Differential Geometry IV
Lecture Notes

Prof. Dr. Thomas Walpuski

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Contents

Lecture 1

Why should you care about spin geometry? There is a plethora of reasons, but here are three:

(1) Modern physics requires spinors, Dirac operators, etc.

(2) The topology of manifolds (or at least certain aspects of it) is deeply intertwined with differential operators, and certain questions can only be answered using Dirac operators.

(3) Seiberg–Witten theory requires spin^c–structure and Dirac operators.

In the following, I will discuss the first point briefly and elaborate on an instance of the second point. I will discuss Seiberg–Witten theory towards the end of the semester.

1 Dirac's problem: a square root of the Laplace operator?

Dirac [Dir28, §2] came across the following question when trying to find a relativistic theory of the electron. Consider a free particle of energy E, momentum $\mathbf{p}$, and mass m. According to Special Relativity,

$$E = \sqrt{\mathbf{p}^2 + m^2}.$$

The **rules of quantisation** dictate that E and $\mathbf{p}$ are to be replaced by the differential operators $i\partial_t$ and $-i\nabla$. Therefore,

$$i\partial_t = \sqrt{\Delta + m^2}$$

with $\Delta := -\sum_{i=1}^3 \partial_{x_i}^2$.

Question 1.1. Is there a differential operator $\not{D}$ satisfying $\Delta = \not{D}^2$?

The *ansatz*

$$\not{D} = \sum_{i=1}^{3} \gamma_i \partial_{x_i}$$

with γ_i constant for the **Dirac operator** leads to system of algebraic equations

$$\gamma_i^2 = -1 \quad \text{and} \quad \gamma_i\gamma_j + \gamma_j\gamma_i = 0.$$

This does not have any solutions in $\mathbf{R}$ or $\mathbf{C}$; but it does have solutions in $\mathbf{H}^{\oplus 2}$ and $\mathrm{M}_2(\mathbf{C})^{\oplus 2}$ (which can be found by hand). However, it is important to observe that $\not{D}$ does not act on functions but rather more complicated objects: **spinors**; cf. Cartan [Car13]. Brauer and Weyl [BW35] realised that the construction of the Dirac operator and spinors is closely related with the Clifford algebra.

2 The signature of a manifold

Situation 2.1. Let X be an closed equidimensional oriented smooth manifold. ×

Definition 2.2. The **intersection form** of X is the bilinear form $b_X\colon \mathrm{H}_{\mathrm{dR}}(X) \to \mathbf{R}$ defined by

$$b_X([\alpha],[\beta]) \coloneqq \int_X \alpha \wedge \beta.$$ •

Proposition 2.3.

(1) *If* $\dim X = 4k$, *then* $b_X|_{\mathrm{H}^{2k}_{\mathrm{dR}}(X)}$ *is symmetric.*

(2) *If* $\dim X = 4k+2$, *then* $b_X|_{\mathrm{H}^{2k+1}_{\mathrm{dR}}(X)}$ *is alternating.* ■

Definition 2.4. If $\dim X = 4k$, then **intersection form** is the quadratic form $q_X\colon \mathrm{H}^{2k}_{\mathrm{dR}}(X) \to \mathbf{R}$ defined by

$$q_X([\alpha]) \coloneqq b_X([\alpha],[\alpha]).$$

The **signature** of X is

$$\sigma(X) \coloneqq \begin{cases} \sigma(q_X) & \text{if } \dim X = 0 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

with $\sigma(q_X)$ denoting the signature of q_X. •

Remark 2.5. By Poincaré duality, q_X is non-degenerate. Therefore, q_X is determined by its signature up to isometry according to Sylvester's Law of Inertia. ♣

Remark 2.6. Hirzebruch [Hir72] says that the concept was introduced by Weyl [Wey24] (but I cannot get a hold of the latter). ♣

3 Relative de Rham cohomology

Situation 3.1. Let Y be a smooth manifold. Let $\iota\colon X \hookrightarrow Y$ be a closed submanifold. ×

Definition 3.2. The **relative de Rham complex** of $\iota\colon X \hookrightarrow Y$ is

$$(\Omega(Y,X) \coloneqq \ker(\iota^*\colon \Omega(X) \to \Omega(Y)), \mathrm{d}).$$ •

The **relative de Rham cohomology** of $\iota\colon X \hookrightarrow Y$ is

$$\mathrm{H}_{\mathrm{dR}}(Y,X) \coloneqq \mathrm{H}(\Omega(Y,X),\mathrm{d}).$$

Proposition 3.3. *The sequence of differential graded algebras*

$$0 \to (\Omega^\bullet(Y,X),\mathrm{d}) \hookrightarrow (\Omega^\bullet(Y),\mathrm{d}) \xrightarrow{\iota^*} (\Omega^\bullet(X),\mathrm{d}) \to 0$$

is exact. ■

Proposition 3.4. *There is a exact sequence*

$$\cdots \to \mathrm{H}^k_{\mathrm{dR}}(Y,X) \to \mathrm{H}^k_{\mathrm{dR}}(Y) \xrightarrow{i^*} \mathrm{H}^k_{\mathrm{dR}}(X) \xrightarrow{\delta} \mathrm{H}^{k+1}_{\mathrm{dR}}(Y,X) \to \cdots;$$

moreover, the connecting homomorphism δ satisfies $\delta[i^*\alpha] = [\mathrm{d}\alpha]$.

Proof. This is an immediate consequence of Proposition 3.3 and the Snake Lemma. ■

Remark 3.5. It is possible to construct a relative de Rham cohomology $\mathrm{H}_{\mathrm{dR}}(f)$ for every smooth map $f\colon X \to Y$ such that the analogue of Proposition 3.4 holds; cf. [BT82, pp. 78–79]. ♣

4 Poincaré–Lefschetz duality

Situation 4.1. Let Y be a compact connected smooth manifold of dimension n. Let $\iota\colon X \hookrightarrow Y$ be a closed submanifold. ×

Theorem 4.2 (Poincaré–Lefschetz duality). *Let $k \in \{0,\dots,n\}$. The bilinear map $\mathrm{H}^k_{\mathrm{dR}}(Y,X) \otimes \mathrm{H}^{n-k}_{\mathrm{dR}}(Y) \to \mathbf{R}$, $[\alpha]\otimes[\beta] \mapsto \int_Y \alpha\wedge\beta$ is a perfect pairing; that is: it induces an isomorphism*

$$\mathrm{H}^k_{\mathrm{dR}}(Y,X) \to \mathrm{H}^{n-k}_{\mathrm{dR}}(Y)^*.$$

Remark 4.3. See [Sch95, Corollary 2.6.2] for a proof. (In my opinion the proof in [Sch95] is needlessly complicated and circuitous, but at least it does have most of the details.) ♣

5 Bordism invariance of the signature

Situation 5.1. Let X_1, X_2 be closed equidimensional oriented smooth manifolds. ×

Proposition 5.2 (Thom [Tho52, Corollaire V.11]). *If X_1 and X_2 are bordant, then $\sigma(X_1) = \sigma(X_2)$.*

Proof. It suffices to show that if $X = \partial Y$ with Y compact and $\dim Y = 4k+1$, then $\sigma(X) = 0$. Denote by $\mathrm{res}\colon \mathrm{H}^{2k}_{\mathrm{dR}}(Y) \to \mathrm{H}^{2k}_{\mathrm{dR}}(X)$ the restriction map. The upcoming argument shows that $\operatorname{im}\mathrm{res} \subset \mathrm{H}^{2k}_{\mathrm{dR}}(X)$ is totally isotropic and $\dim\operatorname{im}\mathrm{res} = \frac{1}{2}b^{2k}(X)$. Therefore, by Proposition 13.3, q_X has vanishing signature.

Since

$$q_x \circ \mathrm{res}([\alpha]) = \int_X \alpha\wedge\alpha = \int_Y \mathrm{d}(\alpha\wedge\alpha) = 2\int_Y \mathrm{d}\alpha\wedge\alpha = 0,$$

$q_X \circ \mathrm{res} = 0$. Therefore, $\operatorname{im}\mathrm{res}$ is isotropic. Moreover, res fits into the commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^{2k}_{\mathrm{dR}}(Y) & \xrightarrow{\mathrm{res}} & \mathrm{H}^{2k}_{\mathrm{dR}}(X) & \xrightarrow{\delta} & \mathrm{H}^{2k+1}_{\mathrm{dR}}(Y,X) \\
 & & \downarrow\cong & & \downarrow\cong \\
 & & \mathrm{H}^{2k}_{\mathrm{dR}}(X)^* & \xrightarrow{\mathrm{res}^*} & \mathrm{H}^{2k}_{\mathrm{dR}}(Y)^*.
\end{array}$$

Here the vertical isomorphism are Poincaré duality and Poincaré–Lefschetz duality. The diagram commutes because of Proposition 3.4 and Stokes' theorem. Therefore,

$$\dim \operatorname{res} = \dim \ker \operatorname{res}^* = b^{2k}(X) - \dim \operatorname{im} \operatorname{res}^* = b^{2k}(X) - \dim \operatorname{im} \operatorname{res}\,.$$

Hence, $\dim \operatorname{res} = \frac{1}{2} b^{2k}(X)$. ∎

Proposition 5.3.

$$\sigma(X_1 \times X_2) = \sigma(X_1)\sigma(X_2).$$

Proof. Without loss of generality $\dim X_1 + \dim X_2 = 4k$. The Künneth theorem identifies

$$\mathrm{H}_{\mathrm{dR}}(X_1 \times X_2) = \mathrm{H}_{\mathrm{dR}}(X_1) \otimes \mathrm{H}_{\mathrm{dR}}(X_2) \quad \text{and} \quad b_{X_1 \times X_2} = b_{X_1} \otimes b_{X_2}.$$

(Here ⊗ denotes the graded tensor product.)

Set $n \coloneqq \dim X_1$. Set

$$V_0 \coloneqq \begin{cases} \mathrm{H}^{n/2}_{\mathrm{dR}}(X_1) \otimes \mathrm{H}^{2k-n/2}_{\mathrm{dR}}(X_2) & \text{if } n \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

For $\ell = 0, \dots, \lfloor n/2 \rfloor$ set

$$V_\ell \coloneqq I_\ell \oplus I_{n-\ell} \quad \text{with} \quad I_\ell \coloneqq \mathrm{H}^{\ell}_{\mathrm{dR}}(X_1) \otimes \mathrm{H}^{2k-\ell}_{\mathrm{dR}}(X_2).$$

The decomposition

$$\mathrm{H}^{2k}_{\mathrm{dR}}(X_1 \times X_2) = V_0 \oplus \bigoplus_{\ell=0}^{\lfloor n/2 \rfloor} V_\ell$$

is perpendicular with respect to $q_{X_1 \times X_2}$; that is:

$$q_{X_1 \times X_2} = q_0 \perp \cdots \perp q_{\lfloor n/2 \rfloor} \quad \text{with} \quad q_\ell \coloneqq q_{X_1 \times X_2}|_{V_\ell}.$$

Therefore, it remains to determine $\sigma(q_\ell)$:

(1) Evidently, If $n = 0 \bmod 4$, then $\sigma(q_0) = \sigma(X_1) q(X_2)$.

(2) If $n = 2 \bmod 4$, then $b_{X_1}|_{\mathrm{H}^{n/2}_{\mathrm{dR}}(X_1)}$ and $b_{X_2}|_{\mathrm{H}^{2k-n/2}_{\mathrm{dR}}(X_2)}$ are alternating. Therefore, there is a totally isotropic $I \subset \mathrm{H}^{n/2}_{\mathrm{dR}}(X_1)$ with $\dim I = \frac{1}{2}\mathrm{H}^{n/2}_{\mathrm{dR}}(X_1)$. Since $I \otimes \mathrm{H}^{2k-n/2}_{\mathrm{dR}}(X_2)$ is totally isotropic, by Proposition 13.3, $\sigma(q_0) = 0$.

(3) For $\ell = 0, \dots, \lfloor n/2 \rfloor$, $I_\ell \subset V_\ell$ is totally isotropic and $\dim I_\ell = \frac{1}{2} V_\ell$. Therefore, by Proposition 13.3, $\sigma(q_\ell) = 0$. ∎

Corollary 5.4 (Thom [Tho54, paragraph after Théorème IV.1]). *The signature induces a ring homomorphism* $\sigma\colon \Omega^{\mathrm{SO}} \to \mathbf{Z}$. ∎

6 Hirzebruch signature theorem

Situation 6.1. Let X be a closed oriented smooth manifold. ×

Definition 6.2. The L **genus** of a vector bundle V over X is

$$L(V) := p_\ell(V) \in \mathrm{H}_{\mathrm{dR}}(X) \quad \text{with} \quad \ell(x) := \frac{\sqrt{x}}{\tanh\sqrt{x}}$$

and p_ℓ denoting the Pontrjagin ℓ–genus. •

Remark 6.3. Since $\tanh(x)$ is odd, $x/\tanh(x)$ is a power series in x^2 and $\ell(x)$ is a power series in x. Indeed,

$$\ell(x) = \sum_{k=0}^{\infty} \frac{2^{2k}B_{2k}}{(2k)!}x^k$$

with B_{2k} denoting the Bernoulli numbers. ♣

Remark 6.4. The L genus is a **multiplicative characteristic class**; that is:

$$L(V_1 \oplus V_2) = L(V_1)L(V_2).$$ ♣

Remark 6.5. $L(V)$ can be expressed in terms of the Pontrjagin classes of V and the L **polynomials**:

$$L(V) = \sum_{k=1}^{\infty} L_k(p_1, \dots, p_k).$$

A computation shows that

$$L_1 = \frac{1}{3}p_1, \quad L_2 = \frac{1}{45}(7p_2 - p_1^2), \quad \dots.$$

see OEIS: A237111. The L polynomials are **multiplicative sequence**; cf. [Hir95, §1; LM89, Chapter III §11]. ♣

Theorem 6.6 (Hirzebruch [Hir53, Theorem 3.1; Hir95, Chapter II Theorem 8.2.2]).

$$\sigma(X) = \langle L(TX), [X]\rangle. \tag{6.7}$$

Remark 6.8. Hirzebruch [Hir72, §2] remarks on the proof of his signature theorem:

> How to prove it? After conjecturing it I went to the library of the Institute of Advanced Studies (June 2, 1953). Thom's *Comptes Rendus* note [Tho53] had just arrived. This finished the proof.

[Tho53] introduced the concept of bordism and announced that $\Omega^{\mathrm{SO}} \otimes_{\mathbf{Z}} \mathbf{R}$ is a polynomial algebra generated by $[\mathbf{C}P^{2n}]$; cf. [Tho54, Théorème IV.17]. Since both sides of (6.7), define ring homomorphisms $\Omega^{\mathrm{SO}} \otimes \mathbf{R} \to \mathbf{R}$ it suffices to verify (6.7) on $\mathbf{C}P^{2n}$. Surely, Hirzebruch did this before going to the library. ♣

Remark 6.9. The Hirzebruch signature theorem implies a sequence of dazzling integrality theorems; e.g.: if $\dim X = 4$, then $\frac{1}{3}\langle p_1(TX), [X]\rangle \in \mathbf{Z}$. The same does not hold for arbitrary real vector bundles. ♣

7 Rohklin's theorem

Theorem 7.1 (Rokhlin [Rok52]). *Let X be a closed oriented 4–manifold. If $w_2(X) = 0$, then $\sigma(X)$ is divisible by 16.*

8 The Euler characteristic operator

Situation 8.1. Let (X, g) be an closed oriented Riemannian manifold. ×

Definition 8.2. The **Euler characteristic operator** associated with (X, g) is

$$\delta \coloneqq \mathrm{d} + \mathrm{d}^* \colon \Omega^{\mathrm{even}}(X) \to \Omega^{\mathrm{odd}}(X). \qquad \bullet$$

Proposition 8.3.

$$\operatorname{index} \delta = \chi(X).$$

Proof. This is an immediate consequence of Hodge theory. ■

Remark 8.4. [LM89, Chapter II Example 6.1] exhibits δ as a Dirac operator. ♣

9 The signature operator

Situation 9.1. Let (X, g) be an closed oriented Riemannian manifold with $\dim X = 2n$. ×

Definition 9.2. Define $\varepsilon \in \mathrm{End}(\Lambda T^*X \otimes \mathbf{C})$ by

$$\varepsilon\alpha \coloneqq i^{k(k-1)+n} * \alpha \quad \text{with} \quad k \coloneqq \deg \alpha. \qquad \bullet$$

Proposition 9.3.

(1) $\varepsilon^2 = \mathbf{1}$.

(2) $\mathrm{d} + \mathrm{d}^*$ *and* ε *anti-commute.*

Proof. To prove (1), observe that for every $\alpha \in \Lambda^k T^*X \otimes \mathbf{C}$

$$\varepsilon^2\alpha = (-1)^{k(2n-k)} i^{k(k-1)+n} i^{(2n-k)(2n-k-1)+n} \alpha = \alpha$$

because

$$2k(2n-k) + k(k-1) + n + (2n-k)(2n-k-1) + n = 4n^2 = 0 \bmod 4.$$

To prove (2) observe that for every $\alpha \in \Omega^k(X, \mathbf{C})$

$$\begin{aligned} \mathrm{d}^*\alpha &= (-1)^{2n(k-1)+1} * \mathrm{d} * \alpha = - * \mathrm{d} * \alpha \\ &= -(-i)^{(2n-k+1)(2n-k)+k(k-1)+2n} \varepsilon \mathrm{d} \varepsilon \alpha = -\varepsilon \mathrm{d} \varepsilon \alpha \end{aligned}$$

because

$$(2n-k+1)(2n-k) + k(k-1) + 2n = 2k(k-1) + 4(n - kn + n^2) = 0 \bmod 4. \qquad ■$$

Definition 9.4. Set
$$\Omega_\pm(X,\mathbf{C}) \coloneqq \{\alpha \in \Omega(X,\mathbf{C}) : \varepsilon\alpha = \pm\alpha\}.$$
•

The **signature operator** associated with (X,g) is
$$D \coloneqq \mathrm{d} + \mathrm{d}^* \colon \Omega_+(X,\mathbf{C}) \to \Omega_-(X).$$

Proposition 9.5.
$$\mathrm{index}_{\mathbf{C}}\, D = \sigma(X).$$

Proof. Since $(\mathrm{d}+\mathrm{d}^*)^2 = \Delta$,
$$\ker D = \mathscr{H}_+(X,\mathbf{C}) \coloneqq \mathscr{H}(X,\mathbf{C}) \cap \Omega_+(X,\mathbf{C}) \quad \text{and}$$
$$\operatorname{coker} D \cong \mathscr{H}_-(X,\mathbf{C}) \coloneqq \mathscr{H}(X,\mathbf{C}) \cap \Omega_-(X,\mathbf{C}).$$

The inclusions $\mathscr{H}^k(X,\mathbf{C}) \hookrightarrow \mathscr{H}_\pm(X,C), \alpha \mapsto \alpha \pm \varepsilon\alpha$ ($k = 0,\ldots,n-1$) and $\mathscr{H}^n_\pm(X,\mathbf{C}) \hookrightarrow \mathscr{H}_\pm(X,C)$ assemble into an isomorphism
$$\mathscr{H}^0(X,\mathbf{C}) \oplus \cdots \oplus \mathscr{H}^{n-1}(X,\mathbf{C}) \oplus \mathscr{H}^n_\pm(X,\mathbf{C}) \cong \mathscr{H}_\pm(X,\mathbf{C}).$$

Therefore,
$$\mathrm{index}_{\mathbf{C}}\, D = \dim_{\mathbf{C}} \mathscr{H}^n_+(X,\mathbf{C}) - \dim_{\mathbf{C}} \mathscr{H}^n_-(X,\mathbf{C}).$$

If n is even, then
$$\mathscr{H}^n_\pm(X,\mathbf{C}) = \mathscr{H}^n_\pm(X) \otimes \mathbf{C} \quad \text{with} \quad \mathscr{H}^n_\pm(X) \coloneqq \{\alpha \in \mathscr{H}^n(X) : *\alpha = \pm\alpha\}.$$

Therefore,
$$\mathrm{index}_{\mathbf{C}}\, D = \dim \mathscr{H}^n_+(X) - \dim \mathscr{H}^n_-(X) = \sigma(X)$$
by Hodge theory.

If n is odd and $\alpha \in \Lambda^n T^*X \otimes \mathbf{C}$, then
$$\varepsilon\alpha = i * \alpha.$$

Therefore,
$$\overline{\mathscr{H}}^n_+(X,\mathbf{C}) = \mathscr{H}^n_-(X,\mathbf{C})$$
and
$$\mathrm{index}_{\mathbf{C}}\, D = 0 = \sigma(X).$$
■

Remark 9.6. Hirzebruch signature theorem predates the Atiyah–Singer index theorem by about decade—but, of course, it can be derived from it because of the above. ♣

Remark 9.7. Is there a differential operator $\not{D}$ whose index explains Rokhlin's theorem? Yes! If X is a closed oriented 4–manifold with $w_2(X) = 0$, then the **Atiyah–Singer operator** $\not{D}$ satisfies
$$\mathrm{index}\, \not{D} = \frac{1}{8}\sigma(X) \quad \text{and} \quad \mathrm{index}\, \not{D} = 0 \bmod 2.$$
♣

Remark 9.8. [LM89, Chapter II Example 6.2] exhibits D as a Dirac operator. ♣

Lecture 2

A problem at the root of spin geometry is to find solutions to the algebraic equation

$$\gamma_i^2 = -1 \quad \text{and} \quad \gamma_i\gamma_j + \gamma_j\gamma_i = 0.$$

The systematic answer to this question is the theory of Clifford algebras. Although it is possible to quite directly attack the problem of determining the Clifford algebras over **R** and **C**, I would find it uncultured to so. There is a Clifford algebra associated with every quadratic form and Clifford algebras play an important role in the theory of quadratic forms. The main purpose of this lecture is to show that every quadratic form can be decomposed (possibly non-uniquely) into simple elementary pieces. At the end, I will mention the Cartan–Dieudonné Theorem, which plays an important role in many developments of the spin group in the literature (but can be replaced by the super Skolem–Noether theorem).

10 Quadratic forms

Situation 10.1. Let k be a field. ×

Definition 10.2. Let V be a k–vector space. A **quadratic form** on V is a map $q\colon V \to k$ such that:

(1) it is homogenous of degree 2; i.e.: for every $\lambda \in k$ and $v \in V$

$$q(\lambda v) = \lambda^2 q(v),$$

and

(2) the map

$$(v, w) \mapsto q(v + w) - q(v) - q(w)$$

defines a symmetric bilinear form $p \in \mathrm{Hom}(S^2V, k)$—the **polarisation** of q. •

Example 10.3. Let $a_1, \dots, a_n \in k$. The **diagonal quadratic form** $\langle a_1, \dots, a_n\rangle\colon k^{\oplus n} \to k$ is defined by

$$\langle a_1, \dots, a_n\rangle(x_1, \dots, x_n) \coloneqq a_1x_1^2 + \cdots + a_nx_n^2.$$ ♠

Example 10.4. Let $a, b \in k$. The quadratic form $[a, b]\colon k \oplus k \to k$ is defined by

$$[a, b](x, y) \coloneqq ax^2 + xy + by^2.$$ ♠

Example 10.5. The **hyperbolic plane** is $[0, 0]$. ♠

Definition 10.6. Let $q_i\colon V_i \to k$ $(i = 1, 2)$ be quadratic forms. The **perpendicular sum** of q_1 and q_2 is the quadratic form $q_1 \perp q_2\colon V_1 \oplus V_2 \to k$ defined by

$$q_1 \perp q_2 \coloneqq q_1 + q_2.$$ •

Remark 10.7. The structure theory of quadratic forms shows that every quadratic form can be decomposed into the above pieces. ♣

Proposition 10.8. *Let V be a k-vector space. Define $Q\colon \mathrm{Hom}(V\otimes V, k) \to \{$quadratic forms on $V\}$ by*

$$Q(b)(v) := b(v \otimes v)$$

Q is surjective and $\ker Q = \mathrm{Hom}(\Lambda^2 V, k)$; that is: the sequence

$$\mathrm{Hom}(\Lambda^2 V, k) \hookrightarrow \mathrm{Hom}(V \otimes V, k) \overset{Q}{\twoheadrightarrow} \{\textit{quadratic forms on } V\}$$

is exact.

Proof. Evidently, $\ker Q = \mathrm{Hom}(\Lambda^2 V, k)$. To see that Q is surjective, let $q\colon V \to k$ be a quadratic form on V and denote its polarisation by p. Choose a basis $\{e_i : i \in I\}$ of V and choose an order $\prec$ on the index set I. Define the $b \in \mathrm{Hom}(V \otimes V, k)$ by

$$b(e_i \otimes e_j) := \begin{cases} 0 & \text{if } i < j, \\ q(e_i) & \text{if } i = j, \\ p(e_i, e_j) & \text{if } i > j. \end{cases}$$

Evidently, $q = Q(b)$. ■

Remark 10.9. If $2 \neq 0 \in k$, then $\frac{1}{2}$ of the polarisation defines a right inverse of Q; therefore, the theory of quadratic form is equivalent to the theory of symmetric bilinear forms. If $2 = 0 \in k$, then the polarisation is alternating. ♣

Definition 10.10. Let $q_i\colon V_i \to k$ $(i = 1, 2)$ be quadratic forms. A **quadratic morphism** $f\colon q_1 \to q_2$ is a linear map such that

$$q_2 \circ f = q_1.$$ •

Remark 10.11. Quadratic forms and quadratic morphisms form a category. ♣

Definition 10.12. Let q be a quadratic form. The **orthogonal group** associated with q is the group

$$\mathrm{O}(q) := \{f\colon q \to q : f \text{ is a quadratic isomorphism}\}.$$ •

[Che54, §1.2; Bou07, §3.4; MH73; Lam73; Lam05; KS80; Knu91; EKM08, §7].

11 Structure theory of quadratic forms

Situation 11.1. Let k be a field. Let $q\colon V \to k$ be a quadratic form with $\dim V < \infty$. Denote by $p \in \mathrm{Hom}(S^2 V, k)$ the polarisation of q. ×

Remark 11.2. An isomorphism $f\colon k^{\oplus n} \cong V$ (that is: a choice of basis) identifies q with the polynomial

$$\sum_{i=1}^{n} a_i x_i^2 + \sum_{i<j} a_{ij} x_i x_j \coloneqq q \circ f(x_1, \dots, x_n).$$

A primary objective of the structure theory of quadratic forms is to simplify this expression as much as possible though a particularly clever choice of f. ♣

Example 11.3. If $\operatorname{char} k \neq 2$ and $a \in k^\times$, then $\langle a, a\rangle \cong [0, 0]$. ♠

Example 11.4. For every $a \in k$, $[0, a] \cong [0, 0]$. ♠

Example 11.5. If $\operatorname{char} k = 2$, then $[a, b] \perp [a, c] \cong [a, b + c] \perp [0, 0]$. ♠

Definition 11.6. The quadratic form q is **non-degenerate** if the map $\flat\colon V \to V^*$ defined by

$$v^\flat(w) \coloneqq p(v, w).$$

is an isomorphism. •

Example 11.7. $\langle a_1, \dots, a_n\rangle$ is non-degenerate if and only if $2^n a_1 \cdots a_n \neq 0$ (in particular: $\operatorname{char} k \neq 2$). ♠

Example 11.8. $[a, b]$ is non-degenerate if and only if $4ab - 1 \neq 0$; e.g. if $\operatorname{char} k = 2$ or $a = 0$. ♠

Definition 11.9. The **radical** of p and the **radical** of q are

$$\operatorname{rad} p \coloneqq \ker \flat \quad \text{and} \quad \operatorname{rad} q \coloneqq \operatorname{rad} p \cap q^{-1}(0)$$

respectively. The **nullity** and **defect** of q are

$$n(q) \coloneqq \dim \operatorname{rad} q \quad \text{and} \quad d(q) \coloneqq \dim \operatorname{rad} p - \dim \operatorname{rad} q$$

respectively. •

Remark 11.10. As an immediate consequence of Definition 10.2 (2), $\operatorname{rad} q \subset \operatorname{rad} p$ is a linear subspace. ♣

Proposition 11.11. *If* $V = \operatorname{rad} p \oplus W$, *then* $q = q|_{\operatorname{rad} p} \perp q|_W$ *and* $q|_W$ *is non-degenerate.*

Proof. By definition of $\operatorname{rad} p$, $q = q|_{\operatorname{rad} p} \perp q|_W$. Since $\ker \flat|_W = 0$, $q|_W$ is non-degenerate. ■

Proposition 11.12. *If* $\operatorname{char} k \neq 2$, *then* $\operatorname{rad} q = \operatorname{rad} p$.

Proof. This is a consequence of $q = Q(\frac{1}{2}p)$. ■

Definition 11.13. The quadratic form q is **anisotropic** if $q^{-1}(0) = \{0\}$. •

Proposition 11.14. *If* $\operatorname{rad} p = \operatorname{rad} q \oplus D$, *then* $q|_D$ *is anisotropic and diagonal with respect to every basis.*

Proof. By definition of $\operatorname{rad} q$, $q|_D$ is anisotropic. If $e_1, \ldots, e_n$ is a basis of D, then

$$q|_D(x_1e_1 + \cdots + x_ne_n) = \sum_{i=1}^{n} a_i x_i^2 + \sum_{i<j} \underbrace{p(e_i, e_j)}_{=0} x_i x_j. \qquad \blacksquare$$

Definition 11.15. Let $W \subset V$. The **perpendicular** of W is

$$W^\perp := \{v \in V : p(v, w) = 0 \text{ for every } w \in W\}. \qquad \bullet$$

Proposition 11.16 (Splitting off non-degenerate summands). *Let $W \subset V$. If q and $q|_W$ are non-degenerate, then $q = q|_W \perp q|_{W^\perp}$, and $q|_{W^\perp}$ is non-degenerate.*

Proof. Since $q|_W$ is non-degenerate, $W \cap W^\perp = 0$. Since $W^\perp = \ker(V \xrightarrow{\flat} V^* \to W^*)$ and q is non-degenerate, $\dim W^\perp = \dim V - \dim W$. Therefore, $W + W^\perp = V$. Evidently, $q = q|_W + q|_{W^\perp}$, and $q|_{W^\perp}$ is non-degenerate. ■

Proposition 11.17 (Diagonalisation of non-degenerate quadratic forms if $\operatorname{char} k \neq 2$). *If $\operatorname{char} k \neq 2$ and q is non-degenerate, then q can be diagonalised; that is: there are $a_1, \ldots, a_n \in k$ with*

$$q \cong \langle a_1, \ldots, a_n \rangle.$$

Proof. The proof is by induction on $\dim V$. If $V = 0$, then the assertion is trivial. If $V \neq 0$, then $q \neq 0$ because it is non-degenerate. Choose $v \in V$ with $q(v) \neq 0$. Since $\operatorname{char} k \neq 2$, $q|_{\langle v \rangle}$ is non-degenerate. Therefore, by Proposition 11.16, $q = q|_{\langle v \rangle} \perp q|_{\langle v \rangle^\perp}$, and $q|_{\langle v \rangle^\perp}$ is non-degenerate. Since $\dim \langle v \rangle^\perp = \dim V - 1$, the assertion follows. ■

Remark 11.18. The proof of Proposition 11.17 can be turned into an algorithm—a variant of the Gram–Schmidt process. ♣

Proposition 11.19. *Suppose that* $\dim V \geqslant 2$. *Let* $v \in V \setminus \operatorname{rad} p$.

(1) *If* $q(v) = 0$, *then there is a* $W \subset V$ *with* $v \in W$ *and* $\dim W = 2$ *such that*

$$q|_W \cong [0, 0].$$

(2) *If* $\operatorname{char} k = 2$, *then there are* $W \subset V$ *with* $v \in W$ *and* $\dim W = 2$, *and* $a, b \in k$ *such that*

$$q|_W \cong [a, b].$$

Proof. Since $v \notin \operatorname{rad} p$, there is a $w' \in V$ with $p(v, w') = 1$. Since $q(v) = 0$ or $\operatorname{char} k = 2$, $w' \notin \langle v \rangle$. This proves (2) with $a := q(v)$ and $b := q(w')$. To prove (1), set $w := w' - q(w')v$ and observe that $q(w) = 0$. ■

Definition 11.20. $S \subset V$ is **totally isotropic** if $q|_S = 0$. The **Witt index** of q is

$$i(q) := \max\{\dim S : S \subset W \subset V \text{ is totally isotropic and } \operatorname{rad} p \cap W = 0\}. \qquad \bullet$$

Theorem 11.21 (Structure theorem for quadratic forms). *Set $n := n(q)$, $d := d(q)$, and $i := i(q)$.*

(1) *If* $\operatorname{char} k \neq 2$*, then there are* $a_1, \ldots, a_r \in k^\times$ *such that*

$$q \cong \langle 0 \rangle^{\perp n} \perp [0,0]^{\perp i} \perp \langle a_1, \ldots, a_r \rangle.$$

Moreover, $\langle a_1, \ldots, a_r \rangle$ *is anisotropic.*

(2) *If* $\operatorname{char} k = 2$*, then there are* $a_1, \ldots, a_d, b_1, c_1, \ldots, b_r, c_r \in k^\times$ *such that*

$$q \cong \langle 0 \rangle^{\perp n} \perp \langle a_1, \ldots, a_d \rangle \perp [0,0]^{\perp i} \perp [b_1, c_1] \perp \ldots \perp [b_r, c_r].$$

Moreover, $\langle a_1, \ldots, a_d \rangle$ *and* $[b_1, c_1] \perp \ldots \perp [b_r, c_r]$ *are anisotropic.*

Remark 11.22. At first glance, Theorem 11.21 seems very satisfactory. However, it does not answer the question to what extend $a_1, \ldots, a_r \in k$ (resp. $a_1, \ldots, a_d, b_1, c_1, \ldots, b_r, c_r \in k$) are uniquely determined by q. If k is quadratically closed (e.g., $k = \mathbf{C}$), then there is a simple answer. If k is a Euclidean field (e.g., $k = \mathbf{R}$), then Sylvester's Law of Inertia answers this question. For general k one has to delve into Witt theory; cf. [EKM08, §8]. ♣

Proof of Theorem 11.21. Choose $W \subset V$ with $V = \operatorname{rad} p \oplus W$. By Proposition 11.11, it suffices to analyse $q|_{\operatorname{rad} p}$ and $q|_W$, and the latter is non-degenerate.

If $\operatorname{char} k \neq 2$, then by Proposition 11.12, $q_{\operatorname{rad} p} = \langle 0 \rangle^{\perp n}$. If $\operatorname{char} k = 2$, then, by Proposition 11.14, $q|_{\operatorname{rad} p} \cong \langle 0 \rangle^{\perp n} \perp \langle a_1, \ldots, a_d \rangle$.

Let $S \subset W$ totally isotropic with $\dim S = i$. Repeated application of Proposition 11.19 and Proposition 11.16 constructs a totally isotropic $S' \subset W$ with $S \cap S' = 0$ and $q|_{S \oplus S'} \cong [0,0]^{\perp i}$. Set $R := (S \oplus S')^\perp$. By Proposition 11.16, $q|_W = q|_{S \oplus S'} \perp q|_R$, $q|_R$ is non-degenerate and anisotropic. If $\operatorname{char} k \neq 2$, then, by Proposition 11.17, $q|_R \cong \langle a_1, \ldots, a_r \rangle$. If $\operatorname{char} k = 2$, then, by repeated application of Proposition 11.19 and Proposition 11.16, $q|_R \cong [b_1, c_1] \perp \ldots \perp [b_r, c_r]$. ■

[Che54, §1.3; EKM08, §7].

12 Quadratic forms over quadratically closed fields

Situation 12.1. Let k be a quadratically closed field. ×

Theorem 12.2. *Let $q\colon V \to k$ be a quadratic form with* $\dim V < \infty$*. Set $n := n(q)$, $i := i(q)$, and* $r := \dim V - n - 2i$*. In this situation, $r \in \{0,1\}$ and*

$$q \cong \langle 0 \rangle^{\perp n} \perp [0,0]^{\perp i} \perp \langle 1 \rangle^r.$$

Proof. Since k is quadratically closed, for every $a \in k^\times$, $\langle a \rangle \cong \langle 1 \rangle$. Moreover, $\langle 1 \rangle^{\perp r}$ is anisotropic if and only if $r \in \{0,1\}$. Since k is quadratically closed, if $a, b \in k^\times$, then

$$[a,b](x,y) = ax^2 + xy + by^2$$

has a non-trivial zero; therefore: $[a,b]$ fails to be anisotropic. Therefore, Theorem 11.21 finishes the proof. ■

Remark 12.3. If $\operatorname{char} k \neq 2$, then $[0,0] \cong \langle 1,1 \rangle$ because $x^2 + y^2 = (x+iy)(x-iy)$ with $i^2 = -1$. ♣

Exercise 12.4. Prove that every quadratic form $q\colon V \to \mathbf{C}$ with $\dim V < \infty$ is isomorphic to $\langle 0 \rangle^n \perp \langle 1 \rangle^{r+2i}$ by repeatedly completing the square.

13 Sylvester's Law of Inertia

Situation 13.1. Let k be a ordered field. In particular, $\operatorname{char} k = 0$. Let $q\colon V \to k$ be a quadratic form with $\dim V < \infty$. ×

Definition 13.2. The quadratic form q is **positive definite** (**negative definite**) if $q(v) > 0$ ($q(v) < 0$) for every $0 \neq v \in V$. The **signature** of q is

$$\sigma(q) := r_+(q) - r_-(q)$$

with

$$r_\pm(q) := \max\{\dim W : \pm q|_W \text{ is positive definite}\}$$ •

Proposition 13.3. *If q is non-degenerate and $i(q) \geqslant \frac{1}{2}\dim V$, then $i(q) = \frac{1}{2}\dim V$ and $\sigma(q) = 0$.*

Proof. By Theorem 11.21 (1), $q \cong [0,0]^{i(q)}$. In particular, $i(q) = \frac{1}{2}\dim V$. Since, $x^2 - y^2 = (x+y)(x-y)$, $[0,0] \cong \langle 1, -1\rangle$. Therefore, $\sigma(q) = 0$. ■

Definition 13.4. A **Euclidean field** is an ordered field k such that every $x \in k$ with $x \geqslant 0$ admits a square root. •

Theorem 13.5 (Sylvester's Law of Inertia). *Set $r := r(q)$, $s := s(q)$, and $n := n(q)$. There are $a_1^+, \dots, a_r^+ > 0$ and $a_1^-, \dots, a_s^- < 0$ such that*

$$q \cong \langle a_1^+, \dots, a_r^+\rangle \perp \langle a_1^-, \dots, a_s^-\rangle \perp \langle 0\rangle^{\perp n};$$

moreover, if k is Euclidean, then $a_1^+ = \dots = a_r^+ = 1$ and $a_1^- = \dots = a_s^- = -1$.

Proof. By Proposition 11.11 and Proposition 11.12, without loss of generality, q is non-degenerate; that is: $n = 0$. Let $V_+ \subset V$ with $q|_{V_+}$ positive definite and $\dim V_+ = r$. Set $V_- := V_+^\perp$. The restriction $q|_{V_-}$ is negative definite; indeed: if $0 \neq v \in V_-$ and $q(v) > 0$, then $q_{V_+ \oplus \langle v\rangle}$ is positive definite—a contradiction. By Proposition 11.17, $q|_{V_\pm}$ can be diagonalised. If k is Euclidean, then, for every $a \in k^\times$, $\langle a\rangle \cong \langle \pm 1\rangle$. ■

[MH73, (2.5)]

14 Cartan–Dieudonné Theorem

Situation 14.1. Let k be a field. Let $q\colon V \to k$ be a quadratic form with $\dim V < \infty$. Denote by $p \in \operatorname{Hom}(S^2V, k)$ the polarisation of q. ×

Definition 14.2. A vector $v \in V$ is **isotropic** if $q(v) = 0$ and **anisotropic** if $q(v) \neq 0$. •

Definition 14.3. Let $v \in V$ be anisotropic. The **reflection** $r_v \in \operatorname{End}(V)$ along v is defined by

$$r_v(w) := w - \frac{p(v,w)}{q(v)}v.$$ •

Proposition 14.4. *For every anisotropic $v \in V$, $r_v^2 = \mathrm{id}_V$ and $r_v \in \mathrm{O}(q)$.*

Proof. By direct computation,

$$\begin{aligned} r_v \circ r_v(w) &= r_v(w) - \frac{p(v, r_v(w))}{q(v)} \\ &= w + \frac{2p(v,w)}{q(v)} v - \frac{p(v,w)p(v,v)}{q(v)^2} v = w \end{aligned}$$

and

$$q(r_v(w)) = q(w) - \frac{p(v,w)^2}{q(v)^2} q(v) + \frac{p(v,w)^2}{q(v)} = q(w). \qquad \blacksquare$$

Theorem 14.5 (Cartan–Dieudonné). *Denote by $R(q) < \mathrm{O}(q)$ the subgroup generated by reflections along anisotropic vectors. If q is non-degenerate and $\dim V < \infty$, then $\mathrm{O}(q) = R(q)$— unless $k = \mathbf{F}_2$, $\dim V = 4$, and q is of Witt index $i(q) = 2$; in which case: $R(q) < \mathrm{O}(q)$ has index two.*

Proof. See [Che54, §1.5; Lam05, Chapter I, Theorem 7.1]. $\blacksquare$

Remark 14.6 (The exceptional case in the Cartan–Dieudonné Theorem). By Theorem 11.21, if $k = \mathbf{F}_2$, $\dim V = 4$, and $i(q) = 2$, then $q \cong [0,0] \perp [0,0] \cong [1,1] \perp [1,1]$. The latter model is more convenient to understand why $\mathrm{O}(q) \neq R(q)$.

For every $0 \neq v \in \mathbf{F}_2^{\oplus 2}$, $[1,1](v) = 1$. Therefore, $\mathrm{O}([1,1]) = \mathrm{GL}_2(\mathbf{F}_2)$. Since $\mathbf{F}_2^2$ has three non-zero elements, $\mathrm{GL}_2(\mathbf{F}_2) \subset S_3$. Indeed, for every $0 \neq v \in \mathbf{F}_2^{\oplus 2}$, r_v transposes the remaining two non-zero elements. Therefore, $\mathrm{GL}_2(\mathbf{F}_2) = S_3$.

Let $(v,w) \in \left(\mathbf{F}_2^{\oplus 2}\right)^{\oplus 2}$. Evidently, $q(v,w) = ([1,1] \perp [1,1])(v,w) \neq 0$ if and only if either ($v = 0$ and $w \neq 0$) or ($v \neq 0$ and $w = 0$). Therefore, reflections in anisotropic vectors generate $S_3 \times S_3 \subset \mathrm{O}(q)$. However, $\sigma \in \mathrm{O}(q)$ defined by $\sigma(v,w) \coloneqq (w,v)$ fails to be of this form. By Theorem 14.5,

$$\mathrm{O}(q) \cong C_2 \ltimes (S_3 \times S_3). \qquad ♣$$

Lecture 3

This lectures constructs the Clifford algebra as a functor from quadratic forms to super algebras. At the end of the lecture, I will explain how to (in principle) compute the Clifford algebra of any non-degenerate finite-dimensional quadratic form.

15 The Clifford algebra of a quadratic form

Situation 15.1. Let k be a field. Let $q\colon V \to k$ be quadratic form. Denote by $p \in \mathrm{Hom}(S^2V, k)$ the polarisation of q. ×

Definition 15.2. A **Clifford algebra** of q is pair $(\mathrm{C}\ell(q), \gamma)$ consisting of a k–algebra $\mathrm{C}\ell(q)$ and a linear map $\gamma\colon V \to \mathrm{C}\ell(q)$ such that:

(1) for every $v \in V$

$$\gamma(v)^2 = q(v) \cdot \mathbf{1},$$

and

(2) if A is a k–algebra together with a linear map $\delta\colon V \to A$ such that for every $v \in V$

$$\delta(v)^2 = q(v) \cdot \mathbf{1},$$

then there is a unique algebra homomorphism $f\colon \mathrm{C}\ell(q) \to A$ such that

$$f \circ \gamma = \delta.$$

•

Remark 15.3. By polarisation, $\gamma(v)^2 = q(v) \cdot \mathbf{1}$ implies

$$\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = p(v, w) \cdot \mathbf{1},$$

but these are not equivalent unless $\mathrm{char}\, k = 2$. ♣

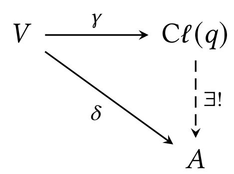


Figure 1: Universal property of the Clifford algebra.

Proposition 15.4 (Construction of the Clifford algebra). *Denote by (TV, i) the tensor algebra of V. Denote by $I_q \subset TV$ the ideal generated by elements of the form*

$$i(v) \otimes i(v) - q(v).$$

Set

$$\mathrm{C}\ell(q) \coloneqq TV/I_q \quad \textit{and} \quad \gamma \coloneqq \pi \circ i\colon V \to \mathrm{C}\ell(q)$$

with $\pi\colon TV \to \mathrm{C}\ell(q)$ denoting the canonical projection. $(\mathrm{C}\ell(q), \gamma)$ is a Clifford algebra of q.

Proof. By construction, $\gamma(v)^2 = q(v) \cdot \mathbf{1}$. By the universal property of the tensor algebra, there exists a unique $\tilde{f}\colon TV \to A$ such that $\tilde{f} \circ \delta = i$. Since $\delta(v)^2 = q(v) \cdot \mathbf{1}$, $\tilde{f}$ factors through $f\colon \mathrm{C}\ell(q) \to A$. Evidently, $f \circ \delta = \gamma$. By the universal property of the quotient, f is unique. ■

Remark 15.5. The proof of Proposition 15.4 is a good exercise to practise the use of universal properties. ♣

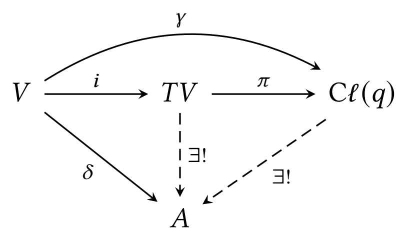


Figure 2: The proof of Proposition 15.4.

Proposition 15.6. *The Clifford algebra* $(\mathrm{C}\ell(q), \gamma)$ *of* q *is unique up to unique isomorphism.* ■

Proposition 15.7. $(\mathrm{C}\ell(0), \gamma) = (\Lambda V, i)$. ■

Proposition 15.8. *Let* q_1, q_2 *be quadratic forms. Denote by* $(\mathrm{C}\ell(q_i), \gamma_i)$ *the Clifford algebra of* q_i *(*$i = 1, 2$*). If* $f\colon q_1 \to q_2$ *is a quadratic morphism, then there is a unique algebra homomorphism* $\mathrm{C}\ell(f)\colon \mathrm{C}\ell(q_1) \to \mathrm{C}\ell(q_2)$ *such that*

$$\mathrm{C}\ell(f) \circ \gamma_1 = \gamma_2 \circ f.$$ ■

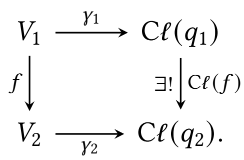


Figure 3: Construction of $\mathrm{C}\ell(f)$.

