,

$$\Delta b_\delta^t(x) \geqslant -\frac{n-1}{d(x, \delta(t))}.$$

Since the right-hand side goes to zero as t tends to infinity, b_γ is superharmonic. □

27 Cheeger–Gromoll Splitting Theorem

Proposition 27.1. *Let (M, g) be a connected, complete, non-compact Riemannian manifold. If M contains a compact subset K with $M \backslash K$ disconnected, then there is a geodesic line passing through K.*

Proof. Since $M \backslash K$ is disconnected, for every $n \in \mathbf{N}$, we can choose a minimal geodesic $\gamma_n \colon [-n, n] \to M$ of the form

$$\gamma_n(t) = \exp_{x_n}(tv_n)$$

with $x_n \in K$ and $|v_n| = 1$. A subsequence of $(x_n)_{n \in \mathbf{N}}$ converges to a limit $x_\infty \in K$, and a further subsequence of $(v_n)_{n \in \mathbf{N}}$ converges to $v_\infty \in T_x M$. The limit geodesic $\gamma_\infty \colon \mathbf{R} \to M$ defined by

$$\gamma_\infty \coloneqq \exp_{x_\infty}(tv_\infty)$$

is the desired geodesic line. □

Theorem 27.2 (Cheeger–Gromoll Splitting Theorem [CG71, Theorem 2]). *Let (M, g) be a connected, complete Riemannian manifold with $\mathrm{Ric}_g \geqslant 0$. If (M, g) contains a geodesic line, then there is a complete Riemannian manifold (N, h) and an isometry*

$$(M, g) \cong (\mathbf{R} \times N, \mathrm{d}t \otimes \mathrm{d}t + h).$$

The following propositions prepare the proof. This argument is due to Eschenburg and Heintze [EH84].

Proposition 27.3. *Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric}_g \geqslant 0$. If (M, g) contains a geodesic line, then there exists a harmonic function $\ell\colon M \to \mathbf{R}$ with*

$$|\nabla \ell| = 1.$$

Proof. Denote the geodesic line by γ. Define $\gamma_\pm\colon [0, \infty) \to M$ by

$$\gamma_\pm(t) \coloneqq \gamma(\pm t).$$

Denote by $b_{\gamma_\pm}$ the Busemann function associated with $\gamma_\pm$. Both $\ell = b_{\gamma_\pm}$ have the asserted properties. To prove this we proceed as follows

Set

$$b \coloneqq b_{\gamma_+} + b_{\gamma_-}.$$

By construction,

$$\begin{aligned} b(\gamma(t)) &= \lim_{s\to\infty} d(\gamma(t), \gamma(s)) + d(\gamma(t), \gamma(-s)) - 2s \\ &= \lim_{s\to\infty} s - t + t + s - 2s \\ &= 0. \end{aligned}$$

By the triangle inequality,

$$\begin{aligned} b(x) &= \lim_{t\to\infty} \left(d(\gamma(-t), x) + d(x, \gamma(t)) - 2t\right) \\ &\geqslant 0. \end{aligned}$$

By Proposition 26.7,

$$\Delta b \geqslant 0.$$

Therefore and by the maximum principle,

$$b = 0; \quad \text{that is:} \quad b_{\gamma_+} = -b_{\gamma_-}.$$

Therefore, $b_{\gamma_\pm}$ is harmonic.

Let $x \in M$. Let $\delta_\pm$ be geodesic rays emanating from x, constructed as in the proof of Proposition 26.1. By (26.8),

$$\begin{aligned} b^t_{\delta_+}(y) &\geqslant b_{\gamma_+}(y) - b_{\gamma_+}(x) \\ &= b_{\gamma_-}(x) - b_{\gamma_-}(y) \geqslant -b^t_{\delta_-}(y). \end{aligned}$$

Lemma 27.4. *Let M be a manifold and $x \in M$. Let $f_+, f, f_-\colon M \to \mathbf{R}$ be functions satisfying*

$$f_+ \geqslant f \geqslant f_- \quad \textit{and} \quad f_+(x) = f(x) = f_-(x).$$

If $f_\pm$ both are differentiable at x, then so is f and, moreover,

$$\mathrm{d}_x f_+ = \mathrm{d}_x f = \mathrm{d}_x f_-.$$

Proof. Without loss of generality, M is an open subset of $\mathbf{R}^n$, $x = 0$, and $f_- = 0$. If $\mathrm{d}_0 f_+ \neq 0$, then there is a $v \in \mathbf{R}^n$ with $\mathrm{d}_0 f_+(v) < 0$. Therefore, for $0 < \varepsilon \ll 1$, $f_+(\varepsilon v) < 0$. Thus $\mathrm{d}_0 f_+ = 0$. For every $v \in \mathbf{R}^n$ with $|v| \ll 1$,

$$\frac{|f_+(v) - f_+(0)|}{|v|} \geqslant \frac{|f(v) - f(0)|}{|v|} \geqslant 0.$$

Since f_+ is differentiable, the limit of the left-hand side as $|v|$ tends to zero vanishes. Thus, the same holds for the limit of the middle, proving that f is differentiable at 0 with $\mathrm{d}_0 f = 0$. □

It follows from the lemma that

$$\nabla b_{\gamma_\pm}(x) = \nabla b^t_{\delta_\pm}(x).$$

This completes the proof since

$$|\nabla b^t_{\delta_\pm}| = 1.$$ □

Proposition 27.5. *Let (M, g) be a complete Riemannian manifold. Suppose $\ell \in C^\infty(M)$ satisfies*

$$|\nabla \ell| = 1 \quad \textit{and} \quad \operatorname{Hess} \ell = 0.$$

Set

$$N := \ell^{-1}(0) \quad \textit{and} \quad h = g|_N.$$

The map $f \colon (\mathbf{R} \times N, \mathrm{d}t \otimes \mathrm{d}t) \to (M, g)$ defined by

$$f(t, x) := \exp_x(t \nabla \ell(x))$$

is an isometry.

Remark 27.6. N is a submanifold by the Regular Value Theorem.

Proof of Proposition 27.5. Since $\nabla\nabla\ell = 0$, for every $x \in N$, $t \mapsto f(t, x)$ is a gradient flow line for the function ℓ. Therefore and since ℓ is no critical points, f is a diffeomorphism.

It remains to prove that f is an isometry. For $v \in T_x N$, let J be the Jacobi field along $t \mapsto f(t, x)$ with

$$J(0) = 0 \quad \text{and} \quad \nabla_t J(0) = v$$

and let V be the parallel vector field along $t \mapsto f(t, x)$ with

$$V(0) = v.$$

Since $\nabla\nabla\ell = 0$, the Jacobi equation for $J(0)$ becomes

$$\nabla_t \nabla_t J = 0; \quad \text{hence:} \quad J(t) = tV(t) \quad \text{and}$$

Therefore,

$$|f_* v| = |J(1)| = |v|.$$

Since, also,
$$|f_*\partial_t| = |\nabla \ell| = 1 = |\partial_t|,$$
f is as isometry. □

Proof of Theorem 27.2. Proposition 27.3 provides us with a harmonic function $\ell\colon M \to \mathbf{R}$ with $|\nabla \ell| = 1$. By Proposition 18.5,
$$\begin{aligned}|\mathrm{Hess}\,\ell|^2 + \mathrm{Ric}(\nabla\ell, \nabla\ell) &= \langle \nabla\Delta\ell, \nabla\ell\rangle - \frac{1}{2}\Delta|\nabla\ell|^2 \\ &= 0.\end{aligned}$$
Therefore and since $\mathrm{Ric} \geqslant 0$,
$$\nabla\nabla\ell = 0.$$
The assertion thus follows from Proposition 27.5. □

Exercise 27.7 (Gallot). Prove that if (M, g) is closed Riemannian manifold with $\mathrm{Ric}_g \geqslant 0$ and $b_1(M) = \dim M$, then it is isometric to a flat torus.

Exercise 27.8 (Gallot). Prove that if (M, g) is closed Riemannian manifold with $\mathrm{Ric}_g \leqslant 0$ and $\dim \mathfrak{iso}(M) = \dim M$, then it is isometric to a flat torus.

Definition 27.9. A subgroup $B_n < \mathrm{O}(n) \ltimes \mathbf{R}^n$ is called a **Bieberbach group** if it acts freely on $\mathbf{R}^n$ and $\mathbf{R}^n/B_n$ is compact.

Remark 27.10. $\mathbf{Z}^n$ obviously is a Bieberbach group. The group B_2 generated by
$$(x, y) \mapsto (x + 1/2, -y) \quad \text{and} \quad (x, y) \mapsto (x, y + 1)$$
also is a Bieberbach group. What is $\mathbf{R}^2/B_2$?

Theorem 27.11 (Bieberbach). *Every Bieberbach group B_k contains $\mathbf{Z}_k$ as finite index subgroup.*

Theorem 27.12 (Structure Theorem for Nonnegative Ricci Curvature [CG71, Theorem 3]). *If (M, g) is a closed, connected Riemannian manifold with $\mathrm{Ric}_g \geqslant 0$, then the following hold:*

1. *The universal cover $\tilde{M}$ is isometric to $\mathbf{R}^k \times N$ with N compact.*
2. *There is a finite subgroup $G < \mathrm{Iso}(N)$, a Bieberbach group B_k, and an exact sequence*
$$0 \to G \to \pi_1(M) \to B_k \to 0.$$

Proof. By Theorem 27.2,
$$(\tilde{M}, g) \cong (\mathbf{R}^k \times N, g_{\mathbf{R}^k} \oplus h)$$
with N containing no geodesic lines. Every geodesic line in $\tilde{M}$ must be of the form $t \mapsto (\gamma(t), x)$. If $g \in \mathrm{Iso}(\tilde{M})$, then it is an isometry of $\tilde{M}$ and thus maps geodesic lines to geodesic lines. In particular, if $v_1, v_2 \in \mathbf{R}^k$, then the N−component of $g(tv_1 + (1-t)v_2, x)$ is independent of $t \in \mathbf{R}$. Therefore, g is of the form
$$g(v, x) = (g_{\mathbf{R}^k}(v, x), g_N(x))$$
Furthermore, for every $v \in \mathbf{R}^k$ and $x \in N$, $d_{(v,x)}g$ preserves $T_v\mathbf{R}^k$. Since $d_{(v,x)}g$ is and isometry, it also preserves T_xN; hence: $\mathrm{d}_{(v,x)}g_{\mathbf{R}^k}|_{T_xN} = 0$. Therefore, $\mathrm{Deck}(\tilde{M}) \subset \mathrm{Iso}(\tilde{M}) \subset \mathrm{Iso}(\mathbf{R}^k) \times \mathrm{Iso}(N)$.

Since M is compact, there is a compact subset $K \subset \tilde{M}$ with
$$\mathrm{Deck}(\tilde{M}) \cdot K = \tilde{M} \quad \text{and thus} \quad \mathrm{Deck}(\tilde{M}) \cdot \pi_{\mathbf{R}^k}(K) = \mathbf{R}^k \quad \text{and} \quad \mathrm{Deck}(\tilde{M}) \cdot \pi_N(K) = N.$$

To prove (1), observe that if N were not compact, then it would contain a geodesic ray γ. By the above, there is a sequence $g_n \in \mathrm{Deck}(\tilde{M})$ such that $g_n(\gamma(n)) \in \pi_N(K)$. Since K and S^{n-1} are compact, after passing to a subsequence, $g_n(\gamma(n))$ converges to a limit $x \in \pi_N(K)$ and $(g_n)_*(\dot{\gamma}(n))$ converges to a limit $v \in T_xM$. This shows that the geodesics $\gamma_n\colon [-n, \infty) \to N$ defined by
$$\gamma_n(t) := g_n(\gamma(n + t))$$
converges to the geodesic $\gamma_\infty\colon \mathbf{R} \to N$ defined by $\gamma_\infty(t) := \exp_x(tv)$. By construction, γ_∞ is a geodesic line in N: a contradiction.

It remains to prove (2). Set
$$B_k := \mathrm{im}(\mathrm{Deck}(\tilde{M}) \to \mathrm{Iso}(\mathbf{R}^k)) \quad \text{and} \quad G := \ker(\mathrm{Deck}(\tilde{M}) \to \mathrm{Iso}(\mathbf{R}^k)).$$
By construction, B_k acts freely. Since $B_k \cdot K = \mathbf{R}^k$, B_k is a Bieberbach group. G acts discretely on N; hence, is finite because N is compact. ☐

28 S.Y. Cheng's first eigenvalue comparison theorem

Definition 28.1. If (M, g) is a closed, connected Riemannian manifold, then $\lambda_1(M, g)$ denotes the first non-zero eigenvalue of the Laplacian. If (M, g) is a compact, connected Riemannian manifold with boundary, then $\lambda_1^D(M, g)$ denotes the first eigenvalue of the Laplacian with Dirichlet boundary conditions.

Definition 28.2. Let $n \in \mathbf{N}$ and $\kappa \in \mathbf{R}$. The function $\Lambda_\kappa^n\colon [0, \infty) \to [0, \infty)$ is defined by
$$\Lambda_\kappa^n(r) := \lambda_1^D(B_r(x), g_\kappa)$$
for $B_r(x) \subset S_\kappa^n$.

Theorem 28.3 (S.Y. Cheng's first eigenvalue comparison theorem). *Let $n \in \mathbf{N}$ and $\kappa \in \mathbf{R}$ Let M be a compact, connected Riemannianian manifold of dimension n and satisfying*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g.$$

The following hold:

1. *Let $x \in M$ and $R > 0$. If $B_R(x)$ is contained within the cut locus of x, then*

$$\lambda_1^D(B_R(x)) \leqslant \Lambda_\kappa^n(R).$$

2. *If $\partial M = \emptyset$, then*

$$\lambda_1(M, g) \leqslant \Lambda_\kappa^n\left(\tfrac{1}{2}\operatorname{diam}(M, g)\right).$$

Proof sketch. To prove (1), let f_κ^n be a Dirichlet eigenfunction on $B_R(x) \subset S_n^\kappa$ with eigenvalue $\Lambda_n^\kappa(R)$. The function f_κ^n has no zeros and thus we can assume $f_\kappa^n \geqslant 0$; moreover, it can be written as $f_\kappa^n(y) = F_\kappa^n(r_x(y))$ for some function F_κ^n. By the maximum principle $(F_\kappa^n)' \leqslant 0$.

Define $f\colon M \supset B_R(x) \to [0, \infty)$ by

$$f(y) = F_\kappa^n(r_x(y)).$$

By Theorem 18.3 and using $(F_n^\kappa)' \leqslant 0$,

$$\begin{aligned}\Delta f &= -(F_\kappa^n)''(r_x)|\nabla r_x|^2 + (F_\kappa^n)'(r_x)\Delta r_x \\ &\leqslant (\Delta_{S_\nu^\kappa} f_\kappa^n)(r_x) \\ &= \Lambda_\kappa^n(R) f.\end{aligned}$$

Denote by r_x the distance function

Therefore,

$$\lambda_1^D(B_R(x)) \leqslant \frac{\int_{B_R(x)} |\nabla f|^2}{\int_{B_R(x)} f^2} \leqslant \Lambda_\kappa^n(R).$$

To prove (2), let $x, y \in M$ with $d(x, y) = \operatorname{diam}(M, g)$. Set $D := \frac{1}{2}\operatorname{diam}(M, g)$. Let f_x be a Dirichlet eigenfunction with eigenvalue $\lambda_1^D(B_D(x))$ and let f_y be a Dirichlet eigenfunction with eigenvalue $\lambda_1^D(B_D(y))$. Assume both are normalized to have L^2–norm equal to 1 and opposite signs. Define f to be equal to f_x and $B_D(x)$, equal to f_y and $B_D(y)$, and zero elsewhere. A short computation shows that

$$\begin{aligned}\lambda_1(M, g) &\leqslant \max\{\lambda_1^D(B_D(x)), \lambda_1^D(B_D(y))\} \\ &\leqslant \Lambda_\kappa^n(D).\end{aligned}$$

□

Exercise 28.4. Fill in the missing details of the above proof.

Exercise 28.5. Using that, on S^n_κ,

$$\Delta r = -(n-1)\frac{\sin_\kappa'(r)}{\sin_\kappa(r)},$$

compute F^n_κ and Λ^n_κ explicitly.

29 Poincaré and Sobolev inequalities

Notation 29.1. Given a Riemannian manifold (M,g), $f \in L^1_{\mathrm{loc}}(M)$, $x \in M$, and $r >$, set

$$\bar{f} = \fint_M f := \frac{1}{\mathrm{vol}(M)} \int_M f \quad \text{and} \quad \bar{f}_{x,r} := \fint_{B_r(x)} f.$$

Theorem 29.2 (Neumann–Poincaré–Sobolev inequality). *Let $\kappa \leqslant 0$ and $D > 0$. Let (M,g) be a complete Riemannian manifold of dimension n with*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g.$$

For all $f \in \mathrm{Lip}(M)$, $x \in M$, and $0 < r \leqslant D$,

$$\left(\fint_{B_r(x)} |f - \bar{f}_{x,r}|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leqslant c(n,\kappa D^2) \fint_{B_r(x)} |\nabla f|. \tag{29.3}$$

The proof presented below goes back to Hajłasz and Koskela [HK95]. It involves a number of steps. Before delving into the proof, let us observe the following consequence.

Theorem 29.4 (Sobolev inequality). *Let $\kappa \leqslant 0$ and $D > 0$. Let (M,g) be a complete Riemannian manifold of dimension n with*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g.$$

For all $f \in \mathrm{Lip}(M)$, $x \in M$, $0 < r \leqslant D$, and $p \in [1,n)$,

$$\left(\fint_{B_r(x)} |f|^{\frac{pn}{n-p}}\right)^{\frac{n-p}{pn}} \leqslant \frac{p(n-1)}{n-p} c(n,\kappa D^2) \left(\fint_{B_r(x)} |\nabla f|^p\right)^{\frac{1}{p}} + \left(\fint_{B_r(x)} |f|^p\right)^{\frac{1}{p}}. \tag{29.5}$$

Proof. Exercise. *Hint:* $p = 1$ uses the triangle inequality; $p > 1$ uses $p = 1$ applied to f^q for an appropriate choice of q. ◻

Remark 29.6. The above shows that uniform lower Ricci bounds do give uniform upper bounds on Poincaré constants, Sobolev constants, etc. However, there are situation in which uniform lower Ricci bounds are not available, but Poincaré upper bounds can be established, for example, using discretization techniques [GS05]. In fact, it is also clear from the proof that Ricci lower bounds are only used for volume doubling estimates and volume lower bounds.

The proof of Theorem 29.2 proceeds by amplifying the following weak L^1 Neumann–Poincaré inequality.

Theorem 29.7 (weak L^1 Neumann–Poincaré inequality). *Let $\kappa \leqslant \mathbf{R}$ and $D > 0$. Let (M, g) be a complete Riemannian manifold of dimension n with*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa.$$

For every $f \in \mathrm{Lip}(M)$, $x \in M$, $0 < r \leqslant D$,

$$\int_{B_r(x)} |f - \bar{f}_{x,r}| \leqslant c(n, \kappa D^2) r \int_{B_{2r}(x)} |\nabla f|.$$

Remark 29.8. This is the *weak* L^1 Neumann–Poincaré inequality because the integral on the right-hand side is over $B_{2r}(x)$ instead of $B_r(x)$. It will be evident from the proof that $B_{2r}(x)$ can be replaced by the convex hull of $B_r(x)$. In particular, if $B_r(x)$ is geodesically convex, the proof establishes the L^1 Neumann–Poincaré inequality.

The proof of Theorem 29.7 will be completely analogous to the classical proof on $\mathbf{R}^n$ once we have the following.

Theorem 29.9 (Cheeger–Colding Segment Inequality). *Let $\kappa \leqslant 0$ and $D > 0$. Let (M, g) be a complete Riemannian manifold of dimension n with*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g.$$

Let $A, B \subset C \subset M$ be measurable and suppose that $\mathrm{diam}(C) \leqslant D$. If, for every $x \in A$ and $y \in B$, let $\gamma_{x,y} \colon [0,1] \to C$ be a minimizing geodesic with with $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$, then, for every $f \in L^\infty(M, [0, \infty))$,

$$\int_A \int_B \int_0^1 f \circ \gamma_{x,y}(t) \mathrm{d}t \mathrm{d}y \mathrm{d}x \leqslant c(n, \kappa D^2)(\mathrm{vol}(A) + \mathrm{vol}(B)) \int_C f. \tag{29.10}$$

Proof. Set

$$c(n, \kappa D^2) \coloneqq \max\left\{ \frac{v_\kappa^n(r)}{v_\kappa^n(r/2)} : r \in (0, D] \right\}.$$

Let $x \in A$ and $y \in B$. In geodesic polar coordinates centered at x, write

$$\mathrm{vol}_g = v \mathrm{d}r \wedge \mathrm{vol}_{S^{n-1}}$$

By Proposition 20.7,

$$\partial_r \left(\frac{v}{v_\kappa^n} \right) \leqslant 0.$$

Therefore, for $t \in [\frac{1}{2}, 1]$,

$$f \circ \gamma_{x,y}(t)\nu(y)\mathrm{d}r \wedge \mathrm{vol}_{S^{n-1}} \leqslant c(n, \kappa D^2) f \circ \gamma_{x,y}(t)\nu(\gamma_{x,y}(t))\mathrm{d}r \wedge \mathrm{vol}_{S^{n-1}},$$

and, thus,

$$\int_{1/2}^{1} \int_A \int_B f \circ \gamma_{x,y}(t)\mathrm{d}y\mathrm{d}x\mathrm{d}t \leqslant \frac{1}{2} c(n, \kappa D^2)\mathrm{vol}(A) \int_C f.$$

Swapping the roles of x and y shows that

$$\int_{0}^{1/2} \int_A \int_B f \circ \gamma_{x,y}(t)\mathrm{d}y\mathrm{d}x\mathrm{d}t \leqslant \frac{1}{2} c(n, \kappa D^2)\mathrm{vol}(B) \int_C f. \qquad \square$$

Proof of Theorem 29.7. For every pair $x, y \in B_r(x)$, choose a minimizing geodesic $\gamma_{x,y} \colon [0, 1] \to B_{2r}(x)$ with $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(0) = y$. By the fundamental theorem of calculus and Theorem 29.9,

$$\begin{aligned}
\int_{B_r(x)} |f(y) - \bar{f}_{x,r}| \,\mathrm{d}y &= \int_{B_r(x)} \left| f(y) - \frac{1}{\mathrm{vol}(B_r(x))} \int_{B_r(x)} f(z)\mathrm{d}z \right| \mathrm{d}y \\
&= \frac{1}{\mathrm{vol}(B_r(x))} \int_{B_r(x)} \left| \int_{B_r(x)} f(y) - f(z)\mathrm{d}z \right| \mathrm{d}y \\
&\leqslant \frac{1}{\mathrm{vol}(B_r(x))} \int_{B_r(x)} \int_{B_r(x)} |f(y) - f(z)| \,\mathrm{d}z\mathrm{d}y \\
&\leqslant \frac{1}{\mathrm{vol}(B_r(x))} \int_{B_r(x)} \int_{B_r(x)} \int_0^1 d(y, z)|\nabla f|(\gamma_{y,z}(t)) \,\mathrm{d}z\mathrm{d}y \\
&\leqslant \frac{2r}{\mathrm{vol}(B_r(x))} \int_{B_r(x)} \int_{B_r(x)} \int_0^1 |\nabla f|(\gamma_{y,z}(t)) \,\mathrm{d}t\mathrm{d}z\mathrm{d}y \\
&\leqslant c(n, \kappa D^2) r \int_{B_{2r}(x)} |\nabla f|. \qquad \square
\end{aligned}$$

Theorem 29.11 (weak type Neumann–Poincaré–Sobolev inequality). *Let $\kappa \leqslant 0$ and $D > 0$. Let (M, g) be a complete Riemannian manifold of dimension n with*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g.$$

For every $f \in \mathrm{Lip}(M)$, $x \in M$, $0 < r \leqslant D$, and $t > 0$,

$$t^{\frac{n}{n-1}} \mathrm{vol}\left(\left\{|f - \bar{f}_{x,r}| > t\right\}\right) \leqslant \frac{c(n, \kappa D^2)}{\mathrm{vol}(B_r(x))^{\frac{1}{n-1}}} \|\nabla f\|_{L^1(B_r(x))}^{\frac{n}{n-1}}.$$

The proof requires a basic result about the maximal function.

Definition 29.12. Let $D > 0$. Let (M, g) be a Riemannian manifold and let U be a bounded subset. Given $f \in L^1_{\mathrm{loc}}(M)$, the **Hardy–Littlewood maximal function** associated with f and D is the function $Mf = M_D f \colon M \to [0, \infty)$ is defined by

$$Mf(x) := \sup_{0<r\leqslant D} \frac{1}{\mathrm{vol}(B_r(x))} \int_{B_r(x)} |f|.$$

Theorem 29.13 (weak type estimate for the maximal function). *Let $\kappa \leqslant 0$ and $D > 0$. Let (M, g) be a complete Riemannian manifold of dimension n with*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g.$$

For every $f \in L^1_{\mathrm{loc}}(M)$,

$$t\mathrm{vol}\left(\{Mf > t\}\right) \leqslant c(n, \kappa D^2) \int_M |f|.$$

Proof. For every $x \in \{Mf > t\}$, choose $0 < r_x \leqslant D$ such that

$$t\mathrm{vol}(B_{r_x}(x)) < \int_{B_{r_x}(x)} |f|.$$

Choose $\{x_a : a \in \Lambda\}$ such that

$$\bigcup_{a\in\Lambda} B_{5r_{x_a}}(x_a) \supset \{Mf > t\} \quad \text{and, for } a \neq b, \quad B_{r_{x_a}}(x_a) \cap B_{r_{x_b}}(x_b) = \varnothing.$$

Exercise 29.14 (Vitali covering lemma). Prove that this can actually be done.

By Theorem 20.1,

$$\begin{aligned} t\mathrm{vol}\left(\{Mf > t\}\right) &\leqslant t \sum_{a\in\Lambda} \mathrm{vol}(B_{5r_{x_a}}(x_a)) \\ &\leqslant c(n, \kappa D^2) \sum_{a\in\Lambda} t\mathrm{vol}(B_{r_{x_a}}(x_a)) \\ &\leqslant c(n, \kappa D^2) \sum_{a\in\Lambda} \int_{B_{r_{x_a}}(x)} |f| \\ &\leqslant c(n, \kappa D^2) \int_M |f|. \end{aligned}$$

□

Proposition 29.15. *Let $\kappa \leqslant 0$ and $D > 0$. If (M, g) is a complete Riemannian manifold of dimension n with*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g,$$

then, for $x \in M$, $y \in B_r(x)$, and $0 < s \leqslant r \leqslant D$,

$$\mathrm{vol}(B_s(y)) \geqslant c(n, \kappa D^2)\mathrm{vol}(B_r(x))s^n.$$

Proof. By Theorem 20.1,

$$\begin{aligned}\frac{\mathrm{vol}(B_r(x))}{\mathrm{vol}(B_s(y))} &\leqslant \frac{\mathrm{vol}(B_{2r}(y))}{\mathrm{vol}(B_s(y))}\\ &\leqslant \frac{V_n^\kappa(2r)}{V_n^\kappa(s)}\\ &\leqslant \frac{c(n,\kappa r^2)}{s^n}.\end{aligned}$$

□

Proof of Theorem 29.11. For $a \in \mathbf{Z}$, set

$$R_a := 2^{-a-1}r.$$

For every $y \in B_r(x)$, set $y_0 := x$, and, for $a \in \mathbf{N}_0$, inductively define y_{a+1} to be the mid-point of a minimal geodesic from y_a to y. By construction:

1. for every $a \in \mathbf{N}_0$, $d(y_a, y) \leqslant 2R_a$,
2. for every $a \in \mathbf{N}_0$, $B_{R_{a-1}}(y_a) \subset B_r(x)$, and
3. for every $a \in \mathbf{N}_0$, there is a point $z_a \in M$ with $B_{R_{a+2}}(z_a) \subset B_{R_a}(y_a) \cap B_{R_{a+1}}(y_{a+1})$.

Since f is continuous,

$$f(y) = \lim_{a\to\infty} \bar{f}_{y_a,R_a}.$$

Therefore, by Theorem 29.7,

$$\begin{aligned}|f(y) - \bar{f}_{x,R_0}| &\leqslant \sum_{a=0}^{\infty} \left|\bar{f}_{y_{a+1},R_{a+1}} - \bar{f}_{y_a,R_a}\right|\\ &\leqslant \sum_{a=0}^{\infty} \left|\bar{f}_{y_{a+1},R_{a+1}} - \bar{f}_{z_a,R_{a+2}}\right| + \left|\bar{f}_{y_a,R_a} - \bar{f}_{z_a,R_{a+2}}\right|\\ &\leqslant c(n,\kappa D^2) \sum_{a=0}^{\infty} \fint_{B_{R_a}(y_a)} \left|f - \bar{f}_{y_a,R_a}\right|\\ &\leqslant c(n,\kappa D^2) \sum_{a=0}^{\infty} R_a \fint_{B_{R_{a-1}}(y_a)} |\nabla f|.\end{aligned}$$

Given $\rho > 0$, for $T \in \mathbf{N}$ such that $\frac{1}{2}\rho \leqslant 2^{-T}R_0 \leqslant \rho$,

$$\begin{aligned}\sum_{a=T+1}^{\infty} R_a \fint_{B_{R_{a-1}}(y_a)} |\nabla f| &\leqslant c(n,\kappa D^2) \sum_{a=T+1}^{\infty} R_a \fint_{B_{R_{3a-1}}(y)} |\nabla f|\\ &\leqslant c(n,\kappa D^2) \sum_{a=T+1}^{\infty} R_a M|\nabla f|(y)\\ &\leqslant c(n,\kappa D^2) 2^{-T} RM|\nabla f|(y)\\ &\leqslant c(n,\kappa D^2)\rho M|\nabla f|(y)\end{aligned}$$

and, by Proposition 29.15,

$$\begin{aligned}
\sum_{a=0}^{T} R_a \fint_{B_{R_{a-1}}(y_a)} |\nabla f| &\leqslant c(n,\kappa D^2) \sum_{a=0}^{T} \frac{R_a}{R_{a-1}^n} \fint_{B_r(x)} |\nabla f| \\
&= c(n,\kappa D^2) R_0^{1-n} \sum_{a=0}^{T} 2^{n(a-1)-a} \fint_{B_r(x)} |\nabla f| \\
&\leqslant c(n,\kappa D^2) R_0^{1-n} \sum_{a=0}^{T} 2^{(n-1)(a-1)} \fint_{B_r(x)} |\nabla f| \\
&\leqslant c(n,\kappa D^2) \left(2^{-T} R_0\right)^{1-n} \fint_{B_r(x)} |\nabla f| \\
&\leqslant c(n,\kappa D^2) \rho^{1-n} \fint_{B_r(x)} |\nabla f|.
\end{aligned}$$

Therefore,

$$|f(y) - f_{x,r}| \leqslant c(n,\kappa D^2) \left(\rho M|\nabla f|(y) + \rho^{1-n} \fint_{B_r(x)} |\nabla f| \right).$$

Choosing

$$\rho = \left(\frac{\fint_{B_r(x)} |\nabla f|}{M|\nabla f|(y)} \right)^{1/n},$$

yields

$$|f(y) - f_{x,r}| \leqslant c(n,\kappa D^2) \left(\fint_{B_r(x)} |\nabla f| \right)^{\frac{1}{n}} M|\nabla f|(y)^{\frac{n-1}{n}}.$$

By Theorem 29.13,

$$\begin{aligned}
\operatorname{vol}\left(\{|f - f_{x,r}| > t\}\right) &\leqslant \operatorname{vol}\left(\{|f - f_{x,R_0}| > t\}\right) \\
&= \operatorname{vol}\left(\left\{|f - f_{x,R_0}|^{\frac{n}{n-1}} > t^{\frac{n}{n-1}}\right\}\right) \\
&\leqslant \operatorname{vol}\left(\left\{c(n,\kappa D^2) \left(\fint_{B_r(x)} |\nabla f| \right)^{\frac{1}{n-1}} M|\nabla f| > t^{\frac{n}{n-1}}\right\}\right) \\
&\leqslant t^{-\frac{n}{n-1}} \frac{c(n,\kappa D^2)}{\operatorname{vol}(B_r(x))^{\frac{1}{n-1}}} \|\nabla f\|_{L^1(B_r(x))}^{\frac{n}{n-1}}.
\end{aligned}$$

□

Theorem 29.16 (Hajłasz and Koskela [HKoo, Theorem 2.1]). *Let (M, g) be a complete Riemannian manifold with finite volume. If there is a constant $c > 0$ such that, for every $f \in \mathrm{Lip}(M)$ and $t > 0$,*

$$t^{\frac{n}{n-1}} \mathrm{vol}\left(\left\{|f - \bar{f}| > t\right\}\right) \leqslant c_S^* \|\nabla f\|_{L^1}^{\frac{n}{n-1}}, \tag{29.17}$$

then, for every $f \in \mathrm{Lip}(M)$,

$$\|f - \bar{f}\|_{L^{\frac{n}{n-1}}} \leqslant c_S \|\nabla f\|_{L^1} \tag{29.18}$$

with $c_S := 16(c_S^*)^{\frac{n-1}{n}}$.

