Seminar Riemannian Convergence Theory

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This seminar will consists of 13 meetings. Here is a list of proposed topics.

(1) Introduction and planning 14.4.2021

(2) Cheeger's finiteness theorem, 21.4.2021

Theorem 0.1 (Cheeger [Che70]). *For every $\Lambda > 0$ and $m \in \mathbf{N}_0$ there are only finitely many homeomorphism types of connected, closed m–manifolds admitting a Riemannian metric g with*

$$\operatorname{diam}(X, g) \leqslant \Lambda, \quad \operatorname{vol}(X, g) \leqslant \Lambda, \quad \textit{and} \quad |R_g| \leqslant \Lambda.$$

The work of Kirby–Siebenmann implies diffeomorphism finiteness in $\dim X \neq 4$.

The ideas in this work set the stage for Riemannian Convergence Theory.

(3) Lipschitz topology and Shikata's theorem, 28.4.2021

The set $\mathfrak{M}$ of isometry classes of compact metric spaces carries a the **Lipschitz metric** d_L. This induces a **Shikata's metric** d_S on the set of diffeomorphism types of closed manifolds.

Theorem 0.2 (Shikata [Shi66]). *For every $m \in \mathbf{N}_0$ there is an $\varepsilon = \varepsilon(m) > 0$ such that if $d_S([X], [Y]) \leqslant \varepsilon$, then $[X] = [Y]$.*

It is an interesting problem to investigate the optimal $\varepsilon(m)$. [Shi66] gives $\varepsilon(m) = 1/(m!)^m$. Karcher [Kar72] has an improved lower bound of roughly $\frac{1}{3}m^{-2}$.

[Shi75] discusses the behaviour of d_S with respect to connected sums, etc.; e.g.; the distance between two manifolds differing by connected sum with an exotic sphere.

[This might be suitable for someone with no or little prior knowledge of Riemannian Geometry.]

(4) Gromov–Hausdorff convergence, 5.5.2021

The Lipschitz topology is very fine and consequently has very few compact subsets.

The set of compact subsets $\mathscr{K}(X)$ of a metric space X carries the **Hausdorff metric**. If X is compact, then so is $\mathscr{K}(X)$. The Hausdorff metric can be used to define the **Gromov–Hausdorff topology** on the set of isometry classes of compact metric spaces. (This theory was invented by Edwards [Edw75] and then invented again by Gromov [Gro81b]). There are numerous examples contrasting the Lipschitz and the Gromov–Hausdorff topologies. The Gromov–Hausdorff topology enjoys excellent compactness properties.

Theorem 0.3 (Edwards [Edw75, Theorems III.3 and III.7]). *$(\mathfrak{M}, d_{GH})$ is separable and complete.*

Theorem 0.4 (Gromov's compactness criterion [Gro81b, p.64]). *A subset $\mathfrak{X} \subset \mathfrak{M}$ is relatively compact if and only if it is uniformly bounded and uniformly totally bounded.*

It turns out to be useful to also introduce the pointed Gromov–Hausdorff topology for non-compact metric spaces (because they often appear as rescaling limits).

[This might be suitable for someone with no or little prior knowledge of Riemannian Geometry.]

(5) Bishop–Gromov volume comparison and Gromov's precompactness theorem, 12.5.2021

The criterion for a subset $\mathfrak{X} \subset \mathfrak{M}$ to relatively compact can be translated to a uniform covering number bound. The Bishop–Gromov Volume Comparison Theorem [Bis63; BC64, Section 11.10 Corollary 3 and 4; Gro81a, Section 2.1] yields upper bounds on volume growth assuming lower Ricci bounds. This result combined with a straight-forward covering argument yields Gromov's precompactness theorem.

Theorem 0.5 ([Gro07, Theorem 5.3]). *Let $\kappa \in \mathbf{R}$, $D > 0$, and $m > 0$. The subset of isometry classes of complete Riemannian manifolds (X, g) of dimension X with*

$$\operatorname{diam}(M) \leqslant D \quad \textit{and} \quad \operatorname{Ric}_g \geqslant (n-1)\kappa g$$

is relatively compact in $(\mathfrak{M}, d_{GH})$.

This translates to a corresponding result for the pointed Gromov–Hausdorff topology. This is is significant for the notion of tangent cone (at infinity).

(6) Examples arising from the Gibbons–Hawking ansatz/Hattori's examples of non-unique tangent cones, 19.5.2021

The **Gibbons–Hawking ansatz** [GH78] is a simple but very useful method for constructing explicit hyperkähler metrics in dimension four. These metrics are Ricci flat. The notion of pointed Gromov–Hausdorff convergence can be studied very concretely for sequences of such metrics.