Remark 15.9. The Clifford algebra defines a functor from the category of quadratic forms to the category of algebras. It is an important invariant of a quadratic form and crucial for understanding spin geometry. ♣

16 The Clifford algebras of $\langle a \rangle$, $\langle a, b \rangle$, and $[a, b]$

Situation 16.1. Let k be a field. ×

Example 16.2. For every $a \in k^\times$

$$\mathrm{C}\ell(\langle a \rangle) = k[i]/(i^2 - a) \quad \text{with} \quad \gamma(x) := xi.$$

More concretely:

(1) Suppose that char $k \neq 2$. If a has a square root $\sqrt{a} \in k^\times$, then $k[i]/(i^2 - a) \cong k \oplus k$ via

$$x + yi \mapsto (x + \sqrt{a}y, x - \sqrt{a}y).$$

(2) If a does not have a square root in $k^\times$, then $k[i]/(i^2 - a)$ is the quadratic field extension $k(\sqrt{a})$. In particular, for $k = \mathbf{R}$, $\mathrm{C}\ell(\langle -1 \rangle) \cong \mathbf{C}$. ♠

Example 16.3. For every $a, b \in k$

$$\mathrm{C}\ell([a,b]) = \begin{bmatrix} a,b \\ k \end{bmatrix} := k\langle i, j\rangle/(i^2 - a, j^2 - b, ij + ji - 1) \quad \text{with} \quad \gamma(x,y) := xi + yj.$$

In particular, $\begin{bmatrix} 0,0 \\ k \end{bmatrix} \cong \mathrm{M}_2(k)$ via

$$i \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad j \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ ♠

Example 16.4. For every $a, b \in k$

$$\mathrm{C}\ell(\langle a,b \rangle) = \begin{pmatrix} a,b \\ k \end{pmatrix} := k\langle i, j\rangle/(i^2 - a, j^2 - b, ij + ji) \quad \text{with} \quad \gamma(x,y) := xi + yj.$$

$\begin{pmatrix} a,b \\ k \end{pmatrix}$ is the **quaternion algebra**; cf. WP: Quaternion algebra. Here are some concrete instances:

(1) $\begin{pmatrix} -1,-1 \\ \mathbf{R} \end{pmatrix} = \mathbf{H}$: **Hamilton's quaterions**; cf. WP: History of the quaternions.

(2) $\begin{pmatrix} a^2,\pm b^2 \\ k \end{pmatrix} \cong \mathrm{M}_2(k)$ via

$$i \mapsto \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \quad \text{and} \quad j \mapsto \begin{pmatrix} 0 & b \\ \pm b & 0 \end{pmatrix}.$$ ♠

17 The Clifford algebra of hyperbolic quadratic forms

Let I be a finite-dimensional k–vector space. Set $V := I^* \oplus I$. Define $q\colon V \to k$ by

$$q(\ell, v) := \ell(v).$$

Define $\delta\colon V \to \mathrm{End}(\Lambda I)$ by

$$\delta(\ell, v) = v \wedge \cdot + i_\ell.$$

By direct computation,

$$\delta(\ell, v)^2 = \ell(v)\mathbf{1} = q(\ell, v)\mathbf{1}.$$

This induces an algebra homomorphism $f\colon \mathrm{C}\ell(q) \to \mathrm{End}(\Lambda I)$. Since $\mathrm{End}(\Lambda I)$ is a simple algebra and $\dim \mathrm{C}\ell(q) = \dim \mathrm{End}(\Lambda I)$, f is an isomorphism.

[Can any of this be done more canonical?]

18 Injectivity of $\gamma\colon V \to \mathrm{C}\ell(q)$

Proposition 18.1. *The map $\gamma\colon V \to \mathrm{C}\ell(q)$ is injective.*

Notation 18.2. In the light of Proposition 18.1, it appears excessive not to drop γ from the notation and tacitly identify $v \in V$ and $\gamma(v) \in \mathrm{C}\ell(q)$. ∘

Remark 18.3. In principle, it should be possible to prove that $\operatorname{im} i \cap I_q = \{0\}$ by a combinatorial argument. It might even appear to be obvious [ABS64, (1.1); **Friedrich2000a**]; however, it is difficult not to mess this up in some way; cf. the notorious argument in [LM89, p.8]. ("Let anyone among you who is without sin be the first to throw a stone [..].") *I would be genuinely interested to see a correct proof along those lines.* The following proof constructs a representation of $\mathrm{C}\ell(q)$ in which $\gamma(v)$ obviously acts non-trivially. ♣

Proof of Proposition 18.1. Let $b \in \mathrm{Hom}(V \otimes V, k)$ such that $q = Q(b)$; that is: $q(v) = b(v \otimes v)$; cf. Proposition 10.8. Every $v \in V$ defines an endomorphism $v \wedge \cdot \in \mathrm{End}(\Lambda V)$ of degree 1. Every $\lambda \in V^*$ defines a unique derivation $i_\lambda \in \mathrm{Der}_{-1}(\Lambda V)$ of degree -1 satisfying $i_\lambda(v) = \lambda(v)$; indeed:

$$i_\lambda(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i+1} \lambda(v_i) \cdot v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k.$$

(This is the differential of the **Koszul complex** associated with λ.) Define $\cdot^\flat\colon V \to V^*$ by

$$v^\flat(w) \coloneqq b(v \otimes w).$$

Define $\delta\colon V \to \mathrm{End}(\Lambda V)$ by

$$\delta(v)\alpha \coloneqq v \wedge \alpha + i_{v^\flat}\alpha$$

Evidently, δ is injective. By direct computation,

$$\delta(v)^2 = q(v) \cdot \mathbf{1}.$$

By the universal property of the Clifford algebra, there is a unique $f\colon \mathrm{C}\ell(q) \to \mathrm{End}(\Lambda V)$ such that $f \circ \gamma = \delta$. Therefore, γ is injective. ■

19 Orthogonal maps as automorphisms of the Clifford algebra

Situation 19.1. Let k be a field. Let $q\colon V \to k$ be quadratic form. ×

Proposition 19.2. *The homomorphism $\mathrm{C}\ell\colon \mathrm{O}(q) \to \mathrm{Aut}(\mathrm{C}\ell(q))$ defined by Proposition 15.8 is injective; moreover: $\phi \in \operatorname{im} \mathrm{C}\ell$ if and only if $\phi(V) \subset V$.*

Notation 19.3. In the light of Proposition 19.2, it is convenient to identify $\mathrm{O}(q)$ as a subgroup of $\mathrm{Aut}(\mathrm{C}\ell(q))$. ∘

Proof. Only the sufficiency in the second part requires a proof. Suppose $\phi \in \mathrm{Aut}(\mathrm{C}\ell(q))$ satisfies $\phi(V) \subset V$. Define $\Phi \in \mathrm{End}(V)$ by $\Phi(v) \coloneqq \phi(v)$. Since ϕ is invertible, $\Phi \in \mathrm{GL}(V)$. Moreover, for every $v \in V$

$$q(v)\mathbf{1} = v^2 = \phi(v^2) = \Phi(v)^2 = q(\Phi(v))^2 \cdot 1.$$

Therefore, $\Phi \in \mathrm{O}(q)$. Evidently, $\phi = \mathrm{C}\ell(\Phi)$. ■

20 Graded algebras

Situation 20.1. Let k be a field. Let G, H be monoids. ×

Definition 20.2. A G–**graded algebra** is an k–algebra A together with a G–**grading**; that is: a direct sum decomposition

$$A = \bigoplus_{g \in G} A_g$$

such that

$$A_g \cdot A_h \subset A_{gh}.$$

An element $a \in A$ is **homogeneous of degree** g if $a \in A_g$. For every $a \in A$ the **homogeneous component of degree** g is the projection a_g of a onto A_g. •

Definition 20.3. A **super algebra** is a $\mathbf{Z}/2\mathbf{Z}$–graded algebra. •

Example 20.4. The tensor algebra TV is $\mathbf{N}_0$–graded; indeed:

$$TV = \bigoplus_{k \in \mathbf{N}_0} TV^k \quad \text{with} \quad TV^k := V^{\otimes k}.$$ ♠

Remark 20.5. Let A be a G–graded algebra. Every homomorphism $f\colon G \to H$ induces a H–grading via

$$A = \bigoplus_{h \in H} A_h \quad \text{with} \quad A_h := \bigoplus_{g \in f^{-1}(h)} A_g.$$ ♣

Example 20.6. Since $\mathbf{N}_0 \to \mathbf{Z} \to \mathbf{Z}/2$, the $\mathbf{N}$–grading on the tensor algebra TV induces $\mathbf{Z}$– and $\mathbf{Z}/2\mathbf{Z}$–gradings. ♠

Definition 20.7. Let A be a G–graded k–algebra. An ideal $I \subset A$ is **homogeneous** if, for every $a \in I$ and $g \in G$, $a_g \in I$. •

Proposition 20.8. *Let A be a G–graded k–algebra. Let A/I be a homogenous ideal. A/I has a unique G–grading such that $\pi\colon A_g \to (A/I)_g$ is surjective.* ■

Remark 20.9. If A, B are G–graded algebras, then $A \otimes B$ is G–graded:

$$(A \otimes B)_g := \bigoplus_{g = hk} A_h \otimes B_k.$$ ♣

In the presence of a G–grading with $G \to \mathbf{Z}/2\mathbf{Z}$, the usual multiplication of the tensor product of algebras often is inappropriate because it violates the Koszul sign convention.

Definition 20.10. Let A, B be $\mathbf{Z}/2\mathbf{Z}$–graded algebras. The **super tensor product** $A \hat{\otimes} B$ is tensor product as a vector space and with the above grading but with the multiplication rule

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) := (-1)^{\deg b_1 \deg a_2} a_1 a_2 \hat{\otimes} b_1 b_2.$$ •

Remark 20.11. If G is a finite group and A, B are G–graded algebras, then $\operatorname{Hom}(A,B)$ is a G–graded vector space:

$$\operatorname{Hom}(A,B) = \bigoplus_{g\in G}\left(\bigoplus_{h\in G}\operatorname{Hom}(A_h, B_{gh})\right). \qquad ♣$$

If G is not finite or not a group, then the right-hand side might be a proper subspace of $\operatorname{Hom}(A,B)$.

21 The Z/2Z–grading of the Clifford algebra

Situation 21.1. Let k be a field. Let $q\colon V \to k$ be quadratic form. ×

Definition 21.2. Denote by TV the tensor algebra of V. Denote by $\pi\colon TV \to C\ell(q)$ the projection map. Since I_q is homogeneous with respect to the $\mathbf{Z}/2\mathbf{Z}$ grading of TV, $C\ell(q)$ inherits a $\mathbf{Z}/2\mathbf{Z}$ grading:

$$C\ell(q)^i := \pi(TV^i) \quad \text{with} \quad TV^i := \bigoplus_{k=0}^{\infty} V^{\otimes 2k+i}. \qquad \bullet$$

Remark 21.3. If $\operatorname{char} k \neq 2$, then

$$C\ell(q)^0 = \{x \in C\ell(q) : C\ell(-\mathbf{1})x = x\} \quad \text{and} \quad C\ell(q)^0 = \{x \in C\ell(q) : C\ell(-\mathbf{1})x = -x\}. \qquad ♣$$

Remark 21.4. If $f\colon q_1 \to q_2$ is a quadratic morphism, then $C\ell(f)$ is a super algebra homomorphism of degree 0. Therefore, the Clifford algebra defines a functor from the category of quadratic forms to the category of super algebras. ♣

Remark 21.5. As rule of thumb, most things involving Clifford algebras should be done in the super way. ♣

Remark 21.6. The isomorphism $C\ell(\langle a^2, \pm b^2\rangle) \cong M_2(k) \cong C\ell([0,0])$ typically are not homomorphisms of super algebras; that is: they disrespect the $\mathbf{Z}/2\mathbf{Z}$ grading. ♣

22 The Clifford algebra of a perpendicular sum

Situation 22.1. Let k be a field. ×

Proposition 22.2 (Clifford algebra of a perpendicular sum). *Let $q_i\colon V_i \to k$ be quadratic forms ($i = 1,2$). Denote by $(C\ell(q_i), \gamma_i)$ the Clifford algebra of q_i ($i = 1,2$). Set*

$$\gamma := \gamma_1 \hat{\otimes} \mathbf{1} + \mathbf{1} \hat{\otimes} \gamma_2 \colon V_1 \oplus V_2 \to C\ell(q_1) \hat{\otimes} C\ell(q_2).$$

$(C\ell(q_1) \hat{\otimes} C\ell(q_2), \gamma)$ *is the Clifford algebra of* $q_1 \perp q_2$.

Proof. By direct computation,

$$\begin{aligned}\gamma(v_1 \oplus v_2) &= \big(\gamma_1(v_1) \mathbin{\hat\otimes} \mathbf{1} + \mathbf{1} \mathbin{\hat\otimes} \gamma_2(v_2)\big)^2 \\ &= \gamma_1(v_1)^2 \mathbin{\hat\otimes} \mathbf{1} + \gamma_1(v_1) \mathbin{\hat\otimes} \gamma_2(v_2) - \gamma_1(v_1) \mathbin{\hat\otimes} \gamma_2(v_2) + \mathbf{1} \mathbin{\hat\otimes} \gamma_2(v_2)^2 \\ &= (q_1(v_1) + q_2(v_2))\mathbf{1}.\end{aligned}$$

Let A be an algebra together with a linear map $\delta\colon V_1 \oplus V_2 \to A$ satisfying

$$\delta(v_1 \oplus v_2)^2 = (q_1(v_1) + q_2(v_2))\mathbf{1}.$$

Set $\delta_i \coloneqq \delta|_{V_i}$. By the universal property of the Clifford algebra, there are unique homomorphisms $f_i\colon \mathrm{C}\ell(V_i, q_i) \to A_i$ such that

$$\delta_i = f_i \circ \gamma_i.$$

Define the linear map $f\colon \mathrm{C}\ell(q_1) \mathbin{\hat\otimes} \mathrm{C}\ell(q_2) \to A$ by

$$f(x_1 \mathbin{\hat\otimes} x_2) \coloneqq f_1(x_1)f_2(x_2).$$

By construction,

$$\delta = f \circ \gamma;$$

moreover, f is uniquely determined by this condition. It remains to verify that f is an algebra homomorphism. From

$$\delta_1(v_1)\delta_2(v_2) + \delta_2(v_2)\delta_1(v_1) = 0$$

it follows that

$$f_1(x_1)f_2(x_2) = (-1)^{\deg(x_1)\deg(x_2)} f_2(x_2)f_1(x_1).$$

Therefore, f is an algebra homomorphism. ■

Remark 22.3. Proposition 22.2 together with Theorem 11.21, Example 16.2, and Example 16.3 allows for the computation of $\mathrm{C}\ell(q)$ for every quadratic form q. See, e.g., the Computation of the real Clifford algebras and Computation of the complex Clifford algebras. ♣

Remark 22.4. The literature is littered with proofs of Proposition 22.2 that: (1) suppose that $\dim V_i < \infty$, and (2) use the dimension formula for $\mathrm{C}\ell(q_i)$. The advantage of the above Proposition 22.2 proof is that it can be used to establish $\dim \mathrm{C}\ell(q) = 2^{\dim V}$—provided $\dim V < \infty$ by appealing to Proposition 22.2, Theorem 11.21, Example 16.2, and Example 16.3. According to [Knu91], this observation is due to Kneser (but I could not locate the lecture notes that Knus refers to). ♣

Lecture 4

In this lecture, I introduce the filtration on the Clifford algebra. This will be used to prove the dimension formula. A more elegant approach is via the symbol and quantisation maps. The last part of the lecture prepares the computation of the real and complex Clifford algebras.

23 Filtered algebras

Situation 23.1. Let k be a field. ×

Definition 23.2. Let V be a vector space. A **filtration** on V is a subspace $F^r V \subset V$ for every $r \in \mathbf{N}_0$ such that

$$F^r V \subset F^{r+1} V$$

for all $r \in \mathbf{N}_0$ and

$$V = \bigcup_{r \in \mathbf{N}_0} F^r V.$$

A vector space together with a filtration is called a **filtered vector space**. •

Every graded vector space V has a canonical filtration given by

$$F^r V = V^{\leqslant r} := \bigoplus_{s \leqslant r} V^s.$$

Definition 23.3. Given a filtered vector space V, the **associated graded vector space** $\operatorname{Gr} V$ is

$$\operatorname{Gr} V := \bigoplus_{r=0}^{\infty} \operatorname{Gr}^r V \quad \text{with} \quad \operatorname{Gr}^r V := F^r V / F^{r-1} V.$$

Here we use the convention $F^{-1} V = \{0\}$. •

Remark 23.4. If V is a graded vector space, then the associated graded vector space $\operatorname{Gr} V$ of V with the canonical filtration is isomorphic to V. ♣

Remark 23.5. If V is a filtered vector space, then there is a canonical linear map $i\colon V \to \operatorname{Gr} V$. The map i is injective and, hence, an isomorphism if V is finite-dimensional. ♣

Definition 23.6. Let A be a k–algebra. A **filtration** in A is a filtration on the underlying vector space such that

$$F^r A \cdot F^s A \subset F^{r+s} A.$$

•

Remark 23.7. If A is a filtered algebra, then $\operatorname{Gr} A$ inherits the structure of graded algebra. ♣

Definition 23.8. If A is a filtered algebra, then $\operatorname{Gr} A$ is called the **associated graded algebra**. •

Remark 23.9. Let A be a filtered algebra and let I be an ideal in A. Given $r \in \mathbf{N}_0$, define

$$F^r(A/I) := (F^r A)/(I \cap F^r A).$$

This defines a filtration on A/I. ♣

24 The filtration of the Clifford algebra

Definition 24.1. Denote by $\pi\colon TV \to C\ell(q)$ the projection map. The **filtration** of $C\ell(q)$ is defined by

$$F^r C\ell(q) := \pi(F^r TV) \quad \text{with} \quad F^r TV := \bigoplus_{i=0}^{r} V^{\otimes i}. \qquad \bullet$$

Proposition 24.2. *The homomorphism* $\operatorname{Gr}\pi\colon TV = \operatorname{Gr} TV \to \operatorname{Gr} C\ell(V,q)$ *factors through an isomorphism*

$$\kappa\colon \Lambda V \cong \operatorname{Gr} C\ell(V,q).$$

Proof. The map $\operatorname{Gr}\pi\colon TV = \operatorname{Gr} TV \to \operatorname{Gr} C\ell(V,q)$ is surjective. Since

$$\ker \operatorname{Gr}\pi = \bigoplus_{r=0}^{\infty} TV^r \cap (I_q + F^{r-1}TV) = I_0,$$

$\ker \operatorname{Gr}\pi$ factors through κ, and κ is an isomorphism. ∎

Since there always is a (non-canonical) vector space isomorphism $\operatorname{Gr} A \cong A$, this implies the following.

Theorem 24.3 (Poincaré–Birkhoff–Witt Theorem for Clifford algebras). $\dim C\ell(q) = 2^{\dim V}$. *Moreover, if* $\{e_i : i \in \{1,\dots,\dim V\}\}$ *is a basis of* V*, then*

$$\{e_{i_1}\cdots e_{i_r} : 1 \leqslant i_1 < \cdots i_r \leqslant \dim V\}$$

is a basis of $C\ell(q)$.

Remark 24.4. The first section of Theorem 24.3 holds whether $\dim V < \infty$ or not. ♣

Remark 24.5. The original(?) Poincaré–Birkhoff–Witt Theorem is about Lie algebras and their universal enveloping algebras. The terminology seems to have spread to many related situations. ♣

25 The symbol map and the quantisation map

Situation 25.1. Let k be a field. Let $q\colon V \to k$ be quadratic form. ×

Remark 25.2. Proposition 24.2 implies that there exists an vector space isomorphism $C\ell(q) \to \Lambda V$. Indeed, the algebra isomorphism $\operatorname{Gr}(q) \to \Lambda V$ is canonical, but the vector space isomorphism $C\ell(q) \to \operatorname{Gr}(q)$ might not be. Chevalley [Che54, Proof of II.1.2] already observed that a lift of q to a blinear form $b \in \operatorname{Hom}(V \otimes V, k)$ induces a vector space isomorphism $C\ell(q) \to \Lambda V$. The following elegant construction is due to Bourbaki [Bou07, §9]. ♣

Definition 25.3. For every $\lambda \in V^*$ denote by $i_\lambda \in \operatorname{Der}_{-1}(TV)$ the unique derivation of degree -1 such that

$$i_\lambda(v) = \lambda(v).$$

•

Remark 25.4. If $\lambda, \mu \in V^*$, then i_λ and i_μ anti-commute. Indeed, $[i_\lambda, i_\mu] = i_\lambda i_\mu + i_\mu i_\lambda$ is a graded derivation of degree -2 and, therefore, vanishes. ♣

Definition 25.5. Let $b \in \mathrm{Hom}(V \otimes V, k)$. Set $v^\flat(w) := b(v \otimes w)$. Define the algebra homomorphism $\Psi_b \colon TV \to \mathrm{End}(TV)$ by

$$\Psi_b(v)x := v \otimes x + i_{v^\flat}(x).$$

Define $\Theta_b \in \mathrm{End}(TV)$ by

$$\Theta_b(x) := \Psi_b(x)\mathbf{1}.$$ •

Lemma 25.6. *Let $b, b_1, b_2 \in \mathrm{Hom}(V \otimes V, k)$.*

(1) *$\Theta_b \colon TV \to TV$ is uniquely characterised by $\Theta_b(\mathbf{1}) = \mathbf{1}$ and*

$$\Theta_b(v \otimes x) = v \otimes \Theta_b(x) + i_{v^\flat}\Theta_b(x)$$

for every $v \in V$ and $x \in TV$.

(2) *$\Theta_0 = \mathrm{id}_{TV}$ and $\Theta_{b_1} \circ \Theta_{b_2} = \Theta_{b_1+b_2}$; in particular: Θ_b is an isomorphism.*

(3) *$\Theta_b I_q \subset I_{q-Q(b)}$; in particular, Θ_b descends to a linear isomorphism*

$$\theta_b \colon \mathrm{C}\ell(V,q) \to \mathrm{C}\ell(V, q - Q(b)).$$

Proof. (1) is obvious.

Evidently, $\Theta_0 = \mathrm{id}_{TV}$. The proof of the second assertion in (2) requires the identity

$$\Theta_b i_\lambda = i_\lambda \Theta_b.$$

The proof of this identity is by induction on the degree. Evidently, it holds on $TV^0 = k$. Moreover, if the identity is know to hold on x, then

$$\begin{aligned}
\Theta_b(i_\lambda(v \otimes x)) &= \Theta_b(\lambda(v)x - v \otimes i_\lambda x) \\
&= \lambda(v)\Theta_b(x) - v \otimes \Theta_b(i_\lambda x) - i_{v^\flat}\Theta_b(i_\lambda x) \\
&= \lambda(v)\Theta_b(x) - v \otimes i_\lambda\Theta_b(x) - i_{v^\flat} i_\lambda \Theta_b(x) \\
&= \lambda(v)\Theta_b(x) - v \otimes i_\lambda\Theta_b(x) + i_\lambda i_{v^\flat}\Theta_b(x) \\
&= i_\lambda(v \otimes \Theta_b(x) + i_{v^\flat}\Theta_b(x)) \\
&= i_\lambda \Theta_b(v \otimes x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Theta_{b_1}(\Theta_{b_2}(v \otimes x))) &= \Theta_{b_1}(v \otimes \Theta_{b_2}(x) + i_{v^{\flat_2}}\Theta_{b_2}(x)) \\
&= v \otimes \Theta_{b_1}(\Theta_{b_2}(x)) + i_{v^{\flat_1}}\Theta_{b_1}(\Theta_{b_2}(x)) + \Theta_{b_1} i_{v^{\flat_2}}\Theta_{b_2}(x)
\end{aligned}$$

This proves (2).

To prove (3), observe the following. Since

$$i_{v^\flat}(w \otimes w - q(w)) = 0,$$

if $\Theta_b(x) \in I_{q-Q(b)}$, then (by induction) $\Theta_b(v \otimes x) = v \otimes \Theta_b(x) + i_{v^\flat}\Theta_b(x) \in I_{q-Q(b)}$. Finally,

$$\begin{aligned}\Theta_b((v \otimes v - q(v)) \otimes x) &= v \otimes \Theta_b(v \otimes x) + i_{v^\flat}\Theta_b(v \otimes x) - q(v) \otimes \Theta_b(x) \\ &= (v \otimes v - q(v) + b(v \otimes v)) \otimes \Theta_b(x).\end{aligned}$$ ■

Corollary 25.7. *If $q = Q(b)$, then the* ***symbol map***

$$\sigma_b := \theta_{-b} \colon \mathrm{C}\ell(V, q) \to \Lambda V$$

and the ***quantisation map***

$$\kappa_b := \theta_b \colon \Lambda V \to \mathrm{C}\ell(V, q)$$

are vector space isomorphisms (and inverses of each other). ■

Remark 25.8. See `https://empg.maths.ed.ac.uk/Activities/Spin/SpinNotes.pdf` for some background on on the terminology. ♣

This immediately implies Theorem 24.3.

Remark 25.9. If char $k \neq 2$, then $b = \frac{1}{2}p$ is the canonical choice to define $\sigma := \sigma_\beta$ and $\kappa := \kappa_b$. ♣

26 Artin–Wedderburn Theorem

Remark 26.1. *What does it mean to determine an algebra?* A key reason to care about algebras are their modules. The Artin–Wedderburn theorem says that the structure of finite-dimensional semi-simple algebras is determined by their simple modules. ♣

Situation 26.2. Let k be a field. Let A be a k–algebra. ×

Definition 26.3.

(1) An A**–module** is a k–vector space V together with an algebra homomorphism $A \to \mathrm{End}(V)$.

(2) An A**–submodule** W of V is an is a k–vector subspace such that $xW \subset W$ for every $x \in A$.

(3) An A–module V is **simple** if 0 and V are its only submodule.

(4) If V, W are A–modules, then

$$\mathrm{Hom}_A(V, W) := \{f \colon V \to W : f \text{ is } A\text{–linear}\}.$$

(5) The **commuting algebra** of a module V is

$$\mathrm{End}_A(V) := \mathrm{Hom}_A(V, V).$$ •

Lemma 26.4 (Schur's Lemma). *Let V, W be simple A–modules. If $f \in \mathrm{Hom}_A(V, W)$, then $f = 0$ or f is invertible.* ■

Corollary 26.5. *If V is an simple A–module, then $\mathrm{End}_A(V)$ is a division algebra over k; that is, every non-zero $x \in \mathrm{End}_A(V)$ is invertible.* ■

Theorem 26.6 (Frobenius). *If D is a finite-dimensional division algebra over* $\mathbf{R}$*, then D is isomorphic to either* $\mathbf{R}$, $\mathbf{C}$*, or* $\mathbf{H}$.

Proposition 26.7. *If k is an algebraically closed field, e.g., $k = \mathbf{C}$, then any division algebra over k is isomorphic to k.*

Definition 26.8. The **Jacobson radical** of A is

$$J(A) \coloneqq \{x \in A : xV = 0 \text{ for every simple } A\text{–module } V\}.$$

A is **semi-simple** if $J(A) = 0$. •

Remark 26.9. $J(A)$ is an ideal of A. ♣

Example 26.10. Let V be a vector space. The Jacobson radical of ΛV is

$$J(\Lambda V) = \bigoplus_{k \geqslant 1} \Lambda^k V;$$

that is: the ideal generated by $V \subset \Lambda V$. To see this, let $x \in V$ and let W be an simple module of ΛV. Set $\ker x \coloneqq \{w \in W : xw = 0\}$ and $\mathrm{im}\, x \coloneqq xW$. Since ΛV is graded commutative, $\ker x$ and $\mathrm{im}\, x$ are submodule of W. Since W is simple, $\ker x = 0$ or $\ker x = W$. Since $x^2 = 0$, $\mathrm{im}\, x \subset \ker x$. This forces, $\ker x = W$. ♠

Theorem 26.11 (Artin–Wedderburn Theorem). *Suppose that* $\dim A < \infty$.

(1) *A has only finitely many simple modules $V_1, \ldots, V_r$ (up to isomorphism) and each V_i is finite-dimensional.*

(2) *If D_i denotes the commuting algebra of V_i, then*

$$A/J(A) \cong \prod_{i=1}^{r} \mathrm{End}_{D_i}(V_i).$$ ■

See Representation theory of finite groups.

27 Frobenius' theorem on real division algebras

Definition 27.1. Let k be a field. A k–algebra D is a **division algebra** if it has no zero divisors. •

Theorem 27.2 (Frobenius [Fro77, last paragraph]). *If D is a finite-dimensional real division algebra, then D is isomorphic to either* $\mathbf{R}$, $\mathbf{C}$*, or* $\mathbf{H}$.

Proof. Since D is a division algebra, left multiplication defines an inclusion

$$D \hookrightarrow \mathrm{End}(D).$$

Set $d := \dim D$ and

$$\mathrm{Im}\, D := \{x \in D : x^2 \in \mathbf{R} \text{ and } x^2 \leqslant 0\}.$$

Lemma 27.3. $\mathrm{Im}\, D = \ker(\mathrm{tr}\colon D \to \mathbf{R})$. *In particular:* $D = \mathbf{R} \oplus \mathrm{Im}\, D$.

Proof. Let $x \in D \hookrightarrow \mathrm{End}(D)$. Denote the minimal and the characteristic polynomial of x by $\mu \in \mathbf{R}[\lambda]$ and $\chi \in \mathbf{R}[\lambda]$ respectively. Decompose χ into irreducible factors as follows

$$\chi(\lambda) = \prod_{i=1}^{a}(\lambda - r_i) \cdot \prod_{j=1}^{b}(\lambda - s_i)(\lambda - \bar{s}_i)$$

with

$$r_1, \dots, r_a \in \mathbf{R} \quad \text{and} \quad s_1, \bar{s}_1, \dots, s_b, \bar{s}_b \in \mathbf{C}\backslash\mathbf{R}.$$

By the Cayley–Hamilton theorem,

$$\chi(x) = 0 \in \mathrm{End}(D).$$

Since D is a division algebra, the minimal polynomial μ is one of the irreducible factors of χ. Since μ and χ have the same roots, either:

(1) $\chi = \mu^d$ and $\mu(\lambda) = \lambda - r$ with $r \in \mathbf{R}$, or

(2) $\chi = \mu^{d/2}$ and $\mu(\lambda) = (\lambda - s)(\lambda - \bar{s}) = \lambda^2 - 2\,\mathrm{Re}\, s \cdot \lambda + |s|^2$ with $s \in \mathbf{C}\backslash\mathbf{R}$.

In the former case, for $x \in \mathbf{R}$ the assertion is obvious because $\mathbf{R} \cap \mathrm{Im}\, D = 0 = \ker(\mathrm{tr}\colon \mathbf{R} \to \mathbf{R})$. Since

$$\chi(\lambda) = \lambda^d - \mathrm{tr}(x)\lambda^{d-1} + \cdots,$$

in the latter case,

$$\mathrm{tr}(x) = -d\,\mathrm{Re}(s).$$

Therefore, if $\mathrm{tr}(x) = 0$, then $\mathrm{Re}\, s = 0$; hence: $x^2 = -|s|^2 \leqslant 0$. Conversely, if $x^2 = -t^2$ with $t \in \mathbf{R}$, then $\mu(\lambda) = \lambda^2 + t^2$; hence: $\mathrm{tr}(\lambda) = 0$. ∎

Define the quadratic form $q\colon \mathrm{Im}\, D \to \mathbf{R}$ by

$$q(x) := -x^2.$$

Denote the polarisation of q by p. Set $b := \frac{1}{2}p$. By construction, b is a Euclidean inner product on $\mathrm{Im}\, D$. Let $S \subset \mathrm{Im}\, D$ be a minimal subspace which generates D as an $\mathbf{R}$–algebra. Let $e_1, \dots, e_s$ be an orthonormal basis of S. By construction,

$$e_i^2 = -1 \quad \text{and} \quad e_i e_j + e_j e_i = 0 \quad \text{and} \quad (i \neq j \in \{1, \dots, s\}).$$

Evidently:

(1) If $n = 0$, then $D \cong \mathbf{R}$.

(2) If $n = 1$, then $D \cong \mathbf{C}$.

(3) If $n = 2$, then $D \cong \mathbf{H}$.

Finally, if $n \geqslant 3$, then $x := e_1 e_2 e_3 \neq \pm 1$ satisfies

$$0 = x^2 - 1 = (x+1)(x-1).$$

This contradicts D being a division algebra. ■

Remark 27.4. Palais [Pal68] has another short elementary proof. ♣

Remark 27.5. Frobenius' theorem on real division algebras implies that the Brauer group $B(\mathbf{R})$ agrees with C_2 and is generated by $\mathbf{H}$: $\mathbf{H} \otimes \mathbf{H} = \mathrm{M}_4(\mathbf{R}) \sim \mathbf{R}$ with $\sim$ denoting Morita equivalence. ♣

28 Representation theory of finite groups

Situation 28.1. Let k be a field. Let G be a finite group. ×

Theorem 28.2 (Maschke's Theorem). *If* char k *does not divide* $|G|$, *then* $k[G]$ *is semi-simple.*

Proof. Let V be a $k[G]$–module. Let $W \subset V$ be submodule. Let $\pi \in \mathrm{End}(V)$ be a k–linear projection onto W. By averaging over G, π can be assumed to be G–invariant; that is: $k[G]$–linear. Therefore, $W' := \ker \pi$ is a complementary $k[G]$–submodule.

By the above, $k[G]$ decomposes into simple modules $k[G] = \bigoplus_{i=1}^r V_i$. Every non-zero $x \in k[G]$ acts non-trivially on $k[G]$ and thus on at least one of the simple modules V_i. Consequently, $J(k[G]) = \{0\}$. ■

Corollary 28.3. *If* char k *does not divide* $|G|$, *then* G *has only finitely many irreducible representations* $V_1, \dots, V_r$, *and*

$$k[G] \cong \prod_{i=1}^r \mathrm{End}_{D_i}(V_i).$$ ■

For comparison, Serre [Ser77] is a classical reference on the representation theory of finite groups.

Lecture 5

The goal of this lecture is to compute the Clifford algebras of all non-degenerate quadratic forms over **R** and **C**. Our computation more or less follows [ABS64] and [LM89, Chapter I §4], which is quite straight-forward and a lot of fun. There are many alternative methods to arrive at the same result; see, e.g., Roe [Roe98, p. 59] for an approach using the representation theory of finite groups due to to J.F. Adams [Roe98, p. 68]. (An important consequence of the computation is a mod 8 (resp. mod 2) periodicity—related to Bott periodicity).

As a result of our computation one can read off the simple modules of the Clifford algebra. (These govern the coarse theory of spinors over pseudo-Riemannian manifolds.) Spelling out what they are is a simple exercise whose solution is written up in the lecture notes. You should look at this and make sure you understand why this works. In the lecture, I will only briefly discuss the role of the volume element.

29 Computation of the real Clifford algebras

[LM89, Chapter I §4]

Definition 29.1. For $r, s \in \mathbf{N}_0$ set

$$\mathrm{C}\ell_{r,s} \coloneqq \mathrm{C}\ell(q_{r,s}) \quad \text{with} \quad q_{r,s} \coloneqq \langle 1\rangle^{\perp r} \perp \langle -1\rangle^{\perp s} \quad \text{defined over} \quad \mathbf{R}. \qquad \bullet$$

Theorem 29.2. *For every $r, s \in \{0, \dots, 7\}$, $\mathrm{C}\ell_{r,s}$ is as in Table 1. Moreover, for every $r, s \in \mathbf{N}_0$*

$$\mathrm{C}\ell_{r+8,s} \cong \mathrm{C}\ell_{r,s} \otimes \mathrm{M}_{16}(\mathbf{R}) \quad \text{and} \quad \mathrm{C}\ell_{r,s+8} \cong \mathrm{C}\ell_{r,s} \otimes \mathrm{M}_{16}(\mathbf{R}).$$

Lemma 29.3. *Let $q\colon V \to k$ be a quadratic form with $\dim V < \infty$. For every $a, b \in k$*

$$\mathrm{C}\ell(q \perp \langle a, b\rangle) \cong \mathrm{C}\ell(-(ab)^{-1} \cdot q) \otimes \mathrm{C}\ell(\langle a, b\rangle).$$

Proof. Denote by (e_1, e_2) the standard basis of $k^{\oplus 2}$. Define $\delta\colon V \oplus k^{\oplus 2} \to \mathrm{C}\ell(-(ab)^{-1} \cdot q) \otimes \mathrm{C}\ell(\langle \pm 1\rangle^{\perp 2})$ by

$$\delta(v, x_1, x_2) \coloneqq v \otimes e_1e_2 + 1 \otimes x_1e_1 + 1 \otimes x_2e_2.$$

Since

$$\delta(v, x_1, x_2)^2 = q(v) + ax^2 + by^2,$$

there is a unique algebra homomorphism $f\colon \mathrm{C}\ell(q \perp \langle a, b\rangle) \to \mathrm{C}\ell(-(ab)^{-1} \cdot q) \otimes \mathrm{C}\ell(\langle a, b\rangle)$. Since f maps onto a set of generators, it is surjective. For dimension reasons it also injective and, hence, an algebra isomorphism. ■

Proposition 29.4.

(1) $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C} \oplus \mathbf{C}$,

(2) $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \cong \mathrm{M}_2(\mathbf{C})$, *and*

$C\ell_{r,s}$	$r = 0$	1	2	3	4	5	6	7
$s = 0$	$\mathbf{R}$	$\mathbf{R}^{\oplus 2}$	$M_2(\mathbf{R})$	$M_2(\mathbf{C})$	$M_2(\mathbf{H})$	$M_2(\mathbf{H})^{\oplus 2}$	$M_4(\mathbf{H})$	$M_8(\mathbf{C})$
1	$\mathbf{C}$	$M_2(\mathbf{R})$	$M_2(\mathbf{R})^{\oplus 2}$	$M_4(\mathbf{R})$	$M_4(\mathbf{C})$	$M_4(\mathbf{H})$	$M_4(\mathbf{H})^{\oplus 2}$	$M_8(\mathbf{H})$
2	$\mathbf{H}$	$M_2(\mathbf{C})$	$M_4(\mathbf{R})$	$M_4(\mathbf{R})^{\oplus 2}$	$M_8(\mathbf{R})$	$M_8(\mathbf{C})$	$M_8(\mathbf{H})$	$M_8(\mathbf{H})^{\oplus 2}$
3	$\mathbf{H}^{\oplus 2}$	$M_2(\mathbf{H})$	$M_4(\mathbf{C})$	$M_8(\mathbf{R})$	$M_8(\mathbf{R})^{\oplus 2}$	$M_{16}(\mathbf{R})$	$M_{16}(\mathbf{C})$	$M_{16}(\mathbf{H})$
4	$M_2(\mathbf{H})$	$M_2(\mathbf{H})^{\oplus 2}$	$M_4(\mathbf{H})$	$M_4(\mathbf{C})$	$M_{16}(\mathbf{R})$	$M_{16}(\mathbf{R})^{\oplus 2}$	$M_{32}(\mathbf{R})$	$M_{32}(\mathbf{C})$
5	$M_4(\mathbf{C})$	$M_4(\mathbf{H})$	$M_4(\mathbf{H})^{\oplus 2}$	$M_8(\mathbf{H})$	$M_8(\mathbf{C})$	$M_{32}(\mathbf{R})$	$M_{32}(\mathbf{R})^{\oplus 2}$	$M_{64}(\mathbf{R})$
6	$M_8(\mathbf{R})$	$M_8(\mathbf{C})$	$M_8(\mathbf{H})$	$M_8(\mathbf{H})^{\oplus 2}$	$M_{16}(\mathbf{H})$	$M_{16}(\mathbf{C})$	$M_{64}(\mathbf{R})$	$M_{64}(\mathbf{R})^{\oplus 2}$
7	$M_8(\mathbf{R})^{\oplus 2}$	$M_{16}(\mathbf{R})$	$M_{16}(\mathbf{C})$	$M_{16}(\mathbf{H})$	$M_{16}(\mathbf{H})^{\oplus 2}$	$M_{32}(\mathbf{H})$	$M_{32}(\mathbf{C})$	$M_{128}(\mathbf{R})$

Table 1: The periodic table of Clifford algebras.

(3) $\mathbf{H} \otimes_{\mathbf{R}} \mathbf{H} \cong \mathrm{M}_4(\mathbf{R})$.

Proof. The isomorphism $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \to \mathbf{C} \oplus \mathbf{C}$ is given by

$$z \otimes w \mapsto zw \oplus z\bar{w}.$$

Identifying $\mathbf{H} = \mathbf{C} \oplus \mathbf{C}j = \mathbf{C}^2$, $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H}$ acts on $\mathbf{C}^2$ via

$$(z \otimes q) \cdot v = zv\bar{q}.$$

This action is $\mathbf{C}$–linear. A computation shows that the resulting map $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \to \mathrm{End}_{\mathbf{C}}(\mathbf{C}^2) \cong \mathrm{M}_2(\mathbf{C})$ is an isomorphism.

Identifying $\mathbf{H} = \mathbf{R}^4$, $\mathbf{H} \otimes_{\mathbf{R}} \mathbf{H}$ acts on $\mathbf{R}^4$ via

$$(p \otimes q) \cdot v = pv\bar{q}.$$

This action is $\mathbf{C}$–linear. A computation shows that the resulting map $\mathbf{H} \otimes_{\mathbf{R}} \mathbf{H} \to \mathrm{End}_{\mathbf{R}}(\mathbf{R}^4) \cong \mathrm{M}_4(\mathbf{R})$ is an isomorphism. ■

Corollary 29.5. *For every $r, s \in \mathbf{N}_0$:*

(1) $\mathrm{C}\ell_{r+1,s+1} \cong \mathrm{C}\ell_{r,s} \otimes \mathrm{M}_2(\mathbf{R})$.

(2) $\mathrm{C}\ell_{r+2,s} \cong \mathrm{C}\ell_{s,r} \otimes \mathrm{M}_2(\mathbf{R})$.

(3) $\mathrm{C}\ell_{r,s+2} \cong \mathrm{C}\ell_{s,r} \otimes \mathbf{H}$.

(4) $\mathrm{C}\ell_{r+4,s} \cong \mathrm{C}\ell_{r+4,s} \cong \mathrm{C}\ell_{r,s} \otimes \mathrm{M}_2(\mathbf{H})$.

(5) $\mathrm{C}\ell_{r+8,s} \cong \mathrm{C}\ell_{r+8,s} \cong \mathrm{C}\ell_{r,s} \otimes \mathrm{M}_{16}(\mathbf{R})$. ■

Proof of Theorem 29.2. It remains to determine the entries of Table 1 by the following procedure:

(1) Example 16.2 and Example 16.4 determine

$$\mathrm{C}\ell_{0,0} \cong \mathbf{R}, \quad \mathrm{C}\ell_{1,0} \cong \mathbf{R}^{\oplus 2}, \quad \mathrm{C}\ell_{0,1} \cong \mathbf{C}, \quad \mathrm{C}\ell_{2,0} \cong \mathrm{C}\ell_{1,1} \cong \mathrm{M}_2(\mathbf{R}), \quad \text{and} \quad \mathrm{C}\ell_{0,2} \cong \mathbf{H}.$$

(2) $\mathrm{C}\ell_{r+2,0} \cong \mathrm{C}\ell_{0,r} \otimes \mathrm{M}_2(\mathbf{R}) \cong \mathrm{C}\ell_{r+2,0}$ determines $\mathrm{C}\ell_{3,0}$.

(3) $\mathrm{C}\ell_{0,s+2} \cong \mathrm{C}\ell_{s,0} \otimes \mathbf{H}$ determines $\mathrm{C}\ell_{0,3}$.

(4) $\mathrm{C}\ell_{r+4,0} \cong \mathrm{C}\ell_{0,r} \otimes \mathrm{M}_2(\mathbf{H})$ determines $\mathrm{C}\ell_{r,0}$ for every $r \in \mathbf{N}_0$.

(5) $\mathrm{C}\ell_{0,s+4} \cong \mathrm{C}\ell_{0,s} \otimes \mathrm{M}_2(\mathbf{H})$ determines $\mathrm{C}\ell_{0,s}$ for every $s \in \mathbf{N}_0$.

(6) $\mathrm{C}\ell_{r+1,s+1} \cong \mathrm{C}\ell_{r,s} \otimes \mathrm{M}_2(\mathbf{R})$ determines the $\mathrm{C}\ell_{r,s}$ for every $r, s \in \mathbf{N}_0$. ■

Remark 29.6. Theorem 29.2 immediately determines the commuting algebra of $C\ell_{r,s}$ up to isomorphism (of $\mathbf{R}$–algebras):

$$D_{r,s} := \mathrm{End}_{C\ell_{r,s}}(C\ell_{r,s}) \cong \begin{cases} \mathbf{R} & \text{if } r-s = 0,2 \bmod 8 \\ \mathbf{C} & \text{if } r-s = 3,7 \bmod 8 \\ \mathbf{H} & \text{if } r-s = 4,6 \bmod 8 \\ \mathbf{R}\oplus\mathbf{R} & \text{if } r-s = 1 \bmod 8 \\ \mathbf{H}\oplus\mathbf{H} & \text{if } r-s = 5 \bmod 8. \end{cases}$$

In particular:

(1) If $r-s = 3,6 \bmod 8$, then every irreducible representation P of $C\ell_{r,s}$ admits complex structure; that is: an homomorphism $\rho\colon \mathbf{C} \to \mathrm{End}_{C\ell_{r,s}}(P)$ of $\mathbf{R}$–algebras. Since $\mathrm{Aut}(\mathbf{C}) = \mathrm{O}(2) = C_2$—generated by complex conjugation $\bar{\cdot}$, there is no distinguished complex structure on P. In fact, ρ and $\rho' := \rho \circ \bar{\cdot}$ are not equivalent; that is: there is no $C \in \mathrm{End}_{C\ell_{r,s}}(P)^\times \cong \mathbf{C}^\times$ satisfying $\rho' = C \circ \rho \circ C^{-1}$.

(2) If $r-s = 4,5,6 \bmod 8$, then every irreducible representation P of $C\ell_{r,s}$ admits quaternionic structure $\rho\colon \mathbf{H} \to \mathrm{End}_{C\ell_{r,s}}(P)$. Since $\mathrm{Aut}(\mathbf{H}) = \mathrm{SO}(3)$, there is no distinguished quaternionic structure either. However, for every $\phi \in \mathrm{Aut}(\mathbf{H})$, there is a $y \in \mathbf{H}^\times$ such that $\phi(\cdot) = y \cdot y^{-1}$. Therefore, ρ and $\rho' := \rho \circ \phi$ are equivalent via $\Phi := \rho(y)$.