Proof of Theorem 29.16. The following shows that it suffices to consider a restricted class of functions f.

Proposition 29.19. *If (29.18) holds for every $f \in \mathrm{Lip}(M)$ with*

$$f \geqslant 0 \quad \textit{and} \quad \mathrm{vol}(\{f = 0\}) \geqslant \mathrm{vol}(M)/2, \tag{29.20}$$

then it holds for every $f \in \mathrm{Lip}(M)$.

Proof. To see this, let $s \in \mathbf{R}$ such that

$$\mathrm{vol}(\{f \geqslant s\}) \geqslant \mathrm{vol}(M)/2 \quad \text{and} \quad \mathrm{vol}(\{f \leqslant s\}) \geqslant \mathrm{vol}(M)/2.$$

Both sides of (29.18) are unaffected by shifting f by a constant. Hence, there is no loss in assuming that $s = 0$. Split f into its positive and negative part:

$$f = f^+ - f^- \quad \text{with} \quad f^+ := \max\{f, 0\} \quad \text{and} \quad f^- := \max\{-f, 0\}.$$

By hypothesis, (29.18) hold for f^+ and f^-. Therefore,

$$\begin{aligned} \|f - \bar{f}\|_{L^{\frac{n}{n-1}}} &\leqslant \left\|f^+ - \overline{f^+}\right\|_{L^{\frac{n}{n-1}}} + \left\|f^- - \overline{f^-}\right\|_{L^{\frac{n}{n-1}}} \\ &\leqslant C\|\nabla f^+\|_{L^1} + C\|\nabla f^-\|_{L^1} \\ &= C\|\nabla f\|_{L^1}. \end{aligned}$$

□

Henceforth, let $f \in \mathrm{Lip}(M)$ and suppose that (29.20) holds. Given $0 \leqslant a \leqslant b$, set

$$f_a^b := \max\{\min\{f, b\}, a\} - a.$$

The proof of (29.18) proceeds by applying (29.17) to the shifted truncations f_a^b. These satisfy

$$\nabla f_a^b = \chi_{\{a<f<b\}} \nabla f \tag{29.21}$$

and, for every $c \in \mathbf{R}$,

$$\mathrm{vol}(\{f_a^b > t\}) \leqslant 2\,\mathrm{vol}(\{|f_a^b - c| > t/2\}). \tag{29.22}$$

The latter is trivial if $c \leqslant t/2$. It holds for $c \geqslant t/2$, because then

$$\{|f_a^b - c| > t/2\} \supset \{f_a^b = 0\} \quad \text{and} \quad \mathrm{vol}(\{f_a^b = 0\}) \geqslant \mathrm{vol}(M)/2.$$

By (29.22), (29.21), and (29.17), for every $0 \leqslant a \leqslant b$, $t > 0$, and $c \in \mathbf{R}$,

$$\begin{aligned} t^{\frac{n}{n-1}} \mathrm{vol}(\{f_a^b > t\}) &\leqslant 2^{\frac{2n}{n-1}} \left(\frac{t}{2}\right)^{\frac{n}{n-1}} \mathrm{vol}\left(\left\{f_a^b - \overline{f_a^b} > t/2\right\}\right) \\ &\leqslant 2^{\frac{2n}{n-1}} c_S^* \|\chi_{\{a<f<b\}} \nabla f\|_{L^1}^{\frac{n}{n-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int f^{\frac{n}{n-1}} &\leqslant \sum_{a\in\mathbf{Z}} 2^{\frac{na}{n-1}} \mathrm{vol}(\{2^{a-1} < f \leqslant 2^a\}) \\ &\leqslant \sum_{a\in\mathbf{Z}} 2^{\frac{na}{n-1}} \mathrm{vol}(\{f > 2^{a-1}\}) \\ &\leqslant \sum_{a\in\mathbf{Z}} 2^{\frac{na}{n-1}} \mathrm{vol}(\{f_{2^{a-2}}^{2^{a-1}} > 2^{a-2}\}) \\ &\leqslant c_S^* \sum_{a\in\mathbf{Z}} 2^{\frac{na}{n-1}} 2^{-\frac{n(a-2)}{n-1}} 2^{\frac{2n}{n-1}} \|\chi_{\{a<f<b\}} \nabla f\|_{L^1}^{\frac{n}{n-1}} \\ &\leqslant 2^{\frac{4n}{n-1}} c_S^* \|\nabla f\|_{L^1}^{\frac{n}{n-1}}. \end{aligned}$$

This implies (29.18) with $c_S \coloneqq 16(c_s^*)^{\frac{n-1}{n}}$. □

30 Moser iteration

Definition 30.1. Let (M, g) be a complete Riemannian manifold of finite volume. Let $\nu > 1$. The $(\nu, 2)$–**Sobolev constant** $c_S = c_S(g, \nu)$ of (M, g) is the smallest constant $c > 0$ such that, for every $f \in C^\infty(M)$,

$$\left(⨍_M |f|^{2\nu}\right)^{\frac{1}{2\nu}} \leqslant c\left(⨍_M |\nabla f|^2\right)^{\frac{1}{2}} + \left(⨍_M |f|^2\right)^{\frac{1}{2}}. \tag{30.2}$$

The function $c_\Delta \colon [0, \infty) \to [0, \infty]$ is defined by

$$c_\Delta(g, \lambda) \coloneqq \sup\left\{\|f\|_{L^\infty} : f \in C^\infty(M) \text{ with} f \geqslant 0, ⨍_M f^2 = 1, \text{ and } \Delta f \leqslant \lambda f\right\}.$$

Theorem 30.3 (Moser [Mos60]). *If (M, g) is a complete Riemannian manifold of finite volume and $\nu > 1$, then*

$$c_\Delta(g, \lambda) \leqslant \exp\left(\frac{c_S(g, \nu)\sqrt{\lambda\nu}}{\sqrt{\nu} - 1}\right).$$

Proof. For every $q \in \mathbf{R}$,

$$\begin{aligned}\fint_M f^{2q-1}\Delta f &= \fint_M \left\langle \nabla f^{2q-1}, \nabla f \right\rangle \\ &= (2q-1)\fint_M |\nabla f|^2 f^{2q-2}.\end{aligned}$$

Therefore, for every $q > 1/2$,

$$\begin{aligned}\fint_M |\nabla f^q|^2 &= q^2 \fint_M |\nabla f|^2 f^{2q-2} \\ &= \frac{q^2}{2q-1}\fint_M f^{2q-1}\Delta f \\ &\leqslant \frac{q^2\lambda}{2q-1}\fint_M f^{2q}.\end{aligned}$$

By the Sobolev inequality (30.2),

$$\begin{aligned}\left(\fint_M f^{2q\nu}\right)^{\frac{1}{2\nu}} &\leqslant c_S\left(\fint_M |\nabla f^q|^2\right)^{\frac{1}{2}} + \left(\fint_M f^{2q}\right)^{\frac{1}{2}} \\ &\leqslant \left[c_S\left(\frac{q^2\lambda}{2q-1}\right)^{1/2} + 1\right]\left(\fint_M f^{2q}\right)^{\frac{1}{2}}.\end{aligned}$$

With $q = \nu^k$, this becomes

$$\left(\fint_M f^{2\nu^{k+1}}\right)^{\frac{1}{2\nu^{\kappa+1}}} \leqslant \left[c_S\nu^k\left(\frac{\lambda}{2\nu^k-1}\right)^{1/2} + 1\right]^{\frac{1}{\nu^k}}\left(\fint_M f^{2\nu^k}\right)^{\frac{1}{2\nu^k}}.$$

Therefore,

$$\begin{aligned}\|f\|_{L^\infty} &= \lim_{k\to\infty}\left(\fint_M f^{2\nu^{k+1}}\right)^{\frac{1}{2\nu^{k+1}}} \\ &\leqslant \prod_{k=0}^{\infty}\left[c_S\nu^k\left(\frac{\lambda}{2\nu^k-1}\right)^{1/2} + 1\right]^{\frac{1}{\nu^k}}\left(\fint_M f^2\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\log(1+x) \leqslant x$, the logarithm of the infinite product is bounded as follows:

$$\begin{aligned}\sum_{k=0}^{\infty}\nu^{-k}\log\left[1 + c_S\nu^k\left(\frac{\lambda}{2\nu^k-1}\right)^{1/2}\right] &\leqslant \sum_{k=0}^{\infty} c_S\left(\frac{\lambda}{2\nu^k-1}\right)^{1/2} \\ &\leqslant c_S\sqrt{\lambda}\sum_{k=0}^{\infty}\frac{1}{\nu^{k/2}} \\ &\leqslant \frac{c_S\sqrt{\lambda}}{1-1/\sqrt{\nu}}.\end{aligned}$$

□

31 Betti number bounds

Theorem 31.1 (Gromov [Gro07, Theorem 5.21]; Li [Li80, Theorem 12], Gallot [Gal83, Théorème 3.1]). *Let $\kappa \leqslant 0$ and $D \geqslant 0$. If (M, g) is a closed Riemannian manifold of dimension $n \geqslant 3$ with*

$$\mathrm{Ric}_g \geqslant (n-1)\kappa g \quad \textit{and} \quad \operatorname{diam} M \leqslant D,$$

then

$$b_1(M) \leqslant c(n, \kappa D^2).$$

Moreover, there is a constant $\varepsilon(n) > 0$, such that if $-\kappa D^2 \leqslant \varepsilon(n)$, then $b_1(M) \leqslant n$.

Remark 31.2. This result complements Theorem 15.1. This is an **estimating theorem**. **As far as I know**, the idea for proving such results originates with Li [Li80]. For an survey of such results see Bérard [Bér88].

The following propositions prepare the proof of Theorem 31.1.

Lemma 31.3 (Li [Li80, Lemma 11]). *Let (M, g) be a closed Riemannian manifold and let E be a Euclidean vector bundle over M. Let $V \subset \Gamma(E)$ be a linear subspace. With*

$$c(V) \coloneqq \sup\left\{\|s\|_{L^\infty}^2 : s \in V \textit{ with } \fint_M |s|^2 = 1\right\}$$

the following holds

$$\dim V \leqslant \operatorname{rk} E \cdot c(V).$$

Remark 31.4. The constant $c(V)$ is the best constant c such that, for every $s \in V$,

$$\|s\|_{L^\infty}^2 \leqslant c \fint_M |s|^2.$$

Proof of Lemma 31.3. Without loss of generality, V is finite dimensional and $\mathrm{vol}(M) = 1$. Set $m \coloneqq \dim V$ and $r \coloneqq \operatorname{rk} E$. Let $s_1, \dots, s_m$ be a L^2–orthonormal basis of V. Set

$$f(x) \coloneqq \sum_{a=1}^{m} |s_a(x)|^2.$$

This function only depends on V and not on the choice of $s_1, \dots, s_m$. By construction,

$$m = \fint_M f \leqslant \max_{x \in M} f(x).$$

Let $x_\star \in M$ be a point at which f achieves its maximum. Without loss of generality, the rank of the evaluation map $\mathrm{ev}\colon V \to E_{x_\star}$ is at most r. Thus its kernel has dimension at least $m - r$, and we can assume that $s_{r+1}, \dots, s_m \in \ker \mathrm{ev}$. Therefore,

$$m \leqslant \sum_{a=1}^{r} |s_a(x_\star)|^2 \leqslant r \cdot c(V). \qquad \square$$

Theorem 31.5 (Estimating theorem). *Let (M, g) be a closed Riemannian manifold. Let E be an Euclidean vector bundle, let ∇ be an orthogonal connection on E, and let $\mathfrak{R} \in \mathrm{End}(E)$. With*

$$\Lambda \coloneqq \sup_{x\in M} \sup \operatorname{spec}(-\mathfrak{R}_x)$$

the following holds

$$\dim \ker(\nabla^*\nabla + \mathfrak{R}) \leqslant \operatorname{rk} E \cdot c_\Delta(g, \Lambda).$$

Proof. Set $V \coloneqq \ker(\nabla^*\nabla + \mathfrak{R})$. For $s \in V$,

$$\begin{aligned}\Delta|s|^2 &= 2\langle \nabla^*\nabla s, s\rangle - 2|\nabla s|^2 \\ &= -2\langle \mathfrak{R}s, s\rangle - 2|\nabla s|^2.\end{aligned}$$

Since

$$\Delta|s|^2 = 2(\Delta|s|)|s| - 2|\nabla|s||^2,$$

by Kato's inequality,

$$\begin{aligned}\Delta|s| &\leqslant -\frac{\langle \mathfrak{R}s, s\rangle}{|s|} \\ &\leqslant \Lambda|s|.\end{aligned}$$

The assertion thus follows from Lemma 31.3. □

Proof of Theorem 31.1. By Proposition 15.2, Theorem 31.5 applies with $\Lambda \leqslant -(n-1)\kappa$. Estimating $c_\Delta(g, \Lambda)$ using Theorem 30.3 and Theorem 29.4,

$$b_1(M) \leqslant n \exp\big(c(n, \kappa D^2)\sqrt{-\kappa} D\big).$$

This proves the first part of Theorem 31.1. To prove the second part, observe that if $-\kappa D^2 \ll_n 1$, then

$$\big\lfloor \exp\big(c(n, \kappa D^2)\sqrt{-\kappa} D\big)\big\rfloor = 1.$$

□

Exercise 31.6. What is the generalization of Theorem 31.1 to b_k? What replaces the Ricci lower bound?

32 Metric spaces

Definition 32.1. A metric space (X, d) is called **separable** if it contains a countable, dense subset.

Definition 32.2. A metric space (X, d) is called **totally bounded** if for every $\varepsilon > 0$ there is a finite collection of balls of radius ε covering X.

Theorem 32.3. *A metric space (X, d) is compact if and only if it is complete and totally bounded.*

Exercise 32.4. Prove Theorem 32.3.

Theorem 32.5. *Every totally bounded metric space is separable.*

Exercise 32.6. Prove Theorem 32.5.

Theorem 32.7 (Kuratowski). *Let (X, d) be a metric space. For every $x_\star \in X$, the $\delta\colon X \to L^\infty(X)$ defined by*

$$\delta(x) \coloneqq d(x,\cdot) - d(x_\star,\cdot)$$

is an isometric embedding.

Proof. This immediately follows from the fact that, for $x, y \in X$,

$$\|\delta(x) - \delta(y)\|_{L^\infty} = \sup_{z\in X} d(x,z) - d(z,y) = d(x,y). \qquad \square$$

Theorem 32.8. *Every separable metric space (X, d) admits an isometric embedding in $\ell^\infty(\mathbf{N})$.*

Proof. By Theorem 32.7, every countable subset of X admits an isometric embedding into $\ell^\infty(\mathbf{N})$. This extends extends to an isometric embedding of X if the subset is dense. $\square$

33 Hausdorff distance

Definition 33.1. Let X be a set. The **power set** of X is the set of all subsets of X and denoted by $\mathfrak{P}(X)$.

Definition 33.2. Let (X, d) be a metric space. The **Hausdorff distance** is the map $d_H = d_H^X\colon \mathfrak{P}(X)\times\mathfrak{P}(X) \to [0,\infty]$ defined by

$$d_H(A,B) \coloneqq \inf\{\varepsilon > 0 : A \subset B_\varepsilon(B) \text{ and } B \subset B_\varepsilon(A)\}.$$

Proposition 33.3. *Let (X, d) be a metric space. Denote by $\mathfrak{C}(X) \subset \mathfrak{P}(X)$ the set of all closed subsets of X. Then $(\mathfrak{C}(X), d_H)$ is a metric space.*

Exercise 33.4. Prove Proposition 33.3.

Theorem 33.5. *If (X, d) is a complete metric space, then $(\mathfrak{C}(X), d_H)$ is complete.*

Proof. Let $(A_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in $(\mathfrak{C}(X), d_H)$. Set

$$A_\infty \coloneqq \bigcap_{n=1}^\infty \overline{\bigcup_{m=n}^\infty A_m}.$$

Let $\varepsilon > 0$. Let $N \in \mathbf{N}$ such that, for $n, m \geqslant N$, $d_H(A_n, A_m) \leqslant \varepsilon$. By definition, for every $x \in A_\infty$, there is an $m \geqslant N$ such that $x \in B_\varepsilon(A_m)$. Therefore and since $x \in A_\infty$ was arbitrary, for every $n \geqslant N$, $A_\infty \subset B_{2\varepsilon}(A_n)$

Let $x \in A_n$ with $n \geqslant N$. Choose $(n_k)_{k\in\mathbf{N}}$ with $N_1 = n$ such that, for every $k \in \mathbf{N}$, $d_H(A_{n_k}, A_{n_{k+1}}) < \varepsilon/2^k$. Furthermore, choose $(x_k)_{k\in\mathbf{N}}$ with $x_1 = x$, $x_k \in A_{n_k}$, and such that, for every $k \in \mathbf{N}$, $d(x_k, x_{k+1}) < \varepsilon/2^k$. Since X is complete, the Cauchy sequence $(x_k)_{k\in\mathbf{N}}$ converges to a limit x_∞. By definition, $x_\infty \in A_\infty$. Since

$$\begin{aligned} d(x, x_\infty) &< \sum_{k=1}^{\infty} \varepsilon/2^k \\ &< \varepsilon \end{aligned}$$

and $x \in A_n$ was arbitrary, for every $n \geqslant N$, $A_n \subset B_{2\varepsilon}(A_\infty)$. □

Theorem 33.6. *If (X, d) is a compact metric space, then $(\mathfrak{C}(X), d_H)$ is compact.*

Exercise 33.7. Prove Theorem 33.6.

34 The Gromov–Hausdorff distance

The theory of Gromov–Hausdorff distance was invented by Edwards [Edw75] and then invented again by Gromov [Gro81b].

Definition 34.1 ([Gro81b, Section 6; Gro07, Chapter 3.A]). Let (X, d_X) and (Y, d_Y) be two metric spaces. The **Gromov–Hausdorff distance** between (X, d_X) and (Y, d_X) is denoted by

$$d_{GH}(X, Y)$$

and defined as the infimum of the numbers

$$d_H^Z(i(X), j(Y))$$

for all metric spaces (Z, d_Z) and all isometric embeddings $i\colon X \to Z$ and $j\colon Y \to Z$.

Proposition 34.2. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $\varepsilon > 0$. If $d_{GH}(X, Y) < \varepsilon$, then there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that:*

1. *For every $x_1, x_2 \in X$, $y_1, y_2 \in Y$:*

$$\begin{aligned} |d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| &\leqslant 2\varepsilon \quad \textit{and} \\ |d_Y(y_1, y_2) - d_X(g(y_1), g(y_2))| &\leqslant 2\varepsilon. \end{aligned} \tag{34.3}$$

2. *For every $\varepsilon > 0$, $x \in X$, $y \in Y$:*

$$\begin{aligned} d_X(x, g(f(x))) &\leqslant 2\varepsilon \quad \textit{and} \\ d_Y(y, f(g(y))) &\leqslant 2\varepsilon. \end{aligned} \tag{34.4}$$

Proof. Let (Z, d_Z) be a metric space and let $i\colon X \to Z$ and $j\colon Y \to Z$ be isometric embeddings such that
$$d_H^Z(i(X), j(Y)) \leqslant \varepsilon.$$
By definition, there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that, for every $x \in X$ and $y \in Y$,
$$d_Z(x, f(x)) \leqslant \varepsilon \quad \text{and} \quad d_Z(y, g(y)) \leqslant \varepsilon.$$
By the triangle, (34.3) and (34.4) hold. □

Proposition 34.5. *If (X, d_X) and (Y, d_Y) are separable, then $d_{GH}(X, Y)$ is equal to the infimum of the numbers*
$$d_H^{\ell^\infty(\mathbf{N})}(i(X), j(Y))$$
for all isometric embeddings $i\colon X \to \ell^\infty(\mathbf{N})$ and $j\colon Y \to \ell^\infty(\mathbf{N})$.

Proof. This is a consequence of Theorem 32.8. □

Proposition 34.6. *The Gromov–Hausdorff distance satisfies the triangle inequality: If X, Y, Z are metric spaces, then*
$$d_{GH}(X, Z) \leqslant d_{GH}(X, Y) + d_{GH}(Y, Z).$$

Exercise 34.7. Prove Proposition 34.6.

Proposition 34.8. *Two compact metric space (X, d_X) and (Y, d_Y) are isometric if and only $d_{GH}(X, Y) = 0$.*

Proof. If (X, d_X) and (Y, d_Y), then trivially $d_{GH}(X, Y) = 0$.

If $d_{GH}(X, Y) = 0$, then, for every $\varepsilon > 0$, there are $f_\varepsilon\colon X \to Y$ and $g_\varepsilon\colon Y \to X$ as in Proposition 34.2. Let $\Gamma \subset X$ and $\Delta \subset Y$ be countable, dense subsets. Since X and Y are compact and by a diagonal sequence argument, there is a null-sequence $(\varepsilon_n)_{n\in\mathbf{N}}$ such that, for every $x \in \Gamma$ and $y \in \Delta$, the limits

$$f(x) \coloneqq \lim_{n\to\infty} f_{\varepsilon_n}(x) \quad \text{and} \quad g(y) \coloneqq \lim_{n\to\infty} g_{\varepsilon_n}(y)$$

exist. By (34.3), the maps $f\colon \Gamma \to Y$ and $g\colon \Delta \to X$ are isometric embeddings, and extend to isometric embeddings $f\colon X \to Y$ and $g\colon Y \to X$. By (34.4), f and g are mutual inverses. Therefore, X and Y are isometric. □

Definition 34.9. Denote by $\mathfrak{M}$ the set of isometry classes of non-empty, compact metric spaces. The metric space $(\mathfrak{M}, d_{GH})$ is called **Gromov–Hausdorff space**.

Theorem 34.10 (Edwards [Edw75, Theorems III.3 and III.7]). *$(\mathfrak{M}, d_{GH})$ is separable and complete.*

Proof. The subset of finite metric spaces (X, d) with d taking only rational values is dense in $(\mathfrak{M}, d_{GH})$. Therefore, $(\mathfrak{M}, d_{GH})$ is separable.

Let $(X_n, d_{X_n})_{n\in\mathbf{N}}$ be a Cauchy sequence in $(\mathfrak{M}, d_{GH})$. By Proposition 34.5, there are isometric embeddings $i_n\colon X_n \to \ell^\infty(\mathbf{N})$ such that $A_n \coloneqq i_n(X_n)$ is a Cauchy sequence in $\mathfrak{C}(\ell^\infty(\mathbf{N}))$. By Theorem 33.5, $(A_n)_{n\in\mathbf{N}}$ converges to a limit A_∞ in $\mathfrak{C}(\ell^\infty(\mathbf{N}))$. A_∞ is complete and totally bounded; hence, compact. By Proposition 34.5, X_n converges to $X_\infty \coloneqq A_\infty$ in $(\mathfrak{M}, d_{GH})$. □

Exercise 34.11. Prove that $(\mathfrak{M}, d_{GH})$ is contractible, has infinite diameter, is a length space, and is not locally compact.

Exercise 34.12. Prove that Gromov–Hausdorff limits of length spaces are length spaces.

Definition 34.13. Let $\mathfrak{X}$ be a set of isometry classes of metric spaces. $\mathfrak{X}$ is **uniformly bounded** if $\sup\{\operatorname{diam}(X, d) : [X, d] \in \mathfrak{X}\} < \infty$. $\mathfrak{X}$ is **uniformly totally bounded** if, for every $\varepsilon > 0$, there is an $n \in \mathbf{N}$ such that every $[X, d] \in \mathfrak{X}$ can be covered by at most n balls of radius ε.

Definition 34.14. Let (X, d) be a metric space and $A \subset X$. A is **relatively compact** if $\bar{A}$ is compact.

Theorem 34.15 (Gromov's compactness criterion [Gro81b, p.64]). *A subset $\mathfrak{X} \subset \mathfrak{M}$ is relatively compact if and only if it is uniformly bounded and uniformly totally bounded.*

Proof. By Theorem 34.10, it suffices to show that $\mathfrak{X}$ is totally bounded. Since $\mathfrak{X}$ is uniformly bounded,

$$D \coloneqq \sup\{\operatorname{diam}(X, d) : [X, d] \in \mathfrak{X}\} < \infty$$

Let $\varepsilon > 0$. Since $\mathfrak{X}$ is uniformly totally bounded, there is an $n \in \mathbf{N}$ such that every $[X, d] \in \mathfrak{X}$ contains n points $x_1, \ldots, x_n$ such that

$$X \subset B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_n).$$

Therefore, (X,d) and $(\{x_1,\dots,x_n\},d)$ have Gromov–Hausdorff distance at most ε. There is a metric $\tilde d$ on $\{1,\dots,n\}$ taking values in $\{0,\varepsilon,2\varepsilon,\dots,\lceil D/\varepsilon\rceil\varepsilon\}$ and such that, for $a,b=1,\dots,n$,

$$|d(x_a,x_b)-\tilde d(a,b)|\leqslant\varepsilon.$$

By construction, (X,d) and $(\{1,\dots,n\},\tilde d)$ have Gromov–Hausdorff distance at most 2ε. Since there are finitely many metrics $\tilde d$ on $\{1,\dots,n\}$ as above and $[X,d]$ was arbitrary, it follows that $\mathfrak{X}$ is totally bounded. □

Definition 34.16. Let (X,d) be a metric space and $r>0$. The **r–covering number** of (X,d) is the minimum number of balls of radius r required to cover X and denoted by

$$\mathrm{cov}(X,d;r).$$

Definition 34.17. Let (X,d) be a metric space and $r>0$. The **upper Minkowski dimension** of (X,d) is

$$\overline{\dim}_M(X,d)\coloneqq\limsup_{r\to 0}\frac{\log\mathrm{cov}(X,d;r)}{\log(1/r)}$$

and the **lower Minkowski dimension** of (X,d) is

$$\underline{\dim}_M(X,d)\coloneqq\liminf_{r\to 0}\frac{\log\mathrm{cov}(X,d;r)}{\log(1/r)}.$$

In case both agree, we say that (X,d) has **Minkowski dimension** $\dim_M(X,d)=\overline{\dim}_M(X,d)$.

Theorem 34.18. *A subset $\mathfrak{X}\subset\mathfrak{M}$ is relatively compact if and only if there exists a function $c\colon(0,\infty)\to\mathbf{N}$ such that, for every $[X,d]\in\mathfrak{X}$ and $r>0$,*

$$\mathrm{cov}(X,d;r)\leqslant c(r).$$

Theorem 34.19 ([Gro07, Theorem 5.3]). *Let $\kappa\in\mathbf{R}$, $D>0$, and $n\in\mathbf{N}$. There is a constant $c=c(\kappa D^2,n)>0$ such that if (M,g) is a complete Riemannian manifold of dimension n with*

$$\mathrm{diam}(M,g)\leqslant D\quad\textit{and}\quad\mathrm{Ric}_g\geqslant(n-1)\kappa g,$$

then, for every $0<r\leqslant D$,

$$\mathrm{cov}(M,g;r)\leqslant c(\kappa,D,n)r^{-n}.$$

Proof. Let $x_1,\dots,x_m\in M$ such that

$$\bigcup_{a=1}^m B_r(x_a)=M$$

and, for $a\neq b$,

$$B_{r/2}(x_a)\cap B_{r/2}(x_b)=\varnothing.$$

Denote by $x_\star$ the center of the ball $B_{r/2}(x_a)$ with smallest volume. By construction and by Theorem 20.1,

$$\begin{aligned} m &\leqslant \frac{\mathrm{vol}(M)}{\mathrm{vol}(B_r(x_\star))} \\ &\leqslant \frac{\mathrm{vol}(B_D(x_\star))}{\mathrm{vol}(B_r(x_\star))} \\ &\leqslant \frac{V_\kappa^n(D)}{V_\kappa^n(r)}. \end{aligned}$$ □

Corollary 34.20. *Let $\kappa \in \mathbf{R}$, $D > 0$, and $n > 0$. The subset of isometry classes of complete Riemannian manifolds (M, g) of dimension n with*

$$\mathrm{diam}(M) \leqslant D \quad \textit{and} \quad \mathrm{Ric}_g \geqslant (n-1)\kappa g$$

is relatively compact in $(\mathfrak{M}, d_{GH})$.

Exercise 34.21. Prove that the Minkowski dimension is lower semi-continuous on the closure of the above subset.

Proposition 34.22 (Boileau). *Let (M, g) be a complete Riemannian manifold of dimension n. For every $x \in M$, and $0 < r \leqslant \mathrm{inj}(M, g)$,*

$$\mathrm{vol}(B_r(x)) \geqslant c(n) r^n.$$

Proposition 34.23. *Let (M, g) be a closed Riemannian manifold of dimension n and let $\delta, V > 0$. If $\mathrm{inj}(M, g) \geqslant \delta > 0$ and $\mathrm{vol}(M, g) \leqslant V < \infty$, then, for every $0 < r \leqslant \delta$,*

$$\mathrm{cov}(M, g; r) \leqslant c(n, \delta, V) r^{-n}.$$

Proof. Let $0 < r \leqslant \delta$. If $x_1, \ldots, x_m$ is a maximal set of points such that, for $a \neq b$,

$$B_{r/2}(x_a) \cap B_{r/2}(x_b) = \varnothing,$$

then

$$m c(n) r^n \leqslant V$$

and

$$\bigcup_{a=1}^m B_r(x_a) = M.$$

Therefore,

$$\mathrm{cov}(M, g; r) \leqslant c(n) V r^{-n}.$$ □

Corollary 34.24. *Let $V, \delta > 0$, and $n > 0$. The subset of isometry classes of compact Riemannian manifolds (M, g) of dimension n with*

$$\operatorname{inj}(M, g) \leqslant \delta \quad \textit{and} \quad \operatorname{vol}(M, g) \leqslant V$$

is relatively compact in $(\mathfrak{M}, d_{GH})$.