Hattori [Hat17] used the Gibbons–Hawking ansatz to construct an example of a hyperkähler 4–manifold with non-unique tangent cones at infinity. This should be contrasted with the uniqueness theorem due to Colding and Minicozzi [CM14].

(7) Cheeger–Gromov $C^{k,\alpha}$ topology, 26.5.2021

The Cheeger–Gromov $C^{k,\alpha}$ topology is much stronger than the Lipschitz topology but gives more useful geometric information. The **Fundamental Lemma of Riemannian Convergence Theory** gives a compactness criterion for families of Riemannian manifolds with "uniformly controlled atlases".

The hypotheses of Cheeger's finiteness theorem yield uniform control on atlases constructed from the exponential map/normal coordinates. This observation together with Shikata's theorem gives another proof of diffeomorphism finiteness.

(8) Harmonic coordinates and uniformly controlled atlases, 9.6.2021

To control normal coordinates requires information on the Riemann curvature, which is often not available. To control harmonic coordinates on the other hand it suffices to have information on Ricci curvature. This is particularly useful for (almost) Einstein metrics. [JK82; DK81; GL91] This discussion yields a compactness theorem under Ricci curvature bounds [And90, Theorem 1.1], lower bounds on the injectivity radius, and upper bounds on the diameter.

(9) Anderson's volume pinching theorem, 16.6.2021

The lower bounds on the injectivity radius and upper bounds on the diameter in [And90, Theorem 1.1] can often be traded for other geometric hypothesis. One application of this circle of ideas is the following volume pinching theorem/almost rigidity theorem.

Theorem 0.6 (Anderson [And90, Theorem 1.2]). *Given $m \in \mathbf{N}$ and $\Lambda > 0$, there exists an $\varepsilon = \varepsilon(m, \Lambda) > 0$ such that if (X, g) is a closed Riemannian manifold of dimension m satisfying*

$$(m-1)g \leqslant \mathrm{Ric}_g \leqslant \Lambda g \quad \text{and} \quad \frac{\mathrm{vol}(M,g)}{\mathrm{vol}(S^m, g_1)} \geqslant 1 - \varepsilon, \tag{0.7}$$

then X is diffeomorphic to S^m.

There are numerous related results which could be discussed as well.

(10) ε–regularity theorem and convergence under integral curvature bounds, 23.6.2021

[And90, Theorem 1.1] together with an ε–regularity result yields the following.

Theorem 0.8 (Anderson [And90, Theorem 2.6]). *Given $m \in \mathbf{N}$, $R, c, v > 0$, and $\alpha \in (0,1)$, If (X_ν, g_ν) is a sequence of Riemannian manifolds of dimension n satisfying*

$$|\mathrm{Ric}_{g_\nu}| \leqslant c, \quad \mathrm{vol}(X_\nu, g_\nu) \geqslant v, \quad \text{and} \quad \int_{X_\nu} |R_{g_\nu}|^{n/2} \leqslant c,$$

then, after passing to a subsequence, (X_ν, g_ν) converges in the Gromov–Hausdorff topology to a Riemannian orbifold (M, g) with a finite number of singular points, each of which so modeled on a cone over S^{m-1}/Γ; g is a continuous Riemannian metric and $C^{1,\alpha}$ on the regular part of X.

A refinement of this result has been proved (independently) by Bando, Kasue, and Nakajima [BKN89, Theorem 5.5]. This result is particularly interesting in dimension four.

[This might require two talks.]

(11) Singularities of Ricci flow: tangent flows, 30.6.2021

Unless he vehemently objects, Shubham will explain some applications of Riemannian Convergence Theory to Ricci flow and the study of singularities thereof.

(12) Sketch of of Cheeger–Colding theory and the almost splitting theorem

The theory developed so far requires upper and lower bounds on the Ricci curvature. From Gromov's pre-compactness theorem Gromov–Hausdorff limits can be obtained assuming lower Ricci bounds only but the limiting spaces are a priori extremely irregular. It turns out that under a no-collapsing hypothesis some regularity for the limit can be established. The rough idea is to stratify the limit by tangent cone type and exploit the almost splitting theorem. This talk should give an overview of this approach to Riemannian Convergence Theory, possibly without any proofs at all. [CC96; CC97; CC00b; CC00a].

[This might require numerous talks—even on a sketch level. This will require a fair amount of work to prepare.]

(13) Collapsing, 7.7.2021

The no-collapsing hypothesis does not always hold. Roughly speaking this means that the Gromov–Hausdorff limit is of lower dimension. There are numerous situations where collapsing is unavoidable and even desirable (e.g., SYZ, adiabatic limits of G_2–manifolds). This talk could discuss a number of examples of collapsing, some structural results on collapsing limits, etc.

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