(3) If $r-s = 1,5 \bmod 8$, then $C\ell_{r,s}$ has two nonequivalent irreducible representations P, P'. Again, among those two none is distinguished.

It turns out that a choice of orientation of $\mathbf{R}^{r+s}$ resolves the above ambiguities. ♣

Remark 29.7. Here is a slightly more abstract perspective of the salient point of Remark 29.6.

Let A be an k–algebra. Let V be an A–module. Denote by $D := \mathrm{End}_A(V) \subset \mathrm{End}_k(V)$ the commuting algebra of V. Evidently, V can be regarded as $D \otimes_k A$–module.

By Schur's lemma, D is a division algebra if V is simple. If $k = \mathbf{R}$, then by Frobenius' Theorem on real division algebras D is *isomorphic* to $K \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Therefore, V can be regarded as a $K \otimes_k A$–module after a choice of isomorphism $D \cong K$ has been made: $K \otimes_k A \cong D \otimes_k A \to \mathrm{End}_k(V)$. If $\phi \in \mathrm{Aut}(K)$, then $K \otimes_k A \overset{\phi}{\cong} K \otimes_k A \cong D \otimes_k A \to \mathrm{End}_k(V)$ yields another structure of a $K \otimes_k A$–module on V. If ϕ is an inner automorphism, then these structures are isomorphic. In general, there are (up to) $\mathrm{Out}(K) = \mathrm{Aut}(K)/\mathrm{Inn}(K)$ many non-equivalent choices. ♣

Remark 29.8. The above discussion is somewhat unsatisfactory in that it treats $C\ell_{r,s}$ as an algebra an not a superalgebra. Section 33 determines $C\ell^0_{r,s}$. This ameliorates the situation to an extend. This approach is common in the literature on spin geometry, but it would probably be better to carry out the computation with the supermindeset. Lam [Lam05, Chapter V §4] shows how it should be done. *Oh well!* ♣

30 Computation of the complex Clifford algebras

Definition 30.1. For $r \in \mathbf{N}_0$ set

$$\mathbf{C}\ell_r := C\ell(\langle 1\rangle^{\perp r}) \quad \text{defined over} \quad \mathbf{C}.$$ •

Remark 30.2. By Proposition 11.17 every non-degenerate quadratic form $q\colon V \to \mathbf{C}$ with $r := \dim V < \infty$ is isomorphic to $\langle 1\rangle^{\perp r}$. ♣

Theorem 30.3. *For every $r \in \mathbf{N}_0$*

$$\mathbf{C}\boldsymbol{\ell}_r \cong \begin{cases} \mathrm{M}_{2^{r/2}}(\mathbf{C}) & \text{if } r \text{ is even} \\ \mathrm{M}_{2^{(r-1)/2}}(\mathbf{C})^{\oplus 2} & \text{if } r \text{ is odd.} \end{cases}$$

Lemma 29.3 and Example 16.4 immediately imply the following.

Corollary 30.4. *For every $r \in \mathbf{N}_0$*

$$\mathbf{C}\boldsymbol{\ell}_{r+1} \cong \mathbf{C}\boldsymbol{\ell}_r \otimes_{\mathbf{C}} \mathrm{M}_2(\mathbf{C}).$$ ■

Proof of Theorem 30.3. $\mathbf{C}\boldsymbol{\ell}_0 \cong \mathbf{C}$ and Example 16.2 determines $\mathbf{C}\boldsymbol{\ell}_1 \cong \mathbf{C} \oplus \mathbf{C}$. Therefore, Corollary 30.4 finishes the proof. ■

Remark 30.5. Of course, for $\mathbf{C}\boldsymbol{\ell}_r$ only the ambiguity discussed in Remark 29.6 (3) remains (if r is odd). ♣

31 Pinor modules

Situation 31.1. Let $r, s \in \mathbf{N}_0$. Let $\mathrm{C}\ell_{r,s}$ be as in Definition 29.1. Suppose that an orientation on $\mathbf{R}^{\oplus(r+s)}$ has been chosen. ×

Definition 31.2. The **volume element**

$$\omega := e_1 \cdots e_{r+s} \in \mathrm{C}\ell_{r,s}$$

with $e_1, \dots, e_{r+s}$ denoting a positive orthonormal basis for $q_{r,s}$. (A moment's thought reveals that ω does not depend on the choice.) •

Proposition 31.3. *For every homogeneous $x \in \mathrm{C}\ell_{r,s}$*

$$\omega x = (-1)^{(\deg\omega+1)\deg x} x\omega;$$

in particular: if $r + s$ is odd, then

$$\omega \in Z(\mathrm{C}\ell_{r,s}).$$

Moreover:

$$\omega^2 = \begin{cases} \mathbf{1} & \text{if } r - s = 0, 1 \bmod 4 \\ -\mathbf{1} & \text{if } r - s = 2, 3 \bmod 4. \end{cases}$$

Proof. By direct computation, for every $v \in V$

$$v\omega = (-1)^{r+s-1}\omega v.$$

This implies the first assertion. Moreover,

$$\begin{aligned} e_1 \cdots e_{r+s} \cdot e_1 \cdots e_{r+s} &= (-1)^{r+s-1} e_1^2 e_2 \cdots e_{r+s} \cdot e_2 \cdots e_{r+s} \\ &= (-1)^{\frac{(r+s)(r+s-1)}{2}} e_1^2 e_2^2 \cdots e_{r+s}^2 \\ &= (-1)^{\frac{(r+s)(r+s-1)}{2}+s} \end{aligned}$$

Finally, observe that

$$(r+s)(r+s-1) + 2s = (r-s)(r-s-1) \bmod 4. \qquad \blacksquare$$

Remark 31.4. This resolves the ambiguities in Remark 29.6 (1) and (3). ♣

Definition 31.5.

(1) If $r - s = 0, 2 \bmod 8$, then the **pinor module** P is the irreducible $C\ell_{r,s}$–module.

(2) If $r - s = 1 \bmod 8$, then the **positive pinor module** P^+ and **negative pinor module** P^- are the irreducible $C\ell_{r,s}$–module on which ω acts as $\mathbf{1}$ and $-\mathbf{1}$ respectively.

(3) If $r - s = 3, 7 \bmod 8$, then the **pinor module** P and the **conjugate pinor module** $\bar{P}$ are the irreducible $\mathbf{C} \otimes C\ell_{r,s}$–modules on which ω acts as i and $-i$ respectively.

(4) If $r - s = 4, 6 \bmod 8$, then the **pinor module** P is the irreducible $\mathbf{H} \otimes C\ell_{r,s}$–module.

(5) If $r - s = 5 \bmod 8$, then the **positive pinor module** P^+ and **negative pinor module** P^- are the irreducible $\mathbf{H} \otimes C\ell_{r,s}$–module on which ω acts as $\mathbf{1}$ and $-\mathbf{1}$ respectively. •

Remark 31.6. By Theorem 29.2, the pinor modules exist and are unique up to isomorphism. ♣

32 Complex pinor modules

Situation 32.1. Let $r \in \mathbf{N}_0$. Let $\mathbf{C}\boldsymbol{\ell}_r$ be as in Definition 30.1. Suppose that an orientation on $\mathbf{R}^{\oplus r}$ has been chosen. ×

Definition 32.2. The **complex volume element**

$$\omega^{\mathbf{C}} := i^{\lfloor \frac{r+1}{2} \rfloor} e_1 \cdots e_r \in \mathbf{C}\boldsymbol{\ell}_r$$

with $e_1, \ldots, e_{r+s}$ denoting a positive orthonormal basis. (A moment's though reveals that $\omega^{\mathbf{C}}$ does not depend on the choice.) •

Remark 32.3. If $r = 3, 4 \bmod 8$, then $\omega^{\mathbf{C}} = -\omega$. ♣

Proposition 32.4. *If r is odd, then $\omega^{\mathbf{C}} \in Z(\mathbf{C}\boldsymbol{\ell}_r)$; moreover, for every r, $(\omega^{\mathbf{C}})^2 = \mathbf{1}$.* ■

Definition 32.5.

(1) If r is even, then the **complex pinor module** P is the irreducible $\mathbf{C}\boldsymbol{\ell}_r$–module.

(2) If r is odd, then the **positive complex pinor module** P^+ and **negative complex pinor module** P^- are the irreducible $\mathbf{C}\boldsymbol{\ell}_r$–modules on which $\omega^{\mathbf{C}}$ acts as $\mathbf{1}$ and $-\mathbf{1}$ respectively. •

Remark 32.6. By Theorem 30.3, the complex pinor modules exist and are unique up to isomorphism. ♣

Lecture 6

The pinor modules over $\mathrm{C}\ell_{r,s}$ from the last lecture can also be considered as modules of the even subalgebra $\mathrm{C}\ell^0_{r,s} \subset \mathrm{C}\ell_{r,s}$. The simple modules of $\mathrm{C}\ell^0_{r,s}$ are the spinor modules. It is important to understand how the pinor and spinor modules are related. Again, this is largely an exercise given the classification derived in the last lecture. The lecture notes contain the answer to this exercise, but in the lecture I will only dicuss a few examples to illustrate how this is done.

The second part of this lecture works towards the construction of spin group $\mathrm{Spin}(q)$. First I will explain that if q is non-degenerate, then $\mathrm{C}\ell(q)$ is a CSSA, a central simple superalgebra. Then I will discuss the Clifford group $\Gamma(q)$ and the special Clifford group $S\Gamma(q)$, a precursor of $\mathrm{Spin}(q)$.

33 The even subalgebra of the Clifford algebra

Situation 33.1. Let k be a field. Let $q\colon V \to k$ be a quadratic form with $\dim V < \infty$. ×

Proposition 33.2. *For every $a \in k^\times$*

$$\mathrm{C}\ell(q \perp \langle a\rangle)^0 \cong \mathrm{C}\ell(V, -a^{-1}\cdot q).$$

Proof. Set $e := (0,1) \in V \oplus k$. Define $\delta\colon V \to \mathrm{C}\ell(q \perp \langle a\rangle)^0$ by

$$\delta(v) := ev.$$

Since

$$\delta(v) = evev = -e^2v^2 = q(v),$$

there is a unique algebra homomorphism $f\colon \mathrm{C}\ell(q) \to \mathrm{C}\ell(q \perp \langle a\rangle)^0$ with $\delta = f \circ \gamma$. Evidently, f is surjective (it maps to a set of generators). Multiplication with e induces a vector space isomorphism $\mathrm{C}\ell(q \perp \langle a\rangle)^0 \cong \mathrm{C}\ell(q \perp \langle a\rangle)^1$. Consequently, $\dim \mathrm{C}\ell(q \perp \langle a\rangle)^0 = 2^{\dim V+1}/2 = \dim \mathrm{C}\ell(q)$. Therefore, f is an isomorphism. ■

Proposition 33.3. *For every $r, s \in \mathbf{N}_0$*

$$\mathrm{C}\ell^0_{r+1,s} \cong \mathrm{C}\ell_{s,r} \quad \textit{and} \quad \mathrm{C}\ell^0_{r,s+1} \cong \mathrm{C}\ell_{r,s}. \qquad ■$$

Remark 33.4. Proposition 33.3 explains the symmetry $\mathrm{C}\ell_{r+1,s} \cong \mathrm{C}\ell_{s+1,r}$ apparent from Table 1. ♣

Proposition 33.5. *For every $r, s \in \mathbf{N}_0$*

$$\mathrm{C}\ell^0_{r,s} \cong \mathrm{C}\ell_{s,r-1} \cong \mathrm{C}\ell^0_{s,r}. \qquad ■$$

34 Spinor modules

Situation 34.1. Let $r, s \in \mathbf{N}_0$. Let $\mathrm{C}\ell_{r,s}$ be as in Definition 29.1. Suppose that an orientation on $\mathbf{R}^{\oplus(r+s)}$ has been chosen. Denote the volume element by $\omega \in \mathrm{C}\ell_{r,s}$. ×

Definition 34.2.

(1) If $r - s = 0 \bmod 8$, then the **positive spinor module** S^+ and the **negative spinor module** S^- are the irreducible $\mathrm{C}\ell^0_{r,s}$–module on which ω acts as $\mathbf{1}$ and $-\mathbf{1}$ respectively.

(2) If $r - s = 1, 7 \bmod 8$, then the **spinor module** S is the irreducible $\mathrm{C}\ell^0_{r,s}$–module.

(3) If $r - s = 2, 6 \bmod 8$, then the **spinor module** S and the **conjugate spinor module** $\bar{S}$ are the irreducible $\mathbf{C} \otimes \mathrm{C}\ell^0_{r,s}$–modules on which ω acts as i and $-i$ respectively.

(4) If $r - s = 3, 5 \bmod 8$, then the **spinor module** S is the irreducible $\mathbf{H} \otimes \mathrm{C}\ell^0_{r,s}$–module.

(5) If $r - s = 4 \bmod 8$, then the **positive spinor module** S^+ and the **negative spinor module** S^- are the irreducible $\mathbf{H} \otimes \mathrm{C}\ell^0_{r,s}$–module on which ω acts as $\mathbf{1}$ and $-\mathbf{1}$ respectively. •

Remark 34.3. By Theorem 29.2, the spinor modules exist and are unique up to isomorphism. ♣

Remark 34.4. The symmetry of Definition 34.2 under exchanging r and s is a consequence of the isomorphism $\mathrm{C}\ell^0_{r,s} \cong \mathrm{C}\ell^0_{s,r}$. ♣

35 Decomposition of pinor into spinor modules

Proposition 35.1.

(0) If $r - s = 0 \bmod 8$*, then*

(a) $P = S^+ \oplus S^-$ *as* $\mathrm{C}\ell^0_{r,s}$*–modules, and*

(b) $\gamma\colon\ \mathbf{R}^{\oplus(r+s)} \longrightarrow \mathrm{End}(S^+, S^-) \oplus \mathrm{End}(S^-, S^+)$.

(1) If $r - s = 1 \bmod 8$*, then* $P^\pm = S$ *as* $\mathrm{C}\ell^0_{r,s}$*–modules.*

(2) If $r - s = 2 \bmod 8$*, then*

(a) $\mathbf{C} \otimes_{\mathbf{R}} P = S \oplus \bar{S}$ *as* $\mathbf{C} \otimes \mathrm{C}\ell^0_{r,s}$*–modules and*

(b) $\mathbf{1}_{\mathbf{C}} \otimes_{\mathbf{R}} \gamma\colon\ \mathbf{R}^{\oplus(r+s)} \longrightarrow \mathrm{End}_{\mathbf{C}}(S, \bar{S}) \oplus \mathrm{End}_{\mathbf{C}}(\bar{S}, S)$.

(3) If $r - s = 3 \bmod 8$*, then*

(a) $P = \bar{P} = S$ *as* $\mathbf{C} \otimes \mathrm{C}\ell^0_{r,s}$*–modules and*

(b) $\gamma\colon\ \mathbf{R}^{\oplus(r+s)} \longrightarrow \mathrm{End}_{\mathbf{C}}(S)$.

(4) If $r - s = 4 \bmod 8$*, then*

(a) $P = S^+ \oplus S^-$ *as* $\mathbf{H} \otimes \mathrm{C}\ell^0_{r,s}$*–modules, and*

(b) $\gamma\colon \mathbf{R}^{\oplus(r+s)} \to \mathrm{End}_{\mathbf{H}}(S^+, S^-) \oplus \mathrm{End}_{\mathbf{H}}(S^-, S^+)$.

(5) *If* $r - s = 5 \bmod 8$, *then*

(a) $P^\pm = S$ *as* $\mathbf{H} \otimes \mathrm{C}\ell^0_{r,s}$*–modules, and*

(b) $\gamma\colon \mathbf{R}^{\oplus(r+s)} \to \mathrm{End}_{\mathbf{H}}(S)$.

(6) *If* $r - s = 6 \bmod 8$, *then*

(a) $P = \mathbf{H} \otimes_{\mathbf{C}} S = S \oplus \bar{S}$ *as* $\mathbf{H} \otimes \mathrm{C}\ell^0_{r,s}$*–modules, and*

(b) $\gamma\colon \mathbf{R}^{\oplus(r+s)} \to \mathrm{Hom}_{\mathbf{C}}(S, \bar{S}) \oplus \mathrm{Hom}_{\mathbf{C}}(\bar{S}, S)$

(7) *If* $r - s = 7 \bmod 8$, *then*

(a) $P = \mathbf{C} \otimes_{\mathbf{R}} S$ *as* $\mathbf{C} \otimes \mathrm{C}\ell^0_{r,s}$*–modules, and*

(b) $\gamma\colon \mathbf{R}^{\oplus(r+s)} \to \mathbf{1}_{\mathbf{C}} \otimes \mathrm{End}(S)$.

36 Complex spinor modules

Situation 36.1. Let $r \in \mathbf{N}_0$. Let $\mathbf{C}\boldsymbol{\ell}_r$ be as in Definition 30.1. Suppose that an orientation on $\mathbf{R}^{\oplus r}$ has been chosen. Denote the complex volume element by $\omega^{\mathbf{C}} \in \mathbf{C}\boldsymbol{\ell}_r$. ×

Definition 36.2.

(1) If r is even, then the **positive complex pinor module** S^+ and **negative complex pinor module** S^- are the irreducible $\mathbf{C}\boldsymbol{\ell}^0_r$–modules on which $\omega^{\mathbf{C}}$ acts as $\mathbf{1}$ and $-\mathbf{1}$ respectively.

(2) If r is odd, then the **complex spinor module** S is the irreducible $\mathbf{C}\boldsymbol{\ell}^0_r$–modules. •

Remark 36.3. By Theorem 30.3, the complex spinor modules exist and are unique up to isomorphism. ♣

37 Decomposition of complex pinor into complex spinor modules

Situation 37.1. Let $r \in \mathbf{N}_0$. Let $\mathbf{C}\boldsymbol{\ell}_r$ be as in Definition 30.1. Suppose that an orientation on $\mathbf{R}^{\oplus r}$ has been chosen. Denote the complex volume element by $\omega^{\mathbf{C}} \in \mathbf{C}\boldsymbol{\ell}_r$. Consider the complex pinor and complex spinor modules; see Definition 32.5 and Definition 36.2. ×

Proposition 37.2.

(1) *If* r *is even, then*

(a) $P = S^+ \oplus S^-$ *as* $\mathbf{C}\boldsymbol{\ell}^0_r$*–modules, and*

(b) $\gamma\colon \mathbf{C}^{\oplus r} \to \mathrm{Hom}_{\mathbf{C}}(S^+, S^-) \oplus \mathrm{Hom}_{\mathbf{C}}(S^-, S^+)$.

(2) *If* r *is odd, then* $P^+ = P^- = S$ *as* $\mathbf{C}\boldsymbol{\ell}^0_r$*–modules.*

38 The Lipschitz group

Situation 38.1. Let k be a field. Let $q\colon V \to k$ be a quadratic form. Let S be a $\mathrm{C}\ell(q)$–module. ×

Definition 38.2. The **Lipschitz group** of S is

$$L(S) := \{x \in \mathrm{GL}(S) : x\gamma(V)x^{-1} \subset \gamma(V) \subset \mathrm{End}(S)\}.$$

•

Remark 38.3. For $k = \mathbf{C}$ the Lipschitz group was introduced by **Friedrich2000a**. It mediates the Clifford module perspective and the principal bundle perspective on spin geometry. Trautman [Tra08] contains a very clear and concise discussion of the role of the (complex) Lipschitz group (even for real spinors). Lazaroiu and Shahbazi [LS19] develop the real theory in exuberant detail—indeed; the theory turns out to be suprisingly intricate. ♣

Definition 38.4 (Lazaroiu and Shahbazi [LS19, Definition 3.9]). S is **weakly faithful** if the map $\gamma\colon V \to \mathrm{C}\ell(q) \to \mathrm{End}(S)$ is injective. •

Proposition 38.5. *If S is weakly faithful, then there is a unique homomorphism* $\mathrm{Ad}\colon L(S) \to \mathrm{O}(q)$ *such that*

$$\gamma(\mathrm{Ad}(x)v) = x\gamma(v)x^{-1};$$

that is: $\gamma\colon V \to \mathrm{End}(S)$ is $L(S)$–equivariant.

Proof. Evidently, the above condition defines a homomorphism $L(S) \to \mathrm{GL}(V)$. Since

$$q(\mathrm{Ad}(x)v) = \gamma(\mathrm{Ad}(x)v)^2 = \big(x\gamma(v)x^{-1}\big)^2 = x\gamma(v)^2x^{-1} = q(v),$$

it factors through $\mathrm{O}(q)$. ■

Proposition 38.6. *If S is weakly faithful, then*

$$D^\times = \ker(\mathrm{Ad}\colon L(S) \to \mathrm{O}(q)).$$

with $D := \mathrm{End}_{\mathrm{C}\ell(q)}(S)$ denoting the commuting algebra of S. ■

Remark 38.7. Determining $\mathrm{im}(\mathrm{Ad}\colon L(S) \to \mathrm{O}(q))$ is a bit more involved; cf. [LS19, Theorem 4.9]. ♣

39 The supercentre of $\mathrm{C}\ell(q)$

Situation 39.1. Let k be a field. ×

Definition 39.2. Let A be a k–superalgebra. The **supercommutator** is the bilinear map $[\cdot,\cdot]\colon A\otimes A \to A$ defined by

$$[x, y] := xy - (-1)^{\deg x\cdot\deg y}yx$$

for homogeneous $x, y \in A$. The **supercenter** of A is

$$\hat{Z}(A) := \{x \in A : [x, y] = 0 \text{ for every } y \in A\}.$$

A is **supercentral** if $\hat{Z}(A) = k$. •

Lemma 39.3. *If A, B are k–superalgebras, then*

$$\hat{Z}(A \hat{\otimes} B) = \hat{Z}(A) \hat{\otimes} \hat{Z}(B).$$

Proof. Evidently, $\hat{Z}(A) \hat{\otimes} \hat{Z}(B) \subset \hat{Z}(A \hat{\otimes} B)$.

If $z \in \hat{Z}(A \hat{\otimes} B)$ is homogeneous, then

$$z = \sum_{i=1}^{r} a_i \hat{\otimes} b_i.$$

with $a_1, \ldots, a_r \in A$ and $b_1, \ldots, b_r \in B$ linearly independent and homogeneous. By direct computation, for every $a \in A$ and $b \in B$

$$0 = [z, a \hat{\otimes} \mathbf{1}] = \sum_{i=1}^{r} (-1)^{\deg b_i} [a_i, a] \otimes b_i \quad \text{and} \quad 0 = [z, \mathbf{1} \hat{\otimes} b] = \sum_{i=1}^{r} (-1)^{\deg b_i} a_i \otimes [b_i, b].$$

Therefore, $a_1, \ldots, a_r \in \hat{Z}(A)$ and $b_1, \ldots, b_r \in \hat{Z}(B)$. ■

(The above argument is from [Lam05, Chapter IV Theorem 2.3(1)].)

Theorem 39.4. *If q is non-degenerate and* $\dim V < \infty$*, then*

$$\hat{Z}(\mathrm{C}\ell(q)) = k.$$

Proof. In the light of Lemma 39.3, Proposition 11.17 and Theorem 11.21 (2) it consider the following two cases:

(1) Consider $\langle a \rangle \colon V := k \to k$ with $a \in k^\times$ and $\operatorname{char} k \neq 2$. Set $e := 1 \in V$. By direct computation, for every $x = x_0 + x_1 e \in \mathrm{C}\ell(\langle a \rangle)$ $(x_0, x_1 \in k)$

$$[x, e] = 2x_1 a.$$

Therefore, $\hat{Z}(\mathrm{C}\ell(\langle a \rangle)) = k$.

(2) Consider $[a, b] \colon V := k^{\oplus 2} \to k$ with $a, b \in k$ and $\operatorname{char} k = 2$. Set $e_1 := (1, 0), e_2 := (0, 1) \in V$. By direct computation, for every $x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_1 e_2 \in \mathrm{C}\ell([a, b])$ $(x_0, x_1, x_2, x_3 \in k)$

$$[x, e_1] = x_2 + x_3 e_1 \quad \text{and} \quad [x, e_2] = x_1 + x_3 e_2.$$

Therefore, $\hat{Z}(\mathrm{C}\ell([a, b]) = k$. ■

Remark 39.5. Although the above proof is quite straight-forward, I would be interested in a more conceptual proof. It might not be reasonable to expect one to exists, however. ♣

40 Tensor products of supercentral supersimple superalgebras

Situation 40.1. Let k be a field. $\times$

Definition 40.2. Let A be a k–superalgebra. A is **supersimple** if it has no non-trivial proper homogeneous ideals. $\bullet$

Lemma 40.3. *Let A, B be k–superalgebras. If A is a supercentral and supersimple and B is supercentral, then $A \hat{\otimes} B$ is supersimple.*

Proof. Let $I \subset A \hat{\otimes} B$ be a non-trivial proper homogenous ideal. If $z \in I$ is homogeneous, then

$$z = \sum_{i=1}^{r} a_i \hat{\otimes} b_i.$$

with $a_1, \ldots, a_r \in A$ and $b_1, \ldots, b_r \in B$ linearly independent, homogeneous, and $\deg a_i + \deg b_i = \deg z$. Choose a non-zero homogeneous $z \in I$ of with minimal r.

Since A is supersimple, the homogeneous ideal (a_1) generated by a_1 is A. Therefore, there are homogeneous $c_1, d_1, \cdots, c_s, d_s \in A$ with

$$\mathbf{1} = \sum_{j=1}^{s} c_j a_1 d_j$$

and $\deg c_i + \deg c_i = \deg a_1$. By construction,

$$z' := \sum_{j=1}^{s} (c_j \hat{\otimes} \mathbf{1}) z (d_j \hat{\otimes} \mathbf{1}) = \pm\mathbf{1} \hat{\otimes} b_1 + \sum_{i=2}^{r} a_i' \otimes b_i \in I.$$

Since $\deg c_i + \deg c_i = \deg a_1$, a_i' is homogeneous. Since $b_1, \ldots, b_r$ are linearly independent, $z' \neq 0$. By the minimality of r, $\pm\mathbf{1}, a_2', \ldots, a_r'$ are linearly independent.

An analogous construction with b_1 instead of a_1 construct a non-zero homogeneous

$$z'' = \mathbf{1} \hat{\otimes} \mathbf{1} + \sum_{i=2}^{r} a_i' \otimes b_i' \in I.$$

with $\mathbf{1}, b_2', \ldots, b_r'$ linearly independent and homogeneous. Evidently, $\deg a_i = \deg b_i$.

If $r = 1$, then $I \supset (\mathbf{1} \hat{\otimes} \mathbf{1}) = A \hat{\otimes} B$—contradicting that I is proper. Therefore, $r \geqslant 2$.

For every $b \in B$

$$[z', \mathbf{1} \hat{\otimes} b] = \sum_{i=2}^{r} a_i' \otimes [b_i', b] \in I.$$

By the minimality of r, $[z', \mathbf{1} \hat{\otimes} b] = 0$. Since $a_2', \ldots, a_r'$ are linearly independent, $[b_i', b] = 0$. Therefore and since B is supercentral, $b_i' \in k$—contradicting that $\mathbf{1}, b_2', \ldots, b_r'$ are linearly independent. $\blacksquare$

(The above argument is from [Lam05, Chapter IV Theorem 2.3(2)].)

Theorem 40.4. *If q is non-degenerate, then $\mathrm{C}\ell(q)$ is supersimple.*

Proof. In the light of Lemma 40.3, Proposition 11.17 and Theorem 11.21 (2) it consider the following two cases:

(1) Consider $\langle a\rangle\colon V := k \to k$ with $a \in k^\times$. Set $e := 1 \in V$. Let $I \subset \mathrm{C}\ell(\langle a\rangle)$ be a proper homogeneous ideal. Let $x = x_0 + x_1 e \in I$ ($x_0, x_1 \in k$). Since I is homogeneous,

$$I \supset (x_0) \quad \text{and} \quad I \supset (x_1 e) \supset (x_1 e^2) = (x_1).$$

Therefore and since I is proper, $x = 0$. Hence, $I = 0$.

(2) Consider $[a, b]\colon V := k^{\oplus 2} \to k$ with $a, b \in k$ and $\operatorname{char} k = 2$. Set $e_1 := (1, 0), e_2 := (0, 1) \in V$. Let $I \subset \mathrm{C}\ell([a, b])$ be a proper homogeneous ideal. Let $x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_1 e_2 \in I$ ($x_0, x_1, x_2, x_3 \in k$). Since I is homogeneous,

$$I \supset (x_0), \quad I \supset (x_1 e_1 + x_2 e_3), \quad \text{and} \quad I \supset (x_3 e_1 e_2).$$

Indeed, since

$$[x, e_1] = x_2 + x_3 e_1, \quad [x, e_2] = x_1 + x_3 e_2, \quad \text{and} \quad [[x, e_1], e_2] = x_3,$$

and I is homogeneous, $I \supset (x_0)$, $I \supset (x_1)$, $I \supset (x_2)$, and $I \supset (x_3)$. Therefore and since I is proper, $x = 0$. Hence, $I = 0$. ■

Remark 40.5. Although the above proof is quite straight-forward, I would be interested in a more conceptual proof. It might not be reasonable to expect one to exists, however. ♣

41 The Clifford group

Situation 41.1. Let k be a field. Let $q\colon V \to k$ be a quadratic form. ×

Definition 41.2. Let A be a k–superalgebra. The supergroup of **homogenous units** of A is

$$A^{h\times} := (A^0 \cup A^1) \cap A^\times. \qquad \bullet$$

Definition 41.3. The **twisted adjoint representation** is the homomorphism $\widetilde{\mathrm{Ad}}\colon \mathrm{C}\ell^0(q)^{h\times} \to \operatorname{Aut}(\mathrm{C}\ell(q))$ defined by

$$\widetilde{\mathrm{Ad}}(x)y := (-1)^{\deg x \cdot \deg y} x y x^{-1}.$$

The **Clifford group** $\Gamma(q)$ is

$$\Gamma(q) := \left\{x \in \mathrm{C}\ell(q)^{h\times} : \widetilde{\mathrm{Ad}}(x) \in \mathrm{O}(q)\right\}. \qquad \bullet$$

Proposition 41.4. *The diagram*

$$\begin{array}{ccc} \Gamma(q) & \xrightarrow{\widetilde{\mathrm{Ad}}} & \mathrm{O}(q) \\ \big\downarrow & & \big\downarrow \\ \mathrm{C}\ell(q)^{h\times} & \xrightarrow{\widetilde{\mathrm{Ad}}} & \operatorname{Aut}(\mathrm{C}\ell(q)). \end{array}$$

is a pullback diagram. ■

Proposition 41.5. *If q is non-degenerate and* $\dim V < \infty$, *then*

$$0 \to k^\times \to \Gamma(q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(q) \to 0$$

is exact.

Proof. By Theorem 39.4, $\ker \widetilde{\mathrm{Ad}} = k^\times \subset \Gamma(q)$. If $v \in V$ is anisotopic, then $v \in \Gamma(q)$ and, by direct computation,

$$\widetilde{\mathrm{Ad}}(v) = r_v \in \mathrm{O}(q).$$

Therefore, the Cartan–Dieudonné Theorem implies that $\widetilde{\mathrm{Ad}}(\Gamma(q)) = \mathrm{O}(q)$—except possibly in the case $q \cong [1,1] \perp [1,1]$ with $k = \mathbf{F}_2$. According to Remark 14.6, it remains to find $x \in \Gamma(q)$ such that $\widetilde{\mathrm{Ad}}(x)$ interchanges the summands $[1,1]$. A direct computation reveals that

$$x := \mathbf{1} + e_1e_2 + e_1e_4 + e_2e_3 + e_3e_4$$

does just that.[1] ∎

Remark 41.6. As a consequence of the proof, $\Gamma(q)$ is generated by anisotopic vectors (and x in the exceptional case $q \cong [1,1] \perp [1,1]$ with $k = \mathbf{F}_2$). ♣

Remark 41.7. The use of the Cartan–Dieudonné Theorem in the proof of Proposition 41.5 can be (and probably should be) replaced with an application of a the Skolem–Noether theorem for supercentral supersimple superalgebras; cf. Elduque and Villa [EV08, Proposition 8]. ♣

Remark 41.8.

(1) Chevalley [Che54, §2.3] considered the **adjoint representation** $\mathrm{Ad}\colon \tilde{\Gamma}(q) \to \mathrm{O}(q)$ defined by

$$\gamma(\mathrm{Ad}(x)v) := x\gamma(v)x^{-1}$$

with $\tilde{\Gamma}(q) := Z(\mathrm{C}\ell(q))^\times\Gamma(q)$; cf. [LS19, §1.18]. Of course, if $\deg\colon \Gamma(q) \to \mathbf{Z}/2\mathbf{Z}$ denotes the grading on $\Gamma(q)$, then

$$\widetilde{\mathrm{Ad}} = (-1)^{\deg} \mathrm{Ad}|_{\Gamma(q)}.$$

Therefore, Proposition 41.5 does not (quite) hold with Ad instead of $\widetilde{\mathrm{Ad}}$; cf. [Che54, p. II.3.1].

(2) Atiyah, Bott, and Shapiro [ABS64, Part I §3] observed that signs might be in order because $\mathrm{C}\ell(q)$ is a superalgebra, and introduced $\widetilde{\mathrm{Ad}}\colon \Gamma(q) \to \mathrm{O}(q)$—although with an a priori different definition of $\Gamma(q)$.

(3) Atiyah, Bott, and Shapiro only defined $\widetilde{\mathrm{Ad}}\colon \Gamma(q) \to \mathrm{O}(q)$ and hid the degree by using the involution $\alpha := \mathrm{C}\ell(-\mathbf{1})$. In doing so they have *laid a trap!* Most geometers (e.g.: [LM89, Chapter II (2.11); Har90, (10.9); Roe98, p.57]) define $\widetilde{\mathrm{Ad}}\colon \mathrm{C}\ell(q)^\times \to \mathrm{Aut}(\mathrm{C}\ell(q))$ with two defects: (a) the domain is too large, and (b) $\deg y$ is not taken into account. This is ultimately almost inconsequential, but it fails to take $\mathrm{C}\ell(q)$ seriously as a superalgebra. (I only realised this after reading [Knu91, Chapter IV (6.1)].) ♣

[1] I found this by a brute-force search using Sage.

42 The special Clifford group

Situation 42.1. Let k be a field. Let $q\colon V \to k$ be a quadratic form. ×

Definition 42.2. The **special Clifford group** is

$$S\Gamma(q) \coloneqq \Gamma(q) \cap \mathrm{C}\ell(q)^0.$$

The **special orthogonal group** is

$$\mathrm{SO}(q) \coloneqq \widetilde{\mathrm{Ad}}(S\Gamma(q)) < \mathrm{O}(q).$$ •

Remark 42.3. If char $k \neq 2$, then by Theorem 14.5 $\mathrm{SO}(q) = \ker(\det\colon \mathrm{O}(q) \to \{\pm 1\})$. If char $k = 2$, then the latter is clearly unsatisfactoy and the above is the correct definition. It turns out that even in char $k = 2$, there is a homomorphism $D\colon \mathrm{O}(q) \to \mathbf{Z}/2\mathbf{Z}$, the Dickson invariant, such that $\mathrm{SO}(q) = \ker D$; cf. [Knu91, Chapter IV §5]. ♣

Remark 42.4. On $S\Gamma(q)$, the twisted adjoint representation $\widetilde{\mathrm{Ad}}$ agress with the adjoint representation Ad. ♣

Proposition 42.5. *The diagram*

$$\begin{array}{ccc} S\Gamma(q) & \xrightarrow{\mathrm{Ad}} & \mathrm{SO}(q) \\ \big\downarrow & & \big\downarrow \\ \mathrm{C}\ell(q)^{0\times} & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(\mathrm{C}\ell(q)^0) \end{array}$$

is a pullback diagram. ■

Lecture 7

The goal of this lecture is complete the construction of $\mathrm{Spin}(q)$ and investigate it in a bit more detail for non-degenerate real quadratic forms.

43 The spinor norm

Situation 43.1. Let k be a field. Let $q\colon V \to k$ be a non-degenerate quadratic form. ×

Definition 43.2. The **transposition** $\cdot^t\colon C\ell(q) \to C\ell(q)$ the unique anti-involution that extends id_V. •

Proposition 43.3. *For every $x \in \Gamma(q)$, $x^t x \in k^\times$.*

Proof. Since $\cdot^t$ extends id_V, for every $x \in \Gamma(q)$ and $v \in V$

$$xvx^{-1} = (xvx^{-1})^t = (x^t)^{-1}vx^t.$$

Therefore, $x^t x \in \ker \widetilde{\mathrm{Ad}} = k^\times$. ■

Definition 43.4. The **norm** is the homomorphism $N\colon \Gamma(q) \to k^\times$ defined by

$$N(x) := x^t x. \qquad \bullet$$

Remark 43.5. Since $N(x) \in k^\times$, it evidently is a homomorphism; moreover, $N(x) = xx^t$. For some reason the latter is more commonly used as the initial definition. In the literature, one also finds a version of the norm with $\cdot^t$ replaced by the anti-involution $\bar{\cdot}$ that extends $-\mathrm{id}_V$. I prefer the above, because $N(v) = q(v)$ for every anisotropic $v \in V$. ♣

Definition 43.6. The **spinor norm** $N\colon \mathrm{O}(q) \to k^\times/(k^\times)^2$ is the homomorphism induced by the norm. Set

$$\Omega(q) := \ker N \subset \mathrm{O}(q) \quad \text{and} \quad S\Omega(q) := \mathrm{SO}(q) \cap \Omega(q). \qquad \bullet$$

The norm and the spinor norm fit in the following commutative diagrams with exact rows:

$$\begin{array}{ccccc} k^\times & \hookrightarrow & \Gamma(q) & \twoheadrightarrow & \mathrm{O}(q) \\ \downarrow{\scriptstyle \cdot^2} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} \\ (k^\times)^2 & \hookrightarrow & k^\times & \twoheadrightarrow & k^\times/(k^\times)^2 \end{array} \quad \text{and} \quad \begin{array}{ccccc} k^\times & \hookrightarrow & S\Gamma(q) & \twoheadrightarrow & \mathrm{SO}(q) \\ \downarrow{\scriptstyle \cdot^2} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} \\ (k^\times)^2 & \hookrightarrow & k^\times & \twoheadrightarrow & k^\times/(k^\times)^2. \end{array} \tag{43.7}$$

Example 43.8. Let $r, s \in \mathbf{N}_0$. Consider $q_{r,s} := \langle 1\rangle^{\perp r} \perp \langle -1\rangle^{\perp s}$ defined over $\mathbf{R}$. Set

$$\mathrm{O}_{r,s} := \mathrm{O}(q_{r,s}).$$

$\mathrm{O}_{r,s}$ has $2^{c(r,s)}$ connected components with

$$c(r,s) := \begin{cases} 0 & \text{if } r = s = 0 \\ 1 & \text{if } (r \geqslant 1 \text{ and } s = 0) \text{ or } (r = 0 \text{ and } s \geqslant 1) \\ 2 & \text{if } r, s \geqslant 1. \end{cases}$$

By the Cartan–Dieudonné Theorem, for every $T \in O_{r,s}$ there are $v_i^\pm$ with $\pm q(v_i^\pm) > 0$ ($i \in \{1, \dots, a_\pm\}$) such that

$$T = r_{v_1^+} \dots r_{v_{a_+}^+} r_{v_1^-} \dots r_{v_{a_-}^-}.$$

Identifying $\mathbf{R}^\times/(\mathbf{R}^\times)^2 = \{\pm 1\}$,

$$\det T = (-1)^{a_+ + a_-} \quad \text{and} \quad N(T) = (-1)^{a_-}.$$

Therefore, if $r = 0$, then $N = \det$, if $s = 0$, then $N = \mathbf{1}$. A moment's thought shows that

$$\det \times N\colon \pi_0(O_{r,s}) \hookrightarrow \{\pm 1\}^2.$$

In particular, $S\Omega(q_{r,s})$ is the identity component of $O_{r,s}$. ♠

Example 43.9. Consider the above in the Lorentzian signature $(r, s) = (1, 3)$. A **Lorentz transformation** is a $T \in O_{1,3}$. T is **proper** if $\det T = 0$; that is: $T \in SO_{1,3}$.

A vector $v \in \mathbf{R}^4$ is **time-like**, **space-like**, and **light-like** if $q(v) > 0$, $q(v) < 0$, and $q(v) = 0$ respectively. The **light-cone** $q^{-1}(0)$ separates $\mathbf{R}^4$ into three connected components:

(1) the **future light-cone** $\{v \in \mathbf{R}^4 : q(v) > 0, v_0 > 0\}$,

(2) the **past light-cone** $\{v \in \mathbf{R}^4 : q(v) > 0, v_0 < 0\}$, and

(3) the **elsewhere** $\{v \in \mathbf{R}^4 : q(v) < 0\}$.

A $T \in O_{r,s}$ might swap the future and past light-cones. If it does not, it is **orthochronous**.

A moment's thought shows that $S\Omega(q_{r,s}) = SO^+_{1,3}$, the group of proper, orthochronous Lorentz transformations.

The quotient $O_{1,3}/SO^+_{1,3}$ is isomorphic to the subgroup of $O_{1,3}$ generated by

$$T = \operatorname{diag}(-1, 1, 1, 1) \quad \text{and} \quad P = \operatorname{diag}(1, -1, -1, -1).$$ ♠

44 Pin(q) and Spin(q)

Situation 44.1. Let k be a field. Let $q\colon V \to k$ be a non-degenerate quadratic form. ×

Definition 44.2. The **pin group** associated with q is the group

$$\operatorname{Pin}(q) := \ker(N\colon \Gamma(q) \to k^\times)$$

The **spin group** associated with q is the group

$$\operatorname{Spin}(q) := \operatorname{Pin}(q) \cap S\Gamma(q).$$ •

Proposition 44.3. *Consider $\{\pm 1\} \subset k^\times$. The sequences*

$$0 \to \{\pm 1\} \to \operatorname{Pin}(q) \xrightarrow{\widetilde{\operatorname{Ad}}} \Omega(q) \to 0$$

and

$$0 \to \{\pm 1\} \to \operatorname{Spin}(q) \xrightarrow{\operatorname{Ad}} S\Omega(q) \to 0$$

are exact.

Remark 44.4 ($\mathrm{Pin}(q)$ and $\mathrm{Spin}(q)$ if $\operatorname{char} k = 2$). If $\operatorname{char} k = 2$, then $\{\pm 1\} \subset k^\times$ is trivial; hence, $\mathrm{Pin}(q) = \Omega(q)$ and $\mathrm{Spin}(q) = S\Omega(q)$. ♣

Proof of Proposition 44.3. Since $\{\pm 1\} = \ker(\,\cdot^2\colon k^\times \to (k^\times)^2)$, the commutative diagrams with exact rows (43.7) extend to the commutative diagrams with exact rows:

$$\begin{array}{ccccc} \{\pm1\} & \hookrightarrow & \mathrm{Pin}(q) & \twoheadrightarrow & \Omega(q) \\ \downarrow & & \downarrow & & \downarrow \\ k^\times & \hookrightarrow & \Gamma(q) & \twoheadrightarrow & \mathrm{O}(q) \\ \downarrow{\scriptstyle (\cdot)^2} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} \\ (k^\times)^2 & \hookrightarrow & k^\times & \twoheadrightarrow & k^\times/(k^\times)^2 \end{array} \quad \text{and} \quad \begin{array}{ccccc} \{\pm1\} & \hookrightarrow & \mathrm{Spin}(q) & \twoheadrightarrow & S\Omega(q) \\ \downarrow & & \downarrow & & \downarrow \\ k^\times & \hookrightarrow & S\Gamma(q) & \twoheadrightarrow & \mathrm{SO}(q) \\ \downarrow{\scriptstyle (\cdot)^2} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} \\ (k^\times)^2 & \hookrightarrow & k^\times & \twoheadrightarrow & k^\times/(k^\times)^2. \end{array} \quad \blacksquare$$

Remark 44.5.

(1) There are numerous and sometimes inequivalent definitions of $\mathrm{Pin}(q)$ and $\mathrm{Spin}(q)$ in the literature. The above agrees with that found in most references on algebraic groups. *I believe that this is the correct definition, and I also believe that the other definitions are wrong and misguided.*

(2) Unless $k = \mathbf{F}_2$, $\dim V = 4$, and $i(q) = 2$,

$$\mathrm{Pin}(q) = \left\{ \lambda v_1 \cdots v_a \in \mathrm{C}\ell(q)^\times : \lambda^2 \prod_{i=1}^{a} q(v_i) = 1 \right\} \quad \text{and}$$
$$\mathrm{Spin}(q) = \left\{ \lambda v_1 \cdots v_{2a} \in \mathrm{C}\ell(q)^\times : \lambda^2 \prod_{i=1}^{2a} q(v_i) = 1 \right\}.$$

This is sometimes used to define $\mathrm{Pin}(q)$ and $\mathrm{Spin}(q)$. Indeed, in light of the Cartan–Dieudonné Theorem this is a rather sensible approach. But, of course, it does give the wrong answer in the exceptional case of the Cartan–Dieudonné Theorem.

(3) References on spin geometry (cf. [LM89, (2.25)]) often use

$$\widetilde{\mathrm{Pin}}(q) \coloneqq \ker(N^2\colon \Gamma(q) \to (k^\times)^2) \quad \text{and} \quad \widetilde{\mathrm{Spin}}(q) \coloneqq \widetilde{\mathrm{Pin}}(q) \cap S\Gamma(q)$$

instead of Definition 44.2, sometimes in the guise of a variation of (2). The apparent advantage is that $\widetilde{\mathrm{Ad}}$ maps $\widetilde{\mathrm{Pin}}(q)$ and $\widetilde{\mathrm{Spin}}(q)$ onto $\mathrm{O}(q)$ and $\mathrm{SO}(q)$ respectively. This, however, can also be seen as a disadvantage because it prematurely gives up the freedom the twist over the non-identity component of $\mathrm{SO}_{r,s}$; see Dąbrowski [Dąb88] and Dąbrowski's $\mathrm{Pin}_{r,s}^{abc}$.

(4) Fortunately, in the setting relevant to Riemannian geometry (that is: $k = \mathbf{R}$ and negative definite q), none of the above makes any difference for $\mathrm{Spin}(q)$. However, neither $\widetilde{\mathrm{Pin}}(q)$ nor $\mathrm{Pin}(q)$ might be appropriate for physics; cf. Janssens [Jan20]. ♣

Proposition 44.6. *Let S be weakly faithful $\mathrm{C}\ell(q)$–module. Denote by $L(S)$ the Lipschitz group of S. The action of $\mathrm{C}\ell(q)^{0\times}$ on S induces a group homomorphism $\mathrm{Spin}(q) \to L(S)$ such that*

$$\begin{array}{ccc} \mathrm{Spin}(q) & \longrightarrow & L(s) \\ \downarrow{\scriptstyle \mathrm{Ad}} & & \downarrow{\scriptstyle \mathrm{Ad}} \\ S\Omega(q) & \hookrightarrow & \mathrm{O}(q) \end{array}$$

commutes. ■

45 The spin group of a perpendicular sum

Situation 45.1. Let k be a field. Let $q_i\colon V_i \to k$ $(i = 1, 2)$ be non-degenerate quadratic forms. ×

The following is important to establish the "2 out of 3 principle" for spin structures.