Exercise 34.25. Prove that the Minkowski dimension is lower semi-continuous on the closure of the above subset.

Exercise 34.26 (*). Prove that the Minkowski dimension is continuous on the closure of the above subset.

35 Pointed Gromov–Hausdorff convergence

Definition 35.1. A **pointed metric space** is a metric space (X, d) together with a point $x \in X$.

Definition 35.2. Let (X, x, d_X) and (Y, y, d_Y) be two pointed metric spaces. The **uniform pointed Gromov–Hausdorff distance** between (X, x, d_X) and (Y, y, d_X) is denoted by

$$d_{pGH}(X, Y)$$

and defined as the infimum of the numbers

$$d_H^Z(i(X), j(Y))$$

for all pointed metric spaces (Z, z, d_Z) and all pointed isometric embeddings $i\colon X \to Z$ and $j\colon Y \to Z$.

For non-compact spaces, it is too restrictive to demand convergence with respect to d_{upGH}.

Definition 35.3. A metric space is called **proper** if every closed ball is compact.

Definition 35.4. Denote by $\mathfrak{M}_\star$ the set of isometry classes of proper, pointed metric spaces. The **pointed Gromov–Hausdorff topology** is the topology on $\mathfrak{M}_\star$ generated by the subbasis consisting of the subsets

$$\left\{[Y, y, d_Y] \in \mathfrak{M}_\star : \text{there is an } s > 0 \text{ with } |r - s| < \delta \text{ such that } d_{pGH}(B_s(x), B_r(y)) < \varepsilon\right\}$$

for $[X, x, d_X] \in \mathfrak{M}_\star$ and $r, \delta, \varepsilon > 0$.

Proposition 35.5. *Let $(X_n, x_n, d_{X_n})_{n\in\mathbf{N}}$ be a sequence of proper, pointed metric spaces and let $(X_\infty, x_\infty, d_{X_\infty})$ be a proper, pointed metric space. In the pointed Gromov–Hausdorff topology,*

$$\lim_{n\to\infty}[X_n, x_n, d_{X_n}] = [X_\infty, x_\infty, d_{X_\infty}]$$

if and only if, for every $r > 0$, there is a sequence $(r_n)_{n\in\mathbf{N}}$ converging to r such that

$$\lim_{n\to\infty} d_{pGH}(B_{r_n}(x_n), B_r(x_\infty)) = 0.$$

Remark 35.6. The variation of r is needed to make sure that $((1 + 1/n)\cdot\mathbf{Z}, 0)$ converges to $(\mathbf{Z}, 0)$, etc.

Theorem 35.7 (Gromov's compactness criterion for the pointed Gromov–Hausdorff topology [Gro81b, p.64]). *A subset $\mathfrak{X} \subset \mathfrak{M}_\star$ is relatively sequentially compact if, for every $r > 0$, the set $\{B_r(x) : [X, x, d_X] \in \mathfrak{X}\}$ is uniformly totally bounded.*

36 Topologies on the space of Riemannian manifolds

Definition 36.1. Let $k \in \mathbf{N}_0$ and $\alpha \in (0, 1)$. Two pairs (M, g) and (N, h) consisting of a $C^{k+1,\alpha}$ manifold and a $C^{k,\alpha}$ Riemannian metric on this manifold are said to be equivalent if and only if exists a $C^{k+1,\alpha}$ diffeomorphism $\phi\colon M \to N$ such that $\phi_* g = h$. Denote by $\mathfrak{R}^{k,\alpha}$ the corresponding set of equivalence classes $[M, g]$.

Definition 36.2. Let $k \in \mathbf{N}_0$ and $\alpha \in (0, 1)$. The $C^{k,\alpha}$**–topology** is the coarsest topology on $\mathfrak{R}^{k,\alpha}$ in which, for every $[M, g] \in \mathfrak{R}^{k,\alpha}$ and $\varepsilon > 0$, the set

$$\Big\{[M, h] : \|h - g\|_{C^{k,\alpha}(M,g)} < \varepsilon\Big\}$$

is open.

Proposition 36.3. *A sequence $([M_\nu, g_\nu])_{\nu\in\mathbf{N}} \in \left(\mathfrak{R}^{k,\alpha}\right)^{\mathbf{N}}$ converges to $[M, g]$ in the $C^{k,\alpha}$ topology if and only if, for every $\nu \gg 1$, there exists a $C^{k+1,\alpha}$ diffeomorphism $\phi_\nu\colon M_\nu \to M$ such that*

$$\lim_{\nu\to\infty} \|(\phi_\nu)_* g_\nu - g\|_{C^{k,\alpha}(M,g)} = 0.$$

Definition 36.4. Let $k \in \mathbf{N}_0$ and $\alpha \in (0, 1)$. Two triples (M, x, g) and (N, y, h) consisting of a $C^{k+1,\alpha}$ manifold, a point in this manifolds, and a $C^{k,\alpha}$ Riemannian metric on this manifold are said to be equivalent if and only if exists a $C^{k+1,\alpha}$ diffeomorphism $\phi\colon M \to N$ such that $\phi(x) = 0$ and $\phi_* g = h$. Denote by $\mathfrak{R}^{k,\alpha}_\star$ the corresponding set of equivalence classes $[M, g]$.

Definition 36.5. Let $k \in \mathbf{N}_0$ and $\alpha \in (0,1)$. The **pointed $C^{k,\alpha}$–topology** is the coarsest topology on $\mathfrak{R}_\star^{k,\alpha}$ in which, for every $[M,x,g] \in \mathfrak{R}_\star^{k,\alpha}$, $R > 0$, and $\varepsilon > 0$, the set consisting of those $[N,y,h]$ for which there are an open neighborhood U of y and a pointed $C^{k+1,\alpha}$ diffeomorphism $\phi_\nu \colon (U,y) \to (B_R(x),x)$ with

$$\|\phi_* h - g\|_{C^{k,\alpha}(B_R(x))} < \varepsilon$$

is open.

Proposition 36.6. *A sequence $([M_\nu, x_\nu, g_\nu])_{\nu\in\mathbf{N}} \in \left(\mathfrak{R}^{k,\alpha}\right)^{\mathbf{N}}$ converges to $[M,x,g]$ in the $C^{k,\alpha}$ topology if and only if, for every $R > 0$ and $\nu \gg_R 1$, there exists open subset U_ν of $x_\nu \in M_\nu$ and a pointed $C^{k+1,\alpha}$ diffeomorphism $\phi_{R,\nu} \colon (U_\nu, x_\nu) \to (B_R(x), x)$ such that*

$$\lim_{\nu\to\infty} \|(\phi_{R,\nu})_* g_\nu - g\|_{C^{k,\alpha}(B_R(x))} = 0.$$

There are numerous interesting variations on these; in particular, the (pointed) $W^{k,p}$ topology.

37 Controlled atlases

Definition 37.1. Let $k \in \mathbf{N}_0$, $\alpha \in (0,1)$, $\Lambda \in \mathbf{N}$ and $r, c > 0$. Let (M,g) be a Riemannian manifold of dimension n. An (r,Λ,c)–**controlled $C^{k+1,\alpha}$ atlas** for (M,g) is a collection charts

$$\{\mathbf{x}_\lambda \colon U_\lambda \to B_{4r}(0) \subset \mathbf{R}^n : \lambda \in \{1,\ldots,\Lambda\}\}$$

such that

$$M = \bigcup_{\lambda=1}^{\Lambda} \mathbf{x}_\lambda^{-1}(B_r(0));$$

for every $\lambda \in \{1,\ldots,\Lambda\}$,

1. in the coordinate chart $\mathbf{x}_\lambda$, for every $x \in B_{2r}(0)$ and $v \in S^{n-1}$,

$$\frac{1}{2} \leqslant \sum_{a,b=1}^{n} g_{ab} v^a v^b \leqslant 2;$$

2. in the coordinate chart $\mathbf{x}_\lambda$,

$$\|g_{ab} - \delta_{ab}\|_{C_r^{k,\alpha}} \leqslant 1;$$

and, for every $\lambda, \mu \in \{1,\ldots,\Lambda\}$,

3. for $\tilde{U}_{\lambda\mu} \coloneqq B_{2r}(0) \cap \mathbf{x}_\lambda(U_\lambda \cap U_\mu)$,

$$\|\mathbf{x}_\mu \circ \mathbf{x}_\lambda^{-1}\|_{C_r^{k+1,\alpha}(\tilde{U}_{\lambda\mu})} \leqslant c.$$

Lemma 37.2 (Fundamental Lemma of Riemannian Convergence Theory). *Let $k \in \mathbf{N}_0$, $0 < \beta < \alpha < 1$, $\Lambda \in \mathbf{N}$, and $r, c > 0$. If $(M_\nu, g_\nu)_{\nu=1}^\infty$ be a sequence of Riemannian manifolds of dimension n each of which admits an (r, Λ, c)–controlled $C^{k+1,\alpha}$ atlas, then, after passing to a subsequence, there is a $C^{k+1,\beta}$ manifold M, a $C^{k,\beta}$ Riemannian metric g on M, and a sequence of $C^{k+1,\beta}$ embeddings $\phi_\nu\colon M_\nu \to M$ such that:*

1. *$\bigcup_{\nu\in\mathbf{N}} \phi_\nu(M_\nu) = M$; in fact, if the M_ν are closed, then ϕ_ν are diffeomorphism.*
2. *$\lim_{\nu\to\infty} \|(\phi_\nu)_* g_\nu - g\|_{C^{k,\beta}(\phi_\nu(M_\nu),g)} = 0$.*

The proof relies on the following almost trivial observation.

Definition 37.3. The **graph** of a map $f\colon X \to Y$ is the subset $\Gamma_f \subset X \times Y$ defined by

$$\Gamma_f \coloneqq \{(x, f(x)) : x \in X\}.$$

Proposition 37.4. *Let $n, m \in \mathbf{N}$, $k \in \mathbf{N}_0$, and $0 < \beta < \alpha < 1$. Let $g_\nu \in C^{k,\alpha}(B_2^n(0), \mathrm{Sym}_n(\mathbf{R}))^{\mathbf{N}}$, $(f_\nu) \in C^{k+1,\alpha}(B_2^n(0), \mathbf{R}^m)^{\mathbf{N}}$, and $f \in C^{k+1,\alpha}(B_2^n(0); \mathbf{R}^m)$. Let U be a tubular neighborhood of Γ_f and $\Pi\colon U \to \Gamma_f$ the projection map. Suppose that*

$$\operatorname{spec} g_\nu \in [1/2, 2], \quad \|g_\nu - \mathbf{1}\|_{C^{k,\alpha}} \leqslant 1, \quad \lim_{\nu\to\infty} \|f_\nu - f\|_{C^{k+1,\alpha}} = 0,$$

and that, for every $\nu \in \mathbf{N}$, $\Gamma_{f_\nu} \subset U$. Define $\phi_\nu\colon B_2^n(0) \to B_2^n(0)$ by

$$\phi_\nu(x) \coloneqq \mathrm{pr}_{\mathbf{R}^n} \circ \Pi(x, f_\nu(x))$$

The following hold:

1. *$\lim_{\nu\to\infty} \|\phi_\nu - \mathrm{id}\|_{C^{k+1,\alpha}} = 0$; in particular, for $\nu \gg 1$, $\phi_\nu|_{B_1(0)}$ is a $C^{k+1,\alpha}$ embedding.*
2. *There is a $g \in C^{k,\beta}(B_1(0); \mathrm{Sym}_n(\mathbf{R}))$ satisfying $\operatorname{spec} g \in [\frac{1}{2}, 2]$ and such that, after passing to a subsequence,*
$$\lim_{\nu\to\infty} \|(\phi_\nu)_* g_\nu - g\|_{C^{k,\beta}(B_1(0))} = 0.$$

Proof. The assumed $C^{k+1,\alpha}$ convergence of (f_ν) to f implies the $C^{k+1,\alpha}$ convergence of (ϕ_ν) to

$$\mathrm{pr}_{\mathbf{R}^n} \circ \Pi(x, f(x)) = \mathrm{id}.$$

By hypothesis and construction of ϕ_ν,

$$\sup_{\nu\in\mathbf{N}} \|(\phi_\nu)_* g_\nu\|_{C^{k,\alpha}(B_1(0))} < \infty.$$

This implies the asserted convergence after passing to a subsequence. □

Proof of Lemma 37.2. Choose a smooth function $\chi \in C_0^\infty(B_{4r}(0),[0,1])$ with $\chi|_{B_{2r}(0)} = 1$. For every $\nu \in \mathbf{N}$, choose an (r,Λ,c)–controlled $C^{k+1,\alpha}$ atlas $\{\mathbf{x}_{\lambda,\nu}\colon U_{\lambda,\nu} \to B_{4r}(0) : \lambda \in \{1,\dots,\Lambda\}\}$. For every $\nu \in \mathbf{N}$ and $\lambda \in \{1,\dots,\Lambda\}$, define the embedding $\iota_\nu\colon M_\nu \to \left(\mathbf{R}^{n+1}\right)^\Lambda$ by

$$\iota_\nu(x) := \left(\chi(\mathbf{x}_\lambda^\nu(x)) \cdot \mathbf{x}_\lambda^\nu(x), \chi(\mathbf{x}_\lambda^\nu(x))\right)_{\lambda=1}^\Lambda.$$

By construction,

$$\iota_\nu(M_\nu) = \bigcup_{\lambda=1}^\Lambda V_{\lambda,\nu} \quad \text{with} \quad V_{\lambda,\nu} := \iota_\nu(\mathbf{x}_{\lambda,\nu}^{-1}(B_{2r}(0)))$$

and

$$V_{\lambda,\nu} = \Gamma_{f_{\lambda,\nu}}$$

for $f_{\lambda,\nu}\colon B_{2r}(0) \to \mathbf{R} \times \left(\mathbf{R}^{n+1}\right)^{\Lambda-1}$ defined by

$$f_{\lambda,\nu}(y) := \left[1; \left(\chi(\mathbf{x}_\mu \circ \mathbf{x}_\lambda^{-1}(y)) \cdot \mathbf{x}_\mu \circ \mathbf{x}_\lambda^{-1}(y), \chi(\mathbf{x}_\mu \circ \mathbf{x}_\lambda^{-1}(y))\right)_{\mu \in \{1,\dots,\hat{\lambda},\dots,\Lambda\}}\right].$$

Furthermore,

$$\sup_{\lambda,\nu} \|f_{\lambda,\nu}\|_{C_r^{k+1,\alpha}} < \infty.$$

Let $\gamma \in (\beta,\alpha)$. By the above, there are $f_\lambda\colon B_{2r}(0) \to \mathbf{R} \times \left(\mathbf{R}^{n-1}\right)^{\Lambda-1}$ such that, after passing to a subsequence,

$$\lim_{\nu\to\infty} \|f_{\lambda,\nu} - f_\lambda\|_{C^{k+1,\gamma}} = 0.$$

The union of the graphs of f_λ forms a $C^{k+1,\gamma}$ submanifold

$$M := \bigcup_{\lambda=1}^\Lambda \Gamma_{f_\lambda} \subset \left(\mathbf{R}^{n+1}\right)^\Lambda.$$

In fact, M is already the union of the graphs of the restrictions $f_\lambda|_{B_r(0)}$.

Let U be a tubular neighborhood of $M \subset \mathbf{R}^{\Lambda(n+1)}$ and $\Pi\colon U \to M$ the projection map. For $\nu \gg 1$, $\iota_\nu(M_\nu) \subset U$ and we define $\phi_\nu\colon M_\nu \to M$ by

$$\phi_\nu := \Pi \circ \iota_\nu.$$

Proposition 37.4 constructs a $C^{k,\beta}$ Riemannian metric g on M and shows that, after passing to a subsequence, $(\phi_\nu)_* g_\nu$ converges to g in $C^{k,\beta}$. □

38 Harmonic coordinates

Definition 38.1 (Einstein [Ein16]). Let (M, g) be a Riemannian manifold of dimension n. A coordinate system $\mathbf{x} = (x^1, \dots, x^n) \colon M \supset U \to \tilde{U} \subset \mathbf{R}^n$ is **harmonic** if, for every $a = 1, \dots, n$,

$$\Delta x^a = 0.$$

Proposition 38.2. *Let (M, g) be a Riemannian manifold of dimension n. For every $x \in M$, there is an open neighborhood U and a harmonic coordinate system $\mathbf{x} = (x^1, \dots, x^n) \colon M \supset U \to \tilde{U} \subset \mathbf{R}^n$.*

Proof. Let $y^1, \dots, y^n \colon M \supset V \to \mathbf{R}$ be normal coordinate system with $y^a(x) = 0$. In this coordinate system,

$$\Delta f = -\sum_{a,b=1}^{n} \frac{1}{\sqrt{\det g}} \partial_a\Big(\sqrt{\det g} \cdot g^{ab} \partial_b f\Big).$$

Since $\mathrm{d}g$ vanishes at 0,

$$(\Delta y^a)(x) = 0.$$

For $0 < \varepsilon \ll 1$, let $x^a \colon B_\varepsilon(x) \to \mathbf{R}$ be the unique solution of the Dirichlet problem

$$\Delta x^a = 0 \text{ on } B_\varepsilon(x) \quad \text{and} \quad x^a = y^a \text{ on } \partial B_\varepsilon(x).$$

By Schauder theory (see, e.g., Gilbarg and Trudinger [GT01, Chapter 6]),

$$\begin{aligned} \|x^a - y^a\|_{C^{2,\alpha}(B_\varepsilon(x))} &\lesssim \|\Delta(x^a - y^a)\|_{C^{0,\alpha}(B_\varepsilon(x))} \\ &= \|\Delta y^a\|_{C^{0,\alpha}(B_\varepsilon(x))} \\ &\lesssim \varepsilon^\alpha. \end{aligned}$$

Since $0 < \varepsilon \ll 1$, $x^1, \dots, x^n$ is a coordinate system; by construction, it is harmonic. ∎

Proposition 38.3. *Let (M, g) be a Riemannian manifold of dimension n. In a harmonic coordinate system $x^1, \dots, x^n \colon M \supset U \to \mathbf{R}$,*

$$\Delta f = -\sum_{a,b=1}^{n} g^{ab} \partial_a \partial_b f.$$

Proof. Since x^c is harmonic,

$$\begin{aligned} 0 &= \Delta x^c \\ &= -\sum_{a,b=1}^{n} \frac{1}{\sqrt{\det g}} \partial_a\Big(\sqrt{\det g} \cdot g^{ab} \partial_b x^c\Big) \\ &= -\sum_{a=1}^{n} \frac{1}{\sqrt{\det g}} \partial_a\Big(\sqrt{\det g} \cdot g^{ac}\Big). \end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta f &= -\sum_{a,b=1}^{n} \frac{1}{\sqrt{\det g}} \partial_a\left(\sqrt{\det g} \cdot g^{ab} \partial_b f\right) \\
&= -\sum_{a,b=1}^{n} \frac{1}{\sqrt{\det g}} \partial_a\left(\sqrt{\det g} \cdot g^{ab}\right) \partial_b f - \sum_{a,b=1}^{n} g^{ab} \partial_a \partial_b f \\
&= -\sum_{a,b=1}^{n} g^{ab} \partial_a \partial_b f.
\end{aligned}$$

□

Proposition 38.4 (Lanczos [Lan22]). *Let (M, g) be a Riemannian manifold of dimension n. In a harmonic coordinate system $x^1, \ldots, x^n \colon M \supset U \to \mathbf{R}$,*

$$\frac{1}{2} \Delta g_{ab} = \mathrm{Ric}_{ab} + Q(g, dg).$$

Proof. By Proposition 18.5 for $f = x^a$,

$$\frac{1}{2} \Delta g(\nabla x^a, \nabla x^a) = -|\mathrm{Hess}\, x^a|^2 - \mathrm{Ric}(\nabla x^a, \nabla x^a).$$

Since $\nabla x^a = \sum_{b=1}^n g^{ab} \partial_b$,

$$\frac{1}{2} \Delta g^{aa} = -|\mathrm{Hess}\, x^a|^2 - \mathrm{Ric}^{aa}.$$

Polarizing this identity yields

$$\frac{1}{2} \Delta g^{ab} = -\left\langle \mathrm{Hess}\, x^a, \mathrm{Hess}\, x^b \right\rangle - \mathrm{Ric}^{ab}.$$

From

$$\sum_{b=1}^{n} g_{ab} g^{bc} = \delta_a^c$$

it follows that

$$0 = \sum_{b=1}^{n} (\Delta g_{ab}) g^{bc} - 2\left\langle \nabla g_{ab}, \nabla g^{bc} \right\rangle + g_{ab}(\Delta g^{bc}).$$

Therefore,

$$\begin{aligned}
\Delta g_{ad} &= \sum_{b=1}^{n} (\Delta g_{ab}) g^{bc} g_{cd} \\
&= \sum_{b,c=1}^{n} 2\left\langle \nabla g_{ab}, \nabla g^{bc} \right\rangle g_{cd} - g_{ab} g_{cd} (\Delta g^{bc}) \\
&= 2 \sum_{b,c=1}^{n} \left\langle \nabla g_{ab}, \nabla g^{bc} \right\rangle g_{cd} + g_{ab} g_{cd} \left\langle \operatorname{Hess} x^b, \operatorname{Hess} x^c \right\rangle + g_{ab} g_{cd} \mathrm{Ric}^{bc} \\
&= 2\mathrm{Ric}_{ad} + 2Q_{ad}
\end{aligned}$$

with

$$Q_{ad} = \sum_{b,c=1}^{n} \left\langle \nabla g_{ab}, \nabla g^{bc} \right\rangle g_{cd} + g_{ab} g_{cd} \left\langle \operatorname{Hess} x^b, \operatorname{Hess} x^c \right\rangle. \qquad \square$$

Remark 38.5. See DeTurck and Kazdan [DK81] for a discussion of the use of harmonic coordinates for question of regularity in Riemannian geometry.

Remark 38.6. The relation between harmonic coordinates and the DeTurck trick is clarified in Graham and Lee [GL91].

39 The harmonic radius

Definition 39.1. Let $k \in \mathbf{N}_0$, $\alpha \in (0,1)$, $x \in \mathbf{R}^n$, $r > 0$, and $U \subset \mathbf{R}$. For $f \in C^{k,\alpha}(U)$, we define the **Hölder norm at scale** r by

$$\|f\|_{C_r^{k,\alpha}(U)} \coloneqq \sum_{|I| \leqslant k} r^{|I|} \|\partial^I f\|_{C^0(U)} + r^{|I|+\alpha} [\partial^I f]_{C^{0,\alpha}(U)}.$$

Exercise 39.2. If $f\colon B_r(0) \to R$ and $f_r \coloneqq f(r\,\cdot)\colon B_1(0) \to \mathbf{R}$, then

$$\|f\|_{C_r^{k,\alpha}(B_r(0))} = \|f_r\|_{C^{k,\alpha}(B_1(0))}.$$

In our upcoming discussion, the following refinements of the injectivity radius will play an central role.

Definition 39.3. Let (M, g) be Riemannian manifold. Let $k \in \mathbf{N}_0$ and $\alpha \in (0, 1)$. For $x \in M$, the $C^{k,\alpha}$ **harmonic radius** of (M, g) at x, denoted by

$$r_H(M, g; k, \alpha; x),$$

is the supremum of those $r > 0$ for which there exists a harmonic coordinate system $\mathbf{x}\colon U \to B_r(0)$ such that $\mathbf{x}(x) = 0$, with respect to this coordinate system, for every $v \in S^{n-1}$,

$$\frac{1}{2} \leqslant \sum_{a,b=1}^{n} g_{ab} v^a v^b \leqslant 2 \quad \text{and} \quad \|g_{ab} - \delta_{ab}\|_{C_r^{k,\alpha}} \leqslant 1.$$

The $C^{k,\alpha}$ **harmonic radius** of (M, g) is

$$r_H(M, g; k, \alpha) \coloneqq \inf_{x \in M} r_H(M, g; k, \alpha; x).$$

Remark 39.4. The choice of $\frac{1}{2}$, 1, and 2 is somewhat arbitrary.

Proposition 39.5. *Let $k \in \mathbf{N}_0$, $\alpha \in (0, 1)$. If (M, g) is a Riemannian manifold and $x \in M$, then*

$$r_H(M, g; k, \alpha, x) > 0.$$

In particular, if M is closed, then

$$r_H(M, g; k, \alpha) > 0.$$

Proof. Since M is closed, it suffices to prove that, for every $x \in M$, $r_H(1, \alpha; x) > 0$. By Proposition 38.2, there is some harmonic coordinate system $\mathbf{x}\colon U \to \mathbf{R}^n$ with $x \in U$. After shrinking U and shifting and scaling $\mathbf{x}$, the coordinate system takes values in $B_r(0)$, satisfies $\mathbf{x}(x) = 0$, and $g_{ab}(0) = \delta_{ab}$. For sufficiently small $r > 0$, the estimates hold. □

Proposition 39.6. *Let $n \in \mathbf{N}$, $k \in \mathbf{N}_0$, and $\alpha \in (0, 1)$. Let (M, g) be a Riemannian manifold of dimension n. If $\mathbf{x}\colon U \to \tilde{U}$ and $\mathbf{y}\colon V \to \tilde{V}$ are harmonic coordinates systems such that:*

1. $\frac{1}{2} \leqslant \sum_{a,b=1}^{n} g_{ab} v^a v^b \leqslant 2$,
2. $\|g_{ab} - \delta_{ab}\|_{C_r^{k,\alpha}} \leqslant 1$, *and*
3. $B_{4r}(0) \subset \mathbf{x}(U \cap V)$,

then

$$\|\mathbf{y} \circ \mathbf{x}^{-1}\|_{C_r^{k+2,\alpha}(B_{2r}(0))} \leqslant c(n, k, \alpha).$$

Proof. Denote by g_{ab} the Riemannian metric in the coordinate system $\mathbf{y}$. By Proposition 38.3 and since $\mathbf{x} = (x^1, \dots, x^n)$ is a harmonic coordinate system, for every $a \in \{1, \dots, n\}$,

$$\begin{aligned}\Delta\big(x^a \circ \mathbf{y}^{-1}\big) &= \sum_{b,c=1}^{n} g^{bc}\partial_b\partial_c\big(x^a \circ \mathbf{y}^{-1}\big) \\ &= 0.\end{aligned}$$

Since $|\mathbf{x} \circ \mathbf{y}^{-1}| \leqslant 8r$, by interior Schauder estimates

$$\|\mathbf{x} \circ \mathbf{y}^{-1}\|_{C_r^{k+2,\alpha}(B_{2r}(0))} \leqslant c(n,k,\alpha). \qquad \square.$$

Proposition 39.7. *Let $n \in \mathbf{N}$, $k \in \mathbf{N}_0$, $\alpha \in (0,1)$, and $\delta > 0$. If (M,g) is a closed Riemannian manifold of dimension n with*

$$r_H(M,g;k,\alpha) \geqslant \delta,$$

then, for $r = r(\delta) > 0$, $\Lambda \coloneqq \operatorname{cov}(M,g;r/2)$, and $c = c(n,k,\alpha)$, there exists a (r,Λ,c)–controlled $C^{k+1,\alpha}$ atlas for (M,g).

Proof. It follows directly from the hypothesis, that there is an $r = r(\delta)$ such that, for every $x \in M$, there is a harmonic coordinate system $\mathbf{x}\colon V \to B_{64r}(0)$ satisfying $\mathbf{x}(x) = 0$,

$$\frac{1}{2} \leqslant \sum_{a,b=1}^{n} g_{ab}v^a v^b \leqslant 2 \quad \text{and} \quad \|g_{ab} - \delta_{ab}\|_{C_r^{k,\alpha}} \leqslant 1.$$

These conditions imply that, for every $t \in [0,64]$,

$$B_{\frac{1}{2}tr}(x) \subset \mathbf{x}^{-1}(B_{tr}(0)) \subset B_{2tr}(x).$$

Choose $x_1, \dots, x_\Lambda \in M$ such that

$$\bigcup_{\lambda=1}^{\Lambda} B_{r/2}(x_\lambda) = M.$$

For every $\lambda \in \{1, \dots, \Lambda\}$, choose a harmonic coordinate system $\mathbf{x}_\lambda\colon U_\lambda \to B_{4r}(0)$ which is the restriction of a harmonic coordinate system $b_\lambda\colon V_\lambda \to B_{64r}(0)$ as above. To see that this is an (r,Λ,c)–controled C^{k+1} atlas, observe that

$$\bigcup_{\lambda=1}^{\Lambda} \mathbf{x}_\lambda^{-1}(B_r(0)) \supset \bigcup_{\lambda=1}^{\Lambda} B_{r/2}(x_\lambda) = M$$

and Proposition 39.6 applies because, for every $\lambda, \mu \in \{1, \dots, \Lambda\}$, if

$$\mathbf{x}_\lambda^{-1}(B_{2r}(0)) \cap \mathbf{x}_\mu^{-1}(B_{4r}(0)) \neq \varnothing,$$

then

$$d(x_\lambda, x_\mu) \leqslant 12r;$$

and, therefore,

$$B_{8r}(x_\lambda) \subset B_{20r}(x_\mu);$$

hence,

$$B_{4r}(0) \subset \mathbf{x}_\lambda(V_\lambda \cap V_\mu). \qquad \square$$

40 Compactness under Ricci and injectivity radius bounds

Theorem 40.1 (Anderson [And90a, Theorem 1.1], Hebey and Herzlich [HH97, Main Theorem]). *Let $n \in \mathbf{N}$, $c, D, \delta > 0$, $k \in \mathbf{N}_0$, and $\alpha \in (0,1)$. The set of isometry classes of closed Riemannian manifold (M, g) of dimension n satisfying*

$$\operatorname{diam}(M,g) \leqslant D, \quad \operatorname{inj}(M,g) \geqslant \delta, \quad \textit{and} \quad |\nabla^\ell \mathrm{Ric}_g| \leqslant c \quad \textit{for all} \quad \ell = 0, \dots, k,$$

is relatively compact in the $C^{k+1,\alpha}$–topology.

This result has the following curious corollary.

Corollary 40.2. *There are only countably many diffeomorphism types of closed manifolds.*

However, there is also the following.

Theorem 40.3 (Taubes [Tau87, Theorem 1.1]). *There exists an uncountable family of diffeomorphism classes of oriented 4–manifolds which are homeomorphic to $\mathbf{R}^4$.*

This follows immediately from Lemma 37.2, Proposition 39.7, and the following lower bound for the harmonic radius.