Proposition 45.2.

(1) *The isomorphism $\mathrm{C}\ell(q_1) \hat{\otimes} \mathrm{C}\ell(q_2) \cong \mathrm{C}\ell(q_1 \perp q_2)$ (constructed in Proposition 22.2) induces an inclusion $\iota\colon \mathrm{Spin}(q_1) \times_{\{\pm 1\}} \mathrm{Spin}(q_2) \hookrightarrow \mathrm{Spin}(q_1 \perp q_2)$.*

(2) *The canonical inclusion $\bar{\jmath}\colon \mathrm{O}(q_1)\times\mathrm{O}(q_2) \hookrightarrow \mathrm{O}(q_1 \perp q_2)$ restricts to an inclusion $\jmath\colon S\Omega(q_1)\times S\Omega(q_q) \hookrightarrow S\Omega(q_1 \perp q_2)$.*

(3) *The diagram*

$$\begin{array}{ccc} \mathrm{Spin}(q_1) \times_{\{\pm 1\}} \mathrm{Spin}(q_2) & \xhookrightarrow{\iota} & \mathrm{Spin}(q_1 \perp q_2) \\ \downarrow{\scriptstyle \mathrm{Ad}} & & \downarrow{\scriptstyle \mathrm{Ad}} \\ S\Omega(q_1) \times S\Omega(q_2) & \xhookrightarrow{\jmath} & S\Omega(q_1 \perp q_2). \end{array}$$

is a pullback diagram.

46 $\mathrm{Spin}_{r,s}$

Situation 46.1. Let $r, s \in \mathbf{N}_0$. Set $q_{r,s} := \langle 1\rangle^{\perp r} \perp \langle -1\rangle^{\perp s}$ defined over $\mathbf{R}$. Set $\mathrm{SO}^+_{r,s} := S\Omega(q_{r,s}) = \ker N \cap \mathrm{SO}(q_{r,s})$. ×

Definition 46.2. Set

$$\mathrm{Spin}_{r,s} := \mathrm{Spin}(q_{r,s}).$$ •

Notation 46.3. Abbreviate

$$\mathrm{Spin}_n = \mathrm{Spin}(n) = \mathrm{Spin}_{0,n}.$$ ∘

Proposition 46.4.

(1) $\mathrm{Spin}_{r,s} \subset (\mathrm{C}\ell^0_{r,s})^\times$ *is a closed Lie subgroup.*

(2) *The group homomorphism $\mathrm{Ad}\colon \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$ is a smooth $\{\pm \mathbf{1}\}$–principal covering map.*

(3) $\mathrm{Spin}_{r,s}$ *is connected if and only if* $(r,s) \notin \{(1,0),(0,1),(1,1)\}$.

Proof. (1) and (2) are obvious. To prove (3) observe the following:

(1) If $(r,s) \notin \{(1,0),(0,1),(1,1)\}$, then there are e_1, e_2 with $q(e_1) = q(e_2) = \pm 1$ and $p(e_1,e_2) = 0$. The path $\gamma \colon [0,\pi/2] \to \mathrm{Spin}_{r,s}$ defined by

$$\gamma(t) = (e_1 \cos(t) + e_2 \sin(t))(e_1 \cos(t) - e_2 \sin(t))$$

satisfies

$$\gamma(\pi/2) = e_1^2 = \pm 1 \quad \text{and} \quad \gamma(\pi/2) = -e_2^2 = \mp 1,$$

Therefore and since $\mathrm{SO}^+_{r,s}$ is connected, $\mathrm{Spin}_{r,s}$ is connected.

(2) $\mathrm{SO}^+_{1,0} = \mathrm{SO}^+_{0,1} = \{\mathbf{1}\}$ and

$$\mathrm{SO}^+_{1,1} = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} : t \in \mathbf{R} \right\} \cong \mathbf{R}.$$

Therefore, the covering map Ad must be trivial in these cases; hence: $\pi_0(\mathrm{Spin}_{r,s}) = \{\pm 1\}$. ■

Remark 46.5. The $\{\pm 1\}$–principal covering maps $\mathrm{Ad} \colon \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$ and $\mathrm{Ad} \colon \mathrm{Spin}_{s,r} \to \mathrm{SO}^+_{s,r} = \mathrm{SO}^+_{r,s}$ is isomorphic. ♣

Remark 46.6. If r, s are quite small, then $\mathrm{Spin}_{r,s}$ can often be identified with more familiar groups. For example:

(1) $\mathrm{Spin}_{0,3} \cong \mathrm{Sp}(1) \cong \mathrm{SU}(2)$.

(2) $\mathrm{Spin}_{1,2} \cong \mathrm{SL}_2(\mathbf{R})$.

(3) $\mathrm{Spin}_{0,4} \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$.

(4) $\mathrm{Spin}_{1,3} \cong \mathrm{SL}_2(\mathbf{C})$.

(5) $\mathrm{Spin}_{0,5} \cong \mathrm{Sp}(2)$.

(6) $\mathrm{Spin}_{0,6} \cong \mathrm{SU}(4)$.

Finding the above isomorphisms and the corresponding adjoint representations is an exercise. (It can be solved by reading [Har90, pp. 298–307]; see also `https://en.wikipedia.org/wiki/Exceptional_isomorphism`.) ♣

Remark 46.7. It also is a fun exercise to determine the

$$\mathrm{Ad}^{-1} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$ ♣

47 $\mathfrak{spin}_{r,s}$

Definition 47.1. Set

$$\mathfrak{spin}_{r,s} \coloneqq \mathrm{Lie}(\mathrm{Spin}_{r,s}). \qquad \bullet$$

Since $\mathrm{Spin}_{r,s} \subset (\mathrm{C}\ell^0_{r,s})^\times$ is a Lie subgroup, $\mathfrak{spin}_{r,s} \subset \mathrm{C}\ell^0_{r,s}$ with the Lie bracket agreeing with the commutator. Set $b = b_{r,s} \coloneqq \frac{1}{2}p_{r,s}$. Denote by $\kappa\colon \Lambda^2\mathbf{R}^{r+s} \to \mathrm{C}\ell^0_{r,s}$ the map induced by the quantisation map. Of course, with respect to the standard basis $e_1, \dots, e_{r+s}$,

$$\kappa(e_i \wedge e_j) = e_i e_j.$$

Identify $\Lambda^2\mathbf{R}^{r+s} = \mathfrak{so}_{r,s}$ via

$$(u \wedge v)w \coloneqq ub_{r,s}(v, x) - vb_{r,s}(u, x).$$

Proposition 47.2.

(1) $\mathfrak{spin}_{r,s}$ *agrees with the image of the quantisation map* $\kappa\colon \Lambda^2\mathbf{R}^{r+s} \to \mathrm{C}\ell^0_{r,s}$.

(2) $\mathrm{Lie}(\mathrm{Ad}) \circ \kappa(\alpha) = 2\alpha$ *for every* $\alpha \in \Lambda^2\mathbf{R}^{r+s} = \mathfrak{so}_{r,s}$.

Proof. Denote the standard basis of $\mathbf{R}^{r+s}$ by $e_1, \dots, e_{r+s}$. Define $x_{ij}\colon \mathbf{R} \to \mathrm{Spin}_{r,s}$ by

$$\begin{aligned} x_{ij}(t) &\coloneqq \begin{cases} (e_i\cos(t) - e_j\sin(t) \cdot (e_i\cos(t) + e_j\sin(t)) & \text{if } i < j \leqslant r \\ (e_i\cos(t) + e_j\sin(t)) \cdot (-e_i\cos(t) + e_j\sin(t)) & \text{if } r+1 \leqslant i < j \\ (\sinh(t)e_i - \cosh(t)e_2)(\sinh(t)e_i + \cosh(t)e_2) & \text{if } i \leqslant r < j \end{cases} \\ &= \begin{cases} \cos(2t) + \sin(2t)e_ie_j & \text{if } i < j \leqslant r \text{ or } r+1 \leqslant i < j \\ \cosh(2t) + \sinh(2t)e_ie_j & \text{if } i \leqslant r < j. \end{cases} \end{aligned}$$

Since $\dot{x}_{ij}(0) = e_ie_j = \kappa(e_i \wedge e_j)$, this proves (1).

To prove (2), observe that

$$[e_ie_j, v] = e_ie_jv - ve_ie_j = 2e_ib(e_j, v) - 2e_jb(e_j, v) = 2(e_i \wedge e_j)v. \qquad \blacksquare$$

Lecture 8

The first part of this lectures completes the algebraic discussion. The second part introduces the "axiomatic" approach to Dirac operators.

48 $\mathrm{Spin}^G_{r,s}$

Definition 48.1. Let $r, s \in \mathbf{N}_0$. Let G be a Lie group. Let $\varepsilon\colon \{\pm 1\} \to G$ be a group homomorphism. Set

$$\mathrm{Spin}^G_{r,s} := \mathrm{Spin}_{r,s} \times_{\{\pm 1\}} G. \qquad \bullet$$

Proposition 48.2. *The sequences*

$$1 \to G \to \mathrm{Spin}^G_{r,s} \xrightarrow{\mathrm{Ad}} \mathrm{SO}^+_{r,s} \to 0$$

and

$$0 \to \mathrm{Spin}_{r,s} \to \mathrm{Spin}^G_{r,s} \to (G/\{\pm 1\}) \to 0$$

are exact. ■

Remark 48.3. Of course, there is also a homomorphism $\mathrm{Spin}^G_{r,s} \to G/\{\pm 1\}$. ♣

49 $\mathrm{Spin}^{\mathrm{U}(1)}_{r,s}$

Remark 49.1. Possibly the most important instance is $G = \mathrm{U}(1)$ with $\varepsilon(-1) = -\mathbf{1}$. This group is often denoted by $\mathrm{Spin}^{\mathbf{C}}_{r,s}$ (or similarly). *A warning is in order:* $\mathrm{Spin}^{\mathbf{C}}_{r,s}$ is not the spin group for $q_{r,s}$ over $\mathbf{C}$, but it can be obtained from it using the real structure on $\mathbf{C}^{r+s}$; cf. [ABS64, p. 9]. ♣

The significance of $\mathrm{Spin}^{\mathrm{U}(1)}_{r,s}$ is that for $\mathbf{C} \otimes \mathrm{C}\ell_{r,s}$–modules it can take the role of the spin group.

Proposition 49.2. *Let S be a $\mathbf{C} \otimes \mathrm{C}\ell_{r,s}$–module which is weakly faithful as a $\mathrm{C}\ell_{r,s}$–module. Denote by $L(S)$ the Lipschitz group of S. The action of $\mathbf{C}^\times \times \mathrm{C}\ell^{0\times}_{r,s}$ on S induces a group homomorphism $\mathrm{Spin}^{\mathrm{U}(1)}_{r,s} \to L(S)$ such that*

$$\begin{array}{ccc} \mathrm{Spin}^{\mathrm{U}(1)}_{r,s} & \longrightarrow & L(S) \\ \downarrow{\scriptstyle \mathrm{Ad}} & & \downarrow{\scriptstyle \mathrm{Ad}} \\ \mathrm{SO}^+_{r,s} & \hookrightarrow & \mathrm{O}_{r,s} \end{array}$$

commutes. ■

$\mathrm{Spin}^{\mathrm{U}(1)}_n$ also interacts well with $\mathrm{U}(n)$.

Proposition 49.3 (Atiyah, Bott, and Shapiro [ABS64, pp. 10, 13, 14]). *Let $n \in \mathbf{N}_0$.*

(1) *The map $\rho\colon \mathrm{U}(n) \to \mathrm{SO}(2n)$ does not lift to* Spin_{2n}.

(2) *The map* $\rho \times \det \colon \mathrm{U}(n) \to \mathrm{SO}(2n) \times \mathrm{U}(1)$ *lifts to* $\mathrm{Spin}_{2n}^{\mathrm{U}(1)}$*; that is,*

$$\begin{array}{ccc} & & \mathrm{Spin}_{2n}^{\mathrm{U}(1)} \\ & \nearrow & \big\downarrow \widetilde{\mathrm{Ad}} \times (-)^2 \\ \mathrm{U}(n) & \xrightarrow[\rho \times \det]{} & \mathrm{SO}(2n) \times \mathrm{U}(1). \end{array}$$

(3) *The complex pinor module* P *be can be identified with* $\Lambda_{\mathbf{C}}(\mathbf{C}^n)^*$ *such that the lift* $\mathrm{U}(n) \to \mathrm{Spin}^c(n)$ *makes the following diagram commutative:*

$$\begin{array}{ccc} \mathrm{U}(n) & \longrightarrow & \mathrm{Spin}_n^c \\ \big\downarrow & & \big\downarrow \\ \mathrm{End}_{\mathbf{C}}(\mathbf{C}^n) & \xrightarrow{(-)^* \circ \Lambda} & \mathrm{End}_{\mathbf{C}}(\Lambda_{\mathbf{C}}(\mathbf{C}^n)^*). \end{array}$$

The Clifford multiplication on $\Lambda_{\mathbf{C}}(\mathbf{C}^n)^*$ *is given by*

$$\gamma(v)\alpha = v^* \wedge \alpha - i(v)\alpha.$$

Proof. (1) is a consequence of the fact that $\pi_1(\rho) \colon \pi_1(\mathrm{U}(n)) \to \pi_1(\mathrm{SO}(2n))$ is surjective, but $\pi_1(\widetilde{\mathrm{Ad}}) \colon \pi_1(\mathrm{Spin}(2n)) \to \pi_1(\mathrm{SO}(2n))$ is not.

(2) is proved by constructing the lift explicitly. Given $f \in \mathrm{U}(n)$, chose a unitary basis $(e_1, \dots, e_n)$ of $\mathbf{C}^n$ in which f is diagonal; that is: $f = \mathrm{diag}(e^{i\alpha_1}, \cdots, e^{i\alpha_n})$. An orthonormal basis of the $2n$–dimensional real Euclidean space $\mathbf{C}^n$ is given by $(e_1, ie_1, \dots, e_n, ie_n)$. Define $\tilde{f} \in \mathrm{Spin}^c(2n)$ by

$$\tilde{f} := \prod_{j=1}^{n} [(\cos(\alpha_j/2) + \sin(\alpha_j/2) e_j(ie_j)), e^{\frac{i}{2}\alpha_j}].$$

Observe that $\alpha_j \in \mathbf{R}/2\pi\mathbf{Z}$, so $\alpha_j/2 \in \mathbf{R}/\pi\mathbf{Z}$. Consequently, the both factors individually are only defined up to a sign. Their product, however, is well-defined. Clearly, $\left(\prod_{j=1}^n e^{\frac{i}{2}\alpha_j}\right)^2 = \det(f)$. The fact that $\rho(f) = \widetilde{\mathrm{Ad}}(\tilde{f})$ follows from following observation.

Proposition 49.4. *Let* (e_1, e_2) *be an orthonormal basis of* $\mathbf{R}^2$ *and let* $\alpha \in \mathbf{R}$*. We have*

$$\widetilde{\mathrm{Ad}}(\cos(\alpha/2) + \sin(\alpha/2) e_1 e_2) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Proof. Since

$$\alpha \left(\cos(\alpha/2) + \sin(\alpha/2) e_1 e_2\right)^{-1} = \cos(\alpha/2) - \sin(\alpha/2) e_1 e_2,$$

we have

$$\begin{aligned} \widetilde{\mathrm{Ad}}\left(\cos(\alpha/2) + \sin(\alpha/2) e_1 e_2\right) e_i &= \left(\cos(\alpha/2) + \sin(\alpha/2) e_1 e_2\right)^2 e_i \\ &= \big(\cos(\alpha/2)^2 - \sin(\alpha/2)^2 \\ &\qquad + 2\cos(\alpha/2)\sin(\alpha/2) e_1 e_2 \big) e_i \\ &= \left(\cos(\alpha) + \sin(\alpha) e_1 e_2\right) e_i. \end{aligned}$$

From this the assertion follows directly. ∎

The formula for the Clifford multiplication defines how $\mathrm{Spin}^c(2n)$ acts on $\Lambda_{\mathbf{C}}(\mathbf{C}^n)^*$. Proving (3) is a matter of a calculation using the explicit formula for the lift constructed above. ■

50 Dąbrowski's $\mathrm{Pin}_{r,s}^{abc}$

Situation 50.1. Let $r, s \in \mathbf{N}_0$. ×

Since $\mathrm{Spin}_{r,s}$ only covers $\mathrm{SO}^+_{r,s}$, it is interesting to determine which double covers $\mathrm{Pin}^\star_{r,s} \to \mathrm{O}_{r,s}$ fit in a commutative diagram of the form

$$\begin{array}{ccccc} \{\pm 1\} & = & \{\pm 1\} & & \\ \downarrow & & \downarrow & & \\ \mathrm{Spin}_{r,s} & \longrightarrow & \mathrm{Pin}^\star_{r,s} & & \\ \downarrow{\scriptstyle \mathrm{Ad}} & & \downarrow{\scriptstyle \pi} & & \\ \mathrm{SO}^+_{r,s} & \hookrightarrow & \mathrm{O}_{r,s} & \twoheadrightarrow & \pi_0(\mathrm{O}_{r,s}). \end{array}$$

The discussion (mildly) depends on r, s since

$$\Delta \cong \begin{cases} 1 & \text{if } r = s = 0 \\ \{\pm 1\} & \text{if } (r \geqslant 1, s = 0) \text{ or } (r = 0, s \geqslant 1) \\ \{\pm 1\} \times \{\pm 1\} & \text{if } r \geqslant 1, s \geqslant 1. \end{cases}$$

The case $r = s = 0$ is trivial. In anycase, $\pi_0(\mathrm{O}_{r,s})$ lifts to a subgroup $\Delta < \mathrm{O}_{r,s}$. Conjugation defines a homomorphism $\Delta \to \mathrm{Aut}(\mathrm{SO}^+_{r,s})$ and

$$\mathrm{O}_{r,s} \cong \tilde{\Delta} \ltimes \mathrm{SO}^+_{r,s}.$$

Set

$$\tilde{\Delta} := \pi^{-1}(\Delta) < \mathrm{Pin}^\star_{r,s}.$$

Evidently, conjugation defines a homomorphism $\tilde{\Delta} \to \mathrm{Aut}(\mathrm{SO}^+_{r,s})$, and

$$\mathrm{Pin}^\star_{r,s} \cong \frac{\tilde{\Delta} \ltimes \mathrm{Spin}_{r,s}}{\{\pm 1\}}$$

By covering theory, $\tilde{\Delta} \to \mathrm{Aut}(\mathrm{Spin}_{r,s})$ is the unique lift of $\tilde{\Delta} \to \Delta \to \mathrm{Aut}(\mathrm{SO}^+_{r,s})$. Therefore, $\mathrm{Pin}^\star_{r,s}$ is determined by $\pi\colon \tilde{\Delta} \to \Delta$. Conversely, every $\pi\colon \tilde{\Delta} \to \Delta$ with $\ker \pi \cong \{\pm \mathbf{1}\}$ defines a $\mathrm{Pin}^\star_{r,s}$.

The above can be made more concrete because the homomorphisms π can understood as follows. For every $T \in \Delta$, $T^2 = \mathbf{1}$ for every $T \in \Delta$. Therefore, if $\tilde{T} \in \pi^{-1}(T)$, then $\tilde{T}^2 \in \ker \pi = \{\pm \mathbf{1}\}$; moreover, $\tilde{T}^2$ depends only on T, and $\tilde{\mathbf{1}}^2 = \mathbf{1}$.

(1) The case $r = s = 0$ is trivial.

(2) If either $r = 0$ or $s = 0$, then $\Delta = \langle T \rangle < \mathrm{O}_{r,s}$ with $T \coloneqq -\mathbf{1}_1 \oplus \mathbf{1}_{r+s_1}$. Set

$$\tilde{\Delta}^{\pm} \coloneqq \left\langle -\mathbf{1}, \tilde{T} \,\middle|\, (-\mathbf{1})^2 = \mathbf{1}, \tilde{T}^2 = \pm\mathbf{1}, (-\mathbf{1})T = T(-\mathbf{1}) \right\rangle$$

and, define $\pi^{\pm} \colon \tilde{\Delta}^{\pm} \to \Delta$ by $\pi(-\mathbf{1}) \coloneqq \mathbf{1}$ and $\pi(\tilde{T}) = T$. More explicitly:

(a) $\tilde{\Delta}^{+} = \{\pm 1\}^2$ with $-\mathbf{1} = (-1, 1)$ and $\tilde{T} = (1, -1)$.

(b) $\tilde{\Delta}^{-} = \{\pm 1, \pm i\} < \mathbf{C}^{\times}$ with $-\mathbf{1} = -1$ and $\tilde{T} = i$.

These gives rise to $\mathrm{Pin}^{\pm}_{r,s}$.

(3) If either $r, s \geqslant 1$, then $\Delta = \langle T_1, T_2 \rangle^2 < \mathrm{O}_{r,s}$ with $T_1 \coloneqq -\mathbf{1}_1 \oplus \mathbf{1}_{r+s-1}$ and $T_2 \coloneqq \mathbf{1}_{r+s-1} \oplus -\mathbf{1}_1$, For $a, b, c \in \{\pm\}$ set

$$\tilde{\Delta}^{abc} \coloneqq \left\langle -\mathbf{1}, \tilde{T}_1, \tilde{T}_2 \,\middle|\, R \right\rangle$$

with R denoting the following relations

$$(-\mathbf{1})^2 = \mathbf{1}, \tilde{T}_1^2 = a\mathbf{1}, \tilde{T}_2^2 = b\mathbf{1}, (\tilde{T}_1\tilde{T}_2)^2 = c\mathbf{1}, (-\mathbf{1})\tilde{T}_1 = \tilde{T}_1(-\mathbf{1}), (-\mathbf{1})\tilde{T}_2 = \tilde{T}_2(-\mathbf{1}),$$

and define $\pi \colon \tilde{\Delta}^{abc} \to \Delta$ by $\pi^{abc}(-\mathbf{1}) \coloneqq \mathbf{1}$ and $\pi^{abc}(\tilde{T}_i) \coloneqq T_i$. With $D_4 \coloneqq \langle R, S \mid R^2 = S^4 = RSRS = \mathbf{1} \rangle$ denoting the dihedral group of order 8 these above are more explicitly given as:

(a) $\tilde{\Delta}^{+++} = \{\pm 1\}^3$ with $-\mathbf{1} = (-1, 1, 1)$, $\tilde{T}_1 = (1, -1, 1)$, $\tilde{T}_2 = (1, 1 - 1)$.

(b) $\tilde{\Delta}^{++-} = D_4$ with $-\mathbf{1} = S^2$, $\tilde{T}_1 = R$, $\tilde{T}_2 = RS$.

(c) $\tilde{\Delta}^{+-+} = D_4$ with $-\mathbf{1} = S^2$, $\tilde{T}_1 = R$, and $\tilde{T}_2 = S$.

(d) $\tilde{\Delta}^{-++} = D_4$ with $-\mathbf{1} = S^2$, $\tilde{T}_1 = S$, $\tilde{T}_2 = R$.

(e) $\tilde{\Delta}^{+--} = \{\pm 1\} \times \{\pm 1, \pm i\}$ with $-\mathbf{1} = (1, -1)$, $\tilde{T}_1 = (1, -1)$, $\tilde{T}_2 = (1, i)$

(f) $\tilde{\Delta}^{-+-} = \{\pm 1\} \times \{\pm 1, \pm i\}$ with $-\mathbf{1} = (1, -1)$, $\tilde{T}_1 = (1, i)$, $\tilde{T}_2 = (1, -1)$.

(g) $\tilde{\Delta}^{--+} = \{\pm 1\} \times \{\pm 1, \pm i\}$ with $-\mathbf{1} = (1, -1)$, $\tilde{T}_1 = (1, i)$, $\tilde{T}_2 = (-1, i)$.

(h) $\tilde{\Delta}^{---} = Q_8 = \langle \pm 1, \pm i, \pm j, \pm k \rangle < \mathbf{H}^{\times}$ with $-\mathbf{1} = -1$, $\tilde{T}_1 = i$, $\tilde{T}_2 = j$.

These gives rise to $\mathrm{Pin}^{\pm}_{r,s}$; see Dąbrowski [Dąb88, p. 11].

Remark 50.2. Of course, there are up to 32 non-isomorphic double covers of $\mathrm{O}_{r,s}$. These have been studied by Trautman [Tra01]. ♣

51 Bilinear forms on pinor modules

`I should have probably said the following in an earlier lecture.`

The (complex) (s)pinor modules P (defined in Section 31, Section 34, Section 32, and Section 36) admit non-degenerate bilinear forms $b \in \mathrm{Hom}(P \otimes P, \mathbf{R})$ with respect to which the γv is skew-adjoint for every $v \in \mathbf{R}^{r+s}$. The construction of these b is quite formidable and discussed in [Har90, §13]. If $r = 0$ or $s = 0$, then b can be assumed to be positive definite.

52 Clifford algebra bundles

Situation 52.1. Let $r, s \in \mathbf{N}_0$. Let X be a smooth manifold. Let $\pi\colon V \to X$ be a vector bundle of rank $r+s$. Let $q \in \Gamma(\mathrm{Hom}(S^2V, \mathbf{R}))$ be a symmetric bilinear form on V of signature $r-s$. Denote the $\mathrm{O}_{r,s}$ frame bundle of (V, q) by $(\rho\colon \mathrm{Fr}(V, q) \to X, R)$. Identify, $V = \mathrm{Fr}(V, q) \times_{\mathrm{O}_{r,s}} \mathbf{R}^{r+s}$. ×

Definition 52.2. The **Clifford algebra bundle** associated with q is the algebra bundle

$$\mathrm{C}\ell(q) := \mathrm{Fr}(V, q) \times_{\mathrm{O}_{r,s}} \mathrm{C}\ell_{r,s}.$$

Denote by $\gamma\colon V \to \mathrm{C}\ell(q)$ the canonical bundle map. •

Remark 52.3. The Serre–Swan Theorem identifies vector bundles V over X (more precisely: their spaces of sections $\Gamma(V)$) with finitely-generated projective modules over the ring $R := C^\infty(X)$. The symmetric bilinear form is nothing but a quadratic form $q\colon \Gamma(V) \to R$. Clifford algebras can be defined over arbitrary rings. An the above construction can be carried out in this framework. I do not know if this is useful for anything. ♣

Example 52.4. The most important example is $V = TX$ and $g := -g$ with g a pseudo-Riemannian metric. ♠

Proposition 52.5. *For every orthogonal corvariant derivative* $\nabla\colon \Gamma(V) \to \Omega^1(X, V)$ *there is a unique covariant derivative* $\nabla_{\mathrm{C}\ell}\colon \Gamma(\mathrm{C}\ell(V)) \to \Omega^1(X, \mathrm{C}\ell(V))$ *such that*

$$\nabla_{\mathrm{C}\ell}\gamma(v) = \gamma(\nabla v) \quad \textit{and} \quad \nabla_{\mathrm{C}\ell}(xy) = (\nabla_{\mathrm{C}\ell}x)y + x(\nabla_{\mathrm{C}\ell}y).$$

■

53 Clifford module bundles

Situation 53.1. Let (X, g) be a pseudo-Riemannian manifold. ×

The following concept comes from Atiyah, Bott, and Shapiro [ABS64].

Definition 53.2. A **Clifford module bundle** over (X, g) is a vector bundle $\pi\colon S \to X$ together with a smooth map of algebra bundles $\mathrm{C}\ell(-g) \to \mathrm{End}(S)$. If S is a Clifford module bundle, then the induced map $\gamma\colon TX \to \mathrm{End}(S)$ is called the **Clifford multiplication**. •

Remark 53.3. *The minus sign is not a mistake*, but might appear somewhat awkward. A typical “solution” is to push the minus sign into the definition of the Clifford algebra. Another “solution” is to keep in mind that (in an not unreasonable convention) the symbol of the differential operator Δ is $-g$. ♣

Example 53.4. $\mathrm{C}\ell(-g)$ is a Clifford module bundle. ♠

Example 53.5. The bundle of exterior algebras

$$S := \Lambda T^*X = \bigoplus_{r=0}^{n} \Lambda^r T^*X$$

is a Clifford module bundle with

$$\gamma(v)\alpha := v^\flat \wedge \alpha - i_v\alpha.$$ ♠

54 Dirac bundles

Situation 54.1. Let (X, g) be a pseudo-Riemannian manifold. ×

Definition 54.2. A **Dirac bundle** over (X, g) consists of:

(1) a Clifford module bundle (S, γ),

(2) a non-degenerate bilinear form $b \in \mathrm{Hom}(S \otimes S, \mathbf{R})$, and

(3) a covariant derivative $\nabla = \nabla^S \colon \Gamma(S) \to \Omega^1(X, S)$

such that:

(4) $\nabla^S(\gamma(v)\phi) = \gamma(\nabla^{\mathrm{LC}} v)\phi + \gamma(v)\nabla^S\phi$,

(5) $b(\gamma(v)\phi, \psi) + b(\phi, \gamma(v)\psi) = 0$, and

(6) $\mathrm{d}b(\phi, \psi) = b(\nabla^S\phi, \psi) + b(\phi, \nabla^S\psi)$.

A **complex Dirac bundle** is a Dirac bundle together with an almost complex structure i which commutes with γ and is ∇^S–parallel. A **quaternionic Dirac bundle** is a Dirac bundle together with an almost complex structures i, j, k which commute with γ, are ∇^S–parallel, and satisfy $ij = -ji = k$. •

Remark 54.3. This deviates from [LM89, Part II Definition 5.2] in that b is not required to be a Euclidean inner product; that is: symmetric and positive definite. This is necessary because (in indefinite signature) the pinor modules do not always have $\mathrm{Spin}_{r,s}$–invariant Euclidean inner products. ♣

Once a Dirac bundle has been found, further examples can be obtained by twisting.

Definition 54.4. Let (S, γ, b, ∇_S) be a Dirac bundle. Let (E, h, ∇_E) be a vector bundle together with a bilinear form h and a covariant derivative ∇_E with $\nabla h = 0$. The twist of (S, γ, b, ∇_S) by (E, h, ∇_E) is

$$(S \otimes E, \gamma \otimes \mathrm{id}_E, b \otimes h, \nabla_S \otimes \nabla_E). \qquad \bullet.$$

For index theory, the following concept is fundamental.

Definition 54.5. A **grading** of a Dirac bundle (S, γ, b, ∇_S) is an $\varepsilon \in \Gamma(\mathrm{End}(S))$ such that:

(1) $\varepsilon^2 = \mathrm{id}$,

(2) $\gamma\varepsilon + \varepsilon\gamma = 0$,

(3) $b(\varepsilon\phi, \varepsilon\psi) = 0$, and

(4) $\nabla\varepsilon = 0$. •

Example 54.6. ΛT^*X has an obvious grading (leading to the Euler characteristic), but its complexification $\Lambda T^*X \otimes \mathbf{C}$ has another grading (leading to the signature); see Section 9. ♠

~

I didn't do so in the lectures, but I think (in hindsight) it would have been good to introduce the following concept.

Definition 54.7. A **pre-Dirac bundle** over (X, g) is a Clifford module together with a (non-degenerate) bilinear form $b \in \mathrm{Hom}(S \otimes S, \mathbf{R})$ with respect to which $\gamma(v)$ is skew-adjoint for every $v \in TX$. Set

$$\mathfrak{o}_{C\ell}(S) := \{T \in \mathrm{End}(S) : [\gamma, T] = 0 \quad \text{and} \quad b(T\cdot, \cdot) = b(\cdot, T\cdot)\}. \qquad \bullet$$

Proposition 54.8. *Let (S, γ, b) be a pre-Dirac bundle. The space $\mathscr{A}(S, \gamma, b)$ of covariant derivatives ∇ on S with respect to which γ and b are parallel is an affine space modelled on $\Omega^1(X, \mathfrak{o}_{C\ell}(S))$.* ■

The significance of this concept is it isolates the "discrete", "topological", "algebraic" data of a Dirac bundle from the "continous", "analytical" part.

55 Dirac operators

Situation 55.1. Let (X, g) be a pseudo-Riemannian manifold of signature (r, s). Let (S, γ, b, ∇_S) be a Dirac bundle over (X, g). ×

Definition 55.2. The **Dirac operator** $D\colon \Gamma(S) \to \Gamma(S)$ associated with (S, γ, ∇_S, b) is the composition

$$\Gamma(S) \xrightarrow{\nabla_S} \Gamma(T^*X \otimes S) \xrightarrow{\sharp \otimes \mathrm{id}_S} \Gamma(TX \otimes S) \xrightarrow{\gamma} \Gamma(S).$$

Here the isomorphism $\sharp\colon T^*X \to TX$ defined by $g(\sharp\alpha, v) := \alpha(v)$. •

Remark 55.3. Suppose that $\dim = r + s$ and g has signature $r - s$. If $e_1, \ldots, e_{r+s}$ is a (local) orthonormal frame then

$$D\phi = \sum_{i=1}^{r+s} \varepsilon_i \gamma(e_i) \nabla_{e_i} \phi \quad \text{with} \quad \varepsilon_i := g(e_i, e_i) \in \{\pm 1\}.$$

It is not difficult to see that this expression does not depend on the choice of $e_1, \ldots, e_{r+s}$. The signs ε_i are crucial! ♣

Example 55.4. $S = \Lambda TM$ with its natural Euclidean metric and covariant derivative is a Dirac bundle. The corresponding Dirac operator is

$$D = \mathrm{d} + \mathrm{d}^*\colon \Lambda TM \to \Lambda TM. \qquad ♠$$

Proposition 55.5. *For every $\phi, \psi \in \Gamma(S)$*

$$b(D\phi, \psi) - b(\phi, D\psi) = \operatorname{div} v \quad \textit{with} \quad \langle v, \cdot \rangle := b(\gamma(\cdot)\phi, \psi).$$

In particular, if X is closed, then

$$\int_X b(D\phi, \psi)\, \mathrm{vol}_g = \int_X b(\phi, D\psi)\, \mathrm{vol}_g.$$

Proof. This is proved by direct computation. ∎

Remark 55.6. If ε is a grading, then S be decomposed into the ± 1–eigenbundles $S^{\pm}$ of ε and the Dirac operator decomposes accordingly as

$$D = \begin{pmatrix} 0 & D^{-} \\ D^{+} & 0 \end{pmatrix}.$$

♣

Lecture 9

This lecture discusses two fundamental facts about Dirac operators: the Weitzenböck formula and the conformal invariance. `For some reason, this lecture went really slow and I didn't manage to get to the existence and classification of spin structures.`

56 The Weitzenböck formula

Situation 56.1. Let (X, g) be a pseudo-Riemannian manifold of signature (r, s). Let (S, γ, b, ∇) be a Dirac bundle over (X, g). Denote by D the associated Dirac operator. ×

Definition 56.2. Define $\mathscr{F}_\nabla \in \Gamma(\mathrm{End}(S))$ by

$$\mathscr{F}_\nabla := \frac{1}{2} \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j \gamma(e_i)\gamma(e_j) F_\nabla(e_i, e_j).$$ •

Proposition 56.3 (Weitzenböck formula for Dirac bundles). *D satisfies the* ***Weitzenböck formula***

$$D^2 = \nabla^*\nabla + \mathscr{F}_\nabla.$$

Proof. Let $x \in X$. Pick a local orthonormal frame $(e_1, \ldots, e_{r+s})$ defined in a neighborhood of x with $(\nabla e_i)_x = 0$. By direct computation at $x \in X$,

$$\begin{aligned}\sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j \gamma(e_i)\nabla_{S,e_i}\gamma(e_j)\nabla_{S,e_j} &= \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j \gamma(e_i)\gamma(e_j)\nabla_{S,e_i}\nabla_{S,e_j} \\ &= -\sum_{i=1}^{n} \nabla_{S,e_i}\nabla_{S,e_i} + \sum_{i<j}^{n} \varepsilon_i \varepsilon_j \gamma(e_i)\gamma(e_j)[\nabla_{S,e_i}, \nabla_{S,e_j}] \\ &= \nabla^*\nabla + \sum_{i<j}^{n} \varepsilon_i \varepsilon_j \gamma(e_i)\gamma(e_j) F_s(e_i, e_j).\end{aligned}$$ ■

Remark 56.4. According to Proposition 56.3, the Dirac operator still is almost a square root of the Laplace operator. ♣

Remark 56.5. The Weitzenböck formula is at the heart of vanishing and estimating theorems established using the Bochner technique. Most of these applications require further information on $\mathscr{F}_\nabla$. ♣

57 The curvature of a Dirac bundle

Situation 57.1. Let (X, g) be a pseudo-Riemannian manifold of signature (r, s). Let (S, γ, b, ∇) be a Dirac bundle over (X, g) ×

The following innocent looking lemma is quite important.

Lemma 57.2. *Let S be a $\mathrm{C}\ell_{r,s}$–module together with a (non-degenerate) bilinear form with respect which $\gamma(v)$ is skew-adjoint for every $v \in \mathbf{R}^{r+s}$. The map $\cdot_S \colon \mathfrak{o}_{r,s} \to \mathfrak{o}(S)$ defined by*

$$A_S := \frac{1}{4}\sum_{i=1}^{r+s} \varepsilon_i [\gamma(e_i), \gamma(Ae_i)]$$

satisfies

$$[A_S, \gamma(v)] = \gamma(Av)$$

for every $v \in \mathbf{R}^{r+s}$.

Proof. Pick a local orthonormal frame $(e_1, \dots, e_{r+s})$. By direct inspection

$$[\gamma(e_i)\gamma(e_j), \gamma(e_k)] = -2\gamma(e_i)\varepsilon_k\delta_{jk} + 2\varepsilon_k\gamma(e_j)\delta_{ik} = -2\gamma(e_i)g(e_j, e_k) + 2\gamma(e_j)g(e_i, e_k).$$

Therefore,

$$\frac{1}{2}[[\gamma(u), \gamma(v)], \gamma(w)] = -\gamma(u)g(v, w) + \gamma(v)g(u, w).$$

By direct computation

$$[A_S, \gamma(v)] = \sum_{i=1}^{r+s} \varepsilon_i\gamma(Ae_i)g(e_i, v) = \gamma(Av)$$ ■

Proposition 57.3. *For every $x \in X$ and $u, v, w \in T_xX$*

$$[F_\nabla(u, v), \gamma(w)] = \gamma(R(u, v)w).$$

Definition 57.4. Let (S, γ, b, ∇) be a Dirac bundle. Define the **twisting curvature** $F_\nabla^{\mathrm{tw}} \in \Omega^2(X, \mathfrak{o}_{\mathrm{C}\ell}(S))$ by

$$F_\nabla^{\mathrm{tw}} := F_\nabla - R_S.$$ •

Remark 57.5. The above is useful because $\mathfrak{o}_{\mathrm{C}\ell}(S)$ is usually rather small and therefore F_∇^{tw} is quite restricted. ♣

Remark 57.6. F_∇^{tw} plays an important role in the Atiyah–Singer Index Theorem. ♣

58 The refined Weitzenböck formula

Situation 58.1. Let (X, g) be a pseudo-Riemannian manifold of signature (r, s). Let (S, γ, b, ∇) be a Dirac bundle over (X, g). Denote by D the associated Dirac operator. ×

Proposition 58.2 (refined Weitzenböck formula for Dirac Bundles). *With*

$$\mathscr{F}_\nabla^{\mathrm{tw}} := \frac{1}{2}\sum_{i,j=1}^{r+s} \varepsilon_i\varepsilon_j\gamma(e_i)\gamma(e_j)F_\nabla^{\mathrm{tw}}(e_i, e_j)$$

and scal_g *denoting the scalar curvature of* g

$$D^2 = \nabla^*\nabla + \frac{1}{4}\mathrm{scal}_g + \mathscr{F}_\nabla^{\mathrm{tw}}.$$

Remark 58.3. Proposition 58.2 exhibits the role of scalar curvature for Dirac operators as the universal term in the Weitzenböck formula. The fact that F_S^{tw} is $C\ell(-g)$–linear often restricts it severely and facilitates the computation of $\mathscr{F}_S^{\mathrm{tw}}$. ♣

Remark 58.4. cf. [LM89, Theorem 8.17] where this is proved for $S \otimes E$. ♣

Remark 58.5. Schrödinger [Sch32, pp. 126–128] first computed D^2 and observed the appearance of $\frac{1}{4}\mathrm{scal}_g$. ♣

The proof relies on the following computation.

Proposition 58.6. *If $e_1, \ldots, e_{r+s}$ is an orthonormal basis, then*

$$\sum_{i,j,\ell=1}^{n} \varepsilon_\ell \varepsilon_i \varepsilon_j \gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k, e_\ell)e_i, e_j\rangle = -2\sum_{i=1}^{n} \varepsilon_i \gamma(e_i)\mathrm{Ric}(e_k, e_i) \quad \textit{and}$$

$$\sum_{i,j,k,\ell=1}^{n} \varepsilon_k \varepsilon_\ell \varepsilon_i \varepsilon_j \gamma(e_k)\gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k, e_\ell)e_i, e_j\rangle = 2\mathrm{scal}_g.$$

Proof. The first identity implies the second directly. To prove the first identity, observe the following:

(1) If i, j, ℓ are pairwise distinct, then

$$\gamma(e_\ell)\gamma(e_i)\gamma(e_j) = \gamma(e_i)\gamma(e_j)\gamma(e_\ell) = \gamma(e_j)\gamma(e_\ell)\gamma(e_i).$$

By the algebraic Bianchi identity

$$\langle R(e_k, e_\ell)e_i, e_j\rangle + \langle R(e_k, e_i)e_j, e_\ell\rangle + \langle R(e_k, e_j)e_\ell, e_i\rangle = 0.$$

Therefore, the sum of terms with i, j, ℓ pairwise distinct appearing the the left-hand side vanishes.

(2) The terms with $i = j$ vanish because $R(e_k, e_\ell)$ is skew-symmetric.

(3) If $i \neq j = \ell$, then

$$\begin{aligned}\varepsilon_\ell \varepsilon_i \varepsilon_j \gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k, e_\ell)e_i, e_j\rangle &= \varepsilon_i \gamma(e_i)\langle R(e_k, e_j)e_i, e_j\rangle \\ &= -\varepsilon_i \gamma(e_i)\langle R(e_j, e_k)e_i, e_j\rangle.\end{aligned}$$

Therefore, the sum of these expressions contributes

$$-\sum_{i=1}^{n} \varepsilon_i \gamma(e_i)\mathrm{Ric}(e_k, e_i).$$

(4) If $j \neq i = \ell$, then

$$\varepsilon_\ell \varepsilon_i \varepsilon_j \gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k, e_\ell)e_i, e_j\rangle = -\varepsilon_j \gamma(e_j)\langle R(e_i, e_k)e_j, e_i\rangle$$

Therefore, the sum of these expressions also contributes

$$-\sum_{i=1}^{n} \varepsilon_i \gamma(e_i)\mathrm{Ric}(e_k, e_i).$$ ■

Proof of Proposition 58.2. By the above,

$$\frac{1}{2}\sum_{i,j=1}^{n}\varepsilon_i\varepsilon_j\gamma(e_i)\gamma(e_j)R_S(e_i,e_j)=\frac{1}{4}\mathrm{scal}_g.$$

This finishes the proof. ■

59 Bochner technique

This is really a baby version: the vanishing version. More interesting applications use the estimating technique; see https://walpu.ski/Teaching/RiemannianGeometry.pdf.

Proposition 59.1 (Bochner). *If X is compact and $\mathscr{F}_S$ is non-negative definite (that is: $\langle\mathscr{F}_S\Phi,\Phi\rangle\geqslant 0$), then $D\Phi=0$ implies $\nabla_S\Phi=0$. Moreover, $\mathscr{F}_S$ is positive definite somewhere, then $\Phi=0$.*

Proof. If $D\Phi=0$, then we have

$$\int_M|\nabla\Phi|^2+\langle\mathscr{F}_S\Phi,\Phi\rangle=0.$$ ■

60 Conformal invariance of Dirac operators

Proposition 60.1. *Let (X,g) be pseudo–Riemannian manifold. Let $f\in C^\infty(X)$. Set $\tilde g\colon e^{2f}g$. The Levi-Civita connections ∇ and $\tilde\nabla$ of g and $\tilde g$ are related by*

$$\tilde\nabla_v w=\nabla_v w+(\mathscr{L}_v f)w+(\mathscr{L}_w f)v-g(v,w)\nabla f.$$

Proposition 60.2 (Hitchin [Hit74, §1.4]). *Let (X,g) be a pseudo-Riemannian manifold. Let $f\in C^\infty(X)$. Set $\tilde g\coloneqq e^{2f}g$. Let (S,γ,b,∇) be a Dirac bundle over (X,g).*

(1) *Set*

$$\tilde\gamma\coloneqq e^f\gamma.\quad\textit{and}\quad\tilde\nabla_v\coloneqq\nabla_v+\frac{1}{4}[\gamma(\nabla f),\gamma(v)]$$

$(S,\tilde\gamma,b,\tilde\nabla)$ is a Dirac bundle over $(X,\tilde g)$.

(2) *The Dirac operator D associated with (S,γ,b,∇) and the Dirac operator $\tilde D$ associated with $(S,\tilde\gamma,b,\tilde\nabla)$ satisfy*

$$\tilde D=e^{-\frac{n+1}{2}f}De^{\frac{n-1}{2}f}.$$

Proof. Evidently, $\tilde\gamma(v)^2=e^{2f}g(v,v)$. Moreover, $\frac{1}{4}[\gamma(\nabla f),\gamma(v)]$ is b–skew-adjoint. Therefore, b is $\tilde\nabla$–parallel. It remains to verify that $\tilde\gamma$ is $\tilde\nabla$–parallel; that is:

$$\tilde\nabla_v(\tilde\gamma(w)\phi)-\tilde\gamma(\tilde\nabla_v w)\phi-\tilde\gamma(w)\tilde\nabla_v\phi=0.$$

Since the Levi–Civita connection $\tilde{\nabla}$ of $\tilde{g}$ satisfies

$$\tilde{\nabla}_v w = \nabla_v w + (\mathscr{L}_v f)w + (\mathscr{L}_w f)v - g(v,w)\nabla f,$$

this amounts to verifying that

$$e^f\left(\frac{1}{4}[[\gamma(\nabla f),\gamma(v)],\gamma(w)]\phi - g(\nabla f, w)\gamma(v) + g(v,w)\gamma(\nabla f)\right)\phi = 0.$$

The latter is a consequence of the following lemma. This proves (1)

If $e_1,\dots,e_n$ is g–orthonormal, then $\tilde{e}_1 := e^{-f}e_1,\dots,e_n := e^{-f}e_n$ is $\tilde{g}$–orthonormal. Since $\varepsilon_i\gamma(e_i)[\gamma(e_j),\gamma(e_j)] = 2(1-\delta_{ij})\gamma(e_j)$,

$$\begin{aligned}\tilde{D} &= \sum_{i=1}^n \varepsilon_i\tilde{\gamma}(\tilde{e}_i)\tilde{\nabla}_{\tilde{e}_i}\\ &= e^{-f}\sum_{i=1}^n \varepsilon_i\gamma(e_i)\left(\nabla_{e_i} + \frac{1}{4}[\gamma(\nabla f),\gamma(e_i)]\right)\\ &= e^{-f}\left(D + \frac{n-1}{2}\gamma(\nabla f)\right)\\ &= e^{-\frac{n+1}{2}f}De^{\frac{n-1}{2}f}.\end{aligned}$$

This proves (2). ■

Lemma 60.3. $\frac{1}{4}[[\gamma(u),\gamma(v)],\gamma(w)] = \gamma(v)g(u,w) - \gamma(u)g(v,w)$.