Theorem 40.4 (Anderson [And90a, Main Lemma 2.2], Hebey and Herzlich [HH97, Theorem 6]). *Let $n \in \mathbf{N}$, $k \in \mathbf{N}_0$, $\delta, c > 0$, and $\alpha \in (0,1)$. If (M, g) is a closed Riemannian manifold of dimension n satisfying*

$$\operatorname{inj}(M,g) \geqslant \delta \quad \textit{and} \quad |\nabla^\ell \mathrm{Ric}_g| \leqslant c \quad \textit{for all} \quad \ell = 0, \dots, k,$$

then

$$r_H(M, g; k, \alpha) \geqslant \varepsilon(n, k, \alpha, c, \delta) > 0.$$

Remark 40.5. The case $k = 0$ is due to Anderson [And90a]; the general case is due to Hebey and Herzlich [HH97]. Anderson and Cheeger [AC92] proved that Theorem 40.4 for $k = 0$ holds assuming only a Ricci lower bound.

Proof of Theorem 40.4. If the assertion fails, then there exists a sequence of $(M_\nu, g_\nu)_{\nu\in\mathbf{N}}$ of Riemannian manifolds satisfying

$$\mathrm{inj}(M_\nu, g_\nu) \geqslant \delta \quad \text{and} \quad |\nabla^\ell \mathrm{Ric}_{g_\nu}| \leqslant c,$$

for $\ell = 0, \dots, k$, and such that

$$\varepsilon_\nu \coloneqq r_H(M_\nu, g_\nu; k, \alpha) \to 0 \quad \text{as} \quad \nu \to \infty.$$

For every $\nu \in \mathbf{N}$, choose $x_\nu \in M_\nu$ such that

$$r_H(M_\nu, g_\nu; k, \alpha; x_\nu) = \varepsilon_\nu.$$

Construction of a rescaled limit. For every $\nu \in \mathbf{N}$, define $s_\nu \colon B_{\varepsilon_\nu^{-1}}(0) \to M_\nu$ by

$$s_\nu(y) \coloneqq \exp_{x_\nu}(\varepsilon_\nu y)$$

and set

$$\tilde{g}_\nu \coloneqq \varepsilon_\nu^{-2} s_\nu^* g_\nu.$$

By construction, for every $R > 0$,

1. $\|\mathrm{Ric}_{\tilde{g}_\nu}\|_{C^k(B_R(0))} \to 0$,
2. $\sup_{x\in B_R(0)} \mathrm{inj}_x(B_{\varepsilon_\nu^{-1}}(0), \tilde{g}_\nu) \to \infty$, and
3. $\inf_{x\in B_R(0)} r_H(B_{\varepsilon_\nu^{-1}}(0), \tilde{g}_\nu; k, \alpha) = r_H(B_{\varepsilon_\nu^{-1}}(0), \tilde{g}_\nu; k, \alpha; 0) = 1$.

Therefore, by Proposition 39.7, for $R > 0$, there exists a $\Lambda = \Lambda(R) \in \mathbf{N}$ and, for every $\nu \gg_R 1$, there exists a (k, Λ, c)–controlled atlas $C^{k+2,\alpha}$ harmonic atlas for $(B_R(0), g_\nu)$. By Lemma 37.2, after passing to a subsequence $(B_R(0), g_\nu)$ converges to a Riemannian manifold (M_R, g_R) in the $C^{k,\beta}$–topology for every $\beta \in (0, \alpha)$. Since $R > 0$ is arbitrary, a diagonal sequence argument shows that, there is a Riemannian manifold (M, g) such that, after passing to a subsequence, for every $R > 0$ and $\beta \in (0, \alpha)$, the sequence $(B_R(0), \tilde{g}_\nu)$ converges in the $C^{k+1,\beta}$–topology to an open submanifold $M_R \subset M$; more precisely, there exists a sequence of maps $\phi_\nu \colon B_{\varepsilon_\nu^{-1}}(0) \to M$ such that, for every $\beta \in (0, \alpha)$,

1. ϕ_ν is a $C^{k+2,\beta}$ embedding,
2. $(\phi_\nu)_* \tilde{g}_\nu$ converges to g in the $C^{k+1,\beta}$–topology, and
3. the sequence $\phi_\nu(0)$ converges to a limit $x_\star \in M$.

That is: $(B_{\varepsilon_\nu^{-1}}(0), 0, \tilde{g}_\nu)_{\nu\in\mathbf{N}}$ converges to $(M, x_\star, g)$ in the pointed $C^{k+1,\beta}$ topology.

Improving convergence. By Proposition 38.4, for every $R > 0$ and $\nu \gg_R 1$, in the a chart of the harmonic atlas constructed above,

$$-\frac{1}{2}\sum_{c,d=1}^{n} \tilde{g}_\nu^{cd}\partial_c\partial_d(\tilde{g}_\nu)_{ab} = \big(\mathrm{Ric}_{\tilde{g}_\nu}\big)_{ab} + Q(\tilde{g}_\nu, d\tilde{g}_\nu).$$

Since $|\mathrm{Ric}_{\tilde{g}_\nu}|$ tends to zero, elliptic theory shows that, for every $p \in [1,\infty)$,

$$\|g_\nu\|_{W^{k+2,p}} \leqslant c(p).$$

Therefore, the conclusion of the previous paragraph, actually, hold for every $\beta \in (0,1)$.

Analysis of the limit. A priori, g is only a $C^{k+1,\alpha}$ metric. However, since $|\mathrm{Ric}_{\tilde{g}_\nu}|$ tends to zero, with respect to the limits of the chart of the harmonic atlas constructed above,

$$-\frac{1}{2}\sum_{c,d=1}^{n} g^{cd}\partial_c\partial_d(g_\nu)_{ab} = Q(g, dg).$$

Therefore, elliptic regularity implies that g is a smooth Ricci-flat metric. Furthermore, by (2), $\mathrm{inj}(M,g) = \infty$. By Theorem 27.12, (M,g) is isometric to $(\mathbf{R}^n, g_0)$.

The contradiction. Denote by $\mathbf{x} = (x^1, \dots, x^n)$ the standard coordinates on $\mathbf{R}^n$. Define $\mathbf{y}_\nu \colon B_4(0) \to V_\nu \subset \mathbf{R}^n$ by

$$\mathbf{y}_\nu = \mathbf{x} \circ \phi_\nu.$$

These satisfy

$$\lim_{\nu\to 0}\|\Delta_{g_\nu}\tilde{\mathbf{y}}_\nu^a\|_{C^{k,\alpha}} = 0.$$

Therefore, arguing as in the proof of Proposition 38.2, for $\nu \gg 1$, there exists a harmonic coordinate system $\mathbf{x}_\nu \colon B_4(0) \to U_\nu \subset \mathbf{R}^n$ with

$$\lim_{\nu\to\infty}\mathbf{x}_\nu(0) = 0 \quad \text{and} \quad B_2(0) \subset U_\nu.$$

Therefore, for $\nu \gg 1$,

$$r_H(B_{\varepsilon_\nu^{-1}}(0), \tilde{g}_\nu; k+1, \alpha; 0) \geqslant 2;$$

contradicting (3). □

The proof of Theorem 40.1 has the following consequence.

Theorem 40.6 (Anderson [And90a, Proposition 3.4]). *Given $n \in \mathbf{N}$, $D, \delta > 0$, and $\lambda \in \mathbf{R}$ there exists a constant $\varepsilon = \varepsilon(n, D, \delta, \lambda) > 0$ such that if (M,g) is a closed Riemannian manifold of dimension n with*

$$\mathrm{diam}(M,g) \leqslant D, \quad \mathrm{inj}(M,g) \geqslant \delta, \quad \textit{and} \quad \|\mathrm{Ric}_g - \lambda g\|_{L^\infty} \leqslant \varepsilon,$$

then there exists a Ricci flat metric on M.

41 Compactness under Ricci bounds and volume pinching

Theorem 41.1 (Anderson [And90a, Proof of Theorem 1.2]). *Given $n \in \mathbf{N}$, $k \in \mathbf{N}_0$, $r, c > 0$, and $\alpha \in (0,1)$, exists an $\delta = \delta(r, c, \alpha) > 0$ such that the following holds. If (M, g) is a closed Riemannian manifold of dimension n satisfying*

$$\frac{\mathrm{vol}(B_r(x))}{\mathrm{vol}(B_r^n(0))} \geqslant 1 - \delta \quad \textit{and} \quad |\nabla^\ell \mathrm{Ric}_g| \leqslant c \quad \textit{for all} \quad \ell = 0, \dots, k,$$

then

$$r_H(M, g; k, \alpha) \geqslant \varepsilon(n, k, \alpha, c) > 0.$$

The proof relies on the following result, which replaces the application of the Theorem 27.12 in the proof of Theorem 40.4. The details of the proof are left as an exercise.

Theorem 41.2 (Anderson [And90a, Gap Lemma 3.1]). *Given $n \in \mathbf{N}$, there exists an $\varepsilon = \varepsilon(n) > 0$ such that if (M, g) is complete Riemannian manifold of dimension n which is Ricci-flat and, for some $x \in M$, satisfies*

$$\lim_{r\to\infty} \frac{\mathrm{vol}(B_r(x))}{\mathrm{vol}(B_r^n(0))} \geqslant 1 - \varepsilon,$$

then (M, g) is isometric to $(\mathbf{R}^n, g_0)$.

Proof. Suppose not; then there exists a null-sequence (ε_ν) and a sequence of complete Ricci flat manifolds (M_ν, g_ν) of dimension n none of which are isometric to $(\mathbf{R}^n, g_0)$ and, for every $\nu \in \mathbf{N}$, there exists a $x_\nu \in M_\nu$ with

$$\lim_{r\to\infty} \frac{\mathrm{vol}_{g_\nu}(B_r(x_\nu))}{\mathrm{vol}(B_r^n(0))} \geqslant 1 - \varepsilon_\nu, \tag{41.3}$$

For every $\nu \in \mathbf{N}$, choose $y_\nu \in B_\nu(x_\nu)$ realizing the minimum of the function

$$c_\nu(y) \coloneqq \frac{\mathrm{inj}_y(M_\nu, g_\nu)}{\nu - d(x_\nu, y)}.$$

By Theorem 27.12, $\mathrm{inj}_{y_\nu}(M_\nu, g_\nu) < \infty$. Therefore, after rescaling, which does not affect (41.3) and Ricci-flatness, we can assume that

$$\mathrm{inj}_{y_\nu}(M_\nu, g_\nu) = 1.$$

Since

$$c_\nu(y) \coloneqq \frac{\mathrm{inj}_y(M_\nu, g_\nu)}{\nu - d(x_\nu, y)}.$$

By construction, for every $R > 0$ and $y \in B_R(y_n)$, $\mathrm{inj}_y(M_\nu, g_\nu)$ is bounded below independent of ν. Therefore, (M_ν, x_ν, g_ν) converges in the pointed $C^{1,\alpha}$ topology to a pointed complete Riemannian manifold (M, x, g) which is Ricci flat and satisfies

$$\lim_{r\to\infty} \frac{\mathrm{vol}(B_r(x))}{\mathrm{vol}(B_r^n(0))} = 1.$$

Therefore, (M, g) is isometric to $(\mathbf{R}^n, g_0)$. However, the injectivity radius is lower semi-continuous with respect to the $C^{1,\alpha}$ topology. This implies $\mathrm{inj}_x(\mathbf{R}^n, g_0) = 1$, a contradiction. □

Theorem 41.4 (Anderson [And90a, Theorem 1.2]). *Given $n \in \mathbf{N}$ and $c > 0$, there exists an $\varepsilon = \varepsilon(n, c) > 0$ such that if (M, g) is a closed Riemannian manifold of dimension n satisfying*

$$(n-1)g \leqslant \mathrm{Ric}_g \leqslant cg \quad \textit{and} \quad \frac{\mathrm{vol}(M, g)}{\mathrm{vol}(S^n, g_1)} \geqslant 1 - \varepsilon, \tag{41.5}$$

then (M, g) is diffeomorphic to S^n.

Proof. Suppose not; then there exists a null-sequence (ε_ν) and a sequence (M_ν, g_ν) of closed Riemannian manifold of dimension n satisfying

$$(n-1)g_\nu \leqslant \mathrm{Ric}_{g_\nu} \leqslant cg_\nu \quad \text{and} \quad \frac{\mathrm{vol}(M, g_\nu)}{\mathrm{vol}(S^n, g_1)} \geqslant 1 - \varepsilon_\nu \tag{41.6}$$

and all of which are not diffeomorphic to S^n. By Theorem 17.1,

$$\mathrm{diam}(M_\nu, g_\nu) \leqslant \pi.$$

In fact, by Theorem 20.1,

$$\lim_{\nu\to\infty} \mathrm{diam}(M_\nu, g_\nu) = \pi$$

and, for every $r > 0$,

$$\lim_{\nu\to\infty} \frac{B_r^{M_\nu}(x)}{B_r^{S^n}(\star)} = 1.$$

By Theorem 41.1, after passing to a subsequence (M_ν, g_ν) converges in the $C^{1,\alpha}$ topology to a limit (M, g). A close reading of the proof of Theorem 20.1 reveals that $|\mathrm{Ric}_{g_\nu} - (n-1)g_\nu|$ tends zero almost everywhere; hence, $\lim_{\nu\to\infty} \|\mathrm{Ric}_{g_\nu} - (n-1)g_\nu\|_{L^p} = 0$ for every $p \in [1, \infty)$.

Exercise 41.7. Actually prove this.

Therefore, (M, g) is weakly Einstein; hence, Einstein:

$$\mathrm{Ric}_g = (n-1)g.$$

Hence, by Theorem 41.1, (M, g) is isometric to (S^n, g_1). This implies that, for $\nu \gg 1$, M_ν is already diffeomorphic to S^n; a contradiction. □

42 Harmonic curvature

Proposition 42.1. *For every Riemannian manifold* (M, g)

$$\mathrm{d}_\nabla^* R = \sum_{b,c,d=1}^{n} (\nabla_c \mathrm{Ric}_{bd} - \nabla_d \mathrm{Ric}_{bc}) \cdot e^b \otimes e^c \otimes e_d.$$

Remark 42.2. Thinking of Ric as a T^*M valued 1–form and raising the d–index, the right-hand side can be identified with $\mathrm{d}_\nabla \mathrm{Ric} \in \Omega^2(M, T^*M)$.

Corollary 42.3. *If* (M, g) *is a Riemannian manifold with*

$$\nabla \mathrm{Ric}_g = 0,$$

then it has harmonic curvature; i.e.:

$$\mathrm{d}_\nabla^* R = 0.$$

Remark 42.4. See Derdziński [Der82] for a (outdated) survey of Riemannian manifolds with harmonic curvature.

Proof of Proposition 42.1. Using the symmetry (5.3) and the differential Bianchi identity (5.5),

$$\nabla_{e_a} R_{abc}{}^{d} + \nabla_{e_c} R_{abd}{}^{a} + \nabla_{e_d} R_{aba}{}^{c} = 0.$$

Therefore

$$\begin{aligned}
\mathrm{d}_\nabla^* R &= -\sum_{a,b,c,d=1}^{n} \nabla_{e_a} R_{abc}{}^{d} \cdot e^b \otimes e^c \otimes e_d \\
&= \sum_{a,b,c,d=1}^{n} \left(\nabla_{e_c} R_{abd}{}^{a} - \nabla_{e_d} R_{abc}{}^{a}\right) \cdot e^b \otimes e^c \otimes e_d \\
&= \sum_{b,c,d=1}^{n} \left(\nabla_{e_c} \mathrm{Ric}_{bd} - \nabla_{e_d} \mathrm{Ric}_{bc}\right) \cdot e^b \otimes e^c \otimes e_d.
\end{aligned}$$

□

43 ε–regularity

Theorem 43.1 (Anderson [And89, Lemma 2.1]). *Given $n \in \mathbf{N}$ and $c_S > 0$, there are constants $\varepsilon_0 = \varepsilon_0(n, c_S) > 0$ and $c = c(n, c_S) > 0$ such that the following holds. Let (M, g) be a Riemannian manifold of dimension n with harmonic curvature and $(\frac{n}{n-2}, 2)$–Sobolev constant at most c_S. If $x \in M$ and $r > 0$ are such that*

$$\varepsilon := \int_{B_r(x)} |R|^{n/2} \leqslant \varepsilon_0,$$

then

$$\sup_{y \in B_{r/4}(x)} |R|^2(y) \leqslant c\varepsilon r^{-4}.$$

Remark 43.2. In the context of Yang–Mills theory, a slight improvement of this result is due to Uhlenbeck [Uhl82, Theorem 3.5]. She in turn derives it from an extension of Theorem 30.3 due to Morrey [Mor08, Theorem 5.3.1]. There is an important strengthening of Uhlenbeck's due to Nakajima [Nak88, Lemma 3.1], using Price's monotonicity formula [Pri83] and the Heinz trick [Hei55]. The proof given below is rather similar to Nakajima's argument.

Proof of Theorem 43.1. Since the Levi-Civita connection is a Yang–Mills, by the Bochner–Weitzenböck formula,

$$\nabla^*\nabla R = R \star R.$$

Therefore,

$$\Delta|R|^2 \leqslant c(n)|R|^3.$$

Define $f\colon\ \bar{B}_{\frac{r}{2}}(x) \to [0, \infty)$ by

$$f(y) := \left[\frac{r}{2} - d(x, y)\right]^4 |R|(y)^2.$$

The assertion follows once we prove that

$$f \leqslant c(n, c_S)\varepsilon.$$

Since f is non-negative and vanishes on the boundary of $B_{\frac{r}{2}}(x)$, there is a $y_\star \in B_{\frac{r}{2}}(x)$ with

$$f(y_\star) = \max_{y \in B_{\frac{r}{2}}(x)} f(y).$$

Set

$$s_\star := \frac{1}{2}\left[\frac{r}{2} - d(x, y_\star)\right] \quad \text{and} \quad c_\star := |R|(y_\star).$$

Suppose $y \in B_{s_\star}(y_\star)$. By the triangle inequality,

$$\begin{aligned}\frac{r}{2} - d(x,y) &\geqslant \frac{r}{2} - d(x,y_\star) - s_\star \\ &= s_\star.\end{aligned}$$

Therefore and since $f(y) \leqslant f(y_\star)$,

$$|R|(y) \leqslant 16c_\star.$$

Hence, on $B_{s_\star}(x)$

$$\Delta|R|^2 \leqslant c(n)c_\star^3.$$

By [GT01, Theorem 9.20], for every $s \in (0, s_\star]$,

$$\begin{aligned}c_\star^2 &= |R|(y_\star)^2 \\ &\leqslant c(n,c_S)\left(s^{-n}\int_{B_s(y_0)}|R|^2 + s^2c(n)c_\star^3\right) \\ &\leqslant c(n,c_S)\big(s^{-4}\varepsilon + s^2c_\star^3\big).\end{aligned}$$

This can be rewritten as

$$s^4c_\star^2 \leqslant c(n,c_S)\big(\varepsilon + s^6c_\star^3\big).$$

Setting

$$t = t(s) := s^2c_\star$$

the above inequality reads

$$t^2(1-ct) \leqslant c\varepsilon.$$

with $c = c(n,c_S)$. Since $\varepsilon \leqslant \varepsilon_0 \ll 1$, the corresponding equation $t^2(1-ct) = c\varepsilon$ has two small roots $t_\pm \sim \sqrt{c\varepsilon}$ and one large root. Since $t(0) = 0$, by continuity $t(s) \leqslant |t_\pm| \leqslant 2\sqrt{c\varepsilon}$ for all $s \in [0, s_\star]$. Therefore,

$$f(y_\star) = 8s_\star^4c_\star^2 \leqslant c(n,c_S)\varepsilon.$$ □

44 Compactness under integral curvature bounds

Theorem 44.1 (Anderson [And90a, Proposition 2.5]). *Given $n \in \mathbf{N}$, $R, c, v > 0$, and $\alpha \in (0,1)$, exists an $\varepsilon(n, ca, v) > 0$ such that the following holds. If (M,g) is a Riemannian manifold of dimension n and $x \in M$ such that*

$$|\mathrm{Ric}_g| \leqslant c, \quad \mathrm{vol}(B_R(x)) \geqslant v, \quad \textit{and} \quad \int_{B_{4R}(x)}|R_g|^{n/2} \leqslant \varepsilon,$$

then

$$r_H(M,g;1,\alpha;x) \geqslant r(n,\kappa,v,\alpha) > 0.$$

Proof. Let ε be as in Theorem 43.1. If the assertion does not hold with this choice; then is a sequence of counterexamples whose limit gives rise to a pointed complete Riemannian manifold (M, x, g) with

$$\mathrm{Ric}_g = 0, \quad r_H(M, g, 1, \alpha) \leqslant 1, \quad \mathrm{vol}(B_r(x)) \geqslant c(v)r^n > 0, \quad \text{and} \quad \int_M |R_g|^{n/2} \leqslant \varepsilon.$$

But by Theorem 43.1, for every $r > 0$,

$$\sup_{y \in B_{r/4}(x)} |R_g|^2(y) \leqslant c\varepsilon r^{-4}.$$

Taking the limit $r \to \infty$ proves that (M, g) is flat. Since it also has Euclidean volume growth, it must be $\mathbf{R}^n$. But then its harmonic radius is ∞; a contradiction. □

Theorem 44.2 (Anderson [And90a, Theorem 2.6]). *Given $n \in \mathbf{N}$, $R, c, v > 0$, and $\alpha \in (0, 1)$, If (M_ν, g_ν) is a sequence of Riemannian manifolds of dimension n satisfying*

$$|\mathrm{Ric}_{g_\nu}| \leqslant c, \quad \mathrm{vol}(M_\nu, g_\nu) \geqslant v, \quad \textit{and} \quad \int_{M_\nabla} |R_g|^{n/2} \leqslant c,$$

then, after passing to a subsequence, (M_ν, g_ν) converges in the Gromov–Hausdorff topology to a Riemannian orbifold (M, g) with a finite number of singular points, each of which so modeled on a cone over S^{n-1}/Γ; g is a continuous Riemannian metric and $C^{1,\alpha}$ on the regular part of M.

Proof sketch. For every $r > 0$, cover M_ν balls of radius r. On the balls with

$$\int_{B_r(x)} |R|^{n/2} \leqslant \varepsilon$$

on obtains a harmonic radius bound. As r goes to zero, for fixed ν, these balls cover all of M_ν. This gives the $C^{1,\alpha}$ convergence of an exhaustion of M_ν. The limit of this will be the regular part of the orbifold. For more details see [And89]. □

Remark 44.3. Anderson [And89] proved that in odd dimension orbifold singularities cannot appear since $\mathbf{R}P^{2n}$ does not bound an orientable manifold and is the only relevant space form. Thus in odd dimensions an integral curvature bound of the above form gives compactness.

45 Weyl curvature tensor

Definition 45.1. The **Kulkarni–Nomizu product** is defined as

$$(h \owedge k)(u, v, w, z) := h(u, z)k(v, w) + h(v, w)k(u, z) - h(u, w)k(v, z) - h(v, z)k(u, w).$$

Proposition 45.2.

1. *If* $e_1, \dots, e_n$ *is an orthonormal basis, then*

$$\sum_{a=1}^{n} (h \owedge g)(e_a, v, w, e_a) = (n-2)h(v,w) + \operatorname{tr} h \cdot \langle v, w\rangle.$$

2. $(g \owedge g)(u, v, w, z) = -2\langle u \wedge v, w \wedge z\rangle.$

Proof. If $e_1, \dots, e_n$ is an orthonormal basis, then

$$\begin{aligned}
&\sum_{a=1}^{n} (h \owedge g)(e_a, e_b, e_b, e_a) \\
&\quad = \sum_{a=1}^{n} h(e_a, e_a)g(e_b, e_b) + h(e_b, e_b)g(e_a, e_a) - h(e_a, e_b)g(e_b, e_a) - h(e_b, e_a)g(e_a, e_b) \\
&\quad = \sum_{a=1}^{n} h(e_a, e_a) + (n-2)h(e_b, e_b).
\end{aligned}$$

This implies the assertion. □

Definition 45.3. The **Weyl curvature tensor** $W \in \Gamma(S^2\Lambda^2 T^*M)$ of (M, g) is defined by

$$W(u, v, w, z) := \langle R(u, v)w, z\rangle - \frac{1}{n-2}(\mathrm{Ric}_g^\circ \owedge g)(u, v, w, z) - \frac{\mathrm{scal}_g}{2n(n-1)}(g \owedge g)(u, v, w, z).$$

Definition 45.4. In dimension 4, the **self-dual Weyl curvature tensor** $W_+ \in \Gamma(S^2\Lambda^+ T^*M)$ and the **anti-self-dual Weyl curvature tensor** $W_- \in \Gamma(S^2\Lambda^- T^*M)$ are the corresponding components of W.

46 Hitchin–Thorpe inequality

Theorem 46.1 (Hitchin–Thorpe inequality). *If* (M, g) *is a closed, oriented* 4*–manifold, then*

$$2\chi(M) \pm 3\sigma(M) = \frac{1}{4\pi^2}\int_M 2|W_\pm|^2 - |\mathrm{Ric}_g^\circ|^2 + \frac{\mathrm{scal}_g^2}{24}. \tag{46.2}$$

In particular, if g *is Einstein,*

$$2\chi(M) \pm 3\sigma(M) = \frac{1}{4\pi^2}\int_M 2|W_\pm|^2 + \frac{\mathrm{scal}_g^2}{24} \geqslant 0.$$

Equality holds in (46.1) if and only if g *is flat or the universal cover of* (M, g) *is a K3 surface equipped with a Ricci-flat metric.*

This is a direct consequence of the following (tedious to prove) Chern–Weil formulae.

Proposition 46.3.

$$\chi(M) = \frac{1}{8\pi^2}\int_M |W|^2 - |\mathrm{Ric}_g^\circ|^2 + \frac{\mathrm{scal}_g^2}{24}.$$

Proposition 46.4.

$$\sigma(M) = \frac{1}{12\pi^2}\int_M |W_+|^2 - |W_-|^2.$$

47 The metric uniformization theorem

Definition 47.1. Let (Σ, g) be a Riemannian 2–manifold. The **Gauß curvature** of (Σ, g) is the function $K = K_g \in C^\infty(\Sigma)$ characterized by the property that, for every non-zero $v \wedge w \in \Lambda^2 TM$,

$$K_g \coloneqq \frac{\langle R_g(v,w)w, v\rangle}{|v \wedge w|^2}.$$

The **curvature form** of (Σ, g) is the 2–form $\Omega = \Omega_g \in \Omega^2(M)$ defined by

$$\Omega_g \coloneqq K_g \,\mathrm{vol}_g.$$

Remark 47.2. The Gauß curvature and the scalar curvature of (Σ, g) are related by

$$\mathrm{scal}_g = 2K_g.$$

Theorem 47.3 (Gauß–Bonnet). *For every closed, connected, oriented Riemannian 2–manifold,*

$$\int_\Sigma K_g \,\mathrm{vol}_g = 2\pi\chi(\Sigma).$$

Theorem 47.4 (Wallach and Warner [WW70]). *Let Σ be a closed, connected, oriented 2–manifold and let $\bar\Omega \in \Omega^2(M)$. There exists a Riemannian metric g such that*

$$\Omega_g = \bar\Omega$$

if and only if

$$\int_\Sigma \bar\Omega = 2\pi\chi(\Sigma). \tag{47.5}$$

In fact, if (47.5), then every conformal class contains a Riemannian metric g satisfying $\Omega_g = \bar\Omega$.

Proposition 47.6. *Let (M, g) be a Riemannian manifold of dimension n, $f \in C^\infty(M)$, and*

$$\tilde{g} = e^{2f} g.$$

The following hold:

$$ {}^{\tilde{g}}\nabla^{\mathrm{LC}}_v w = {}^{g}\nabla^{\mathrm{LC}}_v w + \mathrm{d}f(v)w + \mathrm{d}f(w)v - \langle v, w\rangle \nabla f \quad \text{and} \tag{47.7}$$

$$\mathrm{scal}_{\tilde{g}} = e^{-2f}\left(\mathrm{scal}_g + 2(n-1)\Delta_g f + (n-2)(n-1)|\nabla_g f|_g^2\right). \tag{47.8}$$

Proof. A direct computation shows that ${}^{\tilde{g}}\nabla^{\mathrm{LC}}$ as defined above is both metric with respect to $\tilde{g}$ and torsion-free. The computation of $\mathrm{scal}_{\tilde{g}}$ is an exercise. (It is a bit easier if you specialize to $n = 2$) □

Proof of Theorem 47.4. By Theorem 47.3, it suffices to construct g provided (47.5) is satisfied. Let g be an arbitrary Riemannian. By (47.5),

$$\int_\Sigma \bar{\Omega} = \int_\Sigma \Omega_g.$$

Therefore, by Hodge theory, there exists an $f \in C^\infty(\Sigma)$ such that

$$\bar{\Omega} = \Omega_g - \mathrm{d} * \mathrm{d}f.$$

By Proposition 47.6,

$$\begin{aligned}
\Omega_{e^{2f}g} &= \frac{1}{2}\mathrm{scal}_{e^{2f}g}\mathrm{vol}_{e^{2f}g} \\
&= \frac{1}{2}e^{-2f}(\mathrm{scal}_g + \Delta_g f)e^{2f}\mathrm{vol}_g \\
&= \Omega_g + \Delta_g f \cdot \mathrm{vol}_g \\
&= \Omega_g - \mathrm{d} * \mathrm{d}f.
\end{aligned}$$

□

Theorem 47.9 (Metric Uniformization Theorem). *Let (Σ, j) be a closed Riemann surface. In the conformal class determined by j there is a Riemannian metric g satisfying*

$$\mathrm{scal}_g = \begin{cases} 1 & \text{if } \chi(\Sigma) = 2, \\ 0 & \text{if } \chi(\Sigma) = 0, \\ -1 & \text{if } \chi(\Sigma) \leqslant -2. \end{cases}$$

If $\chi(\Sigma) \neq 0$, then g is unique up to scaling by a constant; otherwise, g is unique.

The cases $\chi(\Sigma) = 2$ is best treated separately as follows.