Proof. Pick a local orthonormal frame $(e_1,\dots,e_{r+s})$. It suffices to prove the above for $\{u,v,w\}\subset\{e_1,\dots,e_{r+s}\}$. By direct inspection

$$[\gamma(e_i)\gamma(e_j),\gamma(e_k)] = -2\gamma(e_i)g(e_j,e_k) + 2\gamma(e_j)g(e_i,e_k).$$

Therefore,

$$\begin{aligned}[[\gamma(e_i),\gamma(e_j)],\gamma(e_k)] &= [\gamma(e_i)\gamma(e_j),\gamma(e_k)] - [\gamma(e_j)\gamma(e_i),\gamma(e_k)]\\ &= -4\gamma(e_i)\varepsilon_k\delta_{jk} + 4\gamma(e_j)\varepsilon_k\delta_{ik}\\ &= -4\gamma(e_i)g(e_j,e_k) + 4\gamma(e_j)g(e_i,e_k).\end{aligned}$$ ■

61 Reduction of the structure group

Situation 61.1. Let G, H be Lie groups. Let $\lambda\colon H\to G$ be a Lie group homomorphism. Let $(p\colon P\to B, R\colon P \circlearrowleft G)$ be a G–principal bundle. ×

Definition 61.2. A **λ–reduction** of (p,R) consists of:

(1) a smooth manifold Q,

(2) a smooth right action $S\colon Q \circlearrowleft H$, and

(3) an H–equivariant smooth map $\xi\colon Q \to P$; that is: for every $x \in Q$ and $g \in H$:

$$\xi(xg) = \xi(x)\lambda(g)$$

such that $(q \coloneqq p \circ \xi\colon Q \to B, S)$ is an H–principal fibre bundle. •

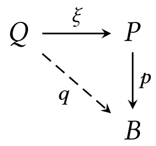


Figure 4: A λ–reduction of (p, R).

Remark 61.3. Let (Q, S, ξ) be a λ–reduction of (p, S). Define $r\colon Q \times_H G \to B$ by $r([x, g]) \coloneqq p(x)$ and $T\colon (Q \times_H G) \times G \to Q \times_H G$ by $T([x, g], h) \coloneqq [x, gh]$. The map $\phi\colon Q \times_H G \to P$ defined by

$$\phi([x, g]) \coloneqq \xi(x)\lambda(g)$$

is an isomorphism $(r, T) \to (p, R)$ of G–principal bundles. ♣

Definition 61.4. Let (Q_i, S_i, ξ_i) $(i = 1, 2)$ be λ–reductions of (p, R). An **isomorphism** $\phi\colon (Q_1, S_1, \xi_1) \to (Q_2, S_2, \xi_2)$ is an isomorphism $\phi\colon (q_1, S_1) \to (q_2, S_2)$ of H–principal bundles such that

$$\xi_2 \circ \phi = \xi_1.$$ •

At this point it would be good to discuss the "gauge group of a reduction of structure group"; and the space of "connections on a reduction of structure group".

62 Spin structures on pseudo-Euclidean vector bundles

Situation 62.1. Let X be a manifold. Let $V \to X$ be an space- and time-oriented Euclidean vector bundle of signature (r, s). Denote the frame bundle of V by $(p\colon \mathrm{Fr}_{\mathrm{SO}^+}(V) \to X, R)$. Denote by $\mathrm{Ad}\colon \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$ the adjoint representation. ×

Definition 62.2. A **spin structure** on V is a Ad–reduction $(\mathfrak{s}, U, \xi)$ of $(p\colon \mathrm{Fr}_{\mathrm{SO}}(V) \to X, R)$. •

Definition 62.3. A **spin manifold** is a pseudo-Riemannian manifold (X, g) together with a spin structure on TX. •

Remark 62.4 (M. Hirsch [Mil63, Alternative Definition 1]). Since $\{\pm 1\} \hookrightarrow \mathrm{Spin}^+_{r,s} \twoheadrightarrow \mathrm{SO}^+_{r,s}$, if $(\mathfrak{s}, U, \xi)$ is a spin structure on V, then $\xi\colon \mathfrak{s} \to \mathrm{Fr}_{\mathrm{SO}^+}(V)$ is a $\{\pm 1\}$–principal covering map.

Moreover, for every $s \in \mathfrak{s}$ the pull-back of $\mathfrak{s}$ via $R_s \colon \mathrm{SO}^+_{r,s} \hookrightarrow \mathrm{Fr}_{\mathrm{SO}^+}(V)$ is isomorphic to $\mathrm{Ad} \colon \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$:

$$\begin{array}{ccc} \mathrm{Spin}_{r,s} & \overset{\exists}{\dashrightarrow} & \mathfrak{s} \\ \downarrow{\scriptstyle \mathrm{Ad}} & & \downarrow{\scriptstyle \xi} \\ \mathrm{SO}^+_{r,s} & \xhookrightarrow{R_s} & \mathrm{Fr}_{\mathrm{SO}^+}(V). \end{array}$$

Conversely, every such $\{\pm 1\}$–principal covering map arises from a spin structure on V. Moreover, $(\mathfrak{s}_i, U_i, \xi_i)$ $(i = 1, 2)$ are isomorphic if and only if ξ_i $(i = 1, 2)$ are isomorphic. ♣

Proposition 62.5. *Let* $(\mathfrak{s}, U, \xi)$ *be a spin structure of* V*. Set* $s \coloneqq p \circ \xi \colon \mathfrak{s} \to X$*. The map* $\mathscr{A}(p, R) \to \mathscr{A}(s, U)$ *defined by*

$$\theta_A \mapsto \mathrm{Lie}(\mathrm{Ad})^{-1} \circ \xi^* \theta_A$$

is a bijection. ■

Remark 62.6. Since $\pi_1(\mathrm{GL}^+_{r+s}(\mathbf{R})) = \mathbf{Z}/2\mathbf{Z}$, $\mathrm{GL}^+_{r+s}(\mathbf{R})$ has a unique non-trivial $\{\pm 1\}$–principal covering map $\rho \colon \widetilde{\mathrm{GL}}^+_{r+s}(\mathbf{R}) \to \mathrm{GL}^+_{r+s}(\mathbf{R})$. The composition of $\mathrm{Ad} \colon \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$ with the inclusion $\mathrm{SO}^+_{r,s} \hookrightarrow \mathrm{GL}^+_{r+s}(\mathbf{R})$ lifts to an inclusion $\mathrm{Spin}_{r,s} \hookrightarrow \widetilde{\mathrm{GL}}^+_{r+s}(\mathbf{R})$. Indeed, these maps form a pullback diagram:

$$\begin{array}{ccc} \mathrm{Spin}_{r,s} & \hookrightarrow & \widetilde{\mathrm{GL}}^+_{r+s}(\mathbf{R}) \\ \downarrow{\scriptstyle \mathrm{Ad}} & & \downarrow{\scriptstyle \rho} \\ \mathrm{SO}^+_{r,s} & \hookrightarrow & \mathrm{GL}^+_{r+s}(\mathbf{R}). \end{array}$$

This can be understood as a very efficient construction of $\mathrm{Spin}_{r,s}$. However, it does not help with understanding the representation theory of $\mathrm{Spin}_{r,s}$. Indeed, the spinor representations do not extend to $\mathrm{GL}^+_{r+s}(\mathbf{R})$. ♣

Remark 62.7. Remark 62.6 induces a bijection between Ad–reductions of $\mathrm{Fr}_{\mathrm{SO}^+}(V)$ (up to isomorphism) and ρ–reduction of $\mathrm{Fr}_{\mathrm{GL}^+}(V)$ (up to isomorphism). This observation is sometimes useful to compare spin structure with respect to different Euclidean metrics on V. ♣

63 Stiefel–Whitney classes

See Milnor and Stasheff [MS74, §4, §8].

Lecture 10

The aim of this lectures is to `(finally!)` study the existence and classification of spin structures. The rest of this lecture is concerned with $\text{spin}^{\text{U}(1)}$ structures and Kähler manifolds.

64 Existence and classification of spin structures

Situation 64.1. Let X be a manifold. Let $V \to X$ be an space- and time-oriented Euclidean vector bundle of signature (r,s). Denote the frame bundle of V by $(p\colon \text{Fr}_{\text{SO}^+}(V) \to X, R)$. Denote by $\pi\colon \text{Spin}_{r,s} \to \text{SO}^+_{r,s}$ the vector representation. ×

Definition 64.2. Let $(\mathfrak{s}, U, \xi)$ be a spin structure on V. Let $\lambda\colon \tilde{X} \to X$ be a $\{\pm 1\}$–principal covering map. The **twist** of $(\mathfrak{s}, U, \xi)$ by λ is the spin structure $(\tilde{\mathfrak{s}}, \tilde{U}, \tilde{\xi})$ defined by

$$\tilde{\mathfrak{s}} := (\mathfrak{s} \times_X \tilde{X})/\{\pm 1\}, \quad \tilde{U}([\sigma, x], g) := [U(\sigma, g), x], \quad \text{and} \quad \tilde{\xi}([\sigma, x]) := \xi(\sigma). \qquad \bullet$$

Remark 64.3. $\{\pm 1\}$–principal covering maps $\tilde{X} \to X$ are classified via their monodromy by $\text{H}^1(X, \mathbf{Z}/2\mathbf{Z})$. Twisting defines an action of $\text{H}^1(X, \mathbf{Z}/2\mathbf{Z})$ on the set of isomorphism classes of spin structure on V. ♣

Proposition 64.4.

(1) *V admits a spin structure if and only if $w_2(V) = 0$.*

(2) *If $w_2(V) = 0$, then the set of isomorphism classes of spin structures on V is a $\text{H}^1(X, \mathbf{Z}/2\mathbf{Z})$–torsor.*

(See also Haefliger [Hae56] and Greub and Petry [GP78].)

Proof. The following argument is due to Milnor [Mil63, p. 199].

Without loss of generality X is connected. By Remark 62.4, it suffices to classify $\{\pm 1\}$–principal covering maps $\xi\colon \mathfrak{s} \to \text{Fr}_{\text{SO}^+}(V)$ whose restriction to every fibre is non-trivial. The 5–term exact sequence associated to the Leray–Serre spectral sequence of $p\colon \text{Fr}_{\text{SO}^+}(V) \to X$ is

$$0 \to \text{H}^1(X, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{p^*} \text{H}^1(\text{Fr}_{\text{SO}^+}(V), \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\text{res}} \text{H}^1(\text{SO}^+_{r,s}, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\delta_V} \text{H}^2(X, \mathbf{Z}/2\mathbf{Z}).$$

(This can also be obtained by applying $\text{Hom}(\cdot, \mathbf{Z}/2\mathbf{Z})$ to the long exact sequence of homotopy groups associated with p.) Since $\{\pm 1\}$–principal covering maps of $\text{Fr}_{\text{SO}^+}(V)$ are classified through their monodromy by

$$\text{H}^1(\text{Fr}_{\text{SO}^+}(V), \mathbf{Z}/2\mathbf{Z}) = \text{Hom}(\pi_1(\text{Fr}_{\text{SO}^+}(V)), \{\pm 1\})$$

and similarly for $\text{SO}^+_{r,s}$. [See, e.g., my lecture notes on Differential Geometry III]. Therefore, the set of isomorphism classes of spin structure on V is $\text{res}^{-1}([\text{Ad}])$ with $[\text{Ad}] \in \text{H}^1(\text{SO}^+_{r,s}, \mathbf{Z}/2\mathbf{Z})$ denoting the class of $\text{Ad}\colon \text{Spin}_{r,s} \to \text{SO}^+_{r,s}$.

Since the above sequence is exact, V admits a spin structure if and only if

$$\delta_V([\mathrm{Ad}]) \in \mathrm{H}^2(X, \mathbf{Z}/2\mathbf{Z})$$

vanishes. This is nothing but $w_2(V)$. This either requires a proof (cf. [Kar68, Proposition (1.1.26)]) or can be taken as the definition of $w_2(V)$.

The above exact sequence exhibits $\mathrm{res}^{-1}([\mathrm{Ad}])$ as a $\mathrm{H}^1(X, \mathbf{Z}/2\mathbf{Z})$–torsor. The $\mathrm{H}^1(X, \mathbf{Z}/2\mathbf{Z})$ action is given by twisting with isomorphism classes of $\{\pm 1\}$–principal covering maps. ■

Remark 64.5. See Milnor's warning about spin structure vs $\mathrm{Spin}_{r,s}$–principal bundles [LM89, Chapter II Remark 1.14]. ♣

65 Spinor bundles and the Atiyah–Singer operator

Situation 65.1. Let X be a manifold. Let $V \to X$ be an space- and time-oriented Euclidean vector bundle signature (r, s). Denote the frame bundle of V by $(p\colon\ \mathrm{Fr}_{\mathrm{SO}^+}(V) \to X, R)$. Denote by $\pi\colon\ \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$ the vector representation. Let $(\mathfrak{s}, U, \xi)$ be a spin structure on V. ×

Definition 65.2. Denote by P an irreducible $\mathrm{C}\ell_{r,s}$–module; cf. Section 31 and b as in Section 51. The **spinor bundle** associated with P and b is the Dirac bundle (S, γ, b, ∇) defined as follows:

(1) Set

$$S \coloneqq \mathfrak{s} \times_{\mathrm{Spin}_{r,s}} P \to X.$$

(2) The Clifford multiplication induces $\gamma\colon\ TX \to \mathrm{End}(S)$.

(3) The bilinear form b is induced by the bilinear form b above compatible with γ.

(4) The Levi-Civita connection of $\mathrm{Fr}_{\mathrm{SO}^+}(V)$ induces a connection on $\mathfrak{s}$ which induces a covariant derivative ∇_S on S. By construction, γ and b are parallel. •

Remark 65.3. As a vector bundle S does not depend on the choice of P, but γ does.

(1) If $r - s = -3, -7 \bmod 8$, then P carries a complex structure.

(2) If $r - s = -4, -5, -6 \bmod 8$, then P carries a quaternionic structure.

(3) If $r - s = 0, -2 \bmod 4$, then P is essentially unique.

(4) If $r - s = -1$, then either the volume element $\omega \in \mathrm{C}\ell(-g)$ acts as $+i$ or as $-i$; but then these are unique.

(5) If $r - s = -5$, then either the volume element $\omega \in \mathrm{C}\ell(-g)$ acts as $+1$ or as -1; but then these are unique.

The discussion in Section 35 governs the further decomposition. ♣

Remark 65.4. If $r - s = 0 \bmod 4$, then S inherits a **canonical grading** from P. ♣

Definition 65.5. Assume the situation of Definition 65.2. The **Atiyah–Singer operator** is the Dirac operator D associated with the spinor bundle (S, γ, b, ∇). •

66 Weitzenböck formula for the Atiyah–Singer operator

`This should probably be called the Schrödinger formula or the Lichnerowicz formula (or both).`

Situation 66.1. Let X be spin manifold. Denote by (S, γ, b, ∇) a spinor bundle. ×

Definition 66.2. Define $R_S \in \Omega^2(X, \mathrm{End}(S))$ by

$$R_S(v,w) := \frac{1}{4}\sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j\gamma(e_i)\gamma(e_j)\langle R(v,w)e_i, e_j\rangle. \qquad \bullet$$

Proposition 66.3. *The curvature F_S of the covariant derivative induced by the Levi-Civita connection is*

$$F_S = R_S.$$

In particular,

$$D^2 = \nabla_S^*\nabla_S + \frac{1}{4}\mathrm{scal}_g.$$

Therefore, if $\mathrm{scal}_g \geqslant 0$, then every harmonic spinor is parallel; if scal_g is positive somewhere, then harmonic spinors must vanish.

Proof. The twisting curvature F_S^{tw} is 2–form with values in skew-symmetric endomorphisms of S which commute with the Clifford multiplication. Since S arises from an irreducible representation, by Schur's Lemma an endomorphism of $\$$ commuting with the Clifford multiplication must be a scalar. A skew-symmetric scalar vanishes. This shows that $F_S^{\mathrm{tw}} = 0$. ■

Alternative proof/Exercise. One can proof directly that $F_S = R_S$. ■

Exercise 66.4. If $S = W$ is a complex spinor bundle, associated to a spin^c–structure prove that $F_S^{\mathrm{tw}} \in \Omega^2(M, i\mathbf{R})$. Identify F_S^{tw} in terms of the curvature of the connection on the characteristic line bundle L. More precisely, prove that $F_S^{\mathrm{tw}} = \frac{1}{2}F_A$ where F_A denotes the curvature of the connection on L.

67 Parallel spinors and Ricci flat metrics

Proposition 67.1 (cf. Hitchin [Hit74, Theorem 1.2]). *Let X be a spin manifold. If there exists a non-zero spinor $\Phi \in \Gamma(\$)$ such that*

$$\nabla\Phi = 0,$$

then X is Ricci flat.

Remark 67.2. This is well-known among physicists, because non-zero parallel spinor are closely related to super symmetry. ♣

Proof. Since Ric is a symmetric tensor, we can chose a local orthonormal frame and functions $\lambda_1, \dots, \lambda_n$ such that

$$\mathrm{Ric}(e_i, e_j) = \lambda_i \delta_{ij}.$$

If Φ is parallel, then in particular $R_S \Phi = 0$. By Definition 66.2 and Proposition 58.6, this means that

$$\begin{aligned}
0 &= \sum_{\ell=1}^{n} \gamma(e_\ell) R_S(e_k, e_\ell)\Phi \\
&= \frac{1}{4} \sum_{i,j,\ell=1}^{n} \gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k, e_\ell)e_i, e_j\rangle\Phi \\
&= -\frac{1}{2} \sum_{i=1}^{n} \gamma(e_i)\mathrm{Ric}(e_k, e_i)\Phi \\
&= -\frac{1}{2}\lambda_k \gamma(e_k)\Phi.
\end{aligned}$$

It follows that $\lambda_1 = \cdots = \lambda_n = 0$ and therefore $\mathrm{Ric} = 0$. ■

All known Ricci flat manifold have special holonomy, that is, $\mathrm{Hol}(g)$ is a strict subgroup of $\mathrm{SO}(n)$. It is a famous open question whether there are any compact Ricci-flat manifolds with $\mathrm{Hol}(g) = \mathrm{SO}(n)$. If M admits a parallel spinor, then it is impossible that $\mathrm{Hol}(g) = \mathrm{SO}(n)$, because the holonomy group of the spin bundle must reduce to a subgroup $\mathrm{Spin}(n-1) \subset \mathrm{Spin}(n)$. The possible holonomy groups have been classified by Berger [Ber55]. The following theorem clarifies the relation between parallel spinors and special holonomy.

Theorem 67.3 (Wang [Wan89]). *Let X be a complete, simply connected, irreducible spin manifold of dimension n. Set $d := \dim\ker \not{D}$. If X is not flat, then one of the following holds:*

(1) $n = 2m$, $\mathrm{Hol}(g) = \mathrm{SU}(m)$ *(that is: M is Calabi–Yau,) and* $d = 2$.

(2) $n = 4m$, $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *(that is: M is hyperkähler), and* $d = m + 1$.

(3) $n = 7$, $\mathrm{Hol}(g) = G_2$, *and* $d = 1$.

(4) $n = 8$, $\mathrm{Hol}(g) = \mathrm{Spin}(7)$, *and* $d = 1$.

Remark 67.4 (Friedrich and Trautman [FT00, Chapter 3, Exercise 4]). For $c > 0$, the metric

$$g = \frac{x_1}{x_1 + c}(\mathrm{d}x_1)^2 + x_1^2(\mathrm{d}x_2)^2 + x_1\sin(x_2)^2(\mathrm{d}x_3)^2 + \frac{x_1 + c}{x_1}(\mathrm{d}x_4)^2$$

is Ricci flat, but does not admit a non-trivial parallel spinor. ♣

68 Spin^G structures on pseudo-Euclidean vector bundles

Situation 68.1. Let X be a manifold. Let $V \to X$ be an space- and time-oriented Euclidean vector bundle of signature (r, s). Denote the frame bundle of V by $(p\colon \mathrm{Fr}_{\mathrm{SO}^+}(V) \to X, R)$. Denote by $\pi\colon \mathrm{Spin}^G_{r,s} \to \mathrm{SO}^+_{r,s}$ the adjoint representation. ×

Definition 68.2. A **spin^G structure** on V is a π–reduction $(\mathfrak{s}, U, \xi)$ of $(p\colon \mathrm{Fr}_{\mathrm{SO}}(V) \to X, R)$. •

Proposition 68.3. *Let $(\mathfrak{s}, U, \xi)$ be a spin^G structure of V. Denote by (q, S) the $G/\{\pm 1\}$–principal bundle associated with $\rho\colon \mathrm{Spin}^G_{r,s} \to G/\{\pm 1\}$. Denote by $\eta\colon \mathfrak{s} \to q$ the induced map. Set $s \coloneqq p \circ u\colon \mathfrak{s} \to X$. The map $\mathscr{A}(p, R) \times \mathscr{A}(q, S) \to \mathscr{A}(s, U)$ defined by*

$$(\theta_A, \theta_B) \mapsto \mathrm{Lie}(\pi \times \rho)^{-1} \circ (\xi^*\theta_A, \eta^*\theta_B)$$

is a bijection. ■

Remark 68.4. Of course, every spin structure induces a spin^G structure. ♣

69 Spin$^{\mathrm{U}(1)}$ structures

Situation 69.1. Let X be a manifold. Let $V \to X$ be an space- and time-oriented Euclidean vector bundle of signature (r, s). Denote the frame bundle of V by $(p\colon \mathrm{Fr}_{\mathrm{SO}^+}(V) \to X, R)$. Denote by $\pi\colon \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$ the vector representation. ×

Definition 69.2. Let $(\mathfrak{s}, U, \xi)$ be a spin$^{\mathrm{U}(1)}$ structure on V. The **characteristic line bundle** of a spin$^{\mathrm{U}(1)}$ structure is the Hermitian line bundle associated with the homomorphism $\mathrm{Spin}^{\mathrm{U}(1)}_{r,s} \to \mathrm{U}(1)/\{\pm 1\} \to \mathrm{U}(1)$. •

Definition 69.3. Let $(\mathfrak{s}, U, \xi)$ be a spin$^{\mathrm{U}(1)}$ structure on V with characteristic line bundle L Let A be a unitary covariant derivative on L. Denote by P an irreducible $\mathbf{C} \otimes C\ell_{r,s}$–module; cf. Section 32 and b as in Section 51. The **complex spinor bundle** associated with P and b is the Dirac bundle (S, γ, b, ∇) defined as follows:

(1) Set
$$S \coloneqq \mathfrak{s} \times_{\mathrm{Spin}^{\mathrm{U}(1)}_{r,s}} P \to X.$$

(2) The Clifford multiplication induces $\gamma\colon TX \to \mathrm{End}(S)$.

(3) The bilinear form b is induced by the bilinear form b above compatible with γ.

(4) The Levi-Civita connection of $\mathrm{Fr}_{\mathrm{SO}^+}(V)$ and A induce a connection on $\mathfrak{s}$ which induces a covariant derivative ∇_S on S. By construction, γ and b are parallel.

If $\dim X$ is even, then a complex spinor bundle inherits a **canonical grading** ε from Section 37. •

Definition 69.4. Assume the situation of Definition 69.3. The **complex Atiyah–Singer operator** is the Dirac operator D associated with the spinor bundle (S, γ, b, ∇). •

Definition 69.5. Let $(\mathfrak{s}, U, \xi)$ be a $\mathrm{spin}^{\mathrm{U}(1)}$ structure on V. Let $(\lambda\colon P \to X, \Lambda)$ be a $\mathrm{U}(1)$–principal bundle. The **twist** of $(\mathfrak{s}, U, \xi)$ by (λ, Λ) is the spin^G structure $(\tilde{\mathfrak{s}}, \tilde{U}, \tilde{\xi})$ defined by

$$\tilde{\mathfrak{s}} := \mathfrak{s} \times_{\mathrm{U}(1)} P, \quad \tilde{U}([\sigma, x], g) := [U(\sigma, g), x], \quad \text{and} \quad \tilde{\xi}([\sigma, x]) := \xi(\sigma). \qquad \bullet$$

Remark 69.6. This construction does not work for arbitrary spin^G structures. It hinges on the fact that $\mathrm{U}(1)$ is abelian $(\mathfrak{s}, \Sigma, \xi)$ ♣

Definition 69.7. Denote by $\beta_2\colon \mathrm{H}^k(X, \mathbf{Z}/2Z) \to \mathrm{H}^{k+1}(X, \mathbf{Z})$ the Bockstein homomorphism induced by the exact sequence $0 \to \mathbf{Z} \xrightarrow{2\times} \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$. Set

$$W_{k+1}(V) := \beta_2 w_k(V). \qquad \bullet$$

Proposition 69.8.

(1) *V admits a $\mathrm{spin}^{\mathrm{U}(1)}$ structure if and only if $w_2(V) \in \mathrm{im}(\mathrm{H}^2(X, \mathbf{Z}) \to \mathrm{H}^2(X, \mathbf{Z}/2\mathbf{Z})$ if and only if $W_3(V) = 0$.*

(2) *If V admits a $\mathrm{spin}^{\mathrm{U}(1)}$ structure, then the set of $\mathrm{spin}^{\mathrm{U}(1)}$ structures is a torsor over $\mathrm{H}^2(X, \mathbf{Z})$.*

Proof. Exercise. ■

70 Spin structures and $\mathrm{spin}^{\mathrm{U}(1)}$ structures on Kähler manifolds

The following is based on Hitchin [Hit74, Section 2.1].

Definition 70.1. If X is a complex manifold, then its **canonical bundle** is

$$\mathscr{K}_X = \Lambda^n_{\mathbf{C}} T^{1,0}X^*$$

and the **anti-canonical bundle** is $\mathscr{K}_X^*$. •

Remark 70.2. If X is a Kähler manifold with volume form vol, then there is a pairing $(\Lambda^n T^{1,0}X^*) \otimes (\Lambda^n T^{0,1}X^*) \to \mathbf{C}$ given by

$$\alpha \otimes \beta \mapsto \frac{\alpha \wedge \beta}{\mathrm{vol}}.$$

In particular,

$$\mathscr{K}_X^* \cong \Lambda^n T^{0,1}X^* \cong \Lambda^n T^{1,0}X. \qquad ♣$$

Proposition 70.3. *Suppose X is a Kähler manifold.*

(1) *For any Hermitian line bundle L, there is a unique $\mathrm{spin}^{\mathrm{U}(1)}$ structure $(\mathfrak{s}, U, \xi)$ on X whose complex spinor bundle is*

$$S = \bigoplus_{k=0}^{n} \Lambda^k T^{0,1}X^* \otimes L$$

whose characteristic line bundle is $L^{\otimes 2} \otimes_{\mathbf{C}} \mathscr{K}_X^*$. Moreover:

$$S^+ = \bigoplus_{k=0}^{\frac{n}{2}} \Lambda^{2k} T^{0,1}X^* \otimes L \quad \text{and} \quad S^- = \bigoplus_{k=0}^{\frac{(n-1)}{2}} \Lambda^{2k+1} T^{0,1}X^* \otimes L$$

(2) *The Clifford multiplication on S is given by*

$$\gamma(v)\alpha = \sqrt{2}(v^{0,1})^* \wedge \alpha - \sqrt{2}i(v^{0,1})\alpha.$$

(3) *If A is a Hermitian connection on L, then the corresponding connection on S induced by the Levi–Civita connection on $\Lambda^k(T^*X)^{0,1}$ is compatible with the Clifford multiplication.*

(4) *If A induces a holomorphic structure $\bar\partial_{\mathscr{L}}$ on L (that is: $F_A^{0,2} = 0$), then*

$$D = \sqrt{2}(\bar\partial_{\mathscr{L}} + \bar\partial_{\mathscr{L}}^*) \colon \Omega^{0,\bullet}(X, \mathscr{L}) \to \Omega^{0,\bullet}(X, \mathscr{L}).$$

In particular, if X is compact, then the space of positive and negative harmonic spinors can be identified with the cohomology groups

$$\bigoplus_{k=0}^{\lfloor n/2 \rfloor} H^{2k}(X, \mathscr{L}) \quad \textit{and} \quad \bigoplus_{k=0}^{\lfloor (n-1)/2 \rfloor} H^{2k+1}(X, \mathscr{L}).$$

Proof. If X is a Kähler manifold, then the structure group of TX is canonically reduced from $\mathrm{SO}(2n)$ to $\mathrm{U}(n)$. It follows from Proposition 49.3, that any Kähler manifold has a canonical $\mathrm{spin}^{\mathrm{U}(1)}$ structure; moreover, the complex spinor bundle is given $\bigoplus_{k=0}^{n} \Lambda^k(T^*X)^{0,1}$ and the Clifford multiplication is as asserted. It is computation to verify that the characteristic line bundle of the canonical $\mathrm{spin}^{\mathrm{U}(1)}$ structure is given by $\mathscr{K}_X^*$. Taking into account that the set of $\mathrm{spin}^{\mathrm{U}(1)}$ structures is a torsor over the group of Hermitian line bundles, the above proves (1) and (2). (3) is obvious and the first half of (4) follows by a direct computation. The second half of (4) follows by Hodge theory. ∎

Proposition 70.4. *A $\mathrm{spin}^{\mathrm{U}(1)}$ structure $(\mathfrak{s}, U, \xi)$ arises from a spin structure if and only if its characteristic line bundle is trivial. The set of spin structures inducing a fixed $\mathrm{spin}^{\mathrm{U}(1)}$ structure is a torsor over* $\ker(H^1(X, \mathbf{Z}_2) \to H^2(X, \mathbf{Z})$ *(that is: the group of Euclidean line bundles with trivial complexification).*

Proof. $\mathrm{Spin}^{\mathrm{U}(1)}(n) = \mathrm{Spin}(n) \times_{\mathbf{Z}_2} \mathrm{U}(1)$ and we have an exact sequence

$$0 \to \mathrm{Spin}(n) \to \mathrm{Spin}^{\mathrm{U}(1)}(n) \to \mathrm{U}(1) \to 0.$$

Since characteristic line bundle is associated to the representation $\mathrm{Spin}^{\mathrm{U}(1)}(n) \to \mathrm{U}(1)$, its triviality is precisely the obstruction to lifting a $\mathrm{spin}^{\mathrm{U}(1)}$ structure to a spin structure. This proves the first section. The second section follows by observing that any two spin structures differ by a Euclidean line bundle $\mathfrak{l}$, while any two $\mathrm{spin}^{\mathrm{U}(1)}$ structures differ by a Hermitian line bundle. ∎

Remark 70.5. Serre duality asserts that for a holomorphic vector bundle $\mathscr{E}$ over a compact complex manifold,

$$H^k(X, \mathscr{E}) \cong H^{n-k}(X, \mathscr{E}^* \otimes \mathscr{K}_X)^*.$$

In terms of the Dolbeault resolution, this duality is induced on chain-level by the pairing

$$\begin{aligned}(\Lambda^k T^{0,1}X^* \otimes \mathscr{E}) \otimes (\Lambda^{n-k}T^{0,1}X^* \otimes \mathscr{E}^* \otimes \mathscr{K}_X) &\cong \mathscr{K}_X^* \otimes \mathscr{K}_X \otimes \mathscr{E} \otimes \mathscr{E}^* \\ &\to \Lambda^{2n}T^*X \otimes \mathbf{C} \\ &\to \mathbf{C}.\end{aligned}$$

This pairing induces an isomorphism

$$\Lambda^k T^{0,1}X^* \otimes \mathscr{E} \cong (\Lambda^{n-k}T^{0,1}X^* \otimes \mathscr{E}^* \otimes \mathscr{K}_X)^*.$$

Using the Hermitian inner product on $\mathscr{K}_X$, we obtain an *anti-linear* isomorphism

$$\sigma\colon \Lambda^k T^{0,1}X^* \otimes \mathscr{E} \cong \Lambda^{n-k}T^{0,1}X^* \otimes \mathscr{E}^* \otimes \mathscr{K}_X.$$

In particular, if $\mathscr{L}$ is a square root of $\mathscr{K}_X$ (that is: $\mathscr{L}^{\otimes 2} \cong \mathscr{K}_X$), then

$$\sigma\colon \Lambda^k T^{0,1}X^* \otimes \mathscr{L} \cong \Lambda^{n-k}T^{0,1}X^* \otimes \mathscr{L}.$$

♣

Proposition 70.6. *Let X be a Kähler manifold.*

(1) *X admits a spin structure if and only if there is a complex line bundle L satisfying $L^{\oplus 2} \cong \mathscr{K}_X$.*

(2) *Suppose X is compact. There is a bijective correspondence between the set of spin structures on X and the set of isomorphism classes of holomorphic line bundles $\mathscr{L}$ satisfying $\mathscr{L}^{\otimes 2} \cong \mathscr{K}_X$. (Each such $\mathscr{L}$ inherits a Hermitian metric from $\mathscr{K}_X$.)*

(3) *Suppose that $\mathscr{L}$ is a square root of $\mathscr{K}_X$ and W denotes the associated complex spinor bundle. There is a spinor bundle S such that:*

 (a) *If $\dim_{\mathbf{C}} X = 1 \bmod 4$, then*

$$\$ = W \quad \textit{and} \quad D\!\!\!/ = \sqrt{2}(\bar\partial + \bar\partial^*).$$

 The is a complex structure J on $\$$ which commutes with Clifford multiplication and anti-commutes with the complex structure i.

 (b) *If $\dim_{\mathbf{C}} X = 2 \bmod 4$, then*

$$\$^\pm = W^\pm \quad \textit{and} \quad D\!\!\!/^\pm = \sqrt{2}(\bar\partial + \bar\partial^*);$$

 moreover, there is a complex structure J on $\$^\pm$ which commutes with Clifford multiplication and anti-commutes with the complex structure i.

 (c) *If $\dim_{\mathbf{C}} X = 3 \bmod 4$, then is a real structure on W which respect to which $\sqrt{2}(\bar\partial + \bar\partial^*)$ is real. With respect to this real structure we have*

$$\$ = \operatorname{Re} W \quad \textit{and} \quad D\!\!\!/ = \sqrt{2}(\bar\partial + \bar\partial^*).$$

(d) *If* $\dim_{\mathbf{C}} X = 4 \bmod 4$, *then is a real structure on* $W^{\pm}$ *With respect to this real structure we have*

$$\slashed{S}^{\pm} = \operatorname{Re} W^{\pm} \quad \text{and} \quad \slashed{D} = \sqrt{2}(\bar{\partial} + \bar{\partial}^{*}).$$

Proof. (1) follows from Proposition 70.4.

(2) Denote $\mathcal{O}^{\times}$ the sheaf of nowhere vanishing holomorphic functions on X. There is a short exact sequence of sheaves

$$1 \to \mathbf{Z}_2 \to \mathcal{O}^{\times} \xrightarrow{x \mapsto x^2} \mathcal{O}^{\times} \to 1.$$

The corresponding long exact sequence in cohomology reads as follows:

$$\mathrm{H}^0(X, \mathcal{O}^{\times}) \to \mathrm{H}^0(X, \mathcal{O}^{\times}) \to \mathrm{H}^1(X, \mathbf{Z}_2) \xrightarrow{\alpha} \mathrm{H}^1(X, \mathcal{O}^{\times}) \to \mathrm{H}^1(X, \mathcal{O}^{\times}) \xrightarrow{\beta} \mathrm{H}^2(X, \mathbf{Z}_2)$$

The map α is injective, because the map $\mathbf{C}^{\times} = \mathrm{H}^0(X, \mathcal{O}^{\times}) \to \mathrm{H}^0(X, \mathcal{O}^{\times}) = \mathbf{C}^{\times}$ is surjective. Recall, that $\mathrm{H}^1(X, \mathcal{O}^{\times})$ classifies a holomorphic line bundles. A holomorphic line bundle $\mathcal{L}$ has a square root if and only if $\beta([\mathcal{L}]) = (c_1(L) \bmod 2) = 0$. If $\beta([\mathcal{L}]) = 0$, then by the above the set of square roots is a torsor over $\mathrm{H}^1(X, \mathbf{Z}_2)$.

For the proof of (3), using 1, one first analyzes the relationship between the spinor representation S and the complex spinor representation W in dimension n and determines the following:

(1) If $n = 2 \bmod 8$, then $S = W$ and W has a complex anti-linear complex structure J. $S = \mathbf{H}$, $W = W^{+} \oplus W^{-} = \mathbf{C} \oplus \mathbf{C}$.

(2) If $n = 4 \bmod 8$, then $S^{\pm} = W^{\pm}$ and $W^{\pm}$ have a complex anti-linear complex structure J.

(3) If $n = 6 \bmod 8$, then there is a real structure on W and $S = \operatorname{Re} W$. This real structure does not respect the splitting $W = W^{+} \oplus W^{-}$. Clifford multiplication is real with respect to this real structure.

(4) If $n = 8 \bmod 8$, then there is a real structure on $W^{\pm}$ and $S^{\pm} = \operatorname{Re} W^{\pm}$. Clifford multiplication is real with respect to this real structure. ■

`The above linear algebra should probably have been discussed earlier.`

Lecture 11

71 Homogeneous pseudo-Riemannian manifolds

Situation 71.1. Let G be a Lie group and $H < G$ a closed subgroup. Since $R\colon G \circlearrowleft H$ is free and proper, $(p\colon G \to X := G/H, R)$ is an H–principal bundle. Set $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$. Denote by $\mu_G \in \Omega^1(G, \mathfrak{g})$ the Maurer–Cartan form. ×

Proposition 71.2. *If $\pi\colon \mathfrak{g} \to \mathfrak{h}$ be an H–equivariant projection, then*

$$\theta := \pi \circ \mu_G$$

is a G–invariant H–principal connection 1–form on (p, R) Moreover, every G–invariant H–principal connection 1–form is of this form.

Proof. By construction, for every $g \in H$ and $\xi \in \mathrm{Lie}(H) \subset \mathrm{Lie}(G)$

$$R_g^*\theta = \pi \circ R_g^*\mu_G = \pi \circ \mathrm{Ad}(g^{-1}) \circ \mu_G = \mathrm{Ad}(g^{-1})\theta$$

and

$$\theta(\xi) = \pi \circ \mu(\xi) = \xi.$$

Therefore, θ is an H–principal connection 1–form. For every $g \in G$

$$L_g^*\theta = \pi \circ L_g^*\mu = \theta.$$

Therefore, θ is G–invariant.

Conversely, if θ is H–principal connection, then $\pi := \theta|_{T_1G}\colon T_1G = \mathfrak{g} \to \mathfrak{h}$ is an H–equivariant projection. Moreover, if θ a G–invariant, then $\theta = \pi \circ \mu_G$. ■

Proposition 71.3. *Let $\pi\colon \mathfrak{g} \to \mathfrak{h}$ be an H–equivariant projection. Set $\theta := \pi \circ \mu \in \mathscr{A}(p, R)^G$ and $\mathfrak{m} := \ker \pi$.*

(1) *The H–equivariant horizontal 1–form $\sigma \in \Omega^1_{\mathrm{hor}}(G, \mathfrak{m})^H$ defined by*

$$\sigma := (\mathbf{1} - \pi) \circ \mu_G$$

is a solder form.

(2) *The torsion of (θ, σ) satisfies*

$$\mathrm{d}\sigma + [\theta \wedge \sigma] = -\frac{1}{2}(\mathbf{1} - \pi)[\sigma \wedge \sigma] \in \Omega^2_{\mathrm{hor}}(G, \mathfrak{m})^H.$$

In particular, (θ, σ) is torsion-free if and only if $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

(3) *The curvature of θ satisfies*

$$\mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta] = -\frac{1}{2}\pi[\sigma \wedge \sigma] \in \Omega^2_{\mathrm{hor}}(G, \mathfrak{h})^H.$$

Proof. This first part is obvious if one can recall what a solder form is. See §4.9 in my lecture notes on gauge theory.

By the Maurer–Cartan equation and since $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$ and $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$,

$$\begin{aligned} \mathrm{d}\sigma + [\theta \wedge \sigma] &= -\frac{1}{2}(\mathbf{1}-\pi)\circ[(\theta+\sigma)\wedge(\theta+\sigma)] + [\theta\wedge\sigma] \\ &= -\frac{1}{2}(\mathbf{1}-\pi)[\sigma\wedge\sigma]. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{d}\theta + [\theta \wedge \theta] &= -\frac{1}{2}\pi\circ[(\theta+\sigma)\wedge(\theta+\sigma)] + \frac{1}{2}[\theta\wedge\theta] \\ &= -\frac{1}{2}\pi[\sigma\wedge\sigma]. \end{aligned}$$

■

Proposition 71.4. *Let $\pi\colon \mathfrak{g} \to \mathfrak{h}$ be an H–equivariant projection. Set $\theta := \pi \circ \mu \in \mathscr{A}(p,R)^G$, $\mathfrak{m} := \ker \pi$, and $\sigma := (\mathbf{1}-\pi)\circ\mu_G \in \Omega^1(G,\mathfrak{m})^H$.*

1. *If $\hat{g} \in \mathrm{Hom}(S^2\mathfrak{m},\mathbf{R})^H$ is an H–invariant pseudo-Euclidean inner product, then $\hat{g}\circ S^2\sigma$ descends to a G–invariant pseudo-Riemannian metric g on X. Moreover, every G–invariant pseudo-Riemannian metric on X is of this form.*
2. *If $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$, then θ induces the Levi-Civita connection of g.* ■

Exercise 71.5. What is the Levi-Civita connection if $[\mathfrak{m},\mathfrak{m}] \not\subset \mathfrak{h}$?

Definition 71.6. A **homogeneous pseudo-Riemannian manifold** is a pseudo-Riemannian manifold of the form $(G/H, g)$ obtained from $(G,H,\pi,\hat{g})$. A **symmetric pseudo-Riemannian manifold** is a pseudo-Riemannian manifold of the form $(G/H,g)$ obtained from $(G,H,\pi,\hat{g})$ with $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$. •

[Bes87, Chapter 7]

72 Spin structures on homogeneous pseudo-Riemannian manifolds

Situation 72.1. Let G be a Lie group and $H < G$ a closed subgroup. Since $R\colon G \circlearrowleft H$ is free and proper, $(p\colon G \to X := G/H, R)$ is an H–principal bundle. Set $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$. Denote by $\mu_G \in \Omega^1(G,\mathfrak{g})$ the Maurer–Cartan form. Let $\pi\colon \mathfrak{g} \to \mathfrak{h}$ be an H–equivariant projection. Set $\mathfrak{m} := \ker\pi$. Let $\hat{g}$ be a pseudo-Euclidean inner product of signature (r,s). Choose an isometry $(\mathfrak{m},\hat{g}) \cong (\mathbf{R}^{r+s}, g_{r,s})$. Denote by g the induces pseudo-Riemannian metric on X induced by $\hat{g}$. ×

Definition 72.2. A **homogeneous spin structure** on X is a homomorphism $H \to \mathrm{Spin}_{r,s}$ which lifts $K \to \mathrm{SO}^+_{r,s}$ along $\mathrm{Ad}\colon \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$. •

The frame bundle $\mathrm{Fr}_{\mathrm{SO}^+}(TX)$ is $G \times_H \mathrm{SO}_{r,s}$. Therefore, a homogeneous spin structure $H \to \mathrm{Spin}_{r,s}$ defined a spin structure $(fs\colon G\times_H \mathrm{Spin}_{r,s}, U\colon \mathfrak{s} \circlearrowleft \mathrm{Spin}_{r,s}, \xi\colon \mathfrak{s} \to \mathrm{Fr}_{\mathrm{SO}^+}(TX)$. If P is a $\mathrm{C}\ell_{r,s}$–module, then the corresponding spinor bundle is

$$S := G \times_K P.$$

If X is symmetric (that is, $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$), then θ induces the spin connection.

A spinor $\phi \in \Gamma(S)$ can be identified with a H–equivariant map

$$\phi\colon G \to P \quad \text{with} \quad \phi(gk) = \mathrm{Ad}(k^{-1})\psi(g).$$

The Clifford multiplication by $v \in T_xX \cong \mathfrak{m}$ is given simply by the Clifford multiplication of $\mathfrak{m}$ on S. The derivative $\nabla\psi \in \Omega^1(X,S)$ can be identified with the H–equivariant 1–form on G with values in S defined by

$$(\nabla\psi)(\xi) = (\mathrm{d}\psi)(\xi) + \tilde{\mathrm{ad}}(\theta(\xi))\psi$$

Therefore, if $(e_1,\dots,e_{r+s})$ is an orthonormal basis for $\mathfrak{m}$, the Dirac operator $D\colon C^\infty(G,P)^H \to C^\infty(G,P)^H$ is is given by

$$D\phi = \sum_{i=1}^{r+s} \varepsilon_i\gamma(e_i)\mathscr{L}_{e_i}\phi.$$

Henceforth, suppose that $\hat{g}$, in fact, arises as the restriction of a from G–invariant Euclidean inner product on $\mathfrak{g}$.

Definition 72.3. The **Casimir operator** of G is the differential operator $\Omega_G\colon C^\infty(G) \to C^\infty(G)$ defined by

$$\Omega_G := -\sum_{i=1}^{n} \mathscr{L}_{e_i}\mathscr{L}_{e_i}$$

for some orthonormal basis $(e_1,\dots,e_n)$ of $\mathfrak{g}$. •

Proposition 72.4.

$$D^2 = \Omega_G + \frac{1}{8}\mathrm{scal}.$$

Sketch of proof. Since $[e_i,e_j] \in \mathfrak{h}$ and, for $\xi \in \mathfrak{h}$, $\mathscr{L}_\xi\psi = -\tilde{\mathrm{ad}}(\xi)$, we have

$$\begin{aligned} D^2\psi &= \sum_{i,j=1}^{m} \gamma(e_i)\gamma(e_j)\mathscr{L}_{e_i}\mathscr{L}_{e_j}\psi \\ &= -\sum_{i=1}^{m} \mathscr{L}_{e_i}^2\psi + \frac{1}{2}\sum_{i,j=1}^{m} \gamma(e_i)\gamma(e_j)\mathscr{L}_{[e_i,e_j]}\psi \\ &= -\sum_{i=1}^{m} \mathscr{L}_{e_i}^2\psi - \frac{1}{2}\sum_{i,j=1}^{m} \gamma(e_i)\gamma(e_j)\tilde{\mathrm{ad}}([e_i,e_j]). \end{aligned}$$

Let $(f_1,\dots,f_k)$ be an orthonormal basis of $\mathfrak{h}$. The above formula can then be written as

$$D^2 = \Omega_G + \sum_{j=1}^{k} \tilde{\mathrm{ad}}(f_j)\tilde{\mathrm{ad}}(f_j) - \frac{1}{2}\sum_{i,j=1}^{m} \gamma(e_i)\gamma(e_j)\tilde{\mathrm{ad}}([e_i,e_j])\psi.$$

A computation identifies the sum of the last two terms with $\frac{1}{8}\mathrm{scal}$. ∎

Here is why the above is useful. The scalar curvature scal of a symmetric space is constant. $L^2\Gamma(S)$ is acted upon by G and can be decomposed into irreducible representations

$$L^2\Gamma(S) = \bigoplus_{\lambda \in \Lambda} V_\lambda.$$

On an irreducible representation, the Casimir operator acts as a constant $c(\lambda)$. Consequently, the spectrum of D^2 is given by

$$\operatorname{spec}(D^2) = \left\{ c(\lambda) + \frac{1}{8}\operatorname{scal} : \lambda \in \Lambda \right\}.$$

This can (in principle) be used to compute the spectrum of D^2 using representation theory.

Example 72.5 (Toy example). Consider the circle $S^1 = \mathbf{R}/2\pi\mathbf{Z}$. It has two spin structures. For one of them, the spinor bundle is the trivial bundle $S = \underline{\mathbf{C}}$ and the Dirac operator is simply $D = i\partial_t$. Consequently,

$$\operatorname{spec} D = \mathbf{Z}$$

with eigenspinors given by $\psi_k(t) = e^{ikt}$.

We can think of S^1 as the symmetric space $\mathrm{U}(1)/\{e\}$. Since $\mathrm{Spin}(1) = \{\pm 1\}$ and $\mathrm{U}(1)$ is connected, there is a unique homogeneous spin structure on S^1. This is the spin structure considered above. The irreducible representation of $\mathrm{U}(1)$ are parametrized by $\mathbf{Z}$: given $k \in \mathbf{Z}$, $\mathrm{U}(1) \to \mathrm{GL}(\mathbf{C}), z \mapsto z^k$ is irreducible. Each of these representations appear with multiplicity one in $L^2\Gamma(S)$ (by Fourier theory). The Casimir operator on the representation parametrized by $k \in \mathbf{Z}$ takes value k^2. Consequently, the above discussion tells us that

$$\operatorname{spec} D^2 = \{k^2 : k \in \mathbf{Z}\}.$$

Of course, this derivation is the same as direct derivation in the previous paragraph. ♠

Remark 72.6. This method has been used by Sulanke to determine the spectrum of D on $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ in her PhD thesis; cf. [Sul80]. A simpler way to determine the spectrum of D on S^n was found by Bär [Bär96]. In fact, Bär's method also determines an explicit eigenbasis with respect to D. ♣

Lecture 12

73 Killing Spinors

Definition 73.1. Let X be a spin manifold. A **Killing spinor** is a spinor $\psi \in \Gamma(S)$ satisfying

$$\nabla_v \psi - \mu\gamma(v)\psi = 0$$

for some constant $\mu \in \mathbf{R}$ and all $v \in TX$. We call μ the **Killing number** of ψ. •

Proposition 73.2. *A Killing spinor with Killing number μ is an eigenspinor with eigenvalue $-n\mu$.*

73.1 Friedrich's lower bound for the first eigenvalue of D

As far as I know, the origin of the study of Killing spinors is the following result.

Theorem 73.3 (Friedrich [Fri80]). *Let X be a compact spin manifold with non-negative but non-vanishing scalar curvature. Denote by λ^+ and λ^- the smallest positive and negative eigenvalues of D respectively. With*

$$\mathrm{scal}_0 \coloneqq \min \mathrm{scal},$$

we have

$$(\lambda^\pm)^2 \geqslant \frac{n}{4(n-1)}\mathrm{scal}_0.$$

If equality holds, then X admits a non-trivial Killing spinor with Killing number $+\frac{n}{4(n-1)}\mathrm{scal}$ or $-\frac{n}{4(n-1)}\mathrm{scal}$.

Remark 73.4. The obvious lower bound on $\lambda^\pm$ arising from Proposition 66.3 is $\lambda^\pm \geqslant \frac{1}{4}\mathrm{scal}_0$. ♣

The proof is based on an important trick. The basic idea is that if $f \in C^\infty(X, \mathbf{R})$, then there is a Weitzenböck formula for $D + f$ which give sharper bounds that Proposition 66.3. More generally, one can replace f with a suitable endomorphism of S.

Definition 73.5. Given $f \in C^\infty(X, \mathbf{R})$, define the covariant derivative ${}^f\nabla$ on S by

$${}^f\nabla_v \Phi \coloneqq \nabla\Phi - f\gamma(v)\Phi.$$

•

Remark 73.6. ${}^f\nabla_v\Phi$ is an orthogonal covariant derivative. ♣

Proposition 73.7. *We have*

$$(D + f)^2 = {}^f\nabla^{*f}\nabla + \frac{1}{4}\mathrm{scal} + (1-n)f^2$$

Proof. Since

$$D(f\phi) = \gamma(\nabla f) + fD(\phi),$$

we have

$$(D+f)^2 = D^2 + 2fD + \gamma(\nabla)f + f^2.$$

By Proposition 66.3, we have

$$(D+f)^2 = \nabla^*\nabla + 2fD + \gamma(\nabla f) + f^2 + \frac{1}{4}\mathrm{scal}.$$

We have

$$\begin{aligned}
{}^f\nabla^{*f}\nabla &= -\sum_{i=1}^n {}^f\nabla_{e_i}{}^f\nabla_{e_i} \\
&= -\sum_{i=1}^n (\nabla_{e_i} - f\gamma(e_i))(\nabla_{e_i} - f\gamma(e_i)) \\
&= -\sum_{i=1}^n \left(\nabla_{e_i}^2 - f^2 - \gamma(e_i)\nabla_{e_i}f - 2f\gamma(e_i)\nabla_{e_i}\right) \\
&= \nabla^*\nabla + nf^2 + \gamma(\nabla f) + 2fD,
\end{aligned}$$

which can be rewritten as

$$\nabla^*\nabla = {}^f\nabla^{*f}\nabla - nf^2 - \gamma(\nabla f) - 2fD.$$

This proves the asserted identity. ■

Corollary 73.8. *If ψ is compactly supported, then*

$$\int_X |(D+f)\psi|^2 = \int_X \left(\frac{1}{4}\mathrm{scal} + (1-n)f^2\right)|\psi|^2 + |{}^f\nabla\psi|^2.$$

■

Proof of Theorem 73.3. Suppose λ is an eigenvalue of D and ψ is an eigenspinor for λ. Using Corollary 73.8 with $f = \mu$ a constant, we obtain

$$0 = \int_X \left(\frac{1}{4}\mathrm{scal} + (1-n)\mu^2 - (\lambda+\mu)^2\right)|\psi|^2 + |{}^f\nabla\psi|^2.$$

Consequently,

$$\frac{1}{4}\mathrm{scal}_0 \leqslant (\lambda+\mu)^2 + (n-1)\mu^2.$$

The minimum of the right-hand side is $\frac{n-1}{n}\lambda^2$; it is achieved at $\mu = -\lambda/n$. This implies the bound. ■

73.2 Killing spinors and Einstein metrics

Proposition 73.9. *If X admits a non-trivial Killing spinor with Killing number μ, then X is Einstein with Einstein constant* $4(n-1)(\mu/n)^2$.

Proof. The curvature of ${}^f\nabla$ is given by

$$\begin{aligned} {}^fF_{k\ell} &= [\nabla_k - f\gamma_k, \nabla_\ell - f\gamma_\ell] \\ &= R^S_{k\ell} - \partial_k f\gamma_k\gamma_\ell + \partial_\ell f\gamma_\ell\gamma_k + f^2[\gamma_k, \gamma_\ell]. \end{aligned}$$

Since Ric is a symmetric tensor, we can chose a local orthonormal frame and functions $\lambda_1, \dots, \lambda_n$ such that

$$\mathrm{Ric}(e_i, e_j) = \lambda_i\delta_{ij}.$$

Arguing as in the proof of Proposition 67.1 with $f = -\lambda/n$, we have

$$\begin{aligned} 0 &= \sum_{\ell=1}^n \gamma(e_\ell)^f F(e_k, e_\ell)\psi \\ &= \frac{1}{4}\sum_{i,j,\ell=1}^n \gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k, e_\ell)e_i, e_j\rangle\psi + \sum_{\ell=1}^n (\lambda/n)^2\gamma(e_\ell)[\gamma(e_k), \gamma(e_\ell)]\psi \\ &= -\frac{1}{2}\sum_{i=1}^n \gamma(e_i)\mathrm{Ric}(e_k, e_i)\psi + 2(n-1)(\lambda/n)^2\gamma(e_k)\psi \\ &= \left(-\frac{1}{2}\lambda_k + 2(n-1)(\lambda/n)^2\right)\gamma(e_k)\psi. \end{aligned}$$

It follows that

$$\mathrm{Ric} = 4(n-1)(\lambda/n)^2.$$

■

73.3 The spectrum of the Atiyah–Singer operator on S^n

Theorem 73.10. *Let $n \geqslant 3$. On S^n, we have*

$$\mathrm{spec}(D) = \{\pm(n/2 + k) : k \in \mathbf{N}_0\}.$$

The multiplicity of $\lambda_{\pm,k} = \pm(n/2 + k)$ is

$$\mathrm{rk}\, S \cdot \binom{k+n-1}{k}.$$

Proof. The following argument goes back to Bär [Bär96].

Proposition 73.11. *Let $n \geqslant 3$. The spinor bundle S of S^n can be trivialized by Killing spinors with Killing number $+1/2$ and also by Killing spinors with Killing number $-1/2$.*

Proof. Consider the covariant derivative ${}^{\pm1/2}\nabla$ defined by ${}^{\pm1/2}\nabla_v\psi = \nabla_v\psi \mp 1/2\gamma(v)\psi$. A computation shows that the curvature of ${}^{\pm1/2}\nabla$ vanishes. Since S^n is simply-connected, it follows that S admits a trivialization by ${}^{\pm1.2}\nabla^{\pm}$–parallel spinors. ■

Proposition 73.12. *We have*

$$(D \pm 1/2)^2 = {}^{\pm 1/2}\nabla^{*\pm 1/2}\nabla + \frac{1}{4}(n-1)^2.$$

Proof. This is Proposition 73.7. ■

Pick Killing spinors $(\psi_1^\pm, \ldots, \psi_m^\pm)$ with Killing number $\pm 1/2$ forming a basis for S point-wise. Here $m = \operatorname{rk} S$. Let (f_k) be a complete L^2 orthonormal basis of eigenfunctions for Δ on S^n. Denote by λ_k the eigenvalue corresponding to f_k.

Proposition 73.13. *We have*

$$\operatorname{spec}(\Delta_{S^n}) = \{k(n+k-1) : k \in \mathbf{N}_0\}.$$

The eigenvalue $\lambda_k = k(n+k-1)$ has multiplicity

$$m_k = \binom{n+k-1}{k}\frac{n+2k-1}{n+k-1}.$$

Clearly $(f_i\psi_j^\pm)$ forms an L^2 orthonormal basis of $L^2\Gamma(S)$. Since $\psi_j^\pm$ is ${}^{\pm 1/2}\nabla$–parallel we have,

$$(D \pm 1/2)^2(f_i\psi_j^\pm) = \left(\lambda_i + \frac{1}{4}(n-1)^2\right) f_i\psi_j^\pm.$$

Therefore, $(f_i\psi_j^\pm)$ is an eigenbasis for $(D \pm 1/2)^2$. Using Proposition 73.13, we can compute the spectrum of $(D \pm 1/2)^2$.

Corollary 73.14. *We have*

$$\operatorname{spec}\left((D \pm 1/2)^2\right) = \left\{k(n+k-1) + (n-1)^2/4 : k \in \mathbf{N}_0\right\}.$$

The eigenvalue $\lambda_k = k(n+k-1) + (n-1)^2/4$ has multiplicity

$$m(\lambda_k) = \binom{n+k-1}{k}\frac{n+2k-1}{n+k-1} \cdot \operatorname{rk} S.$$

■

Proposition 73.15. *If $A^2x = \lambda^2 x$, then $x^\pm = \pm\lambda x + Ax$ satisfy*

$$Ax^\pm = \pm\lambda x^\pm.$$

We have $\sqrt{k(n+k-1) + (n-1)^2/4} = k + \frac{n-1}{2}$. For $\varepsilon = \pm 1$, define

$$\psi_{k\ell}^{\varepsilon\pm} := (D \pm 1/2)(f_k\psi_\ell^\pm) + \varepsilon(k + (n-1)/2)(f_k\psi_\ell^\pm).$$

A brief computation shows that

$$\psi_{k\ell}^{\varepsilon\pm} = \varepsilon(\pm(1-n)/2 + \varepsilon(k + (n-1)/2))(f_k\psi_\ell^\pm) + \gamma(\nabla f_k)\psi_\ell^\pm.$$

Except $\psi_{0,\ell}^{++}$ and $\psi_{0,\ell}^{--}$ these spinors are non-vanishing. It follows that

$$\operatorname{spec}(D \pm 1/2) \subset \{\varepsilon(k + (n-1)/2) : \varepsilon = \pm 1, k \in \mathbf{N}_0\} \backslash \{\pm(n-1)/2)\}.$$

This implies the claim about $\operatorname{spec}(D)$. For the computation of the multiplicities we refer the reader to [Bär96, Lemma 5]. ■

Lecture 13

The purpose of the next couple of lectures is to give a quick but complete discussion of L^2 elliptic theory. The foundation of this approach is the following discussion of the Fourier transform.

74 Fourier transform on L^1

Situation 74.1. Let $n \in \mathbf{N}_0$. Let V be complex Hilbert space. ×

Definition 74.2 (Fourier transform). Let $f \in L^1(\mathbf{R}^n, V)$. The **Fourier transform** of f is the map $\hat{f}\colon \mathbf{R}^n \to X$ defined by

$$\hat{f}(\xi) := \int_{\mathbf{R}^n} e^{-2\pi i\langle \xi,x\rangle} f(x)\,\mathrm{d}x. \tag{74.3}$$ •

Remark 74.4 (The appearance of the factor 2π). The theory of the Fourier transform contains an *inextinguishable* 2π. I have put it in front of the inner product in (74.3). As a consequence it pops out in Proposition 74.8. Another option is to renormalise the measure $\mathrm{d}x$ by a factor of $(2\pi)^{-n/2}$ (and possibly use the notation $\text{đ}x := (2\pi)^{-n/2}\mathrm{d}x$). The factor then resurfaces in Proposition 74.12. Another option is to omit it from (74.3) altogether. The factor then appears in Theorem 79.2 and in the formula for the inverse Fourier transform. ♣

Proposition 74.5 (Fourier transform on L^1). *If $f \in L^1(\mathbf{R}^n, X)$, then $\hat{f} \in C^0_b(\mathbf{R}^n, V)$ and*

$$\|\hat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}. \tag{74.6}$$

Proof. The estimate (74.6) is immediate; indeed,

$$|\hat{f}(\xi)| = \left|\int_{\mathbf{R}^n} e^{-2\pi i\langle \xi,x\rangle} f(x)\,\mathrm{d}x\right| \leqslant \int_{\mathbf{R}^n} |f(x)|\,\mathrm{d}x = \|f\|_{L^1}.$$

To prove that $\hat{f}$ is continuous, observe that for every $\xi, \eta \in \mathbf{R}^n$ and $R > 0$

$$\begin{aligned}|\hat{f}(\xi) - \hat{f}(\eta)| &\leqslant \int_{\mathbf{R}^n} |e^{-2\pi i\langle \xi,x\rangle} - e^{-2\pi i\langle \eta,x\rangle}||f(x)|\,\mathrm{d}x \\ &\leqslant \sup_{x\in B_R(0)} |e^{-2\pi i\langle \xi-\eta,x\rangle} - 1|\|f\|_{L^1} + 2\int_{\mathbf{R}^n\setminus B_R(0)} |f(x)|\,\mathrm{d}x;\end{aligned}$$

and, moreover,

$$\lim_{\zeta\to 0} \sup_{x\in B_R(0)} |e^{-2\pi i\langle \zeta,x\rangle} - 1| = 0 \quad\text{and}\quad \lim_{R\to\infty}\int_{\mathbf{R}^n\setminus B_R(0)} |f(x)|\,\mathrm{d}x = 0.$$ ■

Remark 74.7. The Riemann–Lebesque Lemma strengthens the conclusion of Proposition 74.5; see ??. However, as far as I know is is an open problem to determine the image of the Fourier tranform on L^1. ♣

Proposition 74.8. *Let $k \in \{1, \dots, n\}$. If $f, \partial_k f \in L^1(\mathbf{R}^n, V)$, then*

$$\widehat{\partial_k f} = 2\pi i \xi_k \hat{f}.$$

Proof. By integration by parts,

$$\widehat{\partial_k f}(\xi) = \int_{\mathbf{R}^n} e^{-2\pi i\langle \xi, x\rangle} \partial_k f(x)\, \mathrm{d}x = 2\pi i \xi_k \hat{f}(\xi). \qquad \blacksquare$$

Remark 74.9. Proposition 74.8 can be understood as saying that the Fourier transform diagonalises differentiation. This fact makes the Fourier transform an enormously powerful tool in the theory of linear differential equations. ♣

Proposition 74.10. *Let $k \in \{1, \dots, n\}$. If $f, x_k f \in L^1(\mathbf{R}^n, V)$, then $\partial_k \hat{f} \in C_b^0(\mathbf{R}^n, V)$ and*

$$\widehat{x_k f} = \frac{i}{2\pi} \partial_k \hat{f}.$$

Proof. For every $t \neq 0$ and $\varepsilon > 0$

$$\begin{aligned}\left|\frac{\hat{f}(\xi + te_k) - \hat{f}(\xi)}{2\pi i t} + \widehat{x_k f}(\xi)\right| &\leqslant \int_{\mathbf{R}^n} |(2\pi t x_k)^{-1}(e^{-2\pi i t x_k} - 1 + 2\pi i t x_k)||x_k f(x)|\,\mathrm{d}x \\ &\leqslant \sup_{|s|\leqslant 2\pi\varepsilon} |s^{-1}(e^{-is} - 1 + is)| \cdot \|x_k f\|_{L^1} \\ &\quad + (2\varepsilon^{-1} + 1)\int_{|x_k| \geqslant \varepsilon/t} |x_k f(x)|\,\mathrm{d}x =: \mathrm{I}(\varepsilon) + \mathrm{II}(\varepsilon, t).\end{aligned}$$

Since

$$\lim_{\varepsilon \downarrow 0} \lim_{t \to 0} \mathrm{I}(\varepsilon) + \mathrm{II}(\varepsilon, t) = 0,$$

$\partial_k \hat{f}$ exists and agrees with $\frac{2\pi}{i}\widehat{x_k f}$. By Proposition 74.5, $\partial_k \hat{f} \in C_b^0(\mathbf{R}^n, X)$. ■

Remark 74.11. It is customary to define the Fourier transform directly on Schwartz space $\mathcal{S}(\mathbf{R}^n, V)$. But then one still has to prove that it actually maps $\mathcal{S}(\mathbf{R}^n, V)$ to itself. Proving this basically necessitates Proposition 74.8. ♣

Proposition 74.12. *For every $f, g \in L^1(\mathbf{R}^n, \mathbf{C})$*

$$\widehat{f * g} = \hat{f} \cdot \hat{g}.$$

Proof. By change of variables,

$$\begin{aligned}\widehat{f * g}(\xi) &= \int_{\mathbf{R}^n} e^{-2\pi i \langle \xi, x\rangle} \cdot \left(\int_{\mathbf{R}^n} f(x - y) g(y)\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{-2\pi i\langle \xi, x - y\rangle} f(x - y) \cdot e^{-2\pi i \langle \xi, y\rangle} g(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \hat{f}(\xi) \cdot \hat{g}(\xi).\end{aligned} \qquad \blacksquare$$

75 Fourier transform of a Gaussian

Proposition 75.1 (Fourier transform of a Gaussian). *For every $a > 0$*

$$\widehat{e^{-\pi a|x|^2}}(\xi) = a^{-n/2}e^{-\pi|\xi|^2/a}.$$

Remark 75.2. This simple fact plays an important role. ♣

Proof of Proposition 75.1. For every $a > 0$ and $f \in L^1(\mathbf{R}^n)$

$$\widehat{f(ax)} = a^{-n}\hat{f}(\xi/a).$$

Therefore, it suffices to prove the formula for $a = 1$. By Fubini's theorem, it suffices to consider $n = 1$. By direct computation,

$$\begin{aligned}\widehat{e^{-\pi x^2}}(\xi) &= \int_{\mathbf{R}} e^{-\pi x^2-2\pi ix\xi}\,\mathrm{d}x \\ &= e^{-\pi\xi^2}\int_{\mathbf{R}} e^{-\pi(x+i\xi)^2}\,\mathrm{d}x.\end{aligned}$$

Since the holomorphic function $e^{-\pi z^2}$ has no poles, by Cauchy's integral theorem, the integral is independent of ξ. For $\xi = 0$, it is well-known to evaluate to 1. ■

Proposition 75.3 (Fourier transform of a multivariate Gaussian). *Let $A \in \mathrm{M}_n(\mathbf{R})$ be symmetric and positiv definite. Define $f \in \mathscr{S}(\mathbf{R}^n, \mathbf{C})$ by*

$$f(x) \coloneqq e^{-\pi\langle Ax,x\rangle}$$

Its Fourier transform $\hat{f}$ satisfies

$$\hat{f}(\xi) = \det A^{-1/2}e^{-\pi\langle A^{-1}\xi,\xi\rangle}.$$

Proof. For $A = \mathbf{1}$ this is an immediate consequence of Proposition 75.1. Therefore, since $\langle Ax, x\rangle = |A^{1/2}x|^2$ and by the change of variables $y = A^{1/2}x$,

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbf{R}^n} e^{-2\pi i\langle\xi,x\rangle}e^{-\langle A^{1/2}x\rangle^2}\,\mathrm{d}x \\ &= \det A^{-1/2}\int_{\mathbf{R}^n} e^{-2\pi i\langle A^{-1/2}\xi,y\rangle}e^{-|y|^2}\,\mathrm{d}y \\ &= \det A^{-1/2}e^{-\pi\langle A^{-1}\xi,\xi\rangle}.\end{aligned}$$ ■

76 Schwartz space

Situation 76.1. Let $n \in \mathbf{N}_0$. Let V be a Banach space space. ×

Definition 76.2 (Schwartz space, $\mathscr{S}$).

(1) A smooth map $f \in C^\infty(\mathbf{R}^n, V)$ is a **Schwartz function** (is **rapidly decreasing**) if for every pair of multi-indices $\alpha, \beta \in \mathbf{N}_0^m$

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbf{R}^n} |x^\alpha \partial^\beta f(x)| < \infty.$$

(2) The **Schwartz space** is the Fréchet space $\mathcal{S}(\mathbf{R}^n, V)$ of Schwartz functions defined by the semi-norms $\|\cdot\|_{\alpha,\beta}$ $(\alpha, \beta \in \mathbf{N}_0^m)$. •

Remark 76.3. U is an open in $\mathcal{S}(\mathbf{R}^n, X)$ if and only if for every $f \in U$ there are a finite subset $\Lambda \subset \mathbf{N}_0^m \times \mathbf{N}_0^m$ and $\varepsilon > 0$ such that

$$\bigcap_{(\alpha,\beta) \in \Lambda} \{g \in \mathcal{S}(\mathbf{R}^n, V) : \|g - f\|_{\alpha,\beta} < \varepsilon\} \subset U. \qquad ♣$$

Example 76.4.

(1) $C_c^\infty(\mathbf{R}^n, V) \subset \mathcal{S}(\mathbf{R}^n, V)$.

(2) The function $x^\alpha e^{-|x|^2}$ is Schwartz function.

(3) The function $(1 + |x|)^{-2}$ is not a Schwartz function.

(4) The function $e^{-|x|^2} \sin(|x|^2)$ is not a Schwartz function. ♠

Definition 76.5.

(1) A smooth function $f \in C^\infty(\mathbf{R}^n, \mathbf{C})$ has **moderate growth** if for every $\alpha \in \mathbf{N}_0$ there is a $k = k(\alpha) \in \mathbf{N}_0$ such that

$$\sup \frac{|\partial^\alpha f(x)|}{(1 + |x|^2)^k} < \infty.$$

(2) Denote the ring of smooth functions of moderate growth by O_M^n. •

$\mathcal{S}(\mathbf{R}^n, V)$ is an O_M^n–module.

Proposition 76.6. *Let $f\colon \mathbf{R}^n \to \mathbf{C}$ be measureable. If for every $\phi \in \mathcal{S}(\mathbf{R}^n, V)$ also $f\phi \in \mathcal{S}(\mathbf{R}^n, V)$, then $f \in O_M^n$ and the endomorphism of $f\cdot\colon \mathcal{S}(\mathbf{R}^n, V) \to \mathcal{S}(\mathbf{R}^n, V)$ is continuous, then $f \in O_M^n$.*

77 Fourier transform on Schwartz space

Situation 77.1. Let $n \in \mathbf{N}_0$. Let V be a complex Hilbert space. ×

Proposition 77.2 (Fourier transform on $\mathcal{S}$).

(1) *If $f \in \mathcal{S}(\mathbf{R}^n, V)$, then $\hat{f} \in \mathcal{S}(\mathbf{R}^n, V)$.*

(2) *The map $\hat{\cdot}\colon \mathcal{S}(\mathbf{R}^n, V) \to \mathcal{S}(\mathbf{R}^n, V)$ is continuous.*

Proof. Let $f \in \mathcal{S}(\mathbf{R}^n, X)$. For every $\alpha, \beta \in \mathbf{N}_0^m$, $\partial^\alpha x^\beta f \in L^1(\mathbf{R}^n, X)$. By Proposition 74.8 and Proposition 74.10,

$$\xi^\alpha \partial^\beta \hat{f} = \frac{(2\pi)^{|\beta|-|\alpha|}}{i^{|\alpha|+|\beta|}} \widehat{\partial^\alpha x^\beta f} \in C_b^0(\mathbf{R}^n, X).$$

Therefore, $\hat{f} \in \mathcal{S}(\mathbf{R}^n, X)$.

By Proposition 74.5,

$$\begin{aligned} \|\hat{f}\|_{\alpha,\beta} = \|\xi^\alpha \partial^\beta \hat{f}\|_{L^\infty} &\leqslant (2\pi)^{|\beta|-|\alpha|} \|\partial^\alpha x^\beta f\|_{L^1} \\ &\leqslant (2\pi)^{|\beta|-|\alpha|} \left\|(1+|x|)^{-(n+1)}\right\|_{L^1} \left\|(1+|x|)^{n+1} \partial^\alpha x^\beta f\right\|_{L^\infty}. \end{aligned}$$

Therefore, $\hat{\cdot}$ is continuous. ■

78 Fourier inversion theorem on Schwartz space

Situation 78.1. Let $n \in \mathbf{N}_0$. Let V be a complex Hilbert space. ×

Theorem 78.2 (Fourier inversion theorem on Schwartz space). *The Fourier transform* $\hat{\cdot} \colon \mathcal{S}(\mathbf{R}^n, V) \to \mathcal{S}(\mathbf{R}^n, V)$ *is an isomorphism, and its inverse* $\check{\cdot} \colon \mathcal{S}(\mathbf{R}^n, V) \to \mathcal{S}(\mathbf{R}^n, V)$ *satisfies*

$$\check{g}(x) := \int_{\mathbf{R}^n} e^{2\pi i \langle \xi, x\rangle} g(\xi) \, \mathrm{d}\xi.$$

Remark 78.3. Evidently,

$$\check{g}(x) = \hat{g}(-x).$$ ♣

Proof of Theorem 78.2. By Remark 78.3 and Proposition 77.2, $\check{\cdot}$ is continuous. Therefore, it suffices to prove that $\check{\cdot}$ is the inverse of $\hat{\cdot}$.

Let $f \in \mathcal{S}(\mathbf{R}^n, V)$. By the dominated convergence theorem,

$$\begin{aligned} \check{\hat{f}}(y) &= \lim_{t \downarrow 0} \int_{\mathbf{R}^n} e^{2\pi i \langle \xi, y\rangle - 4\pi^2 t |\xi|^2} \hat{f}(\xi) \, \mathrm{d}\xi \\ &= \lim_{t \downarrow 0} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{-2\pi i \langle \xi, x - y\rangle - 4\pi^2 t |\xi|^2} f(x) \, \mathrm{d}x \mathrm{d}\xi. \end{aligned}$$

For $t > 0$ the integrand is integrable on $\mathbf{R}^n \times \mathbf{R}^n$. Therefore, by Fubini's theorem,

$$\check{\hat{f}}(y) = \lim_{t \downarrow 0} (K_t * f)(y) \quad \text{with} \quad K_t(z) := \int_{\mathbf{R}^n} e^{-2\pi i \langle \xi, z\rangle - 4\pi^2 t |\xi|^2} \mathrm{d}\xi.$$

By Proposition 75.1,

$$K_t(z) = \frac{e^{-|z|^2/4t}}{(4\pi t)^{n/2}}.$$

(This is the heat kernel.) Therefore,

$$\check{\hat{f}} = \lim_{t \to 0} K_t * f = f.$$

Similarly or by Remark 78.3, $\hat{\check{g}} = g$. ■

Here is an important consequence of Theorem 78.2.

Proposition 78.4 (Fourier transform of a product). *For every $f, g \in \mathcal{S}(\mathbf{R}^n, \mathbf{C})$*

$$\widehat{fg} = \hat{f} * \hat{g}.$$

Proof. By Proposition 74.12 with $\check{f}$ and $\check{g}$,

$$\widecheck{fg} = \check{f} * \check{g}.$$

Therefore and by Remark 78.3,

$$\widehat{fg}(\xi) = \int_{\mathbf{R}^n} \check{f}(-\xi - \eta)\check{g}(\eta)\, \mathrm{d}\eta = \int_{\mathbf{R}^n} \hat{f}(\xi + \eta)\hat{g}(-\eta)\, \mathrm{d}\eta = \int_{\mathbf{R}^n} \hat{f}(\xi - \eta)\hat{g}(y)\, \mathrm{d}\eta = (\hat{f} * \hat{g})(\xi). \quad \blacksquare$$

79 Plancherel's theorem

Situation 79.1. Let $n \in \mathbf{N}_0$. Let V be a complex Hilbert space. ×

Theorem 79.2 (Plancherel's theorem). *For every $f, g \in \mathcal{S}(\mathbf{R}^n, V)$*

$$\langle \hat{f}, \hat{g} \rangle_{L^2} = \langle f, g \rangle_{L^2}.$$

Proof. By Theorem 78.2, the assertion is equivalent to

$$\langle \hat{f}, g \rangle_{L^2} = \langle f, \check{g} \rangle_{L^2}.$$

By Fubini's theorem,

$$\begin{aligned}\langle \hat{f}, g \rangle_{L^2} &= \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \langle e^{-2\pi i \langle \xi, x \rangle} f(x), g(\xi) \rangle\, \mathrm{d}x \mathrm{d}\xi \\ &= \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \langle f(x), e^{2\pi i \langle \xi, x \rangle} g(\xi) \rangle\, \mathrm{d}\xi \mathrm{d}x \\ &= \langle f, \check{g} \rangle_{L^2}.\end{aligned} \quad \blacksquare$$

Corollary 79.3. *The Fourier transform extends to an isometry $\hat{\cdot}\colon L^2(\mathbf{R}^n, V) \to L^2(\mathbf{R}^n, V)$.* ■

Lecture 14

80 Tempered distributions

Situation 80.1. Let $n \in \mathbf{N}$. Let V be a complex Hilbert space. ×

Definition 80.2 (tempered distributions, $\mathcal{S}'$).

(1) A **tempered distribution** is a continuous linear map $T\colon \mathcal{S}(\mathbf{R}^n, \bar{V}) \to \mathbf{C}$.

(2) Denote by $\mathcal{S}'(\mathbf{R}^n, V)$ the topological vector space of tempered distributions equipped with the weak–∗–topology. •

Proposition 80.3. *For every $p \in [1,\infty]$ the map $T_\cdot\colon L^p(\mathbf{R}^n, V) \to \mathcal{S}'(\mathbf{R}^n, V)$ defined by*

$$T_f(\phi) \coloneqq \int_{\mathbf{R}^n} \langle \phi(x), f(x)\rangle \,\mathrm{d}x$$

is a continuous injection.

Proof. Evidently, $T_\cdot$ is an injection. It is continuous because for every $\phi \in \mathcal{S}(\mathbf{R}^n, V)$ the map $f \mapsto T_f(\phi)$ is continuous by Hölder's inequality. ■

Notation 80.4. It is convenient to *identify* $L^p(\mathbf{R}^n, V)$ with its image under $T_\cdot$. The statement

$$T \in L^p(\mathbf{R}^n, V)$$

is to be understood as $T = T_f$ for a (unique) $f \in L^p(\mathbf{R}^n, V)$. ○

Proposition 80.5. $\mathcal{S}(\mathbf{R}^n, V) \hookrightarrow \mathcal{S}'(\mathbf{R}^n, V)$ *is a continuous inclusion with dense image.*

Proof of Proposition 80.5. Since $\mathcal{S}(\mathbf{R}^n, V) \hookrightarrow L^2(\mathbf{R}^n, V)$ is continuous, $\mathcal{S}(\mathbf{R}^n, V) \hookrightarrow \mathcal{S}'(\mathbf{R}^n, V)$ is continuous.

Let $T \in \mathcal{S}'(\mathbf{R}^n, V)$. Every open neighborhood U' of T contains an open neighborhood of T the form

$$U \coloneqq \{S \in \mathcal{S}'(\mathbf{R}^n, V) : |T(\phi) - S(\phi)| < \varepsilon \text{ for every } \phi \in \Phi\}$$

with $\Phi \subset \mathcal{S}(\mathbf{R}^n, V)$ finite. In fact, with out loss of generality, Φ is L^2–orthonormal. In this case, evidently,

$$f \coloneqq \sum_{\phi \in \Phi} T(\phi)\phi \in U.$$

Therefore, $\mathcal{S}(\mathbf{R}^n, V) \subset \mathcal{S}'(\mathbf{R}^n, V)$ is dense. ■

Definition 80.6. Let $k \in \{1,\dots,n\}$. The **weak derivative** $\partial_k\colon \mathcal{S}'(\mathbf{R}^n, V) \to \mathcal{S}'(\mathbf{R}^n, V)$ id defined by

$$(\partial_k T)(\phi) \coloneqq -T(\partial_k \phi).$$ •

Proposition 80.7. *Let $k \in \{1,\dots,n\}$. The map $\partial_k\colon \mathcal{S}'(\mathbf{R}^n, V) \to \mathcal{S}'(\mathbf{R}^n, V)$ is continuous and agrees with the usual derivative ∂_k on $\mathcal{S}(\mathbf{R}^n, V) \subset \mathcal{S}'(\mathbf{R}^n, V)$.*

Proof. By integration by parts, the weak derivative agrees with the usual derivative on $\mathcal{S}(\mathbf{R}^n, V)$. It is continuous, because $\partial_k\colon \mathcal{S}(\mathbf{R}^n, V) \to \mathcal{S}(\mathbf{R}^n, V)$ is continuous. ■

Remark 80.8. If $f \in \mathcal{S}'(\mathbf{R}^n, V)$ satisfies $\partial^\alpha f \in C^0(\mathbf{R}^n, V)$ for every $|\alpha| \leqslant k$, then $f \in C^k(\mathbf{R}^n, V)$. ♣

Definition 80.9. Let $f \in O_M^n$ be a smooth function of moderate growth with values in V For every $T \in \mathcal{S}'(\mathbf{R}^n, \mathrm{Hom}(V, W))$ define $Tf \in \mathcal{S}'(\mathbf{R}^n, W)$ by

$$(Tf)(\phi) \coloneqq T(\phi f^*).$$ •

Proposition 80.10. *Let $f \in O_M^n$. The map $\cdot f\colon \mathcal{S}'(\mathbf{R}^n, V) \to \mathcal{S}'(\mathbf{R}^n, V)$ is continuous.*

81 Convolution with tempered distributions

Situation 81.1. Let $n \in \mathbf{N}$. Let V, W be finite-dimensional Hermitian vector spaces. ×

Definition 81.2. Let $T \in \mathcal{S}'(\mathbf{R}^n, \mathrm{Hom}(V, W))$ and $f \in \mathcal{S}(\mathbf{R}^n, V)$. Define the **convolution** $T * f \in \mathcal{S}'(\mathbf{R}^n, W)$ by

$$(T * f)(\phi) \coloneqq T(\phi * \tilde{f}^*).$$ •

Proposition 81.3. *The map $\cdot * \cdot\colon \mathcal{S}'(\mathbf{R}^n, \mathrm{Hom}(V, W)) \times \mathcal{S}(\mathbf{R}^n, V) \to \mathcal{S}'(\mathbf{R}^n, W)$ extends the convolution of Schwartz functions.*

Proof. If $T \in \mathcal{S}(\mathbf{R}^n, \mathrm{Hom}(V, W))$ and $f \in \mathcal{S}(\mathbf{R}^n, V)$, then

$$\begin{aligned}
\int_{\mathbf{R}^n} \langle (T * f)(x), \phi(x)\rangle \,\mathrm{d}x &= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \langle T(y) f(x-y), \phi(x)\rangle \,\mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \langle T(y), \phi(x) f(x-y)^*\rangle \,\mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \langle T(y), \phi(x) f(x-y)^*\rangle \,\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbf{R}^n} \langle T(y), (\phi * \tilde{f}^*)(y)\rangle \,\mathrm{d}y.
\end{aligned}$$ ■

Proposition 81.4. *If $T \in \mathcal{S}'(\mathbf{R}^n, \mathrm{Hom}(V, W))$ and $f \in \mathcal{S}(\mathbf{R}^n, V)$, then $T * f \in C^\infty(\mathbf{R}^n, W)$.* ■

82 Fourier transform on tempered distributions

Definition 82.1 (Fourier transform on tempered distributions). Let $T \in \mathcal{S}'(\mathbf{R}^n)$. The **Fourier transform** of T is the tempered distribution $\hat{T} \in \mathcal{S}'(\mathbf{R}^n)$ defined by

$$\hat{T}(\phi) \coloneqq T(\check{\phi}).$$ •

Proposition 82.2. *The Fourier transform $\hat{\cdot}\colon \mathcal{S}'(\mathbf{R}^n, V) \to \mathcal{S}'(\mathbf{R}^n, V)$ is continuous and extends $\hat{\cdot}\colon \mathcal{S}(\mathbf{R}^n, V) \to \mathcal{S}(\mathbf{R}^n, V)$.*

Proof. $\hat{\cdot}$ is continuous because $\check{\cdot}\colon \mathcal{S}(\mathbf{R}^n, \bar{V}) \to \mathcal{S}(\mathbf{R}^n, \bar{V})$ is. For $T \in \mathcal{S}(\mathbf{R}^n, V)$ and $\phi \in \mathcal{S}(\mathbf{R}^n, \bar{V})$

$$\begin{aligned}\int_{\mathbf{R}^n} \langle \hat{T}(\xi), \phi(\xi)\rangle \,\mathrm{d}\xi &= \int_{\mathbf{R}^n} \langle e^{-2\pi i\langle \xi,x\rangle} T(x), \phi(\xi)\rangle \,\mathrm{d}x\mathrm{d}\xi \\ &= \int_{\mathbf{R}^n} \langle T(x), e^{2\pi i\langle \xi,x\rangle}\phi(\xi)\rangle \,\mathrm{d}\xi\mathrm{d}x \\ &= \int_{\mathbf{R}^n} \langle T(x), \check{\phi}(x)\rangle \,\mathrm{d}x.\end{aligned}$$ ∎

Proposition 82.3 (Fourier inversion on $\mathcal{S}'$). *The Fourier transform* $\hat{\cdot}\colon \mathcal{S}'(\mathbf{R}^n, V) \to \mathcal{S}'(\mathbf{R}^n, V)$ *is an isomorphism, and its inverse* $\check{\cdot}\colon \mathcal{S}'(\mathbf{R}^n, V) \to \mathcal{S}'(\mathbf{R}^n, V)$ *satisfies*

$$\check{T}(\phi) = T(\hat{\phi}).$$

Proof. This is immediate from Theorem 78.2. ∎

Proposition 82.4. *For every* $T \in \mathcal{S}'(\mathbf{R}^n, V)$ *the following hold:*

(1) $\widehat{\partial^\alpha T} = (2\pi i)^{|\alpha|}\xi^\alpha \hat{T}$.

(2) $\widehat{x^\alpha T} = (-2\pi i)^{-|\alpha|}\partial^\alpha \hat{T}$.

(3) $\widehat{T * f} = \hat{T} \cdot \hat{f}$.

Proof. (1) and (2) are obvious. To prove (3), observe that

$$\widehat{T * f}(\phi) = T(\check{\phi} * \tilde{f}^*)$$

and

$$\begin{aligned}\left(\hat{T} \cdot \hat{f}\right)(\phi) &= \hat{T}(\phi \cdot (\hat{f})^*) \\ &= T(\mathcal{F}^{-1}(\phi \cdot (\hat{f})^*)).\end{aligned}$$

Therefore, the assertion follows from

$$\widehat{\check{\phi} * \tilde{f}^*} = \phi \cdot \widehat{\tilde{f}^*}.$$ ∎

83 Sobolev spaces $W^{s,2}$ via Fourier transform

Situation 83.1. Let $n \in \mathbf{N}$. Let $s \in \mathbf{R}$. Let V be a complex Hilbert space. ×

Definition 83.2. Define $(1 + \Delta)^{s/2}\colon \mathcal{S}'(\mathbf{R}^n, V) \to \mathcal{S}'(\mathbf{R}^n, V)$ by

$$(1 + \Delta)^{s/2} f := \mathcal{F}^{-1}(\langle\cdot\rangle^s \mathcal{F}(f))$$

with

$$\langle x\rangle := \left(1 + 4\pi^2|x|^2\right)^{1/2}.$$ •

Remark 83.3. This notation is justified by Proposition 74.8; indeed: $(1+\Delta)^{2k/2} = (1+\Delta)^k$ for every $k \in \mathbf{N}_0$. ♣

Definition 83.4. The **Sobolev space** $W^{s,2}(\mathbf{R}^n, V)$ is the Hilbert space

$$W^{s,2}(\mathbf{R}^n, V) \coloneqq (1+\Delta)^{-s/2} L^2(\mathbf{R}^n, V) \subset \mathcal{S}'(\mathbf{R}^n, V)$$

with

$$\langle \cdot, \cdot \rangle_{W^{s,2}} \coloneqq \langle (1+\Delta)^{s/2} \cdot, (1+\Delta)^{s/2} \cdot \rangle_{L^2}.$$ •

Remark 83.5. Evidently, $H^0(\mathbf{R}^n, V) = L^2(\mathbf{R}^n, V)$. ♣

Proposition 83.6. *Let $s \in \mathbf{R}$. The bilinear form $B \colon W^{-s,2}(\mathbf{R}^n, V) \otimes W^{s,2}(\mathbf{R}^n, V) \to \mathbf{R}$ defined by*

$$B(f \otimes g) \coloneqq \langle (1+\Delta)^{-s/2} f, (1+\Delta)^{s/2} g \rangle_{L^2}$$

induces an isomorphism

$$W^{-s,2}(\mathbf{R}^n, V) \cong W^{s,2}(\mathbf{R}^n, V)^*.$$

Proof. For $s = 0$ this is the the Riesz representation theorem for $L^2(\mathbf{R}^n, V)$. This implies the assertion for every $s \in \mathbf{R}$. ■

Remark 83.7. By Plancherel's theorem,

$$B(f \otimes g) = \langle (1+\Delta)^{-s/2} f, \Delta^{s/2} g \rangle_{L^2} = \langle \langle \cdot \rangle^{-s} \hat{f}, \langle \cdot \rangle^{s} \hat{g} \rangle_{L^2} = \int_{\mathbf{R}^n} \langle \hat{f}, \hat{g} \rangle(\xi) \, \mathrm{d}\xi$$
$$= \int_{\mathbf{R}^n} \langle f, g \rangle(x) \, \mathrm{d}x.$$ ♣

Proposition 83.8. *$\mathcal{S}(\mathbf{R}^n, V) \subset W^{s,2}(\mathbf{R}^n, V)$ is dense.*

Proof. Since $(1+\Delta)^{-s} \mathcal{S}(\mathbf{R}^n, V) = \mathcal{S}(\mathbf{R}^n, V)$, this is a consequence of $\mathcal{S}(\mathbf{R}^n, V) \subset L^2(\mathbf{R}^n, V)$ being dense. ■

Proposition 83.9. *Let $s, t \in \mathbf{R}$. $f \in W^{s+t,2}(\mathbf{R}^n, V)$ if and only if $(1+\Delta)^{t/2} f \in W^{s,2}(\mathbf{R}^n, V)$; moreover,*

$$\|f\|_{W^{s+t,2}} = \|(1+\Delta)^{t/2} f\|_{W^{s,2}}.$$ ■

Proposition 83.10. *If $s \in \mathbf{N}_0$, then $f \in W^{s,2}(\mathbf{R}^n, V)$ if and only if*

$$\partial^\alpha f \in L^2(\mathbf{R}^n, V) \quad \text{for every} \quad \alpha \in \mathbf{N}_0^n \quad \text{with} \quad |\alpha| \leqslant s;$$

moreover,

$$\|f\|_{W^{s,2}}^2 = \sum_{k=0}^{s} \binom{s}{k} \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^2}^2.$$

Proof. By Theorem 79.2 and Proposition 74.8,

$$\|f\|_{W^{s,2}}^2 = \int_{\mathbf{R}^n} \langle\xi\rangle^{2s} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \quad \text{and} \quad \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^2}^2 = \int_{\mathbf{R}^n} |2\pi x|^{2k} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi.$$

By the binomial theorem,

$$\langle x\rangle^{2s} = \sum_{k=0}^{s} \binom{s}{k} |2\pi x|^{2k}.$$

This implies the assertion. ∎

Proposition 83.11. *Let $p \in \mathbf{R}[x_1, \ldots, x_n] \otimes \mathscr{L}(V)$ be a polynomial of degree $\deg(p) = k$. The linear map $p(\partial)\colon \mathcal{S}'(\mathbf{R}^n, V) \to \mathcal{S}'(\mathbf{R}^n, V)$ restricts to a bounded linear operator $p(\partial)\colon W^{s+k,2}(\mathbf{R}^n, V) \to W^{s,2}(\mathbf{R}^n, V)$.*

Proof. There is a $c = c(p) > 0$ such that

$$p(\xi) \leqslant c\langle\xi\rangle^k$$

for every $\xi \in \mathbf{R}^n$. Therefore, by Theorem 79.2,

$$\|p(\partial)f\|_{W^{s,2}} = \|\langle\cdot\rangle^s p(\cdot)\hat{f}\|_{L^2} \leqslant c\|\langle\cdot\rangle^{s+k}\hat{f}\|_{L^2} = \|f\|_{W^{s+k,2}}.$$ ∎

84 Morrey embedding $W^{s,2} \hookrightarrow C^{k,\alpha}$ via Fourier transform

Theorem 84.1 (Morrey embedding). *Let $n \in \mathbf{N}$, $k \in \mathbf{N}_0$, $\alpha \in (0,1)$. Let V be a Banach space. Set*

$$s := \frac{n}{2} + k + \alpha.$$

There is a constant $c = c(n,s) > 0$ such that $W^{s,2}(\mathbf{R}^n, V) \subset C_b^{k,\alpha}(\mathbf{R}^n, V)$ and for every $f \in W^{s,2}(\mathbf{R}^n, V)$

$$\|f\|_{C^{k,\alpha}} \leqslant c\|f\|_{W^{s,2}}. \tag{84.2}$$

Lemma 84.3. *Let $n \in \mathbf{N}$, $p > 0$, $\alpha \in (0,1)$. There is a constant $c = c(n,p,\alpha) > 0$ such that for every $x \in \mathbf{R}^n$*

$$\int_{\mathbf{R}^n} \frac{|e^{2\pi i\langle\xi,x\rangle} - 1|^p}{|\xi|^{n+\alpha p}} \,\mathrm{d}\xi = c \cdot |x|^{\alpha p}.$$

Proof. Denote the integral by $I(x)$. The integrand blows-up like $|\xi|^{-n+p(1-\alpha)}$ near 0, and decays like $|\xi|^{-(n+\alpha p)}$ near ∞. Therefore, $I(x) < \infty$. Evidently, I is $\mathrm{O}(n)$-invariant; hence: it only depends on $|x|$. By change of variables,

$$I(tx) = t^{\alpha p} I(x).$$

This proves the assertion with $c := I(e_1)$. ∎

Proof of Theorem 84.1. It suffices to prove (84.2) for $f \in \mathcal{S}(\mathbf{R}^n, V)$. Indeed, $\mathcal{S}(\mathbf{R}^n, V) \subset W^{s,2}(\mathbf{R}^n, V)$ is dense. If $(f_n) \in \mathcal{S}(\mathbf{R}^n, V)^{\mathbf{N}}$ converges to $f \in W^{s,2}(\mathbf{R}^n, V)$, then, by (84.2), (f_n) is Cauchy in $C_b^{k,\alpha}(\mathbf{R}^n, V)$. Therefore, (f_n) converges to $\tilde{f} \in C_b^{k,\alpha}(\mathbf{R}^n, V)$. Both f and $\tilde{f}$ are limits of (f_n) in $\mathcal{S}'(\mathbf{R}^n, V)$. Since $\mathcal{S}'(\mathbf{R}^n, V)$ is Hausdorff, $f = \tilde{f}$.