Theorem 47.10 (Riemann–Roch). *If Σ is a closed Riemann surface of genus g and D is a divisor on Σ, then*

$$\dim H^0(\Sigma, \mathcal{O}_\Sigma(D)) - \dim H^1(\Sigma, \mathcal{O}_\Sigma(D)) = 1 - g + \deg D.$$

Proposition 47.11. *If Σ is a closed Riemann surface of genus g and $x \in \Sigma$, then there exists a non-constant meromorphic function on Σ which has a pole at x of order at most $g+1$ and is holomorphic away from x.*

Proof. Set

$$D = (g+1)\cdot\{x\}.$$

By Theorem 47.10,

$$\dim H^0(\mathcal{O}_\Sigma(D)) \geqslant 2.$$

Therefore, there is a non-constant $f \in H^0(\mathcal{O}_\Sigma(D))$ with the desired properties. □

Corollary 47.12. *If Σ is a closed Riemann surface of genus g, then there exists a holomorphic map $\pi\colon \Sigma \to \mathbf{C}P^1$ of degree at most $g+1$.*

Corollary 47.13. *If Σ is a closed Riemann surface of genus 0, then it is biholomorphic to $\mathbf{C}P^1$.*

Proof of Theorem 47.9 if $\chi(\Sigma) = 2$. If $\chi(\Sigma) = 2$, then $\Sigma = S^2$ a unique complex structure up to automorphism and the standard metric $g = g_1$ on S^2 satisfies $K_g = 1$. □

Proof of Theorem 47.9 if $\chi(\Sigma) = 0$. Denote by g a representative of the conformal class determined by j. By Theorem 47.3,

$$\int_\Sigma K_g \operatorname{vol}_g = 0.$$

Therefore, there exists a function f, unique up to an additive constant, such that

$$\begin{aligned} K_{e^{2f}g} &= e^{2f}\left(K_g + \Delta_g f\right) \\ &= 0. \end{aligned}$$ □

Proof of Theorem 47.9 if $\chi(\Sigma) = 0$. Denote by g a representative of the conformal class determined by j. By Theorem 47.3,

$$\int_\Sigma K_g \operatorname{vol}_g < 0.$$

The task at hand is to find a function f, such that

$$e^{2f}(K_g + \Delta_g f) = -1;$$

or, equivalently,

$$\Delta f + e^{2f} = -K_g.$$

Theorem 47.14 completes the proof. □

Theorem 47.14 (Kazdan and Warner [KW74]). *Let $n \in \mathbf{N}$ and $p > \frac{n}{2}$. Let (M, g) be a closed, oriented Riemannian manifold of dimension n. Let $a, b \in L^p(M)$ with*

$$\mu \geqslant 0, \qquad \int_M \mu > 0, \qquad \textit{and} \qquad \int_M A > 0.$$

There exists a unique solution $f \in W^{2,p}(M)$ of the ***Kazdan–Warner equation***

$$\Delta f + e^f \mu = A. \tag{47.15}$$

Proof. I learned the following proof from Dietmar Salamon.

Proposition 47.16. *Theorem 47.14 holds if and only if its holds with constant $A > 0$.*

Proof. Set

$$\bar{A} \coloneqq \fint_M A$$

and denote by f_0 the unique solution of

$$\Delta f_0 = A - \bar{A} \quad \text{and} \quad \int_M f_0 = 0.$$

A function $f \in W^{2,p}(M)$ solves (47.15) if and only if $f - f_0$ solves (47.15) with $e^{f_0}\mu$ instead of μ and $\bar{A}$ instead of A. ◻

Proposition 47.17. *If $A > 0$ is a constant, then the unique solution of (47.15) with $\mu = A$ is $f = 0$.*

Proof. Obviously, $f = 0$ is a solution. To prove uniqueness, suppose that

$$\Delta f + e^f A = A.$$

Denote by $x_\star$ a point at which f achieves its maximum and by $x_\dagger$ a point at which f achieves its minimum. Since

$$0 \leqslant \Delta f(x_\star) = A\Big(1 - e^{f(x_\star)}\Big) \quad \text{and} \quad 0 \geqslant \Delta f(x_\star) = A\Big(1 - e^{f(x_\dagger)}\Big)$$

and $A > 0$,

$$0 \leqslant \min f \leqslant \max f \leqslant 0.$$

Therefore, $f = 0$. ◻

Henceforth, fix a constant $A > 0$ and $\mu \in L^p$ as in Theorem 47.14. For $t \in [0, 1]$, set

$$\mu_t \coloneqq t\mu + (1 - t)A$$

and define $F_t\colon\ W^{2,p}(M) \to L^p(M)$ by

$$F_t(f) := \Delta f + e^f \mu_t - A.$$

Consider the parametrized moduli space

$$\mathfrak{M} := \left\{(t, f) \in [0, 1] \times W^{2,p}(M) : F_t(f) = 0\right\}.$$

Denote by $\pi\colon\ \mathfrak{M} \to [0, 1]$ the projection map.

Proposition 47.18. *$\mathfrak{M}$ is a smooth manifold of dimension one with boundary and the map $\pi\colon\ \mathfrak{M} \to [0, 1]$ is a submersion.*

Let me briefly recall the following.

Theorem 47.19 (Implicit Function Theorem). *Let X, Y, Z be Banach spaces. Let $F\colon\ X \times Y \to Z$ be a smooth map. If, for every $(x_\star, y_\star) \in F^{-1}(0)$, the derivative*

$$\mathrm{d}_{(x_\star, y_\star)} F\colon\ X \to Z$$

is invertible, then there exist open neighborhoods U of $x_\star$ and V of $y_\star$ and a smooth map $G\colon\ V \to U$ such that:

1. *for every $y \in V$, $F(G(y), y) = 0$ and*
2. *for every $(x, y) \in U \times V$, if $F(x, y) = 0$, then $x = G(y)$.*

Proof of Proposition 47.18. For every $t \in [0, 1]$ and $f \in W^{2,p}$,

$$\mathrm{d}_f F_t(\hat{f}) = (\Delta + e^f \mu_t)\hat{f}.$$

The operator

$$\mathrm{d}_f F_t\colon\ W^{2,p}(M) \to L^p(M)$$

is injective. To see that, observe that if $\mathrm{d}_f F_t(\hat{f}) = 0$, then

$$\begin{aligned} 0 &= \int_M \hat{f}(\Delta + e^f \mu_t)\hat{f} \\ &= \int_M |\nabla \hat{f}|^2 + e^f \mu_t |\hat{f}|^2. \end{aligned}$$

Since $e^f \mu_t$ is non-negative and positive somewhere, it follows that $\hat{f} = 0$. The derivative $\mathrm{d}_f F_t$ is an injective Fredholm operator of index zero; therefore, it is invertible. Thus the result follows from the Implicit Function Theorem. □

Proposition 47.20. *The map $\pi\colon\ \mathfrak{M} \to [0, 1]$ is proper; that is, if $(t_n, f_n)_{n \in \mathbf{N}}$ is a sequence in $\mathfrak{M}$ with $\lim t_n = t \in [0, 1]$, then a subsequence of (f_n) converges in $W^{2,p}(M)$ to a limit f with $(t, f) \in \mathfrak{M}$.*

The proof relies on the following a priori estimate.

Proposition 47.21. *Set*

$$\bar{\mu} := \fint_M \mu.$$

If f satisfies (47.15) with A constant, then

$$\|f\|_{L^\infty} \leqslant c\frac{A}{\bar{\mu}}\|\mu\|_{L^p} + \log\frac{A}{\bar{\mu}}.$$

Proof. The proof uses a rather clever barrier argument. Denote by v the unique solution of

$$\Delta v = \mu - \bar{\mu} \quad \text{and} \quad \int_M v = 0.$$

By standard elliptic theory,

$$\begin{aligned}
\|v\|_{L^\infty} &\leqslant c\|v\|_{W^{2,p}} \\
&\leqslant c\|\Delta v\|_{L^p} \\
&\leqslant c\|\mu - \bar{\mu}\|_{L^p} \\
&\leqslant c\|\mu\|_{L^p}.
\end{aligned}$$

In particular,

$$v - c\|\mu\|_{L^p} \leqslant 0.$$

For every $\varepsilon \geqslant 0$, set

$$\tilde{f}_\varepsilon := f + \frac{A-\varepsilon}{\bar{\mu}}(v + c_0\|\mu\|_{L^p}) - \log\left(\frac{A-\varepsilon}{\bar{\mu}}\right).$$

If $\tilde{f}_\varepsilon$ achieves its maximum at $x_\star$, then

$$\begin{aligned}
0 &\leqslant \Delta\tilde{f}_\varepsilon(x_\star) \\
&= A - e^{f(x_\star)}\mu(x_\star) + \frac{A-\varepsilon}{\bar{\mu}}(\mu(x_\star) - \bar{\mu}) \\
&= \varepsilon + \mu(x_\star)\left(\frac{A-\varepsilon}{\bar{\mu}} - e^{f(x_\star)}\right).
\end{aligned}$$

If $\varepsilon > 0$, then $\mu(x_\star) > 0$ and

$$f(x_\star) < \log\left(\frac{A-\varepsilon}{\bar{\mu}}\right).$$

Therefore,

$$\tilde{f}_\varepsilon(x_\star) < 0.$$

The above proves that, for every, $\max \tilde{f}_\varepsilon < 0$. Therefore,

$$\tilde{f}_0 \leqslant 0.$$

This implies that

$$\|f\|_{L^\infty} \leqslant c\frac{A}{\bar{\mu}}\|\mu\|_{L^p} + \log\frac{A}{\bar{\mu}}.$$

□

Proof of Proposition 47.20. It follows from Proposition 47.21 that there is a constant $\Lambda > 0$ such that, for every $n \in \mathbf{N}$, $\|f_n\|_{L^\infty} \leqslant \Lambda$. By standard elliptic theory,

$$\begin{aligned}\|f_n\|_{W^{2,p}} &\leqslant c(\|\Delta f_n\|_{L^p} + \|f_n\|_{L^p}) \\ &\leqslant c\|e^{f_n}\mu_t - A\|_{L^p} + c\|f_n\|_{L^p} \\ &\leqslant c(\Lambda).\end{aligned}$$

Therefore, by the sequential Banach–Alaoglou theorem, after passing to a subsequence, (f_n) converges in the weak $W^{2,p}$ topology to a limit f. Since the embedding $W^{2,p} \to L^\infty$ is compact, (f_n) converges to f in the L^∞ topology.

Again, by standard elliptic theory,

$$\begin{aligned}\|f_n - f_m\|_{W^{2,p}} &\leqslant c(\|\Delta(f_n - f_m)\|_{L^p} + \|f_n - f_m\|_{L^p}) \\ &\leqslant c\left\|e^{f_n}\mu_{t_n} - e^{f_m}\mu_{t_m}\right\|_{L^p} + c\|f_n - f_m\|_{L^p} \\ &\leqslant c\|f_n - f_m\|_{L^\infty}.\end{aligned}$$

Therefore, (f_n) is a Cauchy sequence in $W^{2,p}$ as well and converges. □

Proposition 47.18 and Proposition 47.20 imply that $\mathfrak{M}$ is a compact one-dimensional manifold with boundary and the map $\pi\colon \mathfrak{M} \to [0,1]$ is a submersion. This implies that π is a covering map. Since $0 \in [0,1]$ has precisely on preimage under π, it follows that π is a diffeomorphism. □

48 Kähler manifolds

Definition 48.1. A **Kähler manifold** is a Riemannian manifold (X, g) together with a parallel complex structure I which satisfies

$$g(I\cdot, I\cdot) = g.$$

The 2–form $\omega \in \Omega^2(X)$ defined by

$$\omega(\cdot,\cdot) \coloneqq g(I\cdot,\cdot)$$

is called the **Kähler form**.

Proposition 48.2. *If X is a Riemannian manifold and I is an almost complex structure satisfying*

$$g(I\cdot, I\cdot) = g,$$

then

$$\nabla I \Leftrightarrow \begin{cases} N_I = 0 \\ \mathrm{d}\omega = 0. \end{cases}$$

It is customary, in Kähler geometry to emphasize ω over g.

Definition 48.3. If (X, I, ω) is a Kähler manifold with Kähler metric $g = \omega(\cdot, I\cdot)$, then its **Ricci form** is defined by

$$\mathrm{Ric}_\omega = \mathrm{Ric}_g(I\cdot, \cdot). \tag{48.4}$$

Example 48.5 ($\mathbf{C}^n$). On $\mathbf{C}^n$ consider the coordinates $z_a := x_a + iy_a$ and metric

$$g_0 := \sum_{a=1}^{n} (\mathrm{d}x_a)^2 + (\mathrm{d}y_a)^2$$

as well as the complex structure defined by

$$I_0(\partial_{x_a}) := \partial_{y_a}$$

This makes $\mathbf{C}^n$ into a Kähler manifold. The associated Kähler form is

$$\omega_0 := g_0(I_0\cdot, \cdot) = \sum_{a=1}^{n} \mathrm{d}x_a \wedge \mathrm{d}y_a.$$

Define

$$\partial_{z_a} := \frac{1}{2}(\partial_{x_a} - i\partial_{y_a}) \quad \text{and} \quad \bar{\partial}_{z_a} := \frac{1}{2}(\partial_{x_a} + i\partial_{y_a})$$

and

$$\partial := \sum_{a=1}^{n} \mathrm{d}z_a \wedge \partial_{z_a} \quad \text{and} \quad \bar{\partial} := \sum_{a=1}^{n} \mathrm{d}\bar{z}_a \wedge \bar{\partial}_{z_a}$$

with

$$\mathrm{d}z_a = \mathrm{d}x_a + i\mathrm{d}y_a \quad \text{and} \quad \mathrm{d}\bar{z}_a = \mathrm{d}x_a - i\mathrm{d}y_a.$$

We have

$$\begin{aligned} \omega_0 &= -i\bar{\partial}\partial\left(\frac{|z|^2}{2}\right) \\ &= \sum_{a=1}^{n} \frac{i\mathrm{d}z_a \wedge \mathrm{d}\bar{z}_a}{2} \\ &= \sum_{a=1}^{n} \mathrm{d}x_a \wedge \mathrm{d}y_a. \end{aligned}$$

We say that $\frac{1}{2}|z|^2$ is a **Kähler potential**.

The following provides us with a plethora of easy to write down Kähler manifolds.

Proposition 48.6. *If (X, I, g) is a Kähler manifold and $Y \subset X$ is a complex submanifold, then $(Y, I|_Y, g|_Y)$ is a Kähler manifold.*

Corollary 48.7. *Every smooth projective variety is canonically equipped with the structure of a Kähler manifold.*

Define I on T^*X, by
$$I\alpha := \alpha \circ I.$$

Definition 48.8. Let (X, I) is a complex manifold. Set

$$\begin{aligned}
TX^{1,0} &:= \{v \in TX \otimes_{\mathbf{R}} \mathbf{C} : Iv = iv\} \\
TX^{0,1} &:= \{v \in TX \otimes_{\mathbf{R}} \mathbf{C} : Iv = -iv\} \\
T^*X^{1,0} &:= \{\alpha \in T^*X \otimes_{\mathbf{R}} \mathbf{C} : I\alpha = i\alpha\} \\
T^*X^{0,1} &:= \{\alpha \in T^*X \otimes_{\mathbf{R}} \mathbf{C} : I\alpha = -i\alpha\} \\
\Lambda^{p,q}T^*X &:= \Lambda^p_{\mathbf{C}} T^*X^{1,0} \otimes_{\mathbf{C}} \Lambda^q_{\mathbf{C}} T^*X^{0,1}.
\end{aligned}$$

We have
$$\Lambda^k T^*X \otimes_{\mathbf{R}} \mathbf{C} = \bigoplus_{p+q=k} \Lambda^{p,q}T^*X.$$

Set
$$\Omega^{p,q}(X) := \Gamma(\Lambda^{p,q}T^*X).$$

Define $\partial\colon \Omega^{p,q}(X) \to \Omega^{p+1,q}(X)$ and $\bar\partial\colon \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$ to be the composition of d with the projection onto the corresponding summand.

49 Fubini–Study metric

Example 49.1 ($\mathbf{C}P^n$). There is a Kähler form ω_{FS} on $\mathbf{C}P^n$ called the **Fubini–Study form** which when pulled back via $\sigma\colon \mathbf{C}^{n+1}\backslash\{0\} \to \mathbf{C}P^n$ can be written as

$$\begin{aligned}
\sigma^*\omega_{FS} &= -\frac{i}{2\pi}\bar\partial\partial\log|z|^2 \\
&= \frac{i}{2\pi}\partial\sum_{a=0}^n \frac{z_a \mathrm{d}\bar z_a}{|z|^2} \\
&= \frac{i}{2\pi}\left(\sum_{a=0}^n \frac{\mathrm{d}z_a \wedge \mathrm{d}\bar z_a}{|z|^2} - \sum_{a,b=0}^n \frac{\bar z_b \mathrm{d}z_b \wedge z_a \mathrm{d}\bar z_a}{|z|^4}\right).
\end{aligned}$$

Here is another way to think about ω_{FS}. Recall, that $\mathbf{C}P^n$ is equipped with the holomorphic atlas $\{\phi_a\colon \mathbf{C}P^n \supset U_a \to \mathbf{C}^n : a = 0, \dots, n\}$ with

$$\begin{aligned} U_a &\coloneqq \{[z_0 : \cdots : z_n] \in \mathbf{C}P^n : z_a \neq 0\} \quad \text{and} \\ \phi_a([z_0 : \cdots : z_n]) &\coloneqq \left(\frac{z_0}{z_a}, \dots, \frac{\widehat{z_a}}{z_a}, \dots, \frac{z_0}{z_a}\right). \end{aligned}$$

For $a = 0, \dots, n$, define $\hat{\omega}_{FS} \in \Omega^2(\mathbf{C}^n)$ by

$$\begin{aligned} \hat{\omega}_{FS} &\coloneqq -\frac{i}{2\pi}\bar{\partial}\partial \log\left(1 + \sum_{b=1}^{n}|w_b|^2\right) \\ &= \frac{i}{2\pi}\partial\bar{\partial}\log\left(1 + \sum_{b=1}^{n}|w_b|^2\right) \\ &= \frac{i}{2\pi}\partial\sum_{b=1}^{n}\frac{w_b\mathrm{d}\bar{w}_b}{1+|w|^2} \\ &= \frac{i}{2\pi}\left(\sum_{b=1}^{n}\frac{\mathrm{d}w_b \wedge \mathrm{d}\bar{w}_b}{1+|w|^2} - \sum_{b,c=1}^{n}\frac{\bar{w}_c\mathrm{d}w_c \wedge w_b\mathrm{d}\bar{w}_b}{(1+|w|^2)^2}\right). \end{aligned}$$

A computation shows that $\hat{\omega}_{FS}$ is a Kähler form on $\mathbf{C}^n$. A further computation shows that, for every $a, b = 0, \dots, n$, on $U_a \cap U_b$,

$$\phi_a^*\hat{\omega}_{FS} = \phi_b^*\hat{\omega}_{FS}.$$

Therefore, the 2–form $\phi_a^*\hat{\omega}_{FS}$ define a Kähler form on $\mathbf{C}P^n$. A moment's thought shows that under σ this form pulls back to $-\frac{i}{2\pi}\bar{\partial}\partial\log|z|^2$. Thus it agrees with ω_{FS}.

Proposition 49.2. *The Fubini study metric is an Einstein metric; more precisely:*

$$\mathrm{Ric}_{g_{FS}} = 2\pi(n+1)g_{FS}.$$

To prove that g_{FS} is Einstein is easy: just note the metric is $\mathrm{U}(n+1)$–invariant.

Remark 49.3 (Complex polar coordinates). Write the holomorphic coordinate z_b on $\mathbf{C}^n$ as

$$z_b = r_b e^{i\phi_b}$$

with $r_b \geqslant 0$ and $\phi_b \in \mathbf{R}$. Set

$$\begin{aligned} r &\coloneqq \sqrt{\sum_{a=1}^{n} r_a^2} \quad \text{and} \\ \theta &\coloneqq \sum_{a=1}^{n}\frac{r_a^2}{r^2}\mathrm{d}\phi_a. \end{aligned}$$

We compute

$$\begin{aligned}\sum_{a,b=1}^{n} \bar{z}_b \mathrm{d}z_b \wedge z_a \mathrm{d}\bar{z}_a &= \sum_{a,b=1}^{n} (r_b \mathrm{d}r_b + i r_b^2 \mathrm{d}\phi_b) \wedge (r_a \mathrm{d}r_a - i r_a^2 \mathrm{d}\phi_a)\\ &= -2i \sum_{a,b=1}^{n} r_b \mathrm{d}r_b \wedge r_a^2 \mathrm{d}\phi_a\\ &= -2ir^3 \mathrm{d}r \wedge \theta.\end{aligned}$$

Therefore,

$$\omega_0 = \pi r^2 \sigma^* \omega_{FS} + r\mathrm{d}r \wedge \theta.$$

Remark 49.4. Denote by $\rho\colon S^{2n+1} \to \mathbf{C}P^n$ the restriction of $\rho\colon \mathbf{C}^{n+1}\backslash\{0\} \to \mathbf{C}P^n$. This makes S^{2n+1} into a $\mathrm{U}(1)$–principal bundle over $\mathbf{C}P^n$ with the right $\mathrm{U}(1)$–action given by

$$z \cdot \lambda \coloneqq \lambda z.$$

This bundle is often called the **Hopf bundle** and we will denote it by L. The vector field v generating the $\mathrm{U}(1)$–action can be written as

$$v(z) = iz.$$

The radial vector field can be written as

$$\partial_r = z/|z|.$$

Therefore,

$$\begin{aligned}\theta(v) &= r^{-1}\omega_0(\partial_r, v)\\ &= |z|^{-2}\omega_0(z, iz)\\ &= 1.\end{aligned}$$

This proves that $i\theta \in \Omega^1(S^{2n+1}, \mathfrak{u}(1))$ is a connection 1–form on L. By Chern–Weil theory,

$$c_1(L) = \frac{i}{2\pi}[i\mathrm{d}\theta].$$

We compute $d\theta$ as follows,

$$\begin{aligned}\mathrm{d}\theta &= d(r^{-2} i_{r\partial_r}\omega_0)\\ &= -2r^{-3}\mathrm{d}r \wedge i_{r\partial_r}\omega_0 + r^{-2}\mathscr{L}_{r\partial_r}\omega_0\\ &= -2r^{-2}(r\mathrm{d}r \wedge \theta) + 2r^{-2}\omega_0\\ &= 2\pi\sigma^*\omega_{FS}.\end{aligned}$$

Therefore,

$$[\omega_{FS}] = -c_1(L)$$

In particular, $[\omega_{FS}]$ defines an integral cohomology class. This explains why we put a π in the definition of ω_{FS}. We can also think of $\mathbf{C}P^n$ as the Kähler reduction $\mathbf{C}^{n+1}/\!\!/\mathrm{U}(1)$. With the usual(?) conventions the resulting Kähler form will not be integral.

Definition 49.5. Let $n \in \mathbf{N}$ and $k \in \mathbf{Z}$. The holomorphic line $\mathscr{O}_{\mathbf{C}P^n}(k)$ is defined by

$$\mathscr{O}_{\mathbf{C}P^n}(k) := \big(\mathbf{C}^{n+1}\backslash\{0\} \times \mathbf{C}\big)/\mathbf{C}^*$$

with

$$\lambda \cdot (z_0, \dots, z_n, v) := (\lambda z_0, \dots, \lambda z_n, \lambda^k v).$$

Remark 49.6. If $\ell\colon\ \mathbf{C}^{n+1} \to \mathbf{C}$ is a linear function, then the map

$$\mathbf{C}P^n \ni [z_0 : \cdots : z_n] \mapsto [z_0 : \cdots : z_n : \ell(z_0, \dots, z_n] \in \mathscr{O}_{\mathbf{C}P^n}(1)$$

is a section. In fact, every holomorphic section of $\mathscr{O}_{\mathbf{C}P^n}(1)$ is of this form. Invariantly, if V is a complex vector space, then $H^0(\mathscr{O}_{\mathbf{P}V}(1)) = V^*$. Moreover, if $k \geqslant 0$, $H^0(\mathscr{O}_{\mathbf{P}V}(k)) = \mathrm{Sym}^k V^*$ is the space of homogeneous degree k polynomials on V. If $k < 0$, then $H^0(\mathscr{O}_{\mathbf{P}V}(k)) = 0$.

Recall that for a general principal G–bundle P and a representation $G \to \mathrm{GL}(V)$, we define

$$P \times_G V := \frac{P \times V}{G}$$

with the G–action given by

$$g(p, v) := (p \cdot g^{-1}, g \cdot v).$$

Denote by $\bar{L}$, the Hopf bundle but with the right $\mathrm{U}(1)$–action given by

$$z \cdot \lambda := \lambda^{-1} z.$$

Letting $\mathrm{U}(1)$ act on $\mathbf{C}$ via $\lambda \cdot v = \lambda^k v$, we have

$$\mathscr{O}_{\mathbf{C}P^n}(k) = \bar{L} \times_{\mathrm{U}(1)} \mathbf{C}.$$

In particular,

$$\begin{aligned} c_1(\mathscr{O}_{\mathbf{C}P^n}(k)) &= k \cdot c_1(\bar{L}) \\ &= k[\omega_{FS}]. \end{aligned}$$

Example 49.7 ($\mathrm{Bl}_0\, \mathbf{C}^n$). The **blow-up** of $\mathbf{C}^n$ at the origin is

$$\mathrm{Bl}_0\, \mathbf{C}^n := \{(\ell, x) : \ell \subset \mathbf{C}^n \text{ line}, x \in \ell\}.$$

The **exceptional divisor** in $\mathrm{Bl}_0\, \mathbf{C}^n$ is

$$E := \{(\ell, 0) \in \mathrm{Bl}_0(\mathbf{C}^n)\} \cong \mathbf{C}P^{n-1}.$$

Define $\varpi\colon\ \mathrm{Bl}_0\,\mathbf{C}^n \to \mathbf{C}^n$ by

$$\varpi(\ell, x) := x.$$

Define $\varsigma\colon\ \mathrm{Bl}_0\,\mathbf{C}^n \to \mathbf{C}P^{n-1}$ by

$$\varsigma(\ell, x) := \ell.$$

On $\mathrm{Bl}_0\,\mathbf{C}^n \backslash E$,

$$\varsigma = \sigma \circ \varpi.$$

For every $\varepsilon > 0$, there exists a unique Kähler form ω_ε on $\mathrm{Bl}_0\,\mathbf{C}^n$ such that, on $\mathrm{Bl}_0\,\mathbf{C}^n \backslash E$,

$$\omega_\varepsilon = \varpi^* \left[(\varepsilon^2 + \pi r^2)\sigma^* \omega_{FS} + r\mathrm{d}r \wedge \mathrm{d}\theta\right]$$

and, on E,

$$\omega_\varepsilon = \varepsilon^2 \omega_{FS}.$$

Note that as ε tends to zero, $(\mathrm{Bl}_0\,\mathbf{C}^n, g_\varepsilon)$ Gromov–Hausdorff converges to $\mathbf{C}^n$, but fails to converge in the $C^{1,\alpha}$ topology near (and because of) E.

50 Hermitian vector spaces

Definition 50.1. A **Hermitian vector space** is a real vector space V together with an inner product g and an endomorphism $I \in \mathrm{End}(V)$ satisfying

$$I^2 = -\mathrm{id}_V \quad \text{and} \quad g(I\cdot, I\cdot) = g.$$

The **Hermitian form** associated with (V, I, g) is the 2–form $\omega \in \Lambda^2 V^*$ defined by

$$\omega(\cdot, \cdot) = g(I\cdot, \cdot).$$

Proposition 50.2. *If (V, I, g) is a Hermitian vector space, then (V^*, I^*, g^*) is a Hermitian vector space.*

Definition 50.3. Let (V, I, g) be a Hermitian vector space. Set

$$V_{\mathbf{C}} := V \otimes_{\mathbf{R}} \mathbf{C}, \quad V^{1,0} := \{v \in V_{\mathbf{C}} : Iv = iv\}, \quad \text{and} \quad V^{0,1} := \{v \in V_{\mathbf{C}} : Iv = -iv\}.$$

For $p, q \in \mathbf{N}_0$, set

$$\Lambda^{p,q} V := \Lambda^p_{\mathbf{C}} V^{1,0} \otimes_{\mathbf{C}} \Lambda^q_{\mathbf{C}} V^{0,1}.$$

Remark 50.4. Set The extension of I to $\Lambda^\bullet_{\mathbf{C}} V_{\mathbf{C}}$ acts on the summand $\Lambda^{p,q} V$ as i^{p-q}.

Definition 50.5. Let (V, I, g) be a Hermitian vector space. The **Lefschetz operator** $L\colon \Lambda^\bullet V^* \to \Lambda^{\bullet+2}V^*$ is defined by

$$L \coloneqq \omega \wedge \cdot.$$

The **dual Lefschetz operator** $\Lambda\colon \Lambda^\bullet V^* \to \Lambda^{\bullet-2}V^*$ is the adjoint of Λ. A form $\alpha \in \Lambda^\bullet V^*$ is called **primitive** if $\Lambda\alpha = 0$. For $k \in \mathbf{N}$, set

$$P^k V^* \coloneqq \{\alpha \in \Lambda^k V^* : \Lambda\alpha = 0\}.$$

Remark 50.6. For every Hermitian vector space (V, I, g),

$$\Lambda = *^{-1} L * \quad \text{and} \quad \Lambda = *L *^{-1}.$$

Definition 50.7. Let (V, I, g) be a Hermitian vector space of complex dimension n. The **counting operator** $H\colon \Lambda^\bullet V^* \to \Lambda^\bullet V^*$ is defined by

$$H|_{\Lambda^k V^*} \coloneqq (k - n)\mathrm{id}_{\Lambda^k V^*}.$$

Proposition 50.8. *Let (V_1, I_1, g_1) and (V_2, I_2, g_2) be two Hermitian vector spaces. Denote the corresponding Lefschetz operators by L_1 and L_2, the dual Lefschetz operators by Λ_1 and Λ_2, and the counting operators by H_1 and H_2. The Lefschetz operator L, the dual Lefschetz operator Λ, and the counting operator associated with $(V \coloneqq V_1 \oplus V_2, I \coloneqq I_1 \oplus I_2, g \coloneqq g_1 \oplus g_2)$ are given by*

$$L = L_1 \otimes \mathrm{id}_{\Lambda^\bullet V_2^*} + \mathrm{id}_{\Lambda^\bullet V_1^*} \otimes L_2, \quad \Lambda = \Lambda_1 \otimes \mathrm{id}_{\Lambda^\bullet V_2^*} + \mathrm{id}_{\Lambda^\bullet V_1^*} \otimes \Lambda_2, \quad \textit{and}$$
$$H = H_1 \otimes \mathrm{id}_{\Lambda^\bullet V_2^*} + \mathrm{id}_{\Lambda^\bullet V_1^*} \otimes H_2.$$

Proposition 50.9. *For every Hermitian vector space (V, I, g) of complex dimension n,*

$$[H, L] = 2L, \quad [H, \Lambda] = -2\Lambda, \quad \textit{and} \quad [L, \Lambda] = H.$$

Proof. By Proposition 50.8, it suffices to prove the result for $V = \mathbf{C}$. In this case, it follows by a simple direct computation. □

Remark 50.10. The above shows that for every $\Lambda^\bullet V$ is a representation of the Lie algebra $\mathfrak{sl}_2(\mathbf{C})$.

Proposition 50.11. *Let (V, I, g) be a Hermitian vector space of complex dimension n. For every $\alpha \in \Lambda^k V^*$,*

$$[L^j, \Lambda]\alpha = j(k - n + j - 1)L^{j-1}\alpha.$$

Exercise 50.12. Proof this.

Theorem 50.13 (Lefschetz decomposition). *Let (V, I, g) be a Hermitian vector space of dimension n. For every $k \in \mathbf{N}_0$,*

$$\Lambda^k V^* = \bigoplus_{j=0}^{\lfloor k/2 \rfloor} L^j P^{k-2j} V^*.$$

Exercise 50.14. Proof this. Hint: you can use the representation theory of $\mathfrak{sl}_2(\mathbf{C})$.

Theorem 50.15 (Weil [Wei58, Théorème I.2]). *Let (V, I, g) be a Hermitian vector space of complex dimension n. For $j, k \in \mathbf{N}_0$ and every primitive form $\alpha \in \Lambda^k V^*$,*

$$*L^j\alpha = (-1)^{\binom{k+1}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j} I\alpha.$$

Definition 50.16. For a Hermitian vector space (V, I, g), define the operator $T\colon\ \Lambda^\bullet V^* \to \Lambda^\bullet V^*$ by

$$T|_{\Lambda^k V^*} = (-1)^{\binom{k+1}{2}} *^{-1} I.$$

Lemma 50.17 (Anthes [Ant, Proposition 1.6]). *For every Hermitian vector space (V, I, g),*

$$\exp(L)\exp(-\Lambda)\exp(L) = T.$$

Proof. In the situation of Proposition 50.8,

$$\exp(L)\exp(-\Lambda)\exp(L) = \exp(L_1)\exp(-\Lambda_1)\exp(L_1) \otimes \exp(L_2)\exp(-\Lambda_2)\exp(L_2).$$

Let T_1, T_2, and T be the operators associated with (V_1, I_1, g_1), (V_2, I_2, g_2), and (V, I, g) as in Definition 50.16. For $\alpha \in \Lambda^k V_1^*$ and $\beta \in \Lambda^\ell V_2^*$,

$$*^{-1}(\alpha \otimes \beta) = (-1)^{k\ell}(*^{-1}\alpha) \otimes (*^{-1}\beta);$$

therefore,

$$T(\alpha\otimes\beta) = (-1)^{\binom{k+\ell+1}{2}} *^{-1} I(\alpha\otimes\beta) = (-1)^{\binom{k+\ell+1}{2}+k\ell}(*^{-1}I_1\alpha)\otimes(*^{-1}I_2\beta) = (T_1\alpha)\otimes(T_2\beta).$$

As a consequence of the above, it suffices to verify the assertion for $V = \mathbf{C}$. In this case,

$$\exp(L)\exp(-\Lambda)\exp(L) = (1+L)(1-\Lambda)(1+L) = (1 + L - \Lambda + L - L\Lambda - \Lambda L - L\Lambda L).$$

Therefore,

$$\exp(L)\exp(-\Lambda)\exp(L)1 = \omega = T1 \quad \text{and} \quad \exp(L)\exp(-\Lambda)\exp(L)\omega = -1 = T\omega;$$

moreover, for every $\alpha \in V^*$, $*\alpha = I\alpha$ and thus

$$\exp(L)\exp(-\Lambda)\exp(L)\alpha = \alpha = T\alpha. \qquad \square.$$

Proof of Theorem 50.15. By Remark 50.6 and Lemma 50.17, for every $\alpha \in \Lambda^k V^*$,

$$*\exp(L)\alpha = (-1)^{\binom{k+1}{2}} \exp(L)\exp(-\Lambda)I\alpha.$$

Therefore, for every primitive $\alpha \in \Lambda^k V^*$,

$$*\sum_{j=0}^{\infty}\frac{L^j}{j!}\alpha = *\exp(L)\alpha = (-1)^{\binom{k+1}{2}}\exp(L)I\alpha = (-1)^{\binom{k+1}{2}}\sum_{\ell=0}^{\infty}\frac{L^\ell}{\ell!}I\alpha.$$

Comparing degrees finishes the proof. $\square$

51 The Kähler identities

Definition 51.1. For a Kähler manifold (X, I, g), $d^c\colon \Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$ is defined by

$$\mathrm{d}^c := I^{-1}\mathrm{d}I$$

Remark 51.2. The operator d^c satisfies

$$\mathrm{d}^c = -i(\partial - \bar\partial) \quad \text{and} \quad (\mathrm{d}^c)^* = i(\partial^* - \bar\partial^*).$$

Theorem 51.3 (Kähler identities). *For every Kähler manifold* (X, I, ω),

$$i[\Lambda, \bar\partial] = \partial^* \quad \textit{and} \quad i[\Lambda, \partial] = -\bar\partial^*.$$

Remark 51.4. The proof presented here is from Huybrecht's book. It has the feature of being essentially coordinate free, while most proofs rely crucially on proving the Kähler identities for $\mathbf{C}^n$.

These identities are equivalent to

$$[\Lambda, \mathrm{d}] = -(\mathrm{d}^c)^*.$$

Therefore, Theorem 51.3 follows from the following.

Proposition 51.5. *Let* (X, I, g) *be a Kähler manifold. For every primitive* $\alpha \in \Omega^k(X)$, *there are primitive forms* $\delta_0 \in \Omega^{k+1}(X)$ *and* $\delta_1 \in \Omega^{k-1}(X)$ *such that*

$$\mathrm{d}\alpha = \delta_0 + L\delta_1 \tag{51.6}$$

$$[\Lambda, d]L^j\alpha = -jL^{j-1}\delta_0 - (k-n+j-1)L^j\delta_1, \quad \textit{and} \tag{51.7}$$

$$-(\mathrm{d}^c)^*L^j\alpha = -jL^{j-1}\delta_0 - (k-n+j-1)L^j\delta_1. \tag{51.8}$$

Proof. Decompose $\mathrm{d}\alpha$ according to Theorem 50.13 as

$$\mathrm{d}\alpha = \sum_{\ell=0}^{\lfloor k/2 \rfloor} L^\ell \delta_\ell.$$

Since α is primitive, $L^{n-k+1}\alpha = 0$ and thus

$$\sum_{\ell=0}^{\lfloor k/2 \rfloor} L^{n-k+1+\ell}\delta_\ell = 0.$$

L^j is injective on $\Omega^k(X)$ for $j \leqslant n-k$. Therefore, for $\ell \geqslant 2$, $\delta_\ell = 0$. This proves (51.6).

By Proposition 50.11,

$$\begin{aligned}-\mathrm{d}\Lambda L^j\alpha &= \mathrm{d}[L^j,\Lambda]\alpha\\ &= j(k-n+j-1)L^{j-1}\mathrm{d}\alpha\\ &= j(k-n+j-1)L^{j-1}\delta_0 + j(k-n+j-1)L^j\delta_1\end{aligned}$$

and

$$\begin{aligned}\Lambda\mathrm{d}L^j\alpha &= \Lambda L^j\delta_0 + \Lambda L^{j+1}\delta_1\\ &= -[L^j,\Lambda]\delta_0 - [L^{j+1},\Lambda]\delta_1\\ &= -j(k-n+j)L^{j-1}\delta_0 - (j+1)(k-n+j-1)L^j\delta_1.\end{aligned}$$

This implies (51.7).

By Theorem 50.15,

$$\begin{aligned}-(\mathrm{d}^c)^*\alpha &= *I^{-1}\mathrm{d}I * L^j\alpha\\ &= (-1)^{\binom{k+1}{2}}\frac{j!}{(n-k-j)!} * I^{-1}\mathrm{d}IL^{n-k-j}I\alpha\\ &= (-1)^{\binom{k+1}{2}+k}\frac{j!}{(n-k-j)!}I^{-1} * L^{n-k-j}\mathrm{d}\alpha\\ &= (-1)^{\binom{k+1}{2}+k}\frac{j!}{(n-k-j)!}I^{-1} * L^{n-k-j}\delta_0 + (-1)^{\binom{k+1}{2}+k}\frac{j!}{(n-k-j)!}I^{-1} * L^{n-k-j+1}\delta_1\\ &= (-1)^{\binom{k+1}{2}+k+\binom{k+2}{2}}\frac{j!}{(n-k-j)!}\frac{(n-k-j)!}{(j-1)!}L^{j-1}\delta_0\\ &\quad + (-1)^{\binom{k+1}{2}+k+\binom{k}{2}}\frac{j!}{(n-k-j)!}\frac{(n-k-j+1)!}{j!}L^j\delta_1\\ &= -jL^{j-1}\delta_0 - (k-n+j-1)L^j\delta_1.\end{aligned}$$

□

52 The Chern connection

Proposition 52.1. *Let (X,I) be a complex manifold and $\mathscr{E} = (E,\bar\partial)$ be a holomorphic vector bundle over X. For every Hermitian metric H on $\mathscr{E}$, there exists a unique connection $A_{\bar\partial,H}$ on E which satisfies*

$$\nabla_{A_{\bar\partial,H}}H = 0 \quad \textit{and} \quad \nabla^{0,1}_{A_{\bar\partial,H}} = \bar\partial.$$

Definition 52.2. The connection ∇_H is called the **Chern connection** of $\mathscr{E}$. Set

$$\partial_H \coloneqq \nabla_H^{1,0} \quad \text{and} \quad F_H \coloneqq F_{A_{\bar\partial,H}}.$$

Proof of Proposition 52.1. It suffices to prove this statement for an open subset $U \subset \mathbf{C}^n$ and the trivial holomorphic vector bundle. The metric on this bundle is equivalent to a map $h\colon U \to i\mathfrak{u}(r)$. If the desired $A_{\bar\partial,H}$ exists, then we can write it as

$$\nabla_{A_{\bar\partial,H}} = \partial_H + \bar\partial$$

with

$$\partial_H\colon C^\infty(U,\mathbf{C}^r) \to \Omega^{1,0}(U,\mathbf{C}^r).$$

For every $s,t \in C^\infty(U,\mathbf{C}^r)$,

$$\begin{aligned} 0 &= \partial\langle hs,t\rangle - \langle \partial hs,t\rangle - \left\langle hs,\bar\partial t\right\rangle \\ &= \partial\langle hs,t\rangle - \left\langle h(\partial s + (h^{-1}\partial h)s),t\right\rangle - \left\langle hs,\bar\partial t\right\rangle. \end{aligned}$$

This proves that

$$\partial_H s = \partial s + (h^{-1}\partial h)s.$$ □

The proof also shows the following.

Proposition 52.3. *Let (X,I) be a complex manifold and $\mathscr{E} = (E,\bar\partial)$ be a holomorphic vector bundle over X. Let H_0 be a Hermitian metric on $\mathscr{E}$. Let h is a positive definite H_0–selfadjoint endomorphism of H_0. Set $H \coloneqq H_0 h$. The Chern connection associated with H_0 and H are related by*

$$\partial_H = \partial_{H_0} + h^{-1}\partial_{H_0}h.$$

Moreover,

$$F_H = F_{H_0} + \bar\partial(h^{-1}\partial_{H_0}h).$$

In particular: if E is a line bundle, then

$$F_H = F_{H_0} + \bar\partial\partial_{H_0}\log h.$$

Example 52.4. Let $n \in \mathbf{N}$ and $k \in \mathbf{Z}$. Define a Hermitian metric H_k on $\mathscr{O}_{\mathbf{CP}^n}(k)$ by

$$H_k([z_0:\cdots,z_n:s],[z_0:\cdots,z_n:t]) \coloneqq \left(\sum_{a=0}^n |z_a|^2\right)^{-k}\cdot s\bar t.$$

Over $U_a = \{[z_0\cdots,z_n] : z_a \neq 0\}$, the section $[z_0\cdots:z_n] \mapsto [z_0:\cdots:z_n:z_a^k]$ trivializes $\mathscr{O}_{\mathbf{CP}^n}(k)$. Pushing forward via $\phi_a\colon U_a \to \mathbf{C}^n$ we obtain the metric on the trivial bundle over $\mathbf{C}^n$ induced by the function

$$h_k(w_1,\ldots,w_n) \coloneqq \left(1+|w|^2\right)^{-k}.$$

This shows that

$$(\phi_a)_*F_{H_k} = -k\bar\partial\partial\log\left(1+|w|^2\right)$$

Therefore,

$$F_{H_k} = \frac{2\pi}{i} k\omega_{FS}.$$

This implies, in particular, that

$$c_1(\mathscr{O}_{\mathbf{C}P^n}(k)) = k[\omega_{FS}].$$

53 The canonical bundle and Ricci curvature

Definition 53.1. Let (X, I) be a complex manifold of complex dimension n. The **canonical bundle** of X is

$$K_X \coloneqq \Lambda^n_{\mathbf{C}} T^*X^{1,0}.$$

The **anti-canonical bundle** of X is

$$K_X^{-1} = \Lambda^n_{\mathbf{C}} TX^{1,0}.$$

Proposition 53.2. *Let $n \in \mathbf{N}$.*

1. *For every $a, b = 0, \dots, n$, the holomorphic vector field $z_a \partial_{z_b}$ on $\mathbf{C}^{n+1}$ is the lift of a holomorphic vector field on $\mathbf{C}P^n$.*

2. *The **Euler sequence***

$$0 \longrightarrow \mathscr{O}_{\mathbf{C}P^n} \xrightarrow{\begin{pmatrix} z_0 \\ \vdots \\ z_n \end{pmatrix}} \mathscr{O}_{\mathbf{C}P^n}(1)^{\oplus(n+1)} \xrightarrow{\begin{pmatrix} \partial_{z_0} & \cdots & \partial_{z_n} \end{pmatrix}} T\mathbf{C}P^n \longrightarrow 0$$

   is exact.

Proposition 53.3. *For every $n \in \mathbf{N}$,*

$$K_{\mathbf{C}P^n} \cong \mathscr{O}_{\mathbf{C}P^n}(-n-1)$$

Proof. By Proposition 53.2,

$$\begin{aligned} K_X^{-1} &\cong \Lambda^n T\mathbf{C}P^n \\ &\cong \Lambda^n T\mathbf{C}P^n \otimes \mathscr{O}_{\mathbf{C}P^n} \\ &\cong \Lambda^{n+1} \mathscr{O}_{\mathbf{C}P^n}(1)^{\oplus(n+1)} \\ &\cong \mathscr{O}_{\mathbf{C}P^n}(n+1). \end{aligned}$$

□

Proposition 53.4 (Adjunction formula). *Let (X, I) be a complex manifold, Let E be a holomorphic vector bundle over X and let $f \in H^0(N)$ be a holomorphic section of E which is transverse to the zero section. The canonical bundle of $Z = Z(f)$ is*

$$K_Z \cong K_X|_Z \otimes \det(E)|_Z.$$

Proof. Since f is transverse to the zero section, df induces an isomorphism

$$NZ \overset{\mathrm{d}f}{\cong} E.$$

Therefore, the short exact sequence

$$0 \to TZ \to TX|_Z \to NZ \to 0$$

induces the desired isomorphism. □

Example 53.5. Let $k > 0$. Let $f \in H^0(\mathcal{O}_{\mathbf{C}P^n}(k))$ be a generic homogeneous degree k polynomial in $n+1$ variables. By Bertini's theorem,

$$\begin{aligned} X &= Z(f) \\ &\coloneqq \{[z_0 : \cdots : z_n] \in \mathbf{C}P^n : f(z_0, \cdots, z_n) = 0\} \end{aligned}$$

is a smooth hypersurface. By Proposition 53.4,

$$\begin{aligned} K_X &= \mathcal{O}_{\mathbf{C}P^n}(-n-1) \otimes \mathcal{O}_{\mathbf{C}P^n}(k) \\ &= \mathcal{O}_{\mathbf{C}P^n}(k-n-1). \end{aligned}$$

Exercise 53.6. Let $f \in H^0(\mathbf{C}P^n, \mathcal{O}_{\mathbf{C}P^n}(2k))$. Set

$$X \coloneqq \{([z], v) \in \mathcal{O}_{\mathbf{C}P^n}(k) : v \otimes v = f([z])\}.$$

This is a cover of $\mathbf{C}P^n$ branched over $Z(f)$. Prove that

$$\begin{aligned} K_X &\cong K_{\mathbf{C}P^n} \otimes \mathcal{O}_{\mathbf{C}P^n}(2k) \\ &\cong \mathcal{O}_{\mathbf{C}P^n}(2k-n-1). \end{aligned}$$

Proposition 53.7. *Let (X, I, g) be a Kähler manifold.*

1. *The formula*
$$h = g - i\omega$$
defines a Hermitian metric on TX.

2. *The Hermitian metric h induces a Hermitian metric H on K_X.*

3. *The connection ∇^{K_X} on K_X induced by the Levi-Civita connection is the Chern connection of H; that is: it is unitary and induces the holomorpic structure.*

4. *The curvature of ∇^{K_X} satisfies*

$$\begin{aligned} F_{\nabla^{K_X}} &= \mathrm{tr}_{\mathbf{C}} F_{\nabla^{T^*X}} \\ &= i\mathrm{Ric}_\omega. \end{aligned}$$

Proof. The only non-trivial statement is the last. To prove it fix $x \in X$ and let $e_1, \dots, e_n$ be a unitary basis of $T_x M$. Set $f_a \coloneqq I e_a$. Then

$$\begin{aligned}
\mathrm{Ric}(Iv, w) &= \sum_{a=1}^{n} \langle R(e_a, w) Iv, e_a \rangle + \langle R(f_a, w) Iv, f_a \rangle \\
&= \sum_{a=1}^{n} -\langle R(e_a, w) v, f_a \rangle + \langle R(f_a, w) v, e_a \rangle \\
&= \sum_{a=1}^{n} -\langle R(v, f_a) e_a, w \rangle + \langle R(f_a, w) v, e_a \rangle \\
&= \sum_{a=1}^{n} \langle R(v, f_a) w, e_a \rangle + \langle R(f_a, w) v, e_a \rangle \\
&= -\sum_{a=1}^{n} \langle R(v, w) e_a, f_a \rangle.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathrm{tr}_{\mathbf{C}}\, F_{\nabla^{T^*X}}(v, w) &= -\,\mathrm{tr}_{\mathbf{C}}\, R(v, w) \\
&= -\sum_{a=1}^{n} \langle R(v, w) e_a, e_a \rangle - i \langle I R(v, w) e_a, e_a \rangle \\
&= i \langle R(v, w) f_a, e_a \rangle.
\end{aligned}$$

□

Proposition 53.8. *For every Kähler manifold (X, I, g, ω),*

$$c_1(X) = \frac{1}{2\pi} [\mathrm{Ric}_\omega].$$

Proof. This follows from the above and Chern–Weil theory. □

Proposition 53.9. *For every closed Kähler manifold (X, I, g, ω) of complex dimension n, then*

$$\int_X \mathrm{scal}_g\, \omega^n = 4\pi n \big\langle c_1(X) \cup [\omega]^{n-1}, [X] \big\rangle.$$

Proof. On $\mathbf{C}^n$,

$$\begin{aligned}
\mathrm{d}z_a \wedge \mathrm{d}\bar{z}_b \wedge \omega_0^{n-1} &= \left(\frac{i}{2}\right)^{n-1} \mathrm{d}z_a \wedge \mathrm{d}\bar{z}_b \wedge \left(\sum_{c=1}^{n} \mathrm{d}z_c \wedge \mathrm{d}\bar{z}_c \right)^{n-1} \\
&= \left(\frac{i}{2}\right)^{n-1} (n-1)! \delta_{ab} \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \dots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n \\
&= \frac{2}{ni} \delta_{ab} \omega_0^n.
\end{aligned}$$

Hence, if $\alpha \in \Lambda^2 T^*{}_x X$, then, for $e_1, \dots, e_n$ a unitary basis of $T_x X$,

$$\begin{aligned}\frac{\alpha \wedge \omega^{n-1}}{\omega^n} &= \frac{2}{ni} \sum_{a=1}^{n} \alpha\left(\tfrac{1}{2}(e_a - iIe_a), \tfrac{1}{2}(e_a + iIe_a)\right) \\ &= \frac{1}{n} \sum_{a=1}^{n} \alpha(e_a, Ie_a).\end{aligned}$$

Since

$$\mathrm{Ric}_g(Ie_a, Ie_a) = \mathrm{Ric}_g(e_a, e_a),$$

it follows that

$$\begin{aligned}\frac{\mathrm{Ric}_\omega \wedge \omega^{n-1}}{\omega^n} &= \frac{1}{n} \sum_{a=1}^{n} \mathrm{Ric}_\omega(e_a, Ie_a) \\ &= \frac{1}{2n} \mathrm{scal}_g.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_X \mathrm{scal}_g\, \omega^n &= 2n \int_X \mathrm{Ric}_\omega \wedge \omega^{n-1} \\ &= 4\pi n \left\langle c_1(X) \cup [\omega]^{n-1}, [X] \right\rangle.\end{aligned}$$

□

Example 53.10. The round metric on S^2 has constant scalar curvature 2, volume 4π, and

$$\left\langle c_1(S^2), [S^2] \right\rangle = \left\langle e(S^2), [S^2] \right\rangle = 2.$$

Proposition 53.11. *If (X, I, ω) is a Kähler–Einstein manifold with*

$$\mathrm{Ric}_\omega = \lambda\omega,$$

then

$$c_1(X) = \frac{\lambda}{2\pi}[\omega].$$

Therefore,

$$\lambda = \frac{2\pi n!}{\mathrm{vol}(X)} \left\langle c_1(X) \cup [\omega]^{n-1}, [X] \right\rangle$$

Proof. The Einstein equation implies

$$c_1(X) = -c_1(K_X) = \left[\frac{F_{\nabla^{K_X}}}{2\pi i}\right] = \left[\frac{\mathrm{Ric}_g(I\cdot, \cdot)}{2\pi}\right] = \frac{\lambda}{2\pi}[\omega].$$

□

Proof of Proposition 49.2. We need to compare H_{-n-1} and the Hermitian metric induced by the Fubini–Study metric. We work in the trivialization employed in Example 52.4. The Fubini–Study metric induces the Hermitian metric H

$$H(s,t) := \frac{1}{2\pi}\Big[\big(1+|w|^2\big)^{-1}t^*s - \big(1+|w|^2\big)^{-2}t^*ww^*s\Big].$$

The metric on $\Lambda^n TCP^n$ induced by this is

$$\Lambda^n_{\mathbf{C}} H = \det_{\mathbf{C}} H.$$

This is a nuisance to compute directly; fortunately, we don't need to. By $\mathrm{U}(n+1)$–invariance, H and H_{-n-1} agree up to a constant. Another way to figure out what $\det H$ is to observe that minus its imaginary part must be

$$\hat{\omega}^n_{FS},$$

which, on U_a, is

$$\left[\frac{i}{2\pi}\left(\sum_{b=1}^n \frac{\mathrm{d}w_b\wedge \mathrm{d}\bar{w}_b}{1+|w|^2} - \sum_{b,c=1}^n \frac{\bar{w}_c\mathrm{d}w_c\wedge w_b\mathrm{d}\bar{w}_b}{(1+|w|^2)^2}\right)\right]^n = c(n)\frac{\mathrm{d}w_1\wedge\ldots\wedge \mathrm{d}w_n\wedge \mathrm{d}\bar{w}_1\wedge\ldots\wedge\mathrm{d}\bar{w}_n}{(1+|w|^2)^{n+1}}$$

Therefore,

$$\begin{aligned} i\mathrm{Ric}_{g_{FS}}(I\cdot,\cdot) &= F_{\nabla^{K_X}} \\ &= F_{H_{-n-1}} \\ &= 2\pi i(n+1)\omega_{FS}. \end{aligned}$$

This completes the proof. □

Example 53.12. The following generalized the Poincaré metric on the unit disk in C. Denote by D^n the unit disc in $\mathbf{C}^n$. Define a Kähler form on D by

$$\begin{aligned} \omega &= i\bar{\partial}\partial\frac{|z|^2-1}{2} \\ &= \frac{i}{2}\frac{\sum_{a=1}^n \mathrm{d}z^a\wedge\mathrm{d}\bar{z}^a}{1-|z|^2}. \end{aligned}$$

Up to a constant this induces the metric

$$\big(1+|z|^2\big)^{-2}$$

on K_{D^n} and thus

$$\begin{aligned} \mathrm{Ric}_\omega &= i\partial\bar{\partial}\big(1+|z|^2\big)^{-2} \\ &= -2i\big(1+|z|^2\big)^{-3}\sum_{a=1}^n z^a\mathrm{d}\bar{z}^a. \end{aligned}$$

54 The existence of Kähler–Einstein metrics

Definition 54.1. Let (X, I) be a complex manifold. A cohomology class $\alpha \in H^2_{\mathrm{dR}}(X)$ is called **positive (negative)** if is represented by (minus) a Kähler form on X.

It follows from our earlier discussion that if (X, I) admits a Kähler–Einstein metric, then one of the following holds: $c_1(X)$ is negative, $c_1(X) = 0$, or $c_1(X)$ is positive. It is a reasonable question to ask whether the converse holds. These results answer the question if $c_1(X) \leqslant 0$. If $c_1(X) > 0$, then it depends on a subtle stability condition.

Theorem 54.2 (Aubin [Aub76, Théorème 2]). *If (X, I) is a closed, connected complex manifold with negative first Chern class $c_1(X)$, then it admits a Kähler–Einstein metric; moreover, this metric is unique up to a constant.*

Theorem 54.3 (Yau [Yau78]). *If (X, I) is a closed, connected complex manifold with vanishing first Chern class $c_1(X)$, then every Kähler class $[\omega]$ contains a unique Ricci flat Kähler metric.*

Proposition 54.4. *Let (X, I) be a complex manifold. Let ω_0 and ω be Kähler forms and let $f \in C^\infty(X)$. If*

$$\omega^n = e^f \cdot \omega_0^n,$$

then

$$\mathrm{Ric}_\omega = \mathrm{Ric}_{\omega_0} + i\partial\bar{\partial} f.$$

Proof. By hypothesis the Hermitian metrics on K_X are related by

$$H = e^{-f} H_0.$$

Therefore,

$$\begin{aligned} \mathrm{Ric}_\omega &= -iF_H \\ &= -i(F_{H_0} + \bar{\partial}\partial \log e^{-f}) \\ &= \mathrm{Ric}_{\omega_0} + i\bar{\partial}\partial f. \end{aligned}$$

□

Corollary 54.5. *Let (X, I, ω_0) be a Kähler manifold of complex dimension n. Let $f \in C^\infty(X)$ be such that*

$$\omega = \omega_0 + i\bar{\partial}\partial f$$

is a Kähler class. Then

$$\mathrm{Ric}_\omega = \mathrm{Ric}_{\omega_0} + i\bar{\partial}\partial \log\left[\frac{(\omega_0 + i\bar{\partial}\partial f)^n}{\omega_0^n}\right].$$

Lemma 54.6 ($\bar\partial\partial$ lemma). *Let (X, I, ω) be a closed Kähler manifold. Let $\alpha \in \Omega^{p,q}(X, \mathbf{C})$. The following are equivalent:*

1. *There is a $\beta \in \Omega^{p+q-1}(X, \mathbf{C})$ such that $\alpha = d\beta$.*
2. *There is a $\gamma \in \Omega^{p-1,q-1}(X, \mathbf{C})$ such that $\alpha = \bar\partial\partial\gamma$.*

Proof. Suppose that $\alpha = d\beta$. Decomposing $\beta = \beta^{p-1,q} + \beta^{p,q-1}$,

$$\alpha = \partial\beta^{p-1,q} + \bar\partial\beta^{p,q-1}, \quad \bar\partial\beta^{p-1,q} = 0, \quad \text{and} \quad \partial\beta^{p,q-1} = 0.$$

By Hodge theory,

$$\beta^{p-1,q} = \partial\gamma^{p-2,q} + \bar\partial\gamma_-^{p-1,q-1} + \beta_0^{p-1,q} \quad \text{and} \quad \beta^{p,q-1} = \partial\gamma_+^{p-1,q-1} + \bar\partial\gamma^{p,q-2} + \beta_0^{p,q-1}.$$

Therefore,

$$\alpha = \partial\bar\partial\gamma_-^{p-1,q-1} + \bar\partial\partial\gamma_+^{p-1,q-1} = \bar\partial\partial\left(\gamma_+^{p-1,q-1} - \gamma_-^{p-1,q-1}\right). \qquad \square$$

Let $\lambda \in \mathbf{R}$. Suppose $[\mathrm{Ric}_{\omega_0}] = \lambda[\omega_0]$. Then for some $\rho \in C^\infty(X, \mathbf{R})$

$$\mathrm{Ric}_{\omega_0} = \lambda\omega_0 - i\bar\partial\partial\rho.$$

We call ρ a **Ricci potential**. Since ρ is only unique up to a constant, we impose the normalization

$$\int_X e^\rho\omega_0^n = \int_X \omega_0^n.$$

Thus the condition for $\omega = \omega_0 + i\bar\partial\partial f$ to satisfy

$$\mathrm{Ric}_\omega = \lambda\omega$$

becomes

$$\begin{aligned} 0 &= \mathrm{Ric}_{\omega_0} + i\bar\partial\partial\log\left[\frac{(\omega_0 + i\bar\partial\partial f)^n}{\omega_0^n}\right] - \lambda(\omega_0 + i\bar\partial\partial f) \\ &= i\bar\partial\partial\left[\log\left[\frac{(\omega_0 + i\bar\partial\partial f)^n}{\omega_0^n}\right] - \lambda f - \rho\right]. \end{aligned}$$

This equation is equivalent to

$$\log\left[\frac{(\omega_0 + i\bar\partial\partial f)^n}{\omega_0^n}\right] - \lambda f - \rho = \text{constant}.$$

Let us write this as

$$\frac{(\omega_0 + i\bar\partial\partial f)^n}{\omega_0^n} = e^{\lambda f}e^\rho e^c.$$

The constant c is determined by

$$c = -\log \frac{\int e^{\lambda f} e^{\rho} \omega_0^n}{\int_X \omega_0^n}.$$

If $\lambda = 0$, then, by our normalization of ρ, $c = 0$. If $\lambda \neq 0$, then f solves the above equation if and only if $f - \frac{c}{\lambda}$ satisfies

$$\frac{(\omega_0 + i\bar{\partial}\partial f)^n}{\omega_0^n} = e^{\lambda f} e^{\rho} \tag{54.7}$$

and both define the same 2–form $\omega = \omega_0 + i\bar{\partial}\partial f$. Therefore, in all cases, we are lead to (54.7)—a **complex Monge–Ampère equation**. The solution theory for these equation heavily depends on the sign of λ: the difficulty increases drastically as λ goes from negative to zero to positive. The theory for $\lambda < 0$ is due to Aubin [Aub76, Théorème 1]. The theory for $\lambda = 0$ is due to Yau [Yau78] and also proves the following.

Theorem 54.8 (Yau [Yau78]; Calabi's conjecture). *Let (X, I) be a closed complex manifold. If $\rho \in \Omega^{1,1}(M, \mathbf{R})$ represents $c_1(X)$, then in every Kähler class $[\omega]$ there exists a unique Kähler form ω satisfying*

$$\mathrm{Ric}_\omega = \rho.$$

A clear exposition of the proofs of Theorem 54.2, Theorem 54.3, and Theorem 54.8 can be found in Błocki [Bło12].

The case $\lambda > 0$ is extremely delicate and tied up with an algebro-geometric stability condition; but it has now been solved by Chen, Donaldson, and Sun [CDS15a; CDS15b; CDS15c].

55 Bisectional curvature and complex space forms

Definition 55.1 (Goldberg and Kobayashi [GK67, Equation (2)]). Let (X, I, g) be a Kähler manifold. The **bisectional curvature** $\sec\colon TX \times TX \to \mathbf{R}$ is defined by

$$\sec^{\mathbf{C}}(v, w) = \frac{\langle R(v, Iv)Iw, w\rangle}{|v|^2|w|^2}.$$

Remark 55.2. It is easy to see that $\sec^{\mathbf{C}}(v, w)$ does only depend on the 2–planes $\langle v, Iv\rangle$ and $\langle w, Iw\rangle$.