It suffices to prove (84.2) $k = 0$. By Theorem 78.2, for every $f \in \mathcal{S}(\mathbf{R}^n, V \otimes \mathbf{C})$

$$|f(x)| = \left|\int_{\mathbf{R}^n} e^{2\pi i\langle \xi, x\rangle} \hat{f}(\xi)\,\mathrm{d}\xi\right| \leqslant \|\langle \xi\rangle^{-s}\|_{L^2} \cdot \|\langle \xi\rangle^s \hat{f}(\xi)\|_{L^2};$$

moreover, by Lemma 84.3,

$$\begin{aligned} |f(x) - f(y)| &= \int_{\mathbf{R}^n} \left|(e^{2\pi i\langle \xi, x-y\rangle} - 1) \cdot \hat{f}(\xi)\right| \mathrm{d}\xi \\ &\leqslant \|e^{2\pi i\langle \xi, x-y\rangle}|\xi|^{-s}\|_{L^2} \cdot \|\langle \xi\rangle^s \hat{f}(\xi)\|_{L^2} \\ &\leqslant c \cdot |x-y|^\alpha \cdot \|\langle \xi\rangle^s \hat{f}(\xi)\|_{L^2}. \end{aligned}$$

■

Lecture 15

This lecture finishes the discussion of Sobolev spaces via the Fourier transform. We end with a little digression on the (sharp) trace theorem.

85 Sobolev multiplication $W^{s,2} \otimes W^{s,2} \to W^{s,2}$ via Fourier transform

Theorem 85.1 (Sobolev multiplication). *Let $n \in \mathbf{N}$. Let $s > n/2$. There is a constant $c = c(n,s) > 0$ such that if $f, g \in W^{s,2}(\mathbf{R}^n)$, then $fg \in W^{s,2}(\mathbf{R}^n)$ and*

$$\|fg\|_{W^{s,2}} \leqslant c\|f\|_{W^{s,2}}\|g\|_{W^{s,2}}. \tag{85.2}$$

Remark 85.3. In fact, if $s > n/2$, $0 \leqslant r \leqslant s$, $f \in W^{s,2}(\mathbf{R}^n)$ and $g \in W^{r,s}$, then

$$\|fg\|_{W^{r,2}} \leqslant c(n,s)\|f\|_{W^{s,2}}\|g\|_{W^{r,2}}.$$ ♣

The proof relies on the following elementary observation.

Lemma 85.4 (Triangle inequality for $\langle\cdot\rangle^s$). *Let $n \in \mathbf{N}$. For every $x, y \in \mathbf{R}^n$ and $s \geqslant 0$*

$$\langle x+y\rangle^s \leqslant 2^s\big(\langle x\rangle^s + \langle y\rangle^s\big).$$

Proof. By direct computation,

$$\langle x+y\rangle^s = \big(1+|x+y|^2\big)^{s/2} \leqslant \big(1+2|x|^2+2|y|^2\big)^{s/2} \leqslant 2^{s/2}\big(\langle x\rangle^2 + \langle y\rangle^2\big)^{s/2} \leqslant 2^s(\langle x\rangle^s + \langle y\rangle^s).$$ ∎

Proof of Theorem 85.1. It suffices to prove (85.2) for $f, g, \in \mathcal{S}(\mathbf{R}^n)$. By Proposition 78.4,

$$\|fg\|_{W^{s,2}} = \|\langle\xi\rangle^s\widehat{fg}\|_{L^2} \leqslant \|\langle\xi\rangle^s(\hat f * \hat g)\|_{L^2}.$$

By Lemma 85.4,

$$\begin{aligned}
|\langle\xi\rangle^s(\hat f * \hat g)(\xi)| &\leqslant \int_{\mathbf{R}^n} |\langle\xi\rangle^s \hat f(\xi-\eta)\hat g(\eta)|\,\mathrm{d}\eta \\
&\leqslant 2^s\int_{\mathbf{R}^n} |(\langle\xi-\eta\rangle^s + \langle\eta\rangle^s)\hat f(\xi-\eta)\hat g(\eta)|\,\mathrm{d}\eta \\
&\leqslant 2^s\int_{\mathbf{R}^n} |\langle\xi-\eta\rangle^s\hat f(\xi-\eta)||\hat g(\eta)| + |\hat f(\xi-\eta)||\langle\eta\rangle^s\hat g(\eta)|\,\mathrm{d}\eta \\
&= 2^s(|\langle\xi\rangle^s\hat f| * |\hat g| + |\hat f| * |\langle\xi\rangle^s\hat g|).
\end{aligned}$$

Therefore and by Young's inequality,

$$\|fg\|_{W^{s,2}} \leqslant 2^s\Big(\|f\|_{W^{s,2}}\|\hat g\|_{L^1} + \|\hat f\|_{L^1}\|g\|_{W^{s,2}}\Big).$$

By the Cauchy–Schwarz inequality,

$$\|\hat f\|_{L^1} \leqslant \|\langle x\rangle^{-s}\|_{L^2}\|\langle x\rangle^s\hat f\|_{L^2}.$$

This proves the assertion with $c := 2^{s+1}\|\langle x\rangle^{-s}\|_{L^2}$. ∎

86 Schwartz representation theorem

Theorem 86.1 (Schwartz representation theorem).

$$\mathcal{S}'(\mathbf{R}^n, V) = \bigcup_{k,\ell\in\mathbf{N}_0} \langle x\rangle^k (1+\Delta)^\ell C_b^0(\mathbf{R}^n, V)$$

Theorem 86.2 (Schwartz representation theorem in terms of Sobolev spaces).

$$\mathcal{S}'(\mathbf{R}^n, V) = \bigcup_{\substack{k\in\mathbf{N}_0\\ s\in\mathbf{R}}} \langle x\rangle^k W^{s,2}(\mathbf{R}^n, V).$$

Remark 86.3. Theorem 86.2 is crucial for interior elliptic regularity. It provides the foundation for the bootstrapping argument. ♣

Proof Theorem 86.2. Let $T \in \mathcal{S}'(\mathbf{R}^n, V)$. Since T is continuous, $T^{-1}B_1(0) \subset \mathcal{S}(\mathbf{R}^n, V)$ is an open neighborhood of 0. Therefore, it contains an open set of the form

$$\{\phi \in \mathcal{S}(\mathbf{R}^n, V) : \|\langle x\rangle^k \phi\|_{C^k} \leqslant c^{-1}\}.$$

Consequently,

$$|T(\phi)| \leqslant \|\langle x\rangle^k \phi\|_{C^k}.$$

By Theorem 84.1, for $s > k + n/2$

$$|T(\phi)| \leqslant c\|\phi\|_{\alpha,\beta} \leqslant \|\langle x\rangle^k \phi\|_{W^{s,2}}.$$

for $s > n/2$. Therefore, $T \in \left(\langle x\rangle^{-k} W^{s,2}\right)^* = \langle x\rangle^k W^{-s,2}$. ■

Proof of Theorem 86.1. Let $T \in \mathcal{S}'(\mathbf{R}^n, V)$. By Theorem 86.2, there is a $g \in L^2(\mathbf{R}^n, V)$ with

$$T = \langle x\rangle^{-k}(1+\Delta)^\ell (1+\Delta)^{-s/2} g$$

with $k, \ell \in \mathbf{N}_0$ and $s > n/2$. By Theorem 85.1, $f \coloneqq (1+\Delta)^{-s/2} g \in C_b^0(\mathbf{R}^n, V)$. ■

87 The Bessel kernel

Situation 87.1. Let $n \in \mathbf{N}$. Let $s > 0$. ×

Definition 87.2. The **Bessel kernel** $G_s \in C^\infty(\mathbf{R}^n\backslash\{0\}, (0,\infty))$ is defined by

$$G_s(x) \coloneqq \frac{1}{(4\pi)^{s/2}\Gamma(s/2)} \int_0^\infty \frac{e^{-\frac{\pi|x|^2}{t} - \frac{t}{4\pi}}}{t^{1+\frac{n-s}{2}}}\,\mathrm{d}t. \tag{87.3}$$ •

Proposition 87.4.

(1) *G_s has the following asymptotic as $x \to 0$*

$$G_s(x) \sim c_s^0 \cdot \begin{cases} |x|^{s-n} & \text{if } s < n \\ -\log|x| & \text{if } s = n \\ 1 & \text{if } s > n \end{cases} \qquad \text{with} \quad c_s^0 := \begin{cases} \frac{\Gamma\left(\frac{|n-s|}{2}\right)}{2^s \pi^{s/2}} & \text{if } s \neq n \\ -\frac{1}{2^{d-1}\pi^{d/2}} & \text{if } s = n. \end{cases}$$

(2) *G_s has the following asymptotic as $x \to \infty$*

$$G_s(x) \sim c_s^\infty \cdot \frac{e^{-|x|}}{|x|^{\frac{n+1-s}{2}}} \quad \text{with} \quad c_s^\infty := \frac{1}{2^{\frac{n+s-1}{2}} \pi^{\frac{n-1}{2}} \Gamma(\frac{s}{2})}.$$

(3) *$G_s \in L^1(\mathbf{R}^n)$ and*

$$\mathscr{F}^{-1}(G_s) = \langle x \rangle^{-s}.$$

In particular,

$$(1 + \Delta)^{-s/2} f = G_s * f.$$

Proof. This is not terribly difficult, but quite lengthy to do in detail; see [Ste70, Chapter V §3].
Using $1/a = \int_0^\infty e^{-ta} \mathrm{d}t$ and Proposition 75.1 it easy to compute G_2; see [Eva10, §4.3.1(b) Example 1]. For $s \neq 2$ one uses

$$a^{-s/2} = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-at} t^{\frac{s}{2}-1} \, \mathrm{d}t$$

for $s > 0$ which is easy to prove from the integral formula of the Γ function. ∎

88 Sobolev multiplication: $C^{\lceil s \rceil} \otimes W^{s,2} \to W^{s,2}$

Theorem 88.1 (Sobolev multiplication). *Let $n \in \mathbf{N}_0$. Let $s \geqslant 0$. There is a constant $c = c(n, s) > 0$ such that if $f \in C_b^{\lceil s \rceil}(\mathbf{R}^n)$ and $g \in W^{s,2}(\mathbf{R}^n, V)$, then $fg \in W^{s,2}(\mathbf{R}^n, V)$ and*

$$\|fg\|_{W^{s,2}} \leqslant c \sum_{\ell=0}^{\lceil s \rceil} \|\nabla^\ell f\|_{L^\infty} \cdot \|g\|_{W^{s-\ell,2}}. \tag{88.2}$$

Proof. If $s \in \mathbf{N}_0$, then it follows from Proposition 83.10. Therefore, by Proposition 83.9 it remains to consider $0 < s < 1$.

By Proposition 83.9

$$\|fg\|_{W^{s,2}} = \|(1 + \Delta)^{(s-1)/2}(fg)\|_{W^{1,2}}.$$

By Proposition 87.4,

$$(1 + \Delta)^{(s-1)/2}(fg) = G_{1-s} * (fg).$$

By Young's inequality,

$$\|(1 + \Delta)^{(s-1)/2}(fg)\|_{L^2} \leqslant \|G_{1-s}\|_{L^1} \cdot \|f\|_{C^0} \cdot \|g\|_{L^2}.$$

By direct inspection,

$$\begin{aligned}\nabla(G_{1-s} * (fg))(x) &= \int_{\mathbf{R}^n} (\nabla G_{1-s})(x-y) f(y) g(y)\,\mathrm{d}y \\ &= f(x) \int_{\mathbf{R}^n} (\nabla G_{1-s})(x-y) g(y)\,\mathrm{d}y \\ &\quad - \int_{\mathbf{R}^n} (\nabla G_{1-s})(x-y)(f(x)-f(y)) g(y)\,\mathrm{d}y.\end{aligned}$$

The first term is $f(x)\nabla(1+\Delta)^{(s-1)/2}g$. By the fundamental theorem of calculus,

$$|f(x)-f(y)| \leqslant \|f\|_{C^1} \cdot \max\{|x-y|, 2\}.$$

Moreover, by Proposition 87.4, $|\nabla G_{1-s}| \cdot \max\{|\cdot|, 1\} \leqslant cG_{1-s}$. Therefore,

$$\begin{aligned}\|\nabla(1+\Delta)^{(s-1)/2}(fg)\|_{L^2} \leqslant \|f\|_{C^0} &\cdot \|\nabla(1+\Delta)^{(s-1)/2} g\|_{L^2} \\ &+ \|f\|_{C^1} \cdot \|g\|_{W^{s-1,2}}.\end{aligned}$$

This proves the assertion. ■

89 Rellich's theorem via Fourier transform

Theorem 89.1 (Rellich's theorem). *Let $n \in \mathbf{N}$, $s < t$, $\alpha < \beta$. The inclusion*

$$\langle x\rangle^{-\beta} W^{t,2}(\mathbf{R}^n) \hookrightarrow \langle x\rangle^{-\alpha} W^{s,2}(\mathbf{R}^n)$$

is compact.

Proof. Without loss of generality, $s = \beta = 0$. Let $(f_n) \in W^{t,2}(\mathbf{R}^n)^{\mathbf{N}}$ with $\|f_n\|_{W^{t,2}} \leqslant 1$. By the Banach–Alaoglu theorem, after passing to a subsequence, (f_n) weakly converges in $W^{t,2}(\mathbf{R}^n)^{\mathbf{N}}$. Without loss of generality, the weak limit is 0. It remains to prove that it is the strong limit in $\langle x\rangle^{-\alpha} L^2(\mathbf{R}^n)$ as well. Let $R > 0$. Denote by $\phi_R \in C^\infty(\mathbf{R}^n, [0,1])$ a smooth function with $\phi_R(x) = 1$ for $x \in B_R(0)$ and $\phi_R(x) = 0$ for $x \notin B_{2R}(0)$ and $\|\phi_R\|_{C^{\lceil t\rceil}} \leqslant c = c(\lceil t\rceil) > 0$. By Theorem 79.2,

$$\|f_n\|_{\langle x\rangle^{-\alpha} L^2} \leqslant \|\phi_R f_n\|_{L^2} + 2R^\alpha = \left\|\widehat{\phi_R f_n}\right\|_{L^2} + 2R^\alpha$$

Since (f_n) weakly converges to zero in L^2 and

$$\widehat{\phi_R f_n}(\xi) = \langle \phi_R(x) e^{-2\pi i\langle \xi, x\rangle}, f_n(x)\rangle,$$

$(\widehat{\phi_R f_n})$ converges point-wise to zero and is uniformly bounded. By the dominated convergence theorem,

$$\lim_{n\to\infty} \int_{|\xi|\leqslant R} |\widehat{\phi_R f_n}|^2\,\mathrm{d}\xi = 0.$$

Moreover, by Theorem 88.1,

$$\begin{aligned}\int_{|\xi|\geqslant R} \left|\widehat{\phi_R f_n}\right|^2 \,\mathrm{d}\xi &\leqslant R^{-2t} \|\langle\xi\rangle^t \widehat{\phi_R f_n}\|_{L^2}^2 \\ &= R^{-2t} \|\phi_R f_n\|_{W^{t,2}}^2 \\ &\leqslant cR^{-2t}.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|f_n\|_{\langle x\rangle^{-\alpha} L^2} \leqslant c(R^\alpha + R^{-2t})$$

for every $R > 0$. The assertion follows upon passing to $R \to \infty$. ■

Remark 89.2. There is another proof using Arzela–Ascoli; cf. [LM89, Theorem III.2.6]. ♣

Lecture 16

90 The L^2 trace theorem

Situation 90.1. Let $n \in \mathbf{N}$. Consider $\mathbf{R}^n = \mathbf{R} \times \mathbf{R}^{n-1}$ with coordinates (t, x). ×

Definition 90.2. Let $k \in \mathbf{N}_0$. Define $\iota\colon \mathbf{R}^{n-1} \to \mathbf{R}^n$ by $\iota(x) := (0, x)$. Define the k**–jet restriction map** $\mathrm{res}_k\colon \mathcal{S}(\mathbf{R}^n, V) \to \mathcal{S}(\mathbf{R}^{n-1}, V)^{\oplus k}$ by

$$\mathrm{res}_k(f) := ((\partial_t^\ell f) \circ \iota)_{\ell=0}^k.$$

•

Theorem 90.3 (L^2 Trace Theorem). *Let $s > \frac{1}{2}$ and $k \in \mathbf{N}_0$ with $k > s - \frac{1}{2}$.*

(1) *The k–jet restriction map $\mathrm{res}_k\colon \mathcal{S}(\mathbf{R}^n, V) \to \mathcal{S}(\mathbf{R}^{n-1}, V)^{\oplus k}$ extends to bounded operator*

$$\mathrm{res}_k\colon W^{s,2}(\mathbf{R}^n, V) \to \bigoplus_{\ell=0}^k W^{s-\ell-\frac{1}{2},2}(\mathbf{R}^{n-1}, V).$$

(2) res_k *has a bounded right inverse*

$$\mathrm{ext}_k\colon \bigoplus_{\ell=0}^k W^{s-\ell-\frac{1}{2},2}(\mathbf{R}^{n-1}, V) \to W^{s,2}(\mathbf{R}^n, V).$$

Remark 90.4. Theorem 90.3 fails for $s = k + \frac{1}{2}$; see https://math.stackexchange.com/questions/4431637/the-trace-theorem-for-functions-in-h1-2-omega. ♣

The proof of the L^2 Trace Theorem relies on the following.

Proposition 90.5 (Fourier transform and restriction). *Define $I\colon \mathcal{S}(\mathbf{R}^n, V) \to \mathcal{S}(\mathbf{R}^{n-1}, V)$ by*

$$(If)(x) := \int_{\mathbf{R}} f(t, x)\,\mathrm{d}x.$$

For every $f \in \mathcal{S}(\mathbf{R}^n, V)$

$$\widehat{\mathrm{res}_0(f)} = I\hat{f}.$$

Proof. By the Fourier inversion theorem on Schwartz space,

$$\begin{aligned}
\mathrm{res}_0 f(x) &= \int_{\mathbf{R}^n} e^{2\pi i \langle (\tau,\xi),(0,x)\rangle} \hat{f}(\tau, \xi)\,\mathrm{d}\tau\,\mathrm{d}\xi \\
&= \int_{\mathbf{R}^{n-1}} e^{2\pi i\langle \xi, x\rangle} \int_{\mathbf{R}} \hat{f}(\tau, \xi)\,\mathrm{d}\tau\,\mathrm{d}\xi \\
&= \mathcal{F}^{-1}(I\mathcal{F}(f))(x).
\end{aligned}$$

■

Proof of Theorem 90.3. By Theorem 79.2 and Proposition 90.5,

$$\|\mathrm{res}_0(f)\|^2_{W^{s-1/2,2}} = \int_{\mathbf{R}^{n-1}} \langle\xi\rangle^{2s-1} |\widehat{\mathrm{res}_0(f)}(\xi)|^2 \,\mathrm{d}\xi = \int_{\mathbf{R}^{n-1}} \langle\xi\rangle^{2s-1} \left|\int_{\mathbf{R}} \hat f(\tau,\xi)\,\mathrm{d}\tau\right|^2 \mathrm{d}\xi.$$

By Cauchy–Schwarz,

$$\left|\int_{\mathbf{R}} \hat f(\tau,\xi)\,\mathrm{d}\tau\right|^2 \leqslant \int_{\mathbf{R}} \langle(\tau,\xi)\rangle^{-2s}\,\mathrm{d}\tau \cdot \int_{\mathbf{R}} \langle(\tau,\xi)\rangle^{2s} |\hat f(\tau,\xi)|^2\,\mathrm{d}\tau.$$

By the change of variables $\tau = \langle\xi\rangle\tilde\tau$,

$$\int_{\mathbf{R}} \langle\xi\rangle^{2s-1} \langle(\tau,\xi)\rangle^{-2s}\,\mathrm{d}\tau = \int_{\mathbf{R}} \langle\tau\rangle^{-2s}\,\mathrm{d}\tau =: c(s).$$ [2]

Therefore and by Plancherel's theorem,

$$\|\mathrm{res}_0(f)\|^2_{W^{s-\frac12,2}} \leqslant c(s)\int_{\mathbf{R}^{n-1}}\int_{\mathbf{R}} \langle(\tau,\xi)\rangle^{2s}|\hat f(\tau,\xi)|^2\,\mathrm{d}\tau\,\mathrm{d}\xi = c(s)\|f\|^2_{W^{s,2}}.$$

This proves (1) for $k = 0$ and, therefore, evidently, for every $k \in \mathbf{N}_0$.

Let $\chi \in \mathscr{S}(\mathbf{R})$ with $\chi(0) = 1$ and and $\partial_t^\ell \chi(0) = 0$ for every $\ell \in \mathbf{N}$. For $g = (g_0, \dots, g_k) \in \mathscr{S}(\mathbf{R}^{n-1}, V)^{\oplus k}$ set

$$\mathrm{ext}_k(g)(t,x) := \sum_{\ell=0}^{k} \frac{t^\ell}{\ell!} \int_{\mathbf{R}^{n-1}} e^{2\pi i\langle x,\xi\rangle} \chi(\langle\xi\rangle t)\hat g_\ell(\xi)\,\mathrm{d}\xi.$$

By the Fourier inversion theorem on Schwartz space,

$$\mathrm{res}_k \circ \mathrm{ext}_k(g) = g.$$

Therefore, it remains to estimate $\|\mathrm{ext}_k(g)\|_{W^{s,2}}$. By the Fourier inversion theorem on Schwartz space,

$$\left\| t^\ell \int_{\mathbf{R}^{n}-1} e^{2\pi i\langle x,\xi\rangle}\chi(\langle\xi\rangle t)\hat g_\ell(\xi)\right\|^2_{W^{s,2}} = \int_{\mathbf{R}^{n-1}} \langle\xi\rangle^{2s} \underbrace{\left(\int_{\mathbf{R}} |\widehat{t^\ell\chi(\langle\xi\rangle t)}(\tau)|^2\,\mathrm{d}\tau\right)}_{=:\mathrm{I}(\xi)} |\hat g_\ell(\xi)|^2\,\mathrm{d}\xi.$$

By Plancherel's theorem and the change of variables $\tilde t = \langle\xi\rangle t$,

$$\mathrm{I}(\xi) = \int_{\mathbf{R}} t^{2\ell}\chi(\langle\xi\rangle t)^2\,\mathrm{d}t = c_\ell\langle\xi\rangle^{-2\ell-1} \quad \text{with} \quad c_\ell := \int_{\mathbf{R}} t^{2\ell}\chi(t)^2\,\mathrm{d}t.$$

Therefore,

$$\|\mathrm{ext}_k(g)\|_{W^{s,2}} \leqslant \sum_{\ell=0}^{k} \frac{c_\ell}{\ell!}\|g_\ell\|_{W^{s-\ell-\frac12,2}}.$$

This proves (2).[3] ■

[2]A computation reveals that $c(s) = \sqrt{\pi}\frac{\Gamma(s-\frac12)}{\Gamma(s)}$.

[3]It is a curious (but probably meaningless) observation that ext_k is bounded for every $s \in \mathbf{R}$—although res_k is only for $s > k - \frac12$).

91 Differential operators on open subsets of $\mathbf{R}^n$

Situation 91.1. Let $n \in \mathbf{N}$. Let $\Omega \subset \mathbf{R}^n$ be an open subset. Let V, W be vector spaces. $\times$

Definition 91.2. A **differential operator of order** k is a linear map $D\colon C^k(\Omega, V) \to C^0(\Omega, W)$ of the form

$$Df = \sum_{|\alpha| \leqslant k} a_\alpha \partial^\alpha f$$

with $a_\alpha \in C^0(\Omega, \mathrm{Hom}(V, W))$ ($\alpha \in \mathbf{N}_0^n$). •

Definition 91.3. The **principal symbol** of D is $\sigma_D \in C^0(\Omega, \mathrm{Hom}(S^k(\mathbf{R}^n)^*, \mathrm{Hom}(V, W)))$ defined by

$$\sigma_D(x)(\xi) \coloneqq \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha.$$

•

Remark 91.4. If $p \in \mathrm{Hom}(S^k\mathbf{R}^n, \mathrm{Hom}(V, W))$, then

$$\widehat{p(\partial)f} = p(2\pi i\xi)\hat{f}.$$

♣

Definition 91.5.

(1) A polynomial $p \in \mathrm{Hom}(S^k(\mathbf{R}^n)^*, \mathrm{Hom}(V, W))$ of degree k is **elliptic** if, for every $\xi \in \mathbf{R}^n \backslash \{0\}$, $p(\xi)$ is invertible.

(2) D is **elliptic** if, for every $x \in \Omega$, σ_D is elliptic. •

Proposition 91.6. *Let $p \in \mathrm{Hom}(S^k(\mathbf{R}^n)^*, \mathrm{Hom}(V, W))$. Consider the formal differential operator $D \coloneqq p(\partial)\colon \mathbf{R}[x_1, \dots, x_n] \otimes V \to \mathbf{R}[x_1, \dots, x_n] \otimes W$. If p is elliptic, then D is is surjective.*

Proof. Exercise. ■

92 Interior elliptic estimate

Situation 92.1. Let $n \in \mathbf{N}$. Let $\Omega \subset \mathbf{R}^n$ be an open subset. Let $\phi, \psi \in C_c^\infty(\Omega)$. Suppose that $\psi = 1$ on a neighborhood of $\operatorname{supp} \phi$. Let V, W be finite-dimensional Euclidean vector spaces. Let

$$D = \sum_{|\alpha| \leqslant k} a_\alpha \partial^\alpha$$

be a differential operator on Ω of order k. Let $s \in \mathbf{R}$. Suppose that $a_\alpha \in C^{\lceil |s+k| \rceil}(\Omega)$ for every $\alpha \in \mathbf{N}_0^n$. $\times$

Theorem 92.2 (interior elliptic estimate). *There is a constant $c > 0$ such that if $\phi f \in W^{s+k,2}(\mathbf{R}^n, V)$, then*

$$\|\phi f\|_{W^{s+k,2}} \leqslant c\big(\|\phi Df\|_{W^{s,2}} + \|\psi f\|_{W^{s+k-1,2}}\big).$$

Remark 92.3. This is an *a priori estimate* in that it assumes that $\phi f \in W^{s+k,2}(\mathbf{R}^n, V)$. This is later complemented by interior elliptic regularity. ♣

Proposition 92.4 (elliptic estimate for $p(\partial)$). *Let $p \in \mathrm{Hom}(S^k(\mathbf{R}^n)^*, \mathrm{Hom}(V, W))$ be an elliptic polynomial of order k. There is a constant $c = c(p) > 0$ such that for every $s \in \mathbf{R}$ and $f \in W^{s,2}(\mathbf{R}^n, V)$*

$$\|f\|_{W^{s+k,2}} \leqslant c(\|p(\partial)f\|_{W^{s,2}} + \|f\|_{W^{s,2}}).$$

Proof. Since p is elliptic, there is a constant $c = c(p) > 0$ such that for every $\xi \in \mathbf{R}^n\backslash\{0\}$ and $v \in V$

$$|p(2\pi i\xi)v| \geqslant c|\xi|^k|v|.$$

Therefore and because $\langle\xi\rangle^k \leqslant 2^k(1 + |\xi|^k)$,

$$\|f\|_{W^{s+k,2}} = \|\langle\xi\rangle^{s+k}\hat{f}\|_{L^2} \leqslant 2^k c\|\langle\xi\rangle^s\widehat{p(\partial)f}\|_{L^2} + 2^k\|\langle\xi\rangle^s\hat{f}\|_{L^2} = 2^k c\|p(\partial)f\|_{W^{s,2}} + 2^k\|f\|_{W^{s,2}}. \blacksquare$$

Proposition 92.5. *There are constants $\delta = \delta(D), c = c(D) > 0$ such that for every $f \in W^{s+k,2}$ with* $\operatorname{supp} f \subset \psi^{-1}(1)$ *and* $\operatorname{diam}\operatorname{supp} f \leqslant \delta$

$$\|f\|_{W^{s+k,2}} \leqslant c\big(\|Df\|_{W^{s,2}} + \|f\|_{W^{s+k-1,2}}\big).$$

Proof. Let $x \in \operatorname{supp} f$. Set $p \coloneqq \sigma_D(x)$. Evidently,

$$\|(D - p(\partial))f\|_{W^{s,2}} \leqslant \varepsilon\|f\|_{W^{s+k,2}} + c_1\|f\|_{W^{s+k-1,2}}$$

with

$$\varepsilon \coloneqq \sum_{|\alpha|=k} \|a_\alpha - a_\alpha(0)\|_{L^\infty(\operatorname{supp} f)} \quad\text{and}\quad c_1 \coloneqq \sum_{|\alpha|<k} \|a_\alpha\|_{C^{\{\lceil s+k\rceil\}}(\operatorname{supp} f)}.$$

Therefore, with $c_2 = c(p)$ from Proposition 92.4

$$\|f\|_{W^{s+k,2}} \leqslant c_2\|Df\|_{W^{s,2}} + c_2(c_1 + c_2)\|f\|_{W^{s+k-1,2}} + c_2\varepsilon\|f\|_{W^{s+k,2}}.$$

By uniform continuity, $c_2\varepsilon \leqslant \frac{1}{2}$ if $\operatorname{diam}\operatorname{supp} f \leqslant \delta \ll 1$. Rearranging the last term proves the asserted estimate. ■

Remark 92.6. The argument in the proof of Proposition 92.5 is sometimes called "freezing coefficients". ♣

Proof of Theorem 92.2. With ε as in Proposition 92.5. Choose a finite set $\{\chi_i : i \in I\} \subset C_c^\infty(\Omega)$ with

$$\operatorname{diam}\operatorname{supp} \chi_i \leqslant \delta \quad\text{and}\quad \sum_{i\in I} \chi_i = 1 \quad\text{on}\quad \operatorname{supp} \phi.$$

By Proposition 92.5 and Theorem 88.1,

$$\begin{aligned}
\|\phi f\|_{W^{s+k,2}} &\leqslant \sum_{i\in I}\|\chi_i\phi f\|_{W^{s+k,2}} \\
&\leqslant c_1 \sum_{i\in I}\|\psi D\chi_i\phi f\|_{W^{s,2}} + \|\psi\chi_i\phi f\|_{W^{s+k-1,2}} \\
&\leqslant c_2 \sum_{i}\|\psi\phi\chi_i Df\|_{W^{s,2}} + \|\psi[D, \chi_i\phi]f\|_{W^{s,2}} + \|\chi_i\phi f\|_{W^{s+k-1,2}} \\
&\leqslant c_3\big(\|\phi Df\|_{W^{s,2}} + \|\psi f\|_{W^{s+k-1,2}}\big)
\end{aligned}$$

because $[D, \chi_i\phi]$ is a differential operator of order $k-1$. ■

93 Interior elliptic regularity

Continue assuming the situation of Section 92.

Theorem 93.1. *Let $f \in \mathcal{S}'(\mathbf{R}^n, V)$ compact support such that $Df \in W^{s,2}(\mathbf{R}^n, W)$.*

Proposition 93.2. *If $\eta \in L^1(\mathbf{R}^n)$ and $f \in W^{s,2}(\mathbf{R}^n, V)$, then*

$$\|\eta * f\|_{W^{s,2}} \leqslant \|\eta\|_{L^1}\|f\|_{W^{s,2}}.$$

Proof. By Proposition 82.4, Plancherel's theorem, and Proposition 74.5,

$$\|\eta * f\|_{W^{s,2}} = \|\langle\xi\rangle^s\hat{\eta}\hat{f}\|_{L^2} \leqslant \|\hat{\eta}\|_{L^\infty}\|\langle\xi\rangle^s\hat{f}\|_{L^2} \leqslant \|\eta\|_{L^1}\|\langle\xi\rangle^s\hat{f}\|_{L^2}.$$ ■

Proof of Theorem 93.1. Let η be a Friedrich's mollifier. By Theorem 86.2, there is an $\ell \in \mathbf{N}_0$ with

$$f \in \langle x\rangle^\ell W^{s+k-\ell,2}(\mathbf{R}^n, V).$$

Therefore, $\phi f \in W^{s-\ell,2}(\mathbf{R}^n, V)$.

The convolution $\eta_\varepsilon * \phi f$ is smooth, has compact support, and

$$\|[D, \eta_\varepsilon * \cdot]\phi f\|_{W^{s-\ell+1,2}} \leqslant c\|\phi f\|_{W^{s-\ell,2}}.$$

If $\ell \geqslant 1$, then, by Theorem 92.2,

$$\begin{aligned}\|\eta_\varepsilon * \phi f\|_{W^{s+k-\ell+1,2}} &\leqslant c\big(\|\eta_\varepsilon * D(\phi f)\|_{W^{s-\ell+1,2}} + \|\eta_\varepsilon * \phi f\|_{W^{s+k-\ell+1,2}}\big)\\ &\leqslant c\big(\|\psi Df\|_{W^{s-\ell-1,2}} + \|\phi f\|_{W^{s+k-\ell+1,2}}\big).\end{aligned}$$

Therefore, there is null-sequence ε_n such that $\eta_{\varepsilon_n} * \phi f$ weakly converges in $W^{s+k-\ell+1}(\mathbf{R}^n, V)$. Hence, $\phi f \in W^{s+k-\ell+1}(\mathbf{R}^n, V)$.

Iterating this argument with a nested sequence of cut-offs proves the assertion. ■

Lecture 17

94 Sobolev spaces on manifolds

Situation 94.1. Let (X, g) be a closed oriented Riemannian manifold of dimension n. Let $\mathcal{N}$ be a neighborhood of $\Delta \subset X \times X$ such that for every $(x, y) \in \mathcal{N}$ there is a unique shortest geodesic γ_y^x from x to y. Let E be a Euclidean vector bundle over X. ×

Definition 94.2. For $s \in (0, 1)$ define

$$[\phi]_{W^{s,2}} := \left(\int_{\mathcal{N}} \frac{|\mathrm{tra}_{\gamma_y^x}(\phi(x)) - \phi(y)|^2}{d(x, y)^{n+2s}} \right)^{1/2}.$$

For $s \in \mathbf{N}_0$ set

$$\|\phi\|_{W^{s,2}} := \sum_{\ell=0}^{s} \|\nabla^\ell \phi\|_{L^2}.$$

For $s \in (0, \infty)\backslash\mathbf{N}$ set

$$\|\phi\|_{W^{s,2}} := \sum_{\ell=0}^{\lfloor s \rfloor} \|\nabla^\ell \phi\|_{L^2} + \|\nabla^{\lfloor s \rfloor}\phi\|_{W^{s-\lfloor s \rfloor,2}}.$$

Denote by $W^{s,2}\Gamma(X, E)$ the completion of $\Gamma(E)$ with respect to the norm $\|\cdot\|_{W^{s,2}}$. For $s < 0$ set $W^{s,2}\Gamma(X, E) := W^{-s,2}\Gamma(X, E^*)$. •

Remark 94.3. Sometimes it can be useful to replace E^* with $E^\dagger := E^* \otimes \Lambda^n T^*X$. ♣

From the discussion in ?? it is should be clear that the results proved for $W^{s,2}(\mathbf{R}^n, V)$ carry over to $W^{s,2}\Gamma(X, E)$. In particular, the following results hold.

Proposition 94.4. $\Gamma(X, E) \subset W^{s,2}\Gamma(X, E)$ *is dense.*

Proposition 94.5. *If $t > s$, then the inclusion $W^{t,2}\Gamma(X, E) \to W^{s,2}\Gamma(X, E)$ is compact.*

Proposition 94.6. *If $s = n/2 + k + \alpha$ with $\alpha \in (0, 1)$, then $W^{s,2}\Gamma(X, E) \hookrightarrow C^{k,\alpha}\Gamma(X, E)$ and there is a constant $c > 0$ such that*

$$\|\phi\|_{C^{k,\alpha}} \leqslant \|\phi\|_{W^{s,2}}.$$

Definition 94.7. Let $D\colon \Gamma(X, E) \to \Gamma(X, F)$ be a differential operator of order k. The **principal symbol** of D, $\sigma_D\colon \Gamma(\mathrm{Hom}(S^kT^*X, \mathrm{Hom}(E, F)))$, is characterised by

$$\sigma_D(\mathrm{d}_x f)\phi(x) = \frac{1}{k!}([D, f^k]\phi)(x)$$

with $f(x) = 0$. D is **elliptic** if, for every $x \in X$ and every non-zero $\xi \in T_xX^*$, $\sigma_D(\xi) \in \mathrm{Hom}(E_x, F_x)$ is an isomorphism. •

Proposition 94.8. *If $D\colon \Gamma(X, E) \to \Gamma(X, F)$ is a differential operator of order k, then it extends uniquely to a bounded linear operator $D\colon W^{s+k,2}\Gamma(X, E) \to W^{s,2}\Gamma(X, F)$.*

Remark 94.9. More generally, one can consider differential operators with possibly non-smooth coefficients. These usually don't define maps $\Gamma(X,E) \to \Gamma(X,F)$ but only $W^{s+k,2}\Gamma(X,E) \to W^{s,2}\Gamma(X,F)$ for suitable ranges of s. ♣

Proposition 94.10. *Let $s \in \mathbf{R}$. Let $D\colon \Gamma(X,E) \to \Gamma(X,F)$ be an elliptic differential operator of order k. There is a constant $c = c(D,s) > 0$ such that for every $\phi \in W^{s+k,2}\Gamma(E)$*

$$\|\phi\|_{W^{s+k,2}} \leqslant c\big(\|D\phi\|_{W^{s,2}} + \|\phi\|_{W^{s+k-1,2}}\big).$$

With the help of ?? the norm in last term in the inequality can be replaced by weaker norms; e.g.:

Corollary 94.11. *Let $s \geqslant 0$. Let $D\colon \Gamma(X,E) \to \Gamma(X,F)$ be an elliptic differential operator of order k. There is a constant $c = c(D,s) > 0$ such that for every $\phi \in W^{s+k,2}\Gamma(E)$*

$$\|\phi\|_{W^{s+k,2}} \leqslant c(\|D\phi\|_{W^{s,2}} + \|\phi\|_{L^1}).$$

■

Denote by $\mathscr{D}'\Gamma(X,E)$ the dual space of $\Gamma(X,E)$ equipped with the weak–∗–topology—the space of distributional sections of E. As usual one can make sense of differential operators acting on $\mathscr{D}'\Gamma(X,E)$ etc. Evidently, $W^{s,2}\Gamma(X,E) \subset \mathscr{D}'\Gamma(X,E)$.

Proposition 94.12. *Let $D\colon \Gamma(X,E) \to \Gamma(X,F)$ be an elliptic differential operator of order k. Let $\phi \in \mathscr{D}'\Gamma(X,E)$. If $D\phi \in W^{s,2}\Gamma(X,F)$, then $\phi \in W^{s+k,2}\Gamma(X,E)$.*

95 Chern genera

Definition 95.1. Let $r \in \mathbf{N}_0$. Let V be a complex vector bundle of rank r over X. Denote by $\phi_V\colon \operatorname{Hom}(S\mathfrak{gl}_r(\mathbf{C}),\mathbf{C})^{\mathrm{Ad}} \to \mathrm{H}_{\mathrm{dR}}(X)$ the Chern–Weil homomorphism associated with V defined by

$$\phi_V(f) \coloneqq [f(F_A)].$$

Let $f \in \mathbf{C}[[x]]$. The **Chern f–genus** of V is

$$c_f(V) \coloneqq \phi_V(\det \circ f(\tfrac{i}{2\pi}\cdot)) \in \mathrm{H}_{\mathrm{dR}}(X,\mathbf{C}).$$ •

Remark 95.2. Denote by $\operatorname{res}_\Delta\colon \operatorname{Hom}(S\mathfrak{gl}_r(\mathbf{C}),\mathbf{C})^{\mathrm{Ad}} \cong \mathbf{C}[[x_1,\dots,x_r]]^{S_r}$ isomorphism induced by restriction to diagonal matrices. Evidently,

$$c_f \coloneqq \operatorname{res}_\Delta(\det \circ f(\tfrac{i}{2\pi}\cdot)) = \prod_{j=1}^{r} f\Big(\frac{ix_j}{2\pi}\Big).$$

In particular, the Chern class $c(V)$ corresponds to $f(x) = 1 + x$. ♣

Example 95.3. The **Todd genus** $\operatorname{td}(V)$ is the Chern genus associated with

$$\frac{x}{1-e^{-x}}.$$ ♠

Proposition 95.4.

(1) *If L is a complex line bundle, then $c_f(L) = f(c_1(L))$.*

(2) *If V_1 and V_2 are two complex vector bundles, then $c_f(V_1 \oplus V_2) = c_f(V_1) \cup c_f(V_2)$.* ■

96 Pontrjagin genera

Proposition 96.1. *For every formal power series of the form*

$$f = 1 + \cdots$$

there is a unique $g = 1 + \cdots$ *with* $g^2 = f$. ∎

Definition 96.2. Let $r \in \mathbf{N}_0$. Let V be a real vector bundle of rank r over X. Denote by $\phi_{V\otimes\mathbf{C}}\colon \operatorname{Hom}(S\mathfrak{gl}_r(\mathbf{C}), \mathbf{C})^{\mathrm{Ad}} \to \mathrm{H}_{\mathrm{dR}}(X, \mathbf{C})$ the Chern–Weil homomorphism defined by

$$\phi_V(f) := [f(F_A)].$$

Let $g \in \mathbf{C}[[x]]$ with $g(0) = 1$ The **Pontrjagin g–genus** of V is

$$p_g(V) := \phi_{V\otimes\mathbf{C}}(\det \circ f(\tfrac{i}{2\pi}\,\cdot\,)) \in \mathrm{H}_{\mathrm{dR}}(X, \mathbf{C}). \quad \text{with} \quad f(x) := \sqrt{g(x^2)}.$$ •

Remark 96.3. Denote by $\operatorname{res}_\Delta\colon \operatorname{Hom}(S\mathfrak{so}(r), \mathbf{R})^{\mathrm{Ad}} \cong \mathbf{R}[[y_1^2, \ldots, y_{\lfloor r/2 \rfloor}^2]]^{S_{\lfloor r/2 \rfloor}}$ the isomorphism induced by restriction to block diagonal matrices with blocks of the form

$$\begin{pmatrix} 0 & -y \\ y & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \end{pmatrix}.$$

Since

$$\det f\begin{pmatrix} 0 & \frac{-iy}{2\pi} \\ \frac{iy}{2\pi} & 0 \end{pmatrix} = \left(\det g\begin{pmatrix} \frac{y^2}{4\pi^2} & 0 \\ 0 & \frac{y^2}{4\pi^2} \end{pmatrix}\right)^{1/2} = g\left(\frac{y^2}{4\pi^2}\right).$$

Therefore,

$$p_g := \operatorname{res}_\Delta(\det \circ f(\tfrac{i}{2\pi}\,\cdot\,)) = \prod_{j=1}^{\lfloor r/2 \rfloor} g\left(\frac{y_j^2}{4\pi^2}\right).$$

In particular, the Pontrjagin class $p(V)$ corresponds to $g(x) = 1 + x$. ♣

Proposition 96.4. *If* V_1 *and* V_2 *are two real vector bundles, then* $p_g(V_1 \oplus V_2) = p_g(V_1) \cup p_g(V_2)$. ∎

Example 96.5. The L genus is the Pontrjagin genus associated with $\ell(x) := \frac{\sqrt{x}}{\tanh\sqrt{x}}$. It appears in the Hirzebruch signature theorem. ♠

Example 96.6. The $\hat{A}$ genus is the Pontrjagin genus associated with $\hat{a}(x) := \frac{\sqrt{x}/2}{\sinh\sqrt{x}/2}$. It appears in the Atiyah–Singer Index Theorem. ♠

Remark 96.7. If

$$g(x) = \sum_{k=0}^{\infty} a_k x^k,$$

then

$$p_g = \prod_{j=1}^{n} \sum_{k=0}^{\infty} a_k \left(\frac{y_j^2}{4\pi^2}\right)^k.$$

Evidently, p_g can be expressed in terms of the Pontrjagin classes

$$p_k = \frac{1}{(2\pi)^{2k}} \sum_{1\leqslant j_1 < \cdots < j_k \leqslant \lfloor r/2 \rfloor} y_{j_1}^2 \cdots y_{j_k}^2.$$

Doing this properly means developing the combinatorial theory of multiplicative sequences.