Proposition 55.3. *If v, w are unit-vectors, then*

$$\sec^{\mathbf{C}}(v, w) = \sec(v \wedge w) + \sec(v \wedge Jv).$$

Theorem 55.4. *If (X, I, g) is a simply-connected Kähler manifold with constant bisectional curvature, then its (up to a constant scalar factor) it is isometric to $\mathbf{C}P^n$, $\mathbf{C}^n$, or D^n.*

Proof. The proof is similar to that of Theorem 6.5. See [Tia00, Theorem 1.12] for details. ◻

56 The Miyaoka–Yau inequality

Theorem 56.1 (Guggenheimer [Gug52] and Chen and Ogiue [CO75, Theorem 2]). *If (X, I, ω) is a closed Kähler–Einstein manifold of complex dimension n, then*

$$\left\langle \left(\frac{2(n+1)}{n} c_2(X) - c_1(X)^2\right) \cup [\omega]^{n-2}, [X] \right\rangle \geqslant 0.$$

Moreover, equality holds if only if X has constant bisectional curvature.

Theorem 56.2 (Yau [Yau77, Remark (iii)]). *If (X, I, ω) is a closed Kähler manifold with $c_1(X) \leqslant 0$, then*

$$\left\langle \left(\frac{2(n+1)}{n} c_2(X) - c_1(X)^2\right) \cup [\omega]^{n-2}, [X] \right\rangle \geqslant 0.$$

Moreover, equality holds if and only if the universal cover $\tilde{X}$ is biholomorphic to $\mathbf{C}P^n$, $\mathbf{C}^n$, or D^n.

This relies on a computation similar to the one that proves Proposition 57.4.

Theorem 56.3 (Miyaoka–Yau inequality [Miy77, Theorem 4; Yau77, Theorem 4]). *If (X, I, ω) is a closed Kähler surface with $c_1(X) \leqslant 0$, then*

$$c_1(X)^2 \leqslant 3c_2(X).$$

Moreover, equality holds if and only if the universal cover $\tilde{X}$ is biholomorphic to $\mathbf{C}P^2$, $\mathbf{C}^2$, or D^2.

Remark 56.4. Miyaoka [Miy77, Theorem 4] proved this inequality for complex surfaces of general type. Miyaoka [Miy77, Section 6.A] gives a construction by A. Borel and Hirzebruch of infinitely many ball quotients realizing infinitely many values of $c_1^2 = 3c_2$.

Remark 56.5. The rigidity statement is due to Yau [Yau77, Theorem 4].

As a consequence of Theorem 56.3 and a number of results in algebraic geometry, Yau also proved the following result conjectured by Severi [Sev54].

Theorem 56.6 (Yau [Yau77, Theorem 5]). *Every complex surface that is homotopy equivalent to $\mathbf{C}P^2$ is biholomorphic to $\mathbf{C}P^2$.*

The key point is that it was already known by algebro-geometric methods that such surfaces saturate the Miyaoka–Yau inequality.

57 Hermitian–Einstein metrics

Proposition 57.1. *Let (X, I, ω) be a Kähler manifold. For every $f \in C^\infty(X)$,*

$$\begin{aligned} \Delta f &= 2\partial^* \partial f, \\ \Delta &= 2\bar{\partial}^* \bar{\partial} f, \quad \textit{and} \\ \Delta f &= 2i\Lambda \bar{\partial} \partial f \end{aligned}$$

Proof. By Theorem 51.3,

$$\partial^*\partial f = i\Lambda\bar{\partial}\partial f = -i\Lambda\partial\bar{\partial} f = \bar{\partial}^*\bar{\partial} f. \qquad \square$$

Definition 57.2. Let (X, I, ω) be Kähler manifold and let $\mathscr{E}$ be a holomorphic vector bundle of rank r over X. A Hermitian metric H on $\mathscr{E}$ is **projectively Hermitian Yang–Mills** if

$$i\Lambda F_H - \frac{1}{r}\operatorname{tr}(i\Lambda F_H)\cdot \mathrm{id}_E = 0.$$

A Hermitian metric H on $\mathscr{E}$ is **Hermitian Einstein** if it is projectively Yang–Mills and $\operatorname{tr}(i\Lambda F_H)$ is locally constant.

Proposition 57.3. *Let (X, I, ω) be closed Kähler manifold and let $\mathscr{E}$ be a holomorphic vector bundle of rank r over X. If H is a projectively Hermitian Yang–Mills metric on $\mathscr{E}$, then there exists a function $f \in C^\infty(X, \mathbf{R})$ such that He^f is Hermitian Einstein; moreover, f is unique up to an additive constant.*

Proof. For every $f \in C^\infty(X, \mathbf{R})$,

$$i\Lambda F_{He^f} = i\Lambda F_H + i\Lambda\bar{\partial}\partial f \cdot \mathrm{id}_E = i\Lambda F_H + \frac{1}{2}\Delta f \cdot \mathrm{id}_E$$

Therefore, H is projectively Hermitian Yang–Mills if and only if He^f is. Moreover, He^f is Hermitian Einstein if and only if

$$\frac{r}{2}\Delta f = \operatorname{tr}(i\Lambda F_H) - \fint_X \operatorname{tr}(i\Lambda F_H). \qquad \square$$

Proposition 57.4. *Let $\mathscr{E}$ be a holomorphic bundle of rank r and let H be a Hermitian metric on $\mathscr{E}$.*

$$\begin{aligned}\int_X |F_H - \tfrac{1}{r}\operatorname{tr}(F_H)\cdot \mathrm{id}_E|^2 = &\int_X |i\Lambda F_H - \tfrac{1}{r} i\Lambda\operatorname{tr}(F_H)\cdot \mathrm{id}_E)|^2\\ &+ 4\pi\Big\langle \big(2c_2(E) - \tfrac{r-1}{r}c_1(E)^2\big) \cup \tfrac{[\omega]^{n-2}}{(n-2)!}, [X]\Big\rangle.\end{aligned}$$

Here we use the inner product $-\operatorname{tr}$ on $\mathfrak{u}(E, H)$.

The proof relies on the following.

Proposition 57.5. *Let (X, I, ω) be a Kähler manifold. For every $\alpha \in \Omega^{1,1}(X)$,*

$$\alpha \wedge \alpha \wedge \omega^{n-2} = (n-2)!\big(|\Lambda\alpha|^2 - |\alpha|^2\big)\,\mathrm{vol}_g$$

Proof. For $\alpha = \omega$, the left-hand side of the desired identity is $n!\mathrm{vol}_g$ and the right-hand side is $(n-2)!(n^2-n)\mathrm{vol}_g$. Thus it remains to prove the assertion for $\alpha \in \Lambda^{1,1}_0 T^*X = \ker \Lambda$. Since $\Lambda^{1,1}_0 T^*{}_x X$

irreducible representation of $\mathrm{U}(T_xX)$ and the identity is invariant under this group, it suffices to verify the identity in one case. To this end we compute.

$$\begin{aligned}\frac{i}{2}\mathrm{d}z_a \wedge \mathrm{d}\bar{z}_a \wedge \omega_0^{n-2} &= \left(\frac{i}{2}\right)^{n-1}(n-2)!\sum_{b\neq a}\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \ldots \wedge \widehat{\mathrm{d}z_b \wedge \mathrm{d}\bar{z}_b} \wedge \ldots \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n \\ &= \frac{i}{2}(n-2)!\sum_{b\neq a} *(\mathrm{d}z_b \wedge \mathrm{d}\bar{z}_b).\end{aligned}$$

This implies

$$\begin{aligned}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 - \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2) \wedge \frac{\omega_0^{n-2}}{(n-2)!} &= \sum_{a\neq 1} *(\mathrm{d}z_a \wedge \mathrm{d}\bar{z}_a) - \sum_{b\neq 2} *(\mathrm{d}z_b \wedge \mathrm{d}\bar{z}_b) \\ &= - * (\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 - \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2). \qquad \square\end{aligned}$$

Proof of Proposition 57.4. By Chern–Weil theory

$$c(X) = 1 + c_1(X) + c_2(X) + \cdots = \left[\det\left(\mathbf{1} + \frac{i}{2\pi}F_H\right)\right].$$

In particular,

$$c_1(X) = \frac{i}{2\pi}[\mathrm{tr}(F_H)] \quad \text{and} \quad 2c_2(X) - c_1(X)^2 = \frac{1}{4\pi^2}[\mathrm{tr}(F_H \wedge F_H)].$$

Therefore,

$$2rc_2(E) - (r-1)c_1(E)^2 = \frac{r}{4\pi^2}[\mathrm{tr}(F_H \wedge F_H)] - \frac{1}{4\pi^2}[\mathrm{tr}(F_H)]^{\cup 2}.$$

A short computation shows that

$$\frac{r}{4\pi^2}\,\mathrm{tr}\big((F_H - \tfrac{1}{r}\,\mathrm{tr}(F_H)\mathrm{id}_E) \wedge (F_H - \tfrac{1}{r}\,\mathrm{tr}(F_H)\mathrm{id}_E)\big) = \frac{r}{4\pi^2}\,\mathrm{tr}(F_H \wedge F_H) - \frac{1}{4\pi^2}\,\mathrm{tr}(F_H)^{\wedge 2}.$$

Therefore and by Proposition 57.5,

$$\begin{aligned}\big\langle\big(2rc_2(E) - (r-1)c_1(E)^2\big) \cup [\omega]^{n-2}, [X]\big\rangle &= \frac{(n-2)!r}{4\pi}\int_X |F_H - \tfrac{1}{r}\,\mathrm{tr}(F_H)\cdot \mathrm{id}_E|^2 \\ &\quad - \frac{(n-2)!r}{4\pi}\int_X |i\Lambda F_H - \tfrac{1}{r}i\Lambda\,\mathrm{tr}(F_H)\cdot\mathrm{id}_E)|^2. \qquad \square\end{aligned}$$

Definition 57.6. Let (X, I, g) be a closed Kähler manifold of complex dimension n with Kähler form ω. Let $\mathscr{E}$ be a coherent sheaf on X. The **degree** of $\mathscr{E}$ is

$$\deg_\omega(\mathscr{E}) := \left\langle c_1(\mathscr{E}) \cup [\omega]^{n-1}, [X] \right\rangle$$

and its **slope** is

$$\mu(\mathscr{E}) := \frac{\deg_\omega(\mathscr{E})}{\operatorname{rk} \mathscr{E}}.$$

$\mathscr{E}$ is called μ–**stable** if, for every proper torsion-free subsheaf $\mathscr{F} \subset \mathscr{E}$,

$$\mu(\mathscr{F}) < \mu(\mathscr{E}).$$

$\mathscr{E}$ is called μ–**polystable** if

$$\mathscr{E} \cong \bigoplus_{k=1}^{K} \mathscr{E}_k$$

with $\mathscr{E}_k$ μ–stable and $\mu(\mathscr{E}_k) = \mu(\mathscr{E})$.

Theorem 57.7 (Donaldson [Don85; Don87] and Uhlenbeck and Yau [UY86]). *Let (X, I, g) be a closed, connected Kähler manifold. Let $\mathscr{E}$ be a holomorphic vector bundle over X. $\mathscr{E}$ admits a Hermitian–Einstein metric if and only if it is μ–polystable. Furthermore, the Hermitian–Einstein metric is unique up to multiplication with a positive constant.*

Remark 57.8. If X is a Riemann surface, then a torsion-free subsheaf is nothing but a subbundle. Furthermore, in this case, H induces a projectively flat connection on E. This is essentially Narasimhan–Seshadri's theorem [NS65], which they initially proved using algebraic geometry. Donaldson [Don83] reproved this result using analytical techniques.

58 Hyperkähler manifolds

Definition 58.1. A **hyperkähler manifold** is a Riemannian manifold (X, g) together with a triple of complex structures I, J, K with respect to which g is a Kähler metric and which satisfy the relations

$$IJ = K = -IJ. \tag{58.2}$$

Proposition 58.3. *If (X, g, I, J, K) is a hyperkähler manifold, then g is Ricci flat.*

Proof. Set $n := \dim X/2$ and define

$$\theta_I := (\omega_J + i\omega_K)^{n/2}. \tag{58.4}$$

A computation shows that $\theta_I \in \Omega^{n,0}(X)$ and is nowhere vanishing. Moreover, it is parallel. This shows that the Levi-Civita connection induces a flat connection on K_X. Therefore, $\mathrm{Ric}_g = 0$. □

Proposition 58.5 (Hitchin [Hit92, Theorem 2]). *Let (X, g) be a Riemannian manifold together with a triple of almost complex structures I, J, K which are compatible with g and which satisfy (58.2). If the 2–forms*

$$\omega_I \coloneqq g(I\cdot, \cdot), \quad \omega_J \coloneqq g(J\cdot, \cdot), \quad \text{and} \quad \omega_K \coloneqq g(K\cdot, \cdot) \tag{58.6}$$

are closed, then (X, g, I, J, K) is a hyperkähler manifold.

Proof. We need to prove that I, J, and K are integrable. Again, set $n \coloneqq \dim X/2$ and define

$$\theta_I \coloneqq (\omega_J + i\omega_K)^{n/2}. \tag{58.7}$$

We have

$$TM_I^{0,1} = \ker\Big(i(\cdot)\theta_I \colon\ TM \otimes \mathbf{C} \to \Lambda_{\mathbf{C}}^{n-1} TM_I^{1,0}\Big). \tag{58.8}$$

Therefore, for $v, w \in \Gamma(T_I^{0,1}M)$,

$$i([u, v])\theta_I = i(u)i(v)\mathrm{d}\theta_I = 0. \tag{58.9}$$

This shows that $[v, w] \in \Gamma(T_I^{0,1}M)$. By the Newlander–Nirenberg Theorem, I is integrable. The same argument proves that J and K are integrable as well. □

59 The Gibbons–Hawking ansatz

Let U be an open subset of $\mathbf{R}^3$. Denote by $g_{\mathbf{R}^3}$ the restriction of the standard metric on $\mathbf{R}^3$ to U. Let $\pi\colon\ X \to U$ be a principal U(1)–bundle. Denote by $\partial_\alpha \in \mathrm{Vect}(X)$ the generator of the U(1)–action. Let $i\theta \in \Omega^1(X, i\mathbf{R})$ be a U(1)–connection 1–form and let $f \in C^\infty(U, (0, \infty))$ be a positive smooth function such that

$$\mathrm{d}\theta = - *_3 \mathrm{d}f. \tag{59.1}$$

Set

$$g \coloneqq f\pi^* g_{\mathbf{R}^3} + \frac{1}{f}\theta \otimes \theta \tag{59.2}$$

and define complex structures I_1, I_2, I_3 by

$$I_i\partial_\alpha = f^{-1}\partial_{x_i} \quad \text{and} \quad I_i\partial_{x_j} = \sum_{k=1}^{3} \varepsilon_{ijk}\partial_{x_k}. \tag{59.3}$$

The corresponding Hermitian forms are

$$\omega_i \coloneqq \mathrm{d}\theta \wedge \mathrm{d}x_i + \frac{1}{2}\sum_{j,k=1}^{3} \varepsilon_{ijk} f \mathrm{d}x_j \wedge \mathrm{d}x_k. \tag{59.4}$$

Writing (59.1) as

$$\mathrm{d}\theta = -\frac{1}{2}\sum_{\ell,j,k=1}^{3} \varepsilon_{\ell jk}\partial_{x_\ell} f\,\mathrm{d}x_j \wedge \mathrm{d}x_k,$$

we see that

$$\mathrm{d}\omega_i = \mathrm{d}\theta \wedge \mathrm{d}x_i + \frac{1}{2}\sum_{j,k=1}^{3} \varepsilon_{ijk}\mathrm{d}f \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k = 0. \tag{59.5}$$

Therefore, we have proved the following.

Proposition 59.6. *(X, g, I_1, I_2, I_3) is hyperkähler manifold.*

This construction is called the **Gibbons–Hawking ansatz**.

Remark 59.7. By construction, the length of the U(1)–orbit over $x \in U$ is $f(x)^{-1/2}$.

Remark 59.8. The fact that

$$i(\partial_\alpha)\omega_i = -\mathrm{d}x_i \tag{59.9}$$

means that the map $\pi\colon X \to U \subset \mathbf{R}^3$ is a hyperkähler moment map for the action of U(1) on X (with $\mathbf{R}^3$ and $(\mathfrak{u}(1) \otimes \operatorname{Im}\mathbf{H})^*$ identified suitably.

Remark 59.10. By (59.1),

$$\Delta f = 0. \tag{59.11}$$

Conversely, suppose that $f\colon U \to \mathbf{R}$ is harmonic and the cohomology class of $*_3\mathrm{d}f$ lies in $\operatorname{im}(H^2(U, 2\pi\mathbf{Z}) \to H^2(U, \mathbf{R}))$, then there is a U(1)–bundle X over U together a connection $i\theta$ satisfying

$$\mathrm{d}\theta = -*_3\,\mathrm{d}f. \tag{59.12}$$

Example 59.13 ($\mathbf{R}^4$). Let $U = \mathbf{R}^3\backslash\{0\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) = \frac{1}{2|x|}. \tag{59.14}$$

This function is harmonic and satisfies

$$-*_3\mathrm{d}f = \frac{1}{2}\mathrm{vol}_{S^2}. \tag{59.15}$$

Since $\mathrm{vol}(S^2) = 4\pi$, there is a U(1)–bundle X over U together with a connection $i\theta$ such that (59.1). Therefore, the Gibbins–Hawking ansatz yields a hyperkähler metric on X.

By Chern–Weil theory the first Chern number of the restriction of X to S^2 is

$$\int_{S^2} i\frac{i}{4\pi}\mathrm{vol}_{S^2} = -1. \tag{59.16}$$

Up to is isomorphism, there is only one principal U(1)–bundle over S^2: the Hopf bundle $\pi\colon\ S^3 \to S^2$ and the U(1)–action given by $e^{i\alpha}\cdot(z_0, z_1) = (e^{i\alpha}z_0, e^{i\alpha}z_1)$. If g_{S^3} denotes the standard metric on S^3, then

$$\theta = g_{S^3}(\partial_\alpha, \cdot) \tag{59.17}$$

satisfies

$$\mathrm{d}\theta = \pi^*\mathrm{vol}_{S^2}. \tag{59.18}$$

It follows that

$$X = S^3 \times (0, \infty) = \mathbf{R}^4\backslash\{0\} \tag{59.19}$$

and the Gibbons–Hawking ansatz gives the metric

$$g = 2r\,\theta\otimes\theta + \frac{1}{2r}(\mathrm{d}r\otimes\mathrm{d}r + r^2 g_{S^2}). \tag{59.20}$$

The change of coordinates $\rho = \sqrt{2r}$ rewrites this metric as

$$g = \mathrm{d}\rho\otimes\mathrm{d}\rho + \rho^2(\theta\otimes\theta + \frac{1}{4}g_{S^2}) = \mathrm{d}\rho\otimes\mathrm{d}\rho + \rho^2 g_{S^3}.$$

This means that the Gibbons–Hawking ansatz yield the standard metric on $\mathbf{R}^4$.

Example 59.21 (Taub–NUT). Let $U = \mathbf{R}^3\backslash\{0\}$, let $c > 0$, and define $f_c\colon\ U \to \mathbf{R}$ by

$$f_c(x) = \frac{1}{2|x|} + c. \tag{59.22}$$

This function is harmonic and we have

$$\mathrm{d}f_c = \mathrm{d}f. \tag{59.23}$$

By the preceding discussion, $X = S^3 \times (0, \infty)$ and the Gibbons–Hawking ansatz gives the metric

$$g = \left(\frac{1}{2r} + c\right)^{-1}\theta\otimes\theta + \left(\frac{1}{2r} + c\right)(\mathrm{d}r\otimes\mathrm{d}r + r^2 g_{S^2}). \tag{59.24}$$

As r tends to zero this metric is asymptotic to

$$c^{-1}\theta\otimes\theta + g_{\mathbf{R}^3}. \tag{59.25}$$

Although, the metric appears singular at $r = 0$, the coordinate change $\rho = \sqrt{2r}$ rewrites it as

$$(1 + c\rho^2)\mathrm{d}\rho \otimes \mathrm{d}\rho + \rho^2 \left((1 + c\rho^2)^{-1}\theta \otimes \theta + (1 + c\rho^2)\frac{1}{4}g_{S^2}\right) \tag{59.26}$$

which is smooth.

This metric is called the **Taub–NUT metric**. It is non-flat hyperkähler metric on $\mathbf{R}^4$. It was first discovered by Taub [Tau51] and Newman, Tamburino, and Unti [NTU63]. The Taub–NUT space is the archetype of an **ALF space**.

Remark 59.27. It was observed by LeBrun [LeB91] that the Taub–NUT metric is in fact Kähler for the standard complex structure on $\mathbf{C}^2$. Thus it yields a non-flat Ricci-flat Kähler metric on $\mathbf{C}^2$.

Example 59.28 ($(\mathbf{R}^4\backslash\{0\})/\mathbf{Z}_k$). Let $k \in \{1, 2, 3, \dots\}$ Let $U = \mathbf{R}^3\backslash\{0\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) \coloneqq \frac{k}{2|x|}. \tag{59.29}$$

This function is harmonic and it satisfies

$$- *_3\mathrm{d}f = k\mathrm{vol}_{S^2}. \tag{59.30}$$

Thus, the Gibbons–Hawking ansatz applies. Denote by (X_k, g_k) the Riemannian manifold obtained in this way. If $k = 1$, then this $\mathbf{R}^4$ with its standard metric. Let us understand the cases $k \geqslant 2$.

The restriction of X_k to S^2 has Chern number $-k$. This $\mathrm{U}(1)$–bundle is $S^3/\mathbf{Z}_k \to S^2$. Consequently,

$$X_k = S^3/\mathbf{Z}_k \times (0, \infty) = \mathbf{R}^4/\mathbf{Z}_k.$$

We can choose the connection 1-form $i\theta_k$ on X_k such that its pullback to X_1 is $ik\theta_1$. It follows that the pullback of g_k to X_1 can be written as

$$2kr\,\theta \otimes \theta + \frac{k}{2r}(\mathrm{d}r \otimes \mathrm{d}r + r^2 g_{S^2}). \tag{59.31}$$

Up to a coordinate change $r \mapsto kr$ this is the standard metric on $\mathbf{R}^4$. It follows that g_k is the metric induced by the standard metric on $\mathbf{R}^4$.

Example 59.32 (Eguchi–Hanson and multi-center Gibbons–Hawking). Let $x_1, \dots, x_k$ be k distinct points in $\mathbf{R}^3$. Set $U \coloneqq \mathbf{R}^3\backslash\{x_1, \dots, x_k\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) = \sum_{i=1}^{k} \frac{1}{2|x - x_i|}. \tag{59.33}$$

From the in discussion Example 59.13 it is clear that the Gibbons–Hawking ansatz for f produces a Riemannian manifold whose apparent singularities over $x_1, \dots, x_k$ can be removed. Denote the resulting manifold by (X, g).

Since

$$f(x) = \frac{k}{2|x|} + O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty,$$

(X, g) is asymptotic at infinity to $\mathbf{R}^4/\mathbf{Z}_k$. These spaces are called **ALE spaces of type** A_{k-1}. For $k = 2$, this metric was discovered by Eguchi and Hanson [EH79]. The metrics for $k \geqslant 3$ were discovered by Gibbons and Hawking [GH78].

Let us understand the geometry and topology of these spaces somewhat more. Suppose γ is an arc in $\mathbf{R}^3$ from x_i to x_j avoiding all the other points x_k. The pre-image in X of any interior point of γ is an S^1 while the pre-images of the end points are points. Therefore,

$$\pi^{-1}(\gamma) \subset X \tag{59.34}$$

is diffeomorphic to S^2. Suppose γ is straight line segment in $\mathbf{R}^3$ with unit tangent vector

$$v = \sum_{i=1}^{3} a_i \partial_{x_i} \tag{59.35}$$

with $a_1^2 + a_2^2 + a_3^2 = 1$. The tangent spaces to $\pi^{-1}(\gamma)$ are spanned by ∂_α and v. In particular, they are invariant with respect to the complex structure

$$I_v \coloneqq a_1 I_1 + a_2 I_2 + a_3 I_3. \tag{59.36}$$

Its volume is given by

$$\int_{\pi^{-1}(\gamma)} a_1\omega_1 + a_2\omega_2 + a_3\omega_3. \tag{59.37}$$

Therefore,

$$[\pi^{-1}(\gamma)] \neq 0 \in H_2(X, \mathbf{Z}). \tag{59.38}$$

If necessary we can reorder the points x_i so that for $i = 1, \ldots, k-1$, there is a straight-line segment γ_i joining x_i and x_{i+1}. Set

$$\Sigma_i \coloneqq \pi^{-1}(\gamma_i). \tag{59.39}$$

It is not difficult to see that $[\Sigma_1], \ldots, [\Sigma_{k-1}]$ generate $H_2(M; \mathbf{Z})$. It is an exercise to show that

$$[\Sigma_i] \cdot [\Sigma_j] = \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases} \tag{59.40}$$

Remark 59.41. Kronheimer [Kro89b] gave an alternative construction of the ALE spaces of type A_{k-1} (in fact, all ALE spaces) as hyperkähler quotients. He also classified these spaces completely [Kro89a].

Example 59.42. Let $x_1, \dots, x_k$ be k distinct points in $\mathbf{R}^3$ and let $c > 0$. Set $U \coloneqq \mathbf{R}^3 \backslash \{x_1, \dots, x_k\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) = \sum_{i=1}^{k} \frac{1}{2|x - x_i|} + c. \tag{59.43}$$

The Gibbons–Hawking ansatz for f gives rise to the so-called **multi-center Taub–NUT** metric.

Example 59.44. The following is due to Anderson, Kronheimer, and LeBrun [AKL89]. Let $x_1, x_2, \dots$ be an infinite sequence of distinct points in $\mathbf{R}^3$ and denote by U the complement of these points. If

$$\sum_{j=2}^{\infty} \frac{1}{x_1 - x_j} < \infty, \tag{59.45}$$

then

$$f(x) \coloneqq \sum_{j=1}^{\infty} \frac{1}{2|x - x_j|} \tag{59.46}$$

defines a harmonic function on U. The Gibbons–Hawking ansatz gives rise to a hyperkähler manifold X whose second homology $H_2(X, \mathbf{Z})$ is infinitely generated. Anderson, Kronheimer, and LeBrun prove that the metric g is complete.

This is not a complete list of interesting examples of hyperkähler manifold which can be produced using the Gibbons–Hawking ansatz. The most egregious omission is that of the Ooguri–Vafa metric.

60 The Euclidean Schwarzschild metric

Example 60.1. Denote polar coordinates on $\mathbf{R}^2 \backslash \{(0,0)\}$ by α and r define the Riemannian metric g_M by

$$g_{ES} \coloneqq \frac{4r}{r+1} \mathrm{d}\alpha \otimes \mathrm{d}\alpha + \frac{r+1}{r} \mathrm{d}r \otimes \mathrm{d}r + (r+1)^2 g_{S^2}. \tag{60.2}$$

The apparent singularity at $r = 0$ is resolved by the change of variables $r = \rho^2$. This metric is called the **Euclidean Schwarzschild metric**. It is Ricci-flat. The expression (60.2) shows that g_M is asymptotic to the flat metric on $S^1 \times \mathbf{R}^3$. Therefore, g_{ES} has volume growth r^3.

61 Unique continuation and the frequency function

Definition 61.1. Let M be a connected manifold, let E, F be vector bundle over M and let $D\colon \Gamma(E) \to \Gamma(F)$ be a differential operator. We say that D has the **unique continuation property** if the following holds: if $s \in \ker D$ and s vanishes on an open subset, then s vanishes on all of M.

Theorem 61.2. *Let (M,g) be a connected Riemannian manifold, let E be a Euclidean vector bundle, and let A be a connection on E, and let $\mathfrak{R} \in \Gamma(\mathrm{End}(E))$. The differential operator*

$$\nabla_A^* \nabla_A + \mathfrak{R}$$

has the unique continuation property.

Corollary 61.3. *Let (M,g) be a connected Riemannian manifold, If α is a harmonic form which vanishes on an open subset, then α vanishes on all of M.*

Throughout the remainder of this section assume the hypotheses of Theorem 61.2 and fix $s \in \ker D$. Furthermore, denote by $r_0 > 0$ a small constant. Since $s \in \ker D$,

$$\Delta |s|^2 + 2|\nabla_A s|^2 = -\langle \mathfrak{R} s, s\rangle. \tag{61.4}$$

61.1 The frequency function

The proof relies on Almgren's frequency function.

Definition 61.5. For every $x \in M$, define $m_x, D_x \colon (0, r_0] \to [0,\infty)$ by

$$m_x(r) \coloneqq \frac{1}{r^{n-1}} \int_{\partial B_r(x)} |s|^2 \quad \text{and}$$
$$D_x(r) \coloneqq \frac{1}{r^{n-2}} \int_{B_r(x)} |\nabla_A s|^2;$$

and, furthermore, set $r_{-1,x} \coloneqq \sup\{r \in (0,\infty) : m_x(r) = 0\}$ and define the **frequency function** $\mathrm{N}_x \colon (r_{-1,x}, r_0] \to [0,\infty)$ by

$$\mathrm{N}_x(r) \coloneqq \frac{D_x(r)}{m_x(r)}.$$

The key property of the frequency function is the following.

Proposition 61.6. *For every $r \in (r_{-1,x}, r_0]$,*

$$\mathrm{N}_x'(r) \geqslant -cr(1 + \mathrm{N}_x(r)). \tag{61.7}$$

Before embarking on the proof of Proposition 61.6, let us record the following consequence.

Proposition 61.8. *For every $r_{-1,x} < s \leqslant r \leqslant r_0$,*

$$\mathrm{N}_x(s) \leqslant \big(1 + cr^2\big)\mathrm{N}_x(r) + cr^2.$$

Proof. By Proposition 61.6,

$$\frac{\mathrm{d}}{\mathrm{d}r} e^{\frac{1}{2}cr^2}(\mathrm{N}_x(r) + 1) \geqslant 0.$$

This implies

$$\mathrm{N}_x(s) \leqslant e^{\frac{1}{2}c(r^2-s^2)}\mathrm{N}_x(r) + e^{\frac{1}{2}c(r^2-s^2)} - 1. \qquad \square$$

The proof of of Proposition 61.6 relies on the following propositions.

Proposition 61.9. *For every $x \in M$ and $r \in (0, r_0]$,*

$$\int_{B_r(x)} |s|^2 \leqslant cr \int_{\partial B_r(x)} |s|^2.$$

Proof. Denote by $H_{x,r}$ the mean curvature of $\partial B_r(x)$. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r} \int_{\partial B_r(x)} |s|^2 &= \int_{\partial B_r(x)} H_{x,r}|s|^2 + \int_{\partial B_r(x)} \partial_r |s|^2 \\
&= \int_{\partial B_r(x)} H_{x,r}|s|^2 - \int_{\partial B_r(x)} \Delta |s|^2 \\
&= \int_{\partial B_r(x)} H_{x,r}|s|^2 + \int_{B_r(x)} |\nabla_A s|^2 + \langle \Re s, s\rangle.
\end{aligned}$$

Since $H_{x,r} = 1/r + O(r)$ and $r_0 \ll 1$, this term is non-negative. Therefore, $\int_{\partial B_r(x)} |s|^2$ is non-decreasing and this proves the statement. $\square$

Proposition 61.10. *For every $r \in (0, r_0]$,*

$$D_x'(r) = \frac{2}{r^{n-1}} \int_{\partial B_r(x)} |\nabla_{A,\partial_r} s|^2 + \mathfrak{r}_{D'}$$

with

$$|\mathfrak{r}_{D'}| \leqslant cr(D_x(r) + m_x(r)).$$

Proof. Define the tensor field $T \in \Gamma(S^2T^*M)$ by

$$T(v, w) \coloneqq \left\langle \nabla_{A,v} s, \nabla_{A,w} s \right\rangle - \frac{1}{2}\langle v, w\rangle |\nabla_A s|^2.$$

By a straight-forward computation,

$$\operatorname{tr} T = \left(1 - \tfrac{n}{2}\right)|\nabla_A s|^2. \tag{61.11}$$

Let $y \in M$ be an arbitrary point of M and let $e_1, \dots, e_n$ be a local orthonormal frame such that $(\nabla_{e_i} e_j)(y) = 0$. All of the following computations take place at the point y. By direct computation

$$
\begin{aligned}
(\nabla^* T)(e_i) &= -\sum_{j=1}^{n} \left\langle \nabla_{A,e_j} \nabla_{A,e_j} s, \nabla_{A,e_i} s \right\rangle + \left\langle \nabla_{A,e_j} \nabla_{A,e_i} s, \nabla_{A,e_j} s \right\rangle - \left\langle \nabla_{A,e_i} \nabla_{A,e_j} s, \nabla_{A,e_j} s \right\rangle \\
&= \left\langle \nabla_A^* \nabla_A s, \nabla_{A,e_i} s \right\rangle + \sum_{j=1}^{3} \left\langle F_A(e_i, e_j) s, \nabla_{A,e_j} s \right\rangle \\
&= -\left\langle \mathfrak{R} s, \nabla_{A,e_i} s \right\rangle + \sum_{j=1}^{3} \left\langle F_A(e_i, e_j) s, \nabla_{A,e_j} s \right\rangle.
\end{aligned}
$$

Therefore,

$$
|\nabla^* T| \leqslant c(|\mathfrak{R}| + |F_A|)|s||\nabla_A s|. \tag{61.12}
$$

By (61.11), the identity

$$
\int_{B_r(x)} \left\langle \nabla^* T, \mathrm{d}r_x^2 \right\rangle = -2r \int_{\partial B_r(x)} T(\partial_r, \partial_r) + \int_{B_r(x)} \left\langle T, \operatorname{Hess}(r_x^2) \right\rangle
$$

can be rewritten as

$$
\begin{aligned}
\int_{B_r(x)} 2r_x \nabla^* T(\partial_r) = -2r \int_{\partial B_r(x)} |\nabla_{A,\partial_r} s|^2 + r \int_{\partial B_r(x)} |\nabla_A s|^2 \\
+ (n-2) \int_{B_r(x)} |\nabla_A s|^2 + \int_{B_r(x)} \langle T, \mathfrak{r}_{\mathrm{II}} \rangle
\end{aligned}
$$

with

$$
\mathfrak{r}_{\mathrm{II}} \coloneqq \operatorname{Hess}(r_x^2) - 2g.
$$

Since

$$
D_x'(r) = -\frac{1}{r^{n-1}} \int_{B_r(x)} |\nabla_A s|^2 + \frac{n-2}{r^{n-2}} \int_{\partial B_r(x)} |\nabla_A s|^2 + 2\varepsilon^{-2} |\mu(\Phi)|^2,
$$

the inequality (61.12) implies the assertion. □

Proposition 61.13. *For every $r \in (0, r_0]$,*

$$
D_x(r) = \frac{1}{r^{n-2}} \int_{\partial B_r(x)} \left\langle \nabla_{A,\partial_r} s, s \right\rangle + \mathfrak{r}_D
$$

with

$$
|\mathfrak{r}_D| \leqslant c r^2 m_x(r).
$$

Proof. By (61.4),

$$\int_{B_x(r)} 2|\nabla_A s|^2 = -\int_{B_x(r)} \langle \mathfrak{R}s, s\rangle - \int_{B_x(r)} \Delta|s|^2$$
$$= -\int_{B_x(r)} \langle \mathfrak{R}s, s\rangle + \int_{\partial B_x(r)} \partial_r|s|^2.$$

This implies the assertion. ◻

Proposition 61.14. *For every $r \in (0, r_0]$,*

$$m_x'(r) = \frac{2D_x(r)}{r} + \mathfrak{r}_{m'}$$

with

$$|\mathfrak{r}_{m'}| \leqslant crm_x(r).$$

Proof. Denote by $H_{x,r}$ the mean curvature of $\partial B_r(x)$. Be the above computation

$$m_x(r)' = \frac{2}{r^{n-1}}\int_{\partial B_r(x)} \left(H_{x,r} - \tfrac{1}{r}\right)|s|^2 + \frac{1}{r^{n-1}}\int_{\partial B_r(x)} \partial_r|s|^2$$
$$= \frac{2D_x(r)}{r} + \frac{2}{r^{n-1}}\int_{\partial B_r(x)} \left(H_{x,r} - \tfrac{1}{r}\right)|s|^2 - \frac{2\mathfrak{r}_D}{r}.$$

The assertion follows since $\left|H_{x,r} - \frac{1}{r}\right| \leqslant cr$. ◻

Proof of Proposition 61.6. By the above

$$\begin{aligned}
\mathrm{N}_x'(r) &= \frac{D_x'(r)}{m_x(r)} - \frac{D_x(r)m_x'(r)}{m_x(r)^2}\\
&= \frac{2}{m_x(r)}\int_{\partial B_r(x)} |\nabla_{A,\partial_r} s|^2 - \frac{2D_x(r)^2}{rm_x(r)^2} + \frac{\mathfrak{r}_{D'}}{m_x(r)} - \frac{\mathfrak{r}_{m'}}{m_x(r)}\mathrm{N}_x(r)\\
&\geqslant -cr(1+\mathrm{N}_x(r)) + \frac{2}{m_x(r)r^{n-1}}\left[\int_{\partial B_r(x)} |s|^2 \int_{\partial B_r(x)} |\nabla_{A,\partial_r} s|^2 - \left(\int_{\partial B_r(x)} \left\langle \nabla_{A,\partial_r} s, s\right\rangle\right)^2\right].
\end{aligned}$$

By Cauchy–Schwarz, the term in square brackets vanishes. ◻

61.2 Proof of Theorem 61.2

Proposition 61.15. *For every $x \in M$ and $0 < s < r \leqslant r_0$,*

$$\left(\frac{r}{s}\right)^{(2-cr^2)\mathrm{N}_x(r)-cr^2} m_x(s) \leqslant m_x(r) \leqslant \left(\frac{r}{s}\right)^{(2+cr^2)\mathrm{N}_x(r)+cr^2} m_x(s).$$

Proof. By Proposition 61.8 and Proposition 61.14, for $t \in [s, r]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \log m_x(t) &\leqslant \frac{2\mathrm{N}_x(t)}{t} + ct \\ &\leqslant \frac{2(1+cr^2)}{t}\mathrm{N}_x(r) + \frac{cr^2}{t}\end{aligned}$$

as well as

$$\frac{\mathrm{d}}{\mathrm{d}t} \log m_x(t) \geqslant \frac{2(1-cr^2)}{t}\mathrm{N}_x(r) - \frac{cr^2}{t}.$$

These integrate to the asserted inequalities. □

Proposition 61.16. *If $s \neq 0$, then, for every $x \in M$ and $r \in (0, r_0]$,*

$$m_x(r) > 0;$$

in particular, $r_{-1,x} = 0$.

Proof. If $m_x(r) = 0$, for some $r \in (0, r_0]$, then it follows from Proposition 61.15 that $m_x = 0$. Therefore, Φ vanishes on $B_{r_0}(x)$. This in turn implies that $m_y(r_0/2)$ vanishes for all $y \in B_{r_0/2}(x)$. Hence, Φ vanishes on $B_{\frac{3}{2}r_0}(x)$. Repeating this argument shows that s vanishes. □

This proves Theorem 61.2. □

References

[AC92] M. T. Anderson and J. Cheeger. *C^α-compactness for manifolds with Ricci curvature and injectivity radius bounded below.* 35.2 (1992), pp. 265–281. DOI: 10.4310/jdg/1214448075. MR: 1158336. Zbl: 0774.53021 (cit. on p. 77)

[AKL89] M. T. Anderson, P. B. Kronheimer, and C. LeBrun. *Complete Ricci-flat Kähler manifolds of infinite topological type.* *Communications in Mathematical Physics* 125.4 (1989), pp. 637–642. DOI: 10.1007/BF01228345. MR: 1024931. Zbl: 0734.53051 (cit. on p. 122)

[And89] M. T. Anderson. *Ricci curvature bounds and Einstein metrics on compact manifolds.* *Journal of the American Mathematical Society* 2.3 (1989), pp. 455–490. DOI: 10.2307/1990939. MR: 999661. Zbl: 0694.53045 (cit. on pp. 83, 85)

[And90a] M. T. Anderson. *Convergence and rigidity of manifolds under Ricci curvature bounds.* *Inventiones Mathematicae* 102.2 (1990), pp. 429–445. DOI: 10.1007/BF01233434. MR: 1074481. Zbl: 0711.53038 (cit. on pp. 77, 79–81, 84, 85)

[And90b] M. T. Anderson. *On the topology of complete manifolds of non-negative Ricci curvature.* *Topology* 29.1 (1990), pp. 41–55. DOI: 10.1016/0040-9383(90)90024-E. MR: 1046624. Zbl: 0696.53027 (cit. on p. 38)

[And90c] M. T. Anderson. *Short geodesics and gravitational instantons. Journal of Differential Geometry* 31.1 (1990), pp. 265–275. MR: 1030673. Zbl: 0696.53029. (cit. on pp. 39, 40)

[Ant] B. Anthes. *A new proof of the Weil identity.* (cit. on p. 101)

[Aub76] T. Aubin. *Équations du type Monge–Ampère sur les variétés kähleriennes compactes. Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Série A* 283.3 (1976), pp. 119–121. MR: 0433520. Zbl: 0333.53040 (cit. on pp. 110, 112)

[Bas72] H. Bass. *The degree of polynomial growth of finitely generated nilpotent groups. Proceedings of the London Mathematical Society* 25 (1972), pp. 603–614. MR: 0379672. Zbl: 0259.20045 (cit. on p. 38)

[BC64] R. L. Bishop and R. J. Crittenden. *Geometry of manifolds.* Pure and Applied Mathematics. 1964. MR: 0169148. Zbl: 0132.16003 (cit. on p. 30)

[Bér88] P. H. Bérard. *From vanishing theorems to estimating theorems: the Bochner technique revisited.* 19.2 (1988), pp. 371–406. DOI: 10.1090/S0273-0979-1988-15679-0. MR: 956595. Zbl: 0662.53037 (cit. on p. 59)

[BGM71] M. Berger, P. Gauduchon, and E. Mazet. *Le spectre d'une variété riemannienne.* Vol. 194. Lecture Notes in Mathematics. 1971. MR: 0282313. Zbl: 0223.53034 (cit. on p. 29)

[Bis63] R. L. Bishop. *A relation between volume, mean curvature, and diameter. Notices Of The American Mathematical Society* 10.4 (1963), p. 364. (cit. on p. 30)

[Bło12] Z. Błocki. *The Calabi–Yau theorem. Complex Monge–Ampère equations and geodesics in the space of Kähler metrics.* Vol. 2038. Lecture Notes in Mathematics. 2012, pp. 201–227. DOI: 10.1007/978-3-642-23669-3_5. MR: 2932444. Zbl: 1231.32017 (cit. on p. 112)

[BM11] O. Biquard and V. Minerbe. *A Kummer construction for gravitational instantons. Communications in Mathematical Physics* 308.3 (2011), pp. 773–794. MR: 2855540. Zbl: 1234.53020 (cit. on p. 34)

[Boc46] S. Bochner. *Vector fields and Ricci curvature. Bulletin of the American Mathematical Society* 52 (1946), pp. 776–797. DOI: 10.1090/S0002-9904-1946-08647-4. MR: 0018022. Zbl: 0060.38301 (cit. on pp. 18–20)

[Cal58] E. Calabi. *An extension of E. Hopf's maximum principle with an application to Riemannian geometry. Duke Mathematical Journal* 25 (1958), pp. 45–56. MR: 0092069. Zbl: 0079.11801. (cit. on pp. 41, 42)

[CDS15a] X. Chen, S. Donaldson, and S. Sun. *Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities. Journal of the American Mathematical Society* 28.1 (2015), pp. 183–197. DOI: 10.1090/S0894-0347-2014-00799-2. MR: 3264766 (cit. on p. 112)

[CDS15b] X. Chen, S. Donaldson, and S. Sun. *Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π. Journal of the American Mathematical Society* 28.1 (2015), pp. 199–234. DOI: 10.1090/S0894-0347-2014-00800-6. MR: 3264767. (cit. on p. 112)

[CDS15c] X. Chen, S. Donaldson, and S. Sun. *Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof. Journal of the American Mathematical Society* 28.1 (2015), pp. 235–278. DOI: 10.1090/S0894-0347-2014-00801-8. MR: 3264768 (cit. on p. 112)

[CG71] J. Cheeger and D. Gromoll. *The splitting theorem for manifolds of nonnegative Ricci curvature. Journal of Differential Geometry* 6 (1971), pp. 119–128. MR: 0303460. Zbl: 0223.53033. (cit. on pp. 44, 47)

[Che75] S.-Y. Cheng. *Eigenvalue comparison theorems and its geometric applications. Mathematische Zeitschrift* 143 (1975), pp. 289–297. Zbl: 0329.53035 (cit. on p. 34)

[CO75] B.-Y. Chen and K. Ogiue. *Some characterizations of complex space forms in terms of Chern classes. The Quarterly Journal of Mathematics* 26.104 (1975), pp. 459–464. DOI: 10.1093/qmath/26.1.459. MR: 0405303. Zbl: 0315.53034 (cit. on p. 113)

[Der82] A. Derdziński. *On compact Riemannian manifolds with harmonic curvature. Mathematische Annalen* 259.2 (1982), pp. 145–152. DOI: 10.1007/BF01457307. MR: 656660. Zbl: 0489.53042 (cit. on p. 82)

[DK81] D. M. DeTurck and J. L. Kazdan. *Some regularity theorems in Riemannian geometry.* 14.3 (1981), pp. 249–260. MR: 644518. Zbl: 0486.53014. (cit. on p. 74)

[Don83] S. K. Donaldson. *A new proof of a theorem of Narasimhan and Seshadri. Journal of Differential Geometry* 18.2 (1983), pp. 269–277. DOI: 10.4310/jdg/1214437664. MR: 710055. Zbl: 0504.49027 (cit. on p. 116)

[Don85] S. K. Donaldson. *Anti self-dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles. Proceedings of the London Mathematical Society* 50.1 (1985), pp. 1–26. DOI: 10.1112/plms/s3-50.1.1. MR: 765366. Zbl: 0529.53018 (cit. on p. 116)

[Don87] S. K. Donaldson. *Infinite determinants, stable bundles and curvature. Duke Mathematical Journal* 54.1 (1987), pp. 231–247. DOI: 10.1215/S0012-7094-87-05414-7. MR: 885784. Zbl: 0627.53052 (cit. on p. 116)

[dSai16] H. P. de Saint-Gervais. *Uniformization of Riemann surfaces.* 2016. DOI: 10.4171/145. MR: 3494804. Zbl: 1332.30001 (cit. on p. 22)

[Edw75] D. A. Edwards. *The structure of superspace* (1975), pp. 121–133. MR: 0401069. Zbl: 0311.54010 (cit. on pp. 62, 64)

[EH79] T. Eguchi and An. J. Hanson. *Self-dual solutions to Euclidean gravity. Annals of Physics* 120.1 (1979), pp. 82–106. DOI: 10.1016/0003-4916(79)90282-3. MR: 540896. Zbl: 0409.53020 (cit. on p. 121)

[EH84] J.-H. Eschenburg and E. Heintze. *An elementary proof of the Cheeger-Gromoll splitting theorem. Annals of Global Analysis and Geometry* 2.2 (1984), pp. 141–151. DOI: 10.1007/BF01876506. MR: 777905. Zbl: 0548.53041 (cit. on p. 44)

[Ein16] A. Einstein. *Näherungsweise Integration der Feldgleichungen der Gravitation. Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften* 1916 (1916), pp. 688–696. Zbl: 46.1293.02 (cit. on p. 72)

[Gal83] S. Gallot. *Inégalités isopérimétriques, courbure de Ricci et invariants géométriques. II. Comptes Rendus des Séances de l'Académie des Sciences. Série I. Mathématique* 296.8 (1983), pp. 365–368. MR: 699164. Zbl: 0535.53035 (cit. on p. 59)

[GH78] G. W. Gibbons and S. W. Hawking. *Gravitational multi-instantons. Physical Letters* 78B (1978), p. 430. DOI: 10.1016/0370-2693(78)90478-1 (cit. on p. 121)

[GK67] S. I. Goldberg and S. Kobayashi. *Holomorphic bisectional curvature. Journal of Differential Geometry* 1 (1967), pp. 225–233. DOI: 10.4310/jdg/1214428090. MR: 0227901. Zbl: 0169.53202. (cit. on p. 112)

[GL91] C. R. Graham and J. M. Lee. *Einstein metrics with prescribed conformal infinity on the ball. Advances in Mathematics* 87.2 (1991), pp. 186–225. DOI: 10.1016/0001-8708(91)90071-E. MR: 1112625. Zbl: 0765.53034 (cit. on p. 74)

[Gro07] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces.* English. Modern Birkhäuser Classics. 2007, pp. xx+585. MR: 2307192. Zbl: 1113.53001 (cit. on pp. 39, 59, 62, 65)

[Gro81a] M. Gromov. *Curvature, diameter and Betti numbers. Commentarii Mathematici Helvetici* 56.2 (1981). DOI: 10.1007/BF02566208. MR: 630949. Zbl: 0467.53021 (cit. on p. 30)

[Gro81b] M. Gromov. *Groups of polynomial growth and expanding maps. Institut des Hautes Études Scientifiques. Publications Mathématiques* 53 (1981), pp. 53–73. MR: 623534. Zbl: 0474.20018. (cit. on pp. 38, 62, 64, 68)

[Gro81c] M. Gromov. *Structures métriques pour les variétés riemanniennes.* Textes Mathématiques 1. 1981, pp. iv+152. MR: 682063. Zbl: 0509.53034 (cit. on p. 39)

[GS05] A. Grigor'yan and L. Saloff-Coste. *Stability results for Harnack inequalities. Annales de l'Institut Fourier* 55.3 (2005), pp. 825–890. MR: 2149405. (cit. on p. 50)

[GT01] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Reprint of the 1998 edition. Berlin, 2001, pp. xiv+517. MR: MR1814364 (cit. on pp. 72, 84)

[Gug52] H. Guggenheimer. *Über vierdimensionale Einsteinräume. Experientia* 8 (1952), pp. 420–421. MR: 0051559. Zbl: 0047.20707 (cit. on p. 113)

[Gui73] Y. Guivarc'h. *Croissance polynômiale et périodes des fonctions harmoniques. Bulletin de la Société Mathématique de France* 101 (1973), pp. 333–379. MR: 0369608. Zbl: 0294.43003 (cit. on p. 38)

[Hei12] H.-J. Hein. *Gravitational instantons from rational elliptic surfaces. Journal of the American Mathematical Society* 25.2 (2012), pp. 355–393. DOI: 10.1090/S0894-0347-2011-00723-6. MR: 2869021. Zbl: 1239.53066 (cit. on p. 34)

[Hei55] E. Heinz. *Nachrichten der Akademie der Wissenschaften in Göttingen, Mathematisch-Physikalische Klasse. 2a, Mathematisch-Physikalisch-Chemische Abteilung* (1955), pp. 1–12. MR: 0074666. Zbl: 0067.07503 (cit. on p. 83)

[HH97] E. Hebey and M. Herzlich. *Harmonic coordinates, harmonic radius and convergence of Riemannian manifolds. Rendiconti di Matematica e delle sue Applicazioni. Serie VII* 17.4 (1997), pp. 569–605. MR: 1620864. Zbl: 0912.53011 (cit. on p. 77)

[HHN15] M. Haskins, H.-J. Hein, and J. Nordström. *Asymptotically cylindrical Calabi–Yau manifolds. Journal of Differential Geometry* 101.2 (2015), pp. 213–265. DOI: 10.4310/jdg/1442364651. MR: 3399097. Zbl: 1332.32028 (cit. on p. 34)

[Hit92] N. Hitchin. *Hyperkähler manifolds. Astérisque* 206 (1992). Séminaire Bourbaki, Vol. 1991/92, Exp. No. 748, 3, 137–166. MR: 1206066. Zbl: 0979.53051 (cit. on p. 117)

[HK00] P. Hajłasz and P. Koskela. *Sobolev met Poincaré.* Memoirs of the American Mathematical Society 688. 2000. Zbl: 0954.46022 (cit. on p. 56)

[HK95] P. Hajłasz and P. Koskela. *Sobolev meets Poincaré. Comptes Rendus de l'Académie des Sciences* 320.10 (1995), pp. 1211–1215. Zbl: 0837.46024 (cit. on p. 50)

[Hop25] H. Hopf. *Zum Clifford-Kleinschen Raumproblem. Mathematische Annalen* 95 (1925), pp. 313–339. DOI: 10.1007/BF01206614. MR: 1512281. Zbl: 51.0439.05 (cit. on p. 9)

[Hop27] E. Hopf. *Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus. Sitzungsberichte der Preußischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse* (1927), pp. 147–152. Zbl: 53.0454.02 (cit. on p. 41)

[Hur93] A. Hurwitz. *Ueber algebraische Gebilde mit eindeutigen Transformationen in sich. Mathematische Annalen* 41 (1893), pp. 403–442. DOI: 10.1007/BF01443420. MR: 1510753. Zbl: 24.0380.02 (cit. on p. 22)

[Kil91] W. Killing. *Ueber die Clifford–Klein'schen Raumformen. Mathematische Annalen* 39 (1891), pp. 257–278. Zbl: 23.0529.01 (cit. on p. 9)

[Kro89a] P. B. Kronheimer. *A Torelli-type theorem for gravitational instantons. Journal of Differential Geometry* 29.3 (1989), pp. 685–697. MR: 992335. (cit. on p. 121)

[Kro89b] P. B. Kronheimer. *The construction of ALE spaces as hyper-Kähler quotients. Journal of Differential Geometry* 29.3 (1989), pp. 665–683. MR: 992334. (cit. on p. 121)

[KW74] J. L. Kazdan and F. W. Warner. *Curvature functions for compact 2-manifolds. Annals of Mathematics* 99 (1974), pp. 14–47. Zbl: 0273.53034 (cit. on p. 90)

[Lan22] K. Lanczos. *Ein vereinfachendes Koordinatensystem für die Einsteinschen Gravitationsgleichungen. Physikalische Zeitschrift* 23 (1922), pp. 537–539. Zbl: 48.1023.01 (cit. on p. 73)

[LeB91] C. LeBrun. *Complete Ricci-flat Kähler metrics on* $\mathbf{C}^n$ *need not be flat. Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989).* Proceedings of Symposia in Pure Mathematics 52. 1991, pp. 297–304. MR: 1128554 (cit. on p. 120)

[Li80] P. Li. *On the Sobolev constant and the p–spectrum of a compact Riemannian manifold. Annales Scientifiques de l'École Normale Supérieure* 13 (1980), pp. 451–468. Zbl: 0466.53023 (cit. on p. 59)

[Li86] P. Li. *Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature. Annals of Mathematics* 124.1 (1986), pp. 1–21. DOI: 10.2307/1971385. MR: 847950. Zbl: 0613.58032 (cit. on p. 38)

[Lic58] A. Lichnerowicz. *Géométrie des groupes de transformations.* Travaux et Recherches Mathématiques. 1958, pp. ix+193. MR: 0124009. Zbl: 0096.16001 (cit. on p. 27)

[Liu13] G. Liu. 3*–manifolds with nonnegative Ricci curvature. Inventiones Mathematicae* 193.2 (2013), pp. 367–375. DOI: 10.1007/s00222-012-0428-x. MR: 3090181. Zbl: 1279.53035 (cit. on p. 38)

[Mil68] J. Milnor. *A note on curvature and fundamental group. Journal of Differential Geometry* 2 (1968), pp. 1–7. Zbl: 0162.25401. (cit. on pp. 38, 39)

[Miy77] Y. Miyaoka. *On the Chern numbers of surfaces of general type. Inventiones Mathematicae* 42 (1977), pp. 225–237. DOI: 10.1007/BF01389789. MR: 0460343. Zbl: 0374.14007 (cit. on p. 113)

[Mor08] C. B. Morrey Jr. *Multiple integrals in the calculus of variations.* Classics in Mathematics. 2008. DOI: 10.1007/978-3-540-69952-1. MR: 2492985. Zbl: 1213.49002 (cit. on p. 83)

[Mos60] J. Moser. *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. Communications on Pure and Applied Mathematics* 13 (1960), pp. 457–468. DOI: 10.1002/cpa.3160130308. MR: 0170091. Zbl: 0111.09301 (cit. on p. 57)

[Mye41] S. B. Myers. *Riemannian manifolds with positive mean curvature. Duke Mathematical Journal* 8 (1941), pp. 401–404. MR: 0004518. Zbl: 0025.22704. (cit. on p. 23)

[Nak88] H. Nakajima. *Compactness of the moduli space of Yang–Mills connections in higher dimensions. Journal of the Mathematical Society of Japan* 40.3 (1988), pp. 383–392. DOI: 10.2969/jmsj/04030383. MR: 945342. Zbl: 0647.53030 (cit. on p. 83)

[NS65] M. S. Narasimhan and C. S. Seshadri. *Stable and unitary vector bundles on a compact Riemann surface. Annals of Mathematics* 82 (1965), pp. 540–567. MR: 0184252. Zbl: 0171.04803 (cit. on p. 116)

[NTU63] E. Newman, L. Tamburino, and T. Unti. *Empty-space generalization of the Schwarzschild metric. Journal of Mathematical Physics* 4 (1963), pp. 915–923. DOI: 10.1063/1.1704018. MR: 0152345. Zbl: 0115.43305 (cit. on p. 120)

[Oba62] M. Obata. *Certain conditions for a Riemannian manifold to be isometric with a sphere. Journal of the Mathematical Society of Japan* 14 (1962), pp. 333–340. DOI: 10.2969/jmsj/01430333. MR: 0142086. Zbl: 0115.39302 (cit. on p. 27)

[Pan18] J. Pan. *A proof of Milnor conjecture in dimension* 3. *Journal für die reine und angewandte Mathematik* (2018). DOI: 10.1515/crelle-2017-0057 (cit. on p. 38)

[Pri83] P. Price. *A monotonicity formula for Yang–Mills fields. Manuscripta Math.* 43.2-3 (1983), pp. 131–166. DOI: 10.1007/BF01165828. MR: MR707042 (cit. on p. 83)

[Rie68] B. Riemann. *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen. Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen* 13 (1868), pp. 1–20. Zbl: 01.0022.02 (cit. on p. 9)

[Sev54] F. Severi. *Some remarks on the topological characterization of algebraic surfaces. Studies in mathematics and mechanics presented to Richard von Mises.* 1954, pp. 54–61. MR: 0065212. Zbl: 0057.13201 (cit. on p. 113)

[Shi83] K. Shiohama. *A sphere theorem for manifolds of positive Ricci curvature. Transactions of the American Mathematical Society* 275.2 (1983), pp. 811–819. DOI: 10.2307/1999055. Zbl: 0518.53047 (cit. on p. 34)

[Sor00] C. Sormani. *Nonnegative Ricci curvature, small linear diameter growth and finite generation of fundamental groups. Journal of Differential Geometry* 54.3 (2000), pp. 547–559. Zbl: 1035.53045. (cit. on p. 38)

[Tau51] A. H. Taub. *Empty space-times admitting a three parameter group of motions.* 53 (1951), pp. 472–490. DOI: 10.2307/1969567. MR: 0041565. Zbl: 0044.22804 (cit. on p. 120)

[Tau87] C. H. Taubes. *Gauge theory on asymptotically periodic* 4*–manifolds. Journal of Differential Geometry* 25.3 (1987), pp. 363–430. DOI: 10.4310/jdg/1214440981. MR: 882829. Zbl: 0615.57009 (cit. on p. 77)

[Tia00] G. Tian. *Gauge theory and calibrated geometry. I. Annals of Mathematics* 151.1 (2000), pp. 193–268. DOI: 10.2307/121116. arXiv: math/0010015. MR: MR1745014. Zbl: 0957.58013 (cit. on p. 112)

[Uhl82] K. K. Uhlenbeck. *Removable singularities in Yang–Mills fields. Communications in Mathematical Physics* 83.1 (1982), pp. 11–29. MR: MR648355. (cit. on p. 83)

[UY86] K. K. Uhlenbeck and S.-T. Yau. *On the existence of Hermitian–Yang–Mills connections in stable vector bundles. Communications on Pure and Applied Mathematics* 39.S, suppl. (1986). Frontiers of the mathematical sciences: 1985 (New York, 1985), S257–S293. DOI: 10.1002/cpa.3160390714. MR: 861491 (cit. on p. 116)

[Wei58] A. Weil. *Introduction à l'étude des variétés kählériennes.* Actualités scientifiques et industrielles 1267. 1958. MR: 0111056. Zbl: 0137.41103 (cit. on p. 101)

[WW70] N. R. Wallach and F. W. Warner. *Curvature forms for 2-manifolds. Proceedings of the American Mathematical Society* 25 (1970), pp. 712–713. DOI: 10.2307/2036675. MR: 0262976. Zbl: 0197.47901 (cit. on p. 87)

[Xia13] C. Xia. *Eigenvalues on Riemannian manifolds.* Publicações Matemáticas do IMPA. 2013. MR: 3113924. Zbl: 1387.58002 (cit. on p. 35)

[Yau76] S.-T. Yau. *Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. Indiana University Mathematics Journal* 25 (1976), pp. 659–670. DOI: 10.1512/iumj.1976.25.25051. MR: 0417452. Zbl: 0335.53041 (cit. on p. 33)

[Yau77] S.-T. Yau. *Calabi's conjecture and some new results in algebraic geometry. Proceedings of the National Academy of Sciences of the United States of America* 74.5 (1977), pp. 1798–1799. MR: 0451180. Zbl: 0355.32028 (cit. on p. 113)

[Yau78] S.-T. Yau. *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I. Communications on Pure and Applied Mathematics* 31.3 (1978), pp. 339–411. DOI: 10.1002/cpa.3160310304. MR: 480350. Zbl: 0369.53059 (cit. on pp. 110, 112)

[Zhu97] S. Zhu. *The comparison geometry of Ricci curvature. Comparison geometry.* Mathematical Sciences Research Institute Publications 30. 1997, pp. 221–262. MR: 1452876. Zbl: 0896.53036 (cit. on p. 33)

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