Here is how to do this (a little bit) for the $\hat{A}$–genus. recall that

$$\frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)} = 1 - \frac{x}{24} + \frac{7x^2}{5760} + \dots .$$

Therefore,

$$\begin{aligned}\hat{A} &= \prod_{j=1}^{n} \left(1 - \frac{1}{24}\frac{y_j^2}{4\pi^2} + \frac{7}{5760}\left(\frac{y_j^2}{4\pi^2}\right)^2 + \cdots\right) \\ &= 1 - \frac{1}{24}\frac{1}{4\pi^2}\sum_{j=1}^{n} y_j^2 + \left(\frac{1}{4\pi^2}\right)^2 \left(\frac{7}{5670}\sum_{j=1}^{n} y_j^4 + \frac{1}{576}\sum_{1\leqslant j_1 < j_2 \leqslant n} y_{j_1}^2 y_{j_2}^2\right) + \cdots .\end{aligned}$$

Since

$$\left(\frac{1}{4\pi^2}\right)^2 \sum_{j=1}^{n} y_j^4 = p_1^2 - 2p_2,$$

it follows that

$$\begin{aligned}\hat{A} &= 1 - \frac{1}{24}p_1 + \frac{1}{5760}\left(7p_1^2 - 14p_2 + 10p_2\right) + \cdots \\ &= 1 - \frac{1}{24}p_1 + \frac{7p_1^2 - 4p_2}{5760} + \cdots .\end{aligned}$$

See `https://gradmath.org/wp-content/uploads/2020/10/Sati-et-al-GJM-2018.pdf` for more useful results along those lines. ♣

Remark 96.8. The inclusion $\mathfrak{u}(r) \hookrightarrow \mathfrak{o}(2r)$ induces $\mathrm{Hom}(S\mathfrak{o}(r), \mathbf{R})^{\mathrm{Ad}} \to \mathrm{Hom}(S\mathfrak{u}(r), \mathbf{R})^{\mathrm{Ad}}$. The corresponding map $\mathbf{R}[[y_1^2, \dots, y_r^2]] \to \mathbf{R}[[x_1, \dots, x_r]]$ is induced by identifying $x_j = y_j$. Since

$$c_{\mathrm{td}} = \prod_{j=1}^{r} \frac{\frac{ix_j}{2\pi}}{1 - e^{-\frac{ix_j}{2\pi}}}$$

and

$$p_{\hat{a}} = e^{-\frac{1}{2}\sum_{j=1}^{r} \frac{iy_j}{2\pi}} \prod_{j=1}^{r} \frac{\frac{iy_j}{2\pi}}{1 - e^{-\frac{iy_j}{2\pi}}},$$

the Todd genus and the $\hat{A}$ genus of a Hermitian vector bundle V are related by

$$\mathrm{td}(V) = e^{\frac{1}{2}c_1(V)}\hat{A}(V).$$ ♣

97 Atiyah–Singer index theorem for Dirac operators

Situation 97.1. Let (X, g) be a closed Riemannian manifold. ×

Definition 97.2. The $\hat{A}$ **genus** of a vector bundle V over X is

$$\hat{A}(V) := p_{\hat{a}}(V) \in \mathrm{H}_{\mathrm{dR}}(X) \quad \text{with} \quad \hat{a}(x) := \frac{\sqrt{x}/2}{\sinh \sqrt{x}/2}$$

and $p_{\hat{a}}$ denoting the Pontrjagin $\hat{a}$–genus. •

Denote by D the associated Dirac operator.

Definition 97.3. Let $(S, \gamma, b, \nabla, \varepsilon)$ be a complex Dirac bundle over (X, g) together with a grading. Denote the twisting curvature if S by F_∇^{tw}. Denote by $\mathrm{str}\colon \ \mathrm{End}_{\mathbf{C}\ell}(S) \to \mathbf{C}$ the **supertrace** defined by $\mathrm{str}(A) := \mathrm{tr}(\varepsilon A)$. Define $\mathrm{ch}^{\mathrm{tw}}(S) \in \mathrm{H}_{\mathrm{dR}}(X)$ by

$$\mathrm{ch}^{\mathrm{tw}}(S) := \mathrm{str}\exp\left(\frac{iF_\nabla^{\mathrm{tw}}}{2\pi}\right). \qquad \bullet$$

.

Theorem 97.4 (Atiyah–Singer Index Theorem). *Let $(S, \gamma, b, \nabla, \varepsilon)$ be a complex Dirac bundle over (X, g) together with a grading. The index of $D^+\colon \ \Gamma(S^+) \to \Gamma(S^-)$ satisfies*

$$\mathrm{index}_{\mathbf{C}} D^+ = \langle \hat{A}(TX)\mathrm{ch}^{\mathrm{tw}}(S), [X]\rangle.$$

Remark 97.5. If S arises from a spin then, structure $\mathrm{ch}^{\mathrm{tw}}(S) = 1$. In particular, $\mathrm{ch}^{\mathrm{tw}}(S \otimes E) = \mathrm{ch}(E)$. This is useful both for computations and for making guesses. ♣

The Atiyah–Singer Index Theorem is insanely powerful. Here are a few applications.

Example 97.6. Let (X, g) be a Riemannian manifold and $S = \Lambda T^*X \otimes \mathbf{C}$ with the obvious grading. With some work one can show that

$$\mathrm{ch}^{\mathrm{tw}}(S) = \frac{\mathrm{e}(TX)}{\hat{A}(TX)}.$$

Therefore, the Atiyah–Singer Index Theorem implies the Chern–Gauß–Bonnet theorem

$$\chi(X) = \langle \mathrm{e}(TX), [X]\rangle. \qquad ♠$$

To prove the above one first needs to determine F_∇^{tw}.

Proposition 97.7. *Define $\delta\colon \ TX \to \mathrm{End}(\Lambda T^*X)$ by*

$$\delta(v) := v^\flat \wedge \cdot + i(v).$$

The twisting curvature of ΛT^*X satisfies

$$F_\nabla^{\mathrm{tw}} = -\frac{1}{4}\sum_{i,j,k,\ell=1}^{n} R_{ijk}^{\ \ \ \ \ell} \cdot e^i \wedge e^j \otimes \delta(e_k)\delta(e_\ell).$$

with

$$R_{ijk}^{\ \ \ \ \ell} := \langle R(e_i, e_j)e_k, e_\ell\rangle.$$

and $(e_1, \dots, e_n)$ orthonormal.

Proof. If $R \in \Omega^2(X, \mathfrak{o}(TX))$ denotes the Riemann curvature tensor, then $F_\nabla \in \Omega^2(X, \mathfrak{o}(\Lambda T^*X))$ satisfies

$$F_\nabla = -\Lambda R^*.$$

Therefore,

$$\begin{aligned} F_\nabla &= \sum_{i,j,k,\ell=1}^{n} -R_{ijk}^{\ \ \ \ell} \cdot e^i \wedge e^j \otimes e^k \wedge i(e_\ell) \\ &= \frac{1}{4} \sum_{i,j,k,\ell=1}^{n} R_{ijk}^{\ \ \ \ell} \cdot e^i \wedge e^j \otimes (\gamma(e_k) + \delta(e_k))(\gamma(e_\ell) - \delta(e_\ell)) \\ &= \frac{1}{4} \sum_{i,j,k,\ell=1}^{n} R_{ijk}^{\ \ \ \ell} \cdot e^i \wedge e^j \otimes (\gamma(e_k)\gamma(e_\ell) - \delta(e_k)\delta(e_\ell) + \delta(e_k)\gamma(e_\ell) - \gamma(e_k)\delta(e_\ell)). \end{aligned}$$

The contributions from first term are R_S. The last two term vanish since $\gamma(v)\delta(w)+\delta(w)\gamma(v) = 0$. This proves the assertion. ■

The Chern–Gauß–Bonnet theorem finally follows from:

Proposition 97.8. *Let $R \in \mathfrak{o}(2n)$. Define $F^{\mathrm{tw}} \in \mathfrak{o}(\Lambda(\mathbf{R}^{2n})^*)$ by*

$$F^{\mathrm{tw}} := -\frac{1}{4} \sum_{k,\ell=1}^{2n} R_k^\ell \cdot \delta(e_k)\delta(\ell).$$

With respect to the grading $\varepsilon = (-1)^{\deg}$ on ΛT^*X

$$\operatorname{str} \exp\left(\frac{iF^{\mathrm{tw}}}{2\pi}\right) = \frac{\mathrm{Pf}(\frac{R}{2\pi})}{\hat{A}(\frac{iR}{2\pi})}.$$

The proof of this is an exercise.

Example 97.9. The Hirzebruch signature theorem follows from the Atiyah–Singer Index Theorem applied to $\Lambda T^*X \otimes \mathbf{C}$ with the signature grading. Indeed,

$$\mathrm{ch}^{\mathrm{tw}}(S) = \frac{L(TX)}{\hat{A}(TX)}.$$

♠

Example 97.10. Let X be a closed Kähler manifold. Let $\mathscr{E} = (E, \bar{\partial}_E)$ be a holomorphic vector bundle together with a Hermitian metric. The **Todd class** of E is

$$\mathrm{td}(E) := \left[\det\left(\frac{iF_A/2\pi}{e^{iF_A/2\pi} - 1}\right)\right] \in \mathrm{H}_{\mathrm{dR}}(X).$$

Applying Atiyah–Singer Index Theorem to $S = \Lambda_{\mathbf{C}} T^*X \otimes_{\mathbf{C}} E$ with the obvious grading implies the Hirzebruch–Riemann–Roch theorem

$$\begin{aligned} \chi(\mathscr{E}) &= \sum_i (-1)^i \dim \mathrm{H}^i(X, \mathscr{E}) = \operatorname{index}\left(\partial_E + \partial_E^* \colon \Omega^{0,\mathrm{ev}}(X, E) \to \Omega^{0,\mathrm{odd}}(X, E)\right) \\ &= \langle \mathrm{td}(TX)\mathrm{ch}(E), [X] \rangle. \end{aligned}$$

♠

Lecture 18

In this lecture I discussed how to compute/understand the terms in the Atiyah–Singer Index Theorem. (I don't have typed notes for this.)

98 Divisibility of $\hat{A}$

Theorem 98.1 (Atiyah–Singer; Rokhlin [Rok52]). *Let (X, g) be a compact Riemannian manifold with $\dim X = 4 \bmod 8$. If $w_2(X) = 0$, then $\langle \hat{A}(TX), [X] \rangle$ is even; in particular, if $\dim X = 4$, then $\sigma(X)$ is divisible by 16.*

Proof. Choose a spin structure on (X, g). The corresponding real spinor S bundle has a quaternionic structure because $\dim X = 4 \bmod 8$. The complex spinor bundle W agrees with the real spinor bundle but only remembers one of the complex structures; however, because $\omega_{\mathbf{C}} = -\omega$ for $\dim X = 4 \bmod 8$ the gradings of S and W are the opposite of each other. Therefore,

$$\begin{aligned} -\operatorname{index}_{\mathbf{C}}(D^+ \colon \Gamma(W^+) \to \Gamma(W^-)) &= \operatorname{index}_{\mathbf{C}}(D^+ \colon \Gamma(S^+) \to \Gamma(S^-)) \\ &= \frac{1}{2} \operatorname{index}_{\mathbf{R}}(D^+ \colon \Gamma(S^+) \to \Gamma(S^-)) \\ &= 2 \operatorname{index}_{\mathbf{H}}(D^+ \colon \Gamma(S^+) \to \Gamma(S^-)). \end{aligned}$$

By the Atiyah–Singer Index Theorem

$$\langle \hat{A}(TX), [X] \rangle = \operatorname{index}_{\mathbf{C}}(D^+ \colon \Gamma(W^+) \to \Gamma(W^-)).$$

If $\dim X = 4$, then by the Hirzebruch signature theorem, $\langle \hat{A}(TX), [X] \rangle = -\frac{1}{24} \langle p_1(TX), [X] \rangle = -\frac{1}{8}\sigma(X)$. ■

Theorem 98.2 (Freedman). *There exists unique a compact simply connected* topological *4–manifold with $w_2(M) = 0$ and intersection form*

$$E_8 = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}.$$

Since $\sigma(E_8) = 8$, it follows that M cannot admit a smooth structure.

Rohklin's theorem also leads to the following invariant of a spin 3–manifold.

Definition 98.3. Let $(N, \mathfrak{s})$ be spin 3–manifold. The **Rokhlin invariant** of $(M, \mathfrak{s})$ is defined as

$$\mu(N, \mathfrak{s}) = \sigma(M) \bmod 16 \in \mathbf{Z}/16\mathbf{Z}.$$

where M is any compact spin 4–manifold with $\partial M = N$. •

Invariants of this type still play an important role in geometry and topology. One of the recent applications of this idea is the ν−invariant of G_2−manifold due to Crowley and Nordström [CN12].

Lecture 19

This lectures begins the final part of this course. I will introduce the Seiberg–Witten equation and sketch the construction of the Seiberg–Witten invariant.

99 The Seiberg–Witten equation

The Seiberg–Witten equation originates in Seiberg and Witten [SW94]; see also the discussion in [Wit94; Wit07].

Consider $\mathrm{Ad}\colon \mathrm{Spin}(4) \to \mathrm{SO}(4)$. Denote by P the pinor module of $\mathrm{C}\ell_{0,4} \cong \mathrm{M}_2(\mathbf{H})$. The volume element ω decomposes $P = S^+ \oplus S^-$ into irreducible representations of Spin(4). Moreover, $S^\pm$ carry the structure of an $\mathbf{H}$–module. The Clifford multiplication $\gamma\colon \mathbf{R}^4 \to \mathrm{End}(P)$ is equivariant and $\mathbf{H}$–linear and exchanges S^+ and S^-. Define $\gamma\colon \Lambda\mathbf{R}^4 \to \mathrm{End}_{\mathbf{H}}(P)$ by

$$\gamma(v_1 \wedge \cdots \wedge v_k) \coloneqq \gamma(v_1)\cdots\gamma(v_k)$$

for $v_1, \ldots, v_k$ orthogonal. Of course, this is nothing but the quantisation map.

Proposition 99.1.

(1) *If $\alpha \in \Lambda^+\mathbf{R}^4$ and $\phi \in S^+$, then $\gamma(\alpha)\phi = 0$.*

(2) *The image of $\mathbf{R} \oplus \Lambda^+\mathbf{R}^4$ in $\mathrm{C}\ell_{0,4}$ is a subalgebra and isomorphic to $\mathbf{H}$ via*

$$e_1 \wedge e_2 + e_3 \wedge e_4 \mapsto 2i, \quad e_1 \wedge e_3 + e_4 \wedge e_2 \mapsto 2j, \quad e_1 \wedge e_4 + e_2 \wedge e_3 \mapsto 2k.$$

(3) $\gamma(\Lambda^+\mathbf{R}^4) = \mathfrak{sp}(S^-) \subset \mathrm{End}_{\mathbf{H}}(S^-)$.

Proof. This is proved by a direct computation. ∎

$\mathrm{Spin}^{\mathrm{U}(1)} = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{U}(1)$. The complex pinor module of $\mathbf{C}\ell_4 \cong \mathrm{M}_4(\mathbf{C})$ is $\mathbf{C}^4$ and agrees with P considered as a complex with the complex structure induced by $i \in \mathbf{H}$. The *complex volume element* $\omega_{\mathbf{C}} = -i\omega$ (!) decomposes $P = W^+ \oplus W^-$. A moment's though reveals that

$$W^\pm = S^\mp.$$

The above establishes an isomorphism

$$\gamma\colon \Lambda^+\mathbf{R}^4 \to \mathfrak{su}(W^+).$$

Define the quadratic map $\mu\colon W^+ \to \Lambda^+\mathbf{R}^4 \otimes i\mathbf{R}$ by

$$\mu(\phi) \coloneqq \gamma^{-1}\big(\phi\phi^* - \tfrac{1}{2}|\phi|^2 \cdot \mathbf{1}\big).$$

(This goes under various different notations in the literature, but I prefer the notation μ because this is the distinguished moment map for the linear action of U(1) on W^+.)

Define $\chi\colon \mathrm{Spin}^{\mathrm{U}(1)}(4) \to \mathrm{U}(1)$ by $\chi[g, e^{i\alpha}] \coloneqq e^{2i\alpha}$. The representations $\mathrm{Spin}(4) \to \mathrm{Sp}(S^\pm) = \mathrm{Sp}(1)$ descend to representations $\rho_\pm\colon \mathrm{Spin}^{\mathrm{U}(1)}(4) \to \mathrm{U}(W^\mp) = \mathrm{U}(2)$ and

$$\det_{\mathbf{C}} \circ \rho_\pm = \chi.$$

Situation 99.2. Let X be a connected closed oriented 4–manifold ×

Pick a Riemannian metric g and a spin$^{\mathrm{U}(1)}$ structure $\mathfrak{w}$ (for short) on X. Denote the positive and negative spinor bundles by $W^{\pm}$. Denote by L the **characteristic line bundle**, that is, the Hermitian line bundle associate with $\mathfrak{w}$ and χ. A **spin connection** on $\mathfrak{w}$ is a $\mathrm{Spin}^{\mathrm{U}(1)}$–principal connection which induces the Levi-Civita connection on $\mathrm{Fr}_{\mathrm{SO}}(TX)$. Denote the set of these connections by $\mathscr{A}(\mathfrak{w})$. $\mathscr{A}(\mathfrak{w})$ is an affine space modelled on $\Omega^1(X, i\mathbf{R})$. Every $A \in \mathscr{A}(\mathfrak{w})$ induces a unitary covariant derivative A_L on L. The space of unitary covariant derivatives $\mathscr{A}(L)$ also is an affine space modelled on $\Omega^1(X, i\mathbf{R})$; a moment's thought shows that $(A + a)_L = A_L + 2a$. The twisting curvature F_A^{tw} can be identified as an element of $\Omega^2(X, i\mathbf{R})$. Another moment's thought shows that $F_A^{\mathrm{tw}} = \frac{1}{2}F_{A_L}$.

The **Seiberg–Witten equation** is the following equation for $(A, \Phi) \in \mathscr{A}(\mathfrak{w}) \times \Gamma(X, W^+)$.

$$\begin{aligned} D_A^+\Phi &= 0 \quad \text{and} \\ F_A^{\mathrm{tw},+} &= \mu(\Phi). \end{aligned} \tag{99.3}$$

The above discussion shows that this can also be understood as an equation on $(A, \Phi) \in \mathscr{A}(L) \times \Gamma(X, W^+)$. A solution of this equation is a **Seiberg–Witten monopole**.

What is the natural symmetry group of this equation? The gauge group of $\mathfrak{w}$ as a $\mathrm{Spin}^{\mathrm{U}(1)}(4)$–principal bundle are sections of

$$\mathfrak{w} \times_C \mathrm{Spin}^{\mathrm{U}(1)}(4) \to X$$

with C denoting the conjugation action. Such a gauge transformation preserves the Clifford multiplication if and only if it acts trivially on $\mathrm{Fr}_{SO}(TX)$. These gauge transformations are all induced by maps elements of $\mathscr{G}(\mathfrak{w}) := C^\infty(X, \mathrm{U}(1))$ acting via $\mathrm{U}(1) \to \mathrm{Spin}^{\mathrm{U}(1)}(4)(\to \mathrm{SO}(4))$. This is the natural symmetry group of (99.3).

Let $\eta \in \Omega^+(X, i\mathbf{R})$ The η**–perturbed Seiberg–Witten moduli space** for the spin$^{\mathrm{U}(1)}$–structure $\mathfrak{w}$ is

$$\mathscr{M}(\mathfrak{w}, \eta) := \left\{(A, \Phi) \in \mathscr{A}(\mathfrak{w}) \times \Gamma(W^+) : D_A^+\Phi = 0, F_A^{\mathrm{tw},+} = \mu(\Phi)g + \eta\right\}/\mathscr{G}(\mathfrak{w}).$$

The Seiberg–Witten invariant is constructed out of $\mathscr{M}(\mathfrak{w}, \eta)$. This involves the following steps:

(1) Prove that $\mathscr{M}(\mathfrak{w}, \eta)$ is compact.

(2) Prove that, for generic η, $\mathscr{M}(\mathfrak{w}, \eta)$ is a smooth manifold of dimension $d(\mathfrak{w})$ predicted by the Atiyah–Singer Index Theorem (provided $b^+(X) \geqslant 1$).

(3) Orient $\mathscr{M}(\mathfrak{w}, \eta)$.

(4) Prove that $[\mathscr{M}(\mathfrak{w})] := [\mathscr{M}(\mathfrak{w}, \eta)] \in \mathrm{H}_d(\mathscr{B}(\mathfrak{w}))$ with $\mathscr{B}(\mathfrak{w}) := (\mathscr{A}(\mathfrak{w}) \times \Gamma(W^+))/\mathscr{G}(\mathfrak{w})$ is independent of η (and g).

(5) If $d = 0$, then the Seiberg—Witten invariant is simply $[\mathscr{M}(\mathfrak{w})] \in \mathrm{H}_0(\mathscr{B}(\mathfrak{w})) = \mathbf{Z}$. If $d \neq 0$, then needs to figure out how to get a number by finding some cohomology classes on $\mathscr{B}(\mathfrak{w})$.

100 Compactness of $\mathscr{M}(\mathfrak{w}, \eta)$

A key feature of (99.3) is that $\mathscr{M}(\mathfrak{w}, \eta)$ is compact. Very much unlike the situation for ASD instantons or pseudo-holomorphic maps, there is no need to *invent* a compactification.

Theorem 100.1. *Suppose that X is closed. Let $\eta \in \Omega^+(X, i\mathbf{R})$ and $(\eta_n) \in \Omega^+(X, i\mathbf{R})^{\mathbf{N}}$ with $\eta_n \to \eta$ with respect to the C^∞ topology. Let $(A_n, \Phi_n) \in (\mathscr{A}(\mathfrak{w}) \times \Gamma(W^+))^{\mathbf{N}}$ with*

$$D^+_{A_n}\Phi_n = 0 \quad \textit{and} \quad F^{\mathrm{tw},+}_{A_n} = \mu(\Phi_n) + \eta_n.$$

After passing to a subsequence and up to the action of $\mathscr{G}(\mathfrak{w})$, there is a $(A, \Phi) \in \mathscr{A}(\mathfrak{w}) \times \Gamma(W^+)$ satisfying

$$D^+_A\Phi = 0 \quad \textit{and} \quad F^{\mathrm{tw},+}_A = \mu(\Phi) + \eta.$$

and $(A_n, \Phi_n) \to (A, \Phi)$ with respect to the C^∞ topology.

Proposition 100.2. *Let $\eta \in \Omega^+(X, i\mathbf{R})$ and $(A, \Phi) \in \mathscr{A}(\mathfrak{w}) \times \Gamma(W^+)$. If*

$$D^+_A\Phi = 0 \quad \textit{and} \quad F^{\mathrm{tw},+}_A = \mu(\Phi) + \eta,$$

then

$$\frac{1}{2}\Delta|\Phi|^2 - |\nabla_A\Phi|^2 + \frac{1}{4}\mathrm{scal}_g|\Phi|^2 + \frac{1}{2}|\Phi|^4 + \langle\gamma(\eta)\Phi, \Phi\rangle = 0.$$

Proof of Theorem 100.1. By Proposition 58.2,

$$\nabla^*_A\nabla_A\Phi + \frac{1}{4}\mathrm{scal}_g + \gamma(F^{\mathrm{tw}}_A)\Phi = 0.$$

Moreover,

$$\gamma(F^{\mathrm{tw}}_A)\Phi - \gamma(\eta)\Phi = \Phi\Phi^*\Phi - \frac{1}{2}|\Phi|^2\Phi = \frac{1}{2}|\Phi|^2\Phi.$$

Therefore,

$$\nabla^*_A\nabla_A\Phi + \frac{1}{4}\mathrm{scal}_g\Phi + \frac{1}{2}|\Phi|^2\Phi + \gamma(\eta)\Phi = 0.$$

Since

$$\Delta|\Phi|^2 = 2\langle\nabla^*_A\nabla_A\Phi, \Phi\rangle - 2|\nabla_A\Phi|^2,$$

the assertion follows. ■

Proposition 100.3. *Suppose that X is closed. Let $\eta \in \Omega^+(X, i\mathbf{R})$ and $(A, \Phi) \in \mathscr{A}(\mathfrak{w}) \times \Gamma(W^+)$. If*

$$D^+_A\Phi = 0 \quad \textit{and} \quad F^{\mathrm{tw},+}_A = \mu(\Phi) + \eta,$$

then

$$\|\Phi\|^2_{L^\infty} \leqslant c(g, \eta) := \sup_{x_0 \in X}\left(-\tfrac{1}{2}\mathrm{scal}_g(x_0) + 2|\eta|(x_0)\right).$$

Proof. Let $x_0 \in X$ with $|\Phi(x_0)|^2 = \|\Phi\|^2_{L^\infty}$. By the above and the maximum principle,

$$|\Phi|(x_0)^2 \leqslant \frac{1}{2}\mathrm{scal}_g(x_0) + 2|\eta|(x_0).$$ ■

Remark 100.4. The sign of μ is crucial for the above! ♣

Proof of Theorem 100.1. Choose $A_0 \in \mathscr{A}(\mathfrak{w})$. Set $a_n := A_n - A_0 \in \Omega^1(X, i\mathbf{R})$. The η_n–perturbed Seiberg–Witten equation becomes

$$\begin{aligned} D^+_{A_0}\Phi_n &= -\gamma(a_n)\Phi_n \\ \mathrm{d}^+ a_n &= \mu(\Phi_n) + \eta_n - F^{\mathrm{tw},+}_{A_0}. \end{aligned}$$

The left-hand side is not elliptic, and this should be expected in light of the infinite-dimensional symmetry group $\mathscr{G}(\mathfrak{w})$. If $u \in \mathscr{G}(\mathfrak{w}) = C^\infty(X, \mathrm{U}(1))$, then

$$u^*(A_n) - A_0 = a_n + u^*\mu_{\mathrm{U}(1)} = a_n + u^{-1}\mathrm{d}u.$$

Therefore (more or less by Hodge theory), up replacing (A_n, Φ_n) by $u_n^*(A_n, \Phi_n)$ for suitable $u_n \in \mathscr{G}(\mathfrak{w})$, there is no loss in assuming that

$$\begin{aligned} D^+_{A_0}\Phi_n &= -\gamma(a_n)\Phi_n \\ \mathrm{d}^+ a_n &= \mu(\Phi_n) + \eta_n - F^{\mathrm{tw},+}_{A_0} \\ \mathrm{d}^* a_n &= 0. \end{aligned}$$

This is elliptic.

The kernel of $\delta = (\mathrm{d}^+, \mathrm{d}^*) \colon \Omega^1(X) \to \Omega^+(X) \oplus \Omega^0(X)$ is the space of harmonic 1–forms $\mathscr{H}^1 = \mathscr{H}^1(X, g)$. If Π denotes the projection onto $\mathscr{H}^1$, then because δ is Fredholm

$$\|a_n\|_{W^{1,2}} \leqslant c + \|\Pi a_n\|_{L^2}.$$

To obtain a bound on $\|a_n\|_{W^{1,2}}$, it remains to control $\|\Pi a_n\|_{L^2}$. This can be done using the work of Uhlenbeck, but in the present case there is an elementary argument. Let $\alpha \in \mathrm{H}^1(\mathrm{U}(1), \mathbf{Z})$ be a generator. The map $\mathscr{G}(\mathfrak{w}) = C^\infty(X, \mathrm{U}(1)) \to \mathrm{H}^1(X, \mathbf{Z})$ given by $u \mapsto u^*\alpha$ is surjective. Indeed, $[X, \mathrm{U}(1)] \cong \mathrm{H}^1(X, \mathbf{Z})$. Up to sign, the image of α in $\mathrm{H}^1_{\mathrm{dR}}(\mathrm{U}(1))$ is $[\mu_{\mathrm{U}(1)}]$. Therefore, the action by gauge transformations can be used to change Πa_n by the image of $\mathrm{H}^1(X, \mathbf{Z})$ in $\mathrm{H}^1_{\mathrm{dR}}(X) \cong \mathscr{H}^1$. Since the quotient $\mathrm{H}^1(X, \mathbf{R})/\mathrm{H}^1(X, \mathbf{Z}) \cong T^{b_1(X)}$ is compact, $\|\Pi a_n\|_{L^2}$ can be assumed to be uniformly bounded.

At this point $\|a_n\|_{W^{1,2}}$ is uniformly bounded. Therefore, $D^+_{A_0}\Phi_n$ is uniformly bounded in L^2. Together with the L^∞ bound this yields a uniform bound of $\|\Phi\|_{W^{1,2}}$. Since $\|\nabla_{A_0}\mu(\Phi)\| \leqslant c|\nabla_A\Phi||\Phi|$, $\|\delta a_n\|_{W^{1,2}}$ is uniformly bounded. Therefore, again, $\|a_n\|_{W^{2,2}}$ is uniformly bounded. Unfortunately, $W^{2,2}$ does not embed into L^∞. Otherwise, $D^+_A\Phi$ would be uniformly bounded in $W^{1,2}$. To overcome this one has can either work with $W^{k,p}$ or $W^{s,2}$. In any case, soon one arrives at uniform bounds in $W^{3,2}$ which is a Banach algebra. This in turn allows one to uniformly obtain $W^{k,2}$ bounds for arbitrary k.

Compactness then follows by Banach–Alaogolou and Morrey embedding (or Morrey embedding and Arzela–Ascoli). ∎

Lecture 20

101 Construction of $\mathscr{M}(\mathfrak{w},\eta)$ as a manifold

Let $s \in 2 + \mathbf{N}_0$. By Theorem 85.1,

$$W^{s+1,2}\mathscr{G}(\mathfrak{w}) := W^{s+1,2}(X, \mathrm{U}(1)) := \{u \in W^{s+1,2}(X, \mathbf{C}) : u(x) \in \mathrm{U}(1) \text{ for every } x \in X\}.$$

is a Banach Lie group and smoothly acts the right of

$$W^{s,2}\mathscr{C}(\mathfrak{w}) := W^{s,2}\mathscr{A}(\mathfrak{w}) \oplus W^{s,2}\Gamma(W^+)$$

via

$$u^*(A,\Phi) := (A + u^{-1}\mathrm{d}u, u\Phi).$$

The η–perturbed Seiberg–Witten equation can be encoded as the $W^{s+1,2}\mathscr{G}(\mathfrak{w})$–equivariant map $\mathrm{sw}\colon W^{s,2}\mathscr{C}(\mathfrak{w}) \to W^{s-1,2}\Gamma(W^-) \oplus W^{s-1,2}\Omega^+(X, i\mathbf{R})$ defined by

$$\mathrm{sw}(A,\Phi) := \big(D_A^+\Phi, F_A^{\mathrm{tw},+} - \mu(\Phi)\big).$$

With this notation in place

$$\mathscr{M}(\mathfrak{w},\eta) = \mathrm{sw}^{-1}(0,\eta)/W^{s,2}\mathscr{G}(\mathfrak{w}).$$

(This is not completely accurate because $\mathscr{M}(\mathfrak{w},\eta)$ was initially constructed using C^∞ configurations; but the map canonical map $\mathscr{M}(\mathfrak{w},\eta) \to \mathrm{sw}^{-1}(0,\eta)/W^{s,2}\mathscr{G}(\mathfrak{w})$ turns out to be a homeomorphism because of elliptic regularity. Let us not bother with this.) The task at hand is to make this into a manifold.

Proposition 101.1. *The action $W^{s,2}\mathscr{C}(\mathfrak{w}) \circlearrowleft W^{s+1,2}\mathscr{G}(\mathfrak{w})$ is proper.*

Proof. Let $(A_n,\Phi_n) \in \big(W^{s,2}\mathscr{A}(\mathfrak{w}) \times W^{s+1,2}\Gamma(W^+)\big)^{\mathbf{N}}$ and $(u_n) \in W^{s+1,2}\mathscr{G}(\mathfrak{w})^{\mathbf{N}}$. Suppose that $(A_n,\Phi_n) \to (A,\Phi)$ and $u_n^*(A_n,\Phi_n) \to (B,\Psi)$. In particular, $\|u_n^*A_n - A_n\|_{W^{s,2}}$ is uniformly bounded. Therefore and because

$$\mathrm{d}u_n = u_n(u_n^*A_n - A_n),$$

$\|u_n\|_{W^{1,2}}$ is uniformly bounded. A bootstrapping argument shows that $\|\mathrm{d}u_n\|_{W^{s,2}}$ is uniformly bounded. By Banach–Alaogolou, a subsequence of (u_n) converges in $W^{s+1,2}\mathscr{G}(\mathfrak{w})$. ■

Definition 101.2. A configuration (A,Φ) is **irreducible** if $\Phi \neq 0$. Denote the open subset of irreducible configurations by

$$\mathscr{C}^*(\mathfrak{w})$$

and denote by

$$\mathscr{M}^*(\mathfrak{w},\eta) \subset \mathscr{M}(\mathfrak{w},\eta)$$

the subset of irreducible Seiberg–Witten monopoles. •

Proposition 101.3.

(1) *If* $\mathscr{M}(\mathfrak{w};\eta)$ *contains a configuration* $[A,\Phi]$ *which is not irreducible, then* $\eta = F_A^{\mathrm{tw},+}$.

(2) *The image of the map* $W^{s,2}\mathscr{A}(\mathfrak{w}) \to W^{s-1,2}\Omega^+(X, i\mathbf{R}), A \mapsto F_A^{\mathrm{tw},+}$ *has codimension* $b^+(X)$.

Proof. The first assertion is trivial.

Choose a base-point A_0. Since $F_{A_0+a}^{\mathrm{tw},+} = F_{A_0}^{\mathrm{tw},+} + \mathrm{d}^+a$, the image of $A \mapsto F_A^{\mathrm{tw},+}$ is an affine space modelled on $\operatorname{im} \mathrm{d}^+ \colon \Omega^1(X, i\mathbf{R}) \to \Omega^+(X, i\mathbf{R})$. By Hodge theory, the latter has codimension $b^+(X)$. ■

Remark 101.4. As a consequence of the above, if $b^+(X) \geqslant 1$, then a generic choice of η ensures that $\mathscr{M}(\mathfrak{w},\eta)$ contains only irreducible monopoles: $\mathscr{M}^*(\mathfrak{w},\eta) = \mathscr{M}(\mathfrak{w},\eta)$. Moreover, if $b^+(X) \geqslant 2$, then a this is even true for a generic path $(\eta_t)_{t\in[0,1]}$. ♣

The action of $W^{s,2}\mathscr{C}^*(\mathfrak{w}) \circlearrowleft W^{s+1,2}\mathscr{G}(\mathfrak{w})$ is free and proper; hence:

$$\mathscr{B}^*(\mathfrak{w}) \coloneqq W^{s,2}\mathscr{C}^*(\mathfrak{w})/W^{s+1,2}\mathscr{G}(\mathfrak{w})$$

carries the structure of a Banach manifold. The Seiberg–Witten map descends to a smooth section

$$\mathrm{sw} \in \Gamma(V)$$

of

$$V \coloneqq \big(W^{s,2}\mathscr{C}^*(\mathfrak{w}) \times_{W^{s+1,2}\mathscr{G}(\mathfrak{w})} W^{s-1,2}\Gamma(W^-)\big) \oplus W^{s-1,2}\Omega^+(X, i\mathbf{R}).$$

Remark 101.5. Because Seiberg–Witten theory is an abelian gauge theory, one does not just have local slices but, in fact, "global slices". We will come back to this later. ♣

Proposition 101.6. $\mathrm{sw}_\eta \in \Gamma(V)$ *is a Fredholm section of index*

$$\operatorname{index} \mathrm{sw}_\eta = d(\mathfrak{w}) \coloneqq \frac{1}{4}\big(c_1(L)^2 - 2\chi(X) - 3\sigma(X)\big).$$

Proof. Choose a base-point A_0. A local slice for the action $W^{s,2}\mathscr{C}^*(\mathfrak{w}) \circlearrowleft W^{s+1,2}\mathscr{G}(\mathfrak{w})$ is given by imposing the gauge fixing condition $\mathrm{d}^*(A - A_0) = 0$. The Seiberg–Witten map with this gauge fixing condition becomes

$$(A_0 + a, \Phi) \mapsto (D_{A_0}^+\Phi + \gamma(a)\Phi, \mathrm{d}^+a - \mu(\Phi) - \eta, \mathrm{d}^*a).$$

Therefore, its derivative is $D_{A_0}^+ \oplus \delta^+ \colon \Gamma(W^+) \oplus \Omega^1(X, i\mathbf{R}) \to \Gamma(W^-) \oplus \Omega^+(X, i\mathbf{R}) \oplus \Omega^0(X, i\mathbf{R})$. By the Atiyah–Singer Index Theorem and by Hodge theory, the index of this operator is

$$\begin{aligned} 2\operatorname{index}_{\mathbf{C}} D_{A_0}^+ + \operatorname{index}\delta &= 2\langle \hat{A}(TX)e^{\frac{1}{2}c_1(L)}, [X]\rangle + (b^1(X) - b^0(X) - b^+(X)) \\ &= -\frac{1}{4}\sigma(X) + \frac{1}{4}c_1(L)^2 - \frac{1}{2}(\chi(X) + \sigma(X)). \end{aligned}$$ ■

Proposition 101.7. *The map*

$$\overline{\mathrm{sw}}\colon W^{s,2}\mathscr{C}^*(\mathfrak{w}) \times W^{s-1,2}\Omega^+(X, i\mathbf{R}) \to W^{s-1,2}\Gamma(W^-) \oplus W^{s-1,2}\Omega^+(X, i\mathbf{R})$$

defined by

$$\overline{\mathrm{sw}}(A, \Phi; \eta) \coloneqq \mathrm{sw}(A, \Phi) - (0, \eta)$$

has 0 *as a regular value.*

Proof. Evidently, $\dim \operatorname{coker} T_{(A,\Phi;\eta)}\overline{\mathfrak{sw}} < \infty$. Suppose that $(\Psi, \beta) \perp \operatorname{im} T_{(A,\Phi;\eta)}\overline{\mathfrak{sw}}$ (in L^2). Of course, $\beta = 0$ and for every (a, ϕ)

$$\langle D_A^+\phi, \Psi\rangle + \langle \gamma(a)\Phi, \Psi\rangle = 0.$$

Therefore, $D_A^-\Psi = 0$. Since $\Phi \neq 0$ and $W^{s,2} \hookrightarrow C^0$, there is an open subset U on which Φ des not vanish. On this set the $\gamma(\cdot)\Phi\colon T^*X \otimes i\mathbf{R} \to W^-$ is surjective. Consequently, Ψ vanishes on U. By unique continuation this implies that $\Psi = 0$. ■

Proposition 101.8. *The subset of those η for which $(0, \eta)$ is a regular value of* sw $\in \Gamma(V)$ *is comeager.*

Proof. The map

$$\mathrm{pr}_2\colon \overline{\mathrm{sw}}^{-1}(0)/W^{s,2}\mathscr{G}(\mathfrak{w}) \to W^{s-1,2}\Omega^+(X, i\mathbf{R})$$

is Fredholm and η is a regular value of pr_2 if and only if it is a regular value of sw. ■

Corollary 101.9.

(1) *If $b^+(X) \geqslant 1$, then, for a generic η, $\mathscr{M}(\mathfrak{w}; \eta)$ is a compact submanifold of $\mathscr{B}^*(\mathfrak{w})$ of dimension $d(\mathfrak{w})$*

(2) *If $b^+(X) \geqslant 2$ and η_0, η_1 are generic, then for a generic path (η_t) from η_0 to η_1*

$$\coprod_{t\in[0,1]} \mathscr{M}(\mathfrak{w}; \eta_t) \subset \mathscr{B}^*(\mathfrak{w}) \times [0, 1]$$

is a submanifold of dimension $d(\mathfrak{w}) + 1$. ■

Lecture 21

102 Orientations

The above shows that

$$[\mathscr{M}(\mathfrak{w})] := [\mathscr{M}(\mathfrak{w};\eta)] \in \Omega^{\mathrm{O}}_{d(\mathfrak{w})}(\mathscr{B}^*(\mathfrak{w}))$$

is a well-defined *unoriented* bordism class in $\mathscr{B}^*(\mathfrak{w})$. This produces at most $\mathbf{Z}/2\mathbf{Z}$–valued invariants. To lift to $\mathbf{Z}$ it is necessary to orient $\mathscr{M}(\mathfrak{w};\eta)$. By construction,

$$T_{[A,\Phi]}\mathscr{M}(\mathfrak{w};\eta) = \ker T_{[A,\Phi]}\mathfrak{s}\mathfrak{w}.$$

To orient a vector bundle V is to choose a section of the $\{\pm 1\}$–principal bundle

$$\mathrm{or}(V) := \mathbf{P}\det V := (\Lambda^{\mathrm{rk}\,V} V\backslash\{0\})/\mathbf{R}^+.$$

This requires a discussion of "the" determinant line bundle $\det D$ of a family of Fredholm operators D. Consider a family $D\colon P \to \mathscr{F}$ of Fredholm operators. If $\operatorname{coker} D(p) = 0$ for every $p \in P$, then $\ker D \to P$ is a vector bundle and so is

$$\det D = \det\ker D \otimes (\det\operatorname{coker} D)^* = \det\ker D \otimes \mathbf{R}^* \cong \det\ker D.$$

What we need to do is to find a section of

$$\mathbf{P}\det\ker T_{[A,\Phi]}\mathfrak{s}\mathfrak{w} \to \mathscr{M}(\mathfrak{w};\eta)$$

(preferably in some "consistent" way.) The advantage of $\det\ker D \otimes (\det\operatorname{coker} D)^*$ over $\ker D$ is that it has constant dimension and therefore has a chance to become a vector bundle. Actually topologising the the "set-theoretic" vector bundle

$$\det D = \coprod_{p\in D_p} \det\ker D(p) \otimes (\det\operatorname{coker} D(p))^*$$

in a way consistent is not as trivial as it might seem. The exposition in [Boh07, Appendix B] is quite understandable (but possibly not as explicit as one might wish). The definitive reference is Zinger [Zin16].

Instead of $\det D$ it is often better to consider the $\mathbf{Z}/2\mathbf{Z}$–principal bundle.

$$\mathrm{or}(D) := \mathbf{P}(\det D).$$

If such a construction indeed exists and (D_t) is a homotopy of families of Fredholm operators, then it induces an isomorphism

$$\mathbf{P}\det(D_0) \to \mathbf{P}\det(D_1).$$

Let us pretend that we have all looked up a concrete construction of $\det D$ and checked that it is well-defined.

We apply this to

$$T_{[A,\Phi]}\mathfrak{sw}(a,\phi) = \begin{pmatrix} D_A^+\phi + \gamma(a)\Phi \\ \mathrm{d}^+a + T_\Phi\mu(\phi) \\ \mathrm{d}^*a \end{pmatrix}.$$

The family $T\mathrm{sw}$ is linearly homotopic to

$$(a,\phi) \mapsto \begin{pmatrix} D_A^+\phi \\ \mathrm{d}^+a \\ \mathrm{d}^*a \end{pmatrix}.$$

Observe that:

(1) $\ker D_A+$, $\operatorname{coker} D_A^+$ are complex linear and, therefore, have a orientation: if V is a complex vector space, then we declare $(e_1, f_1 := ie_1, \cdot, e_n, f_n := ie_n)$ to be positive. In particular, $\mathbf{P}\det D^+$ is trivialised by this convention.

(2) $\delta = (\mathrm{d}^+, \mathrm{d}^*)$ is independent of (A,Φ) and and $\det\delta = \det \mathrm{H}^1(X) \otimes (\det \mathrm{H}^+(X) \oplus \mathrm{H}^0(X))^*$.

From the above discussion it is clear that (once we decided what $\det D$ means) $\mathbf{P}\det T\mathrm{sw}$ is a trivial $\mathbf{Z}/2\mathbf{Z}$–principal bundle and a section of it is determined by choosing a **homology orientation**; that is: an element of the $\mathbf{Z}/2\mathbf{Z}$–torsor

$$\mathbf{P}\big(\det \mathrm{H}^1(X) \otimes (\det \mathrm{H}^+(X) \oplus \mathrm{H}^0(X))^*\big).$$

This ultimately orients $\mathscr{M}(\mathfrak{w};\eta)$ as well as the bordisms between different choices of η—although it is not necessarily easy to compute the orientation concretely.

The upshot of this discussion is that the *oriented* bordism class

$$[\mathscr{M}(\mathfrak{w})] := [\mathscr{M}(\mathfrak{w};\eta)] \in \Omega^{\mathrm{SO}}_{d(\mathfrak{w})}(\mathscr{B}^*(\mathfrak{w}))$$

is well-defined—provided $b^+(X) \geqslant 2$. In fact, in practice, we only make use of

$$[\mathscr{M}(\mathfrak{w})] := [\mathscr{M}(\mathfrak{w};\eta)] \in \mathrm{H}_{d(\mathfrak{w})}(\mathscr{B}^*(\mathfrak{w}), \mathbf{Z}).$$

From this numbers are obtained by pairing with element of $\mathrm{H}^{d(\mathfrak{w})}(\mathscr{B}^*(\mathfrak{w}), \mathbf{Z})$.

Proposition 102.1. *There are only finitely many spin$^{\mathrm{U}(1)}$ structures $\mathfrak{w}$ for which $[\mathscr{M}(\mathfrak{w})] \neq 0$.*

Proof. If $[\mathscr{M}(\mathfrak{w})] \neq 0$, then there is a η with $\|\eta\|_{L^\infty} \leqslant 1$ such that $(0,\eta)$ is a regular value of $\mathrm{sw} \in \Gamma(V)$. Therefore, $d(\mathfrak{w}) \geqslant 0$ and by the index formula

$$c_1(L)^2 \geqslant 3\sigma(X) + 2\chi(X).$$

Since

$$\pi c_1(L) = [iF_A^{\mathrm{tw}}] \text{and} \int_X (iF_A^{\mathrm{tw}}) \wedge (iF_A^{\mathrm{tw}}) = \|F_A^{\mathrm{tw},+}\|^2_{L^2} - \|F_A^{\mathrm{tw},-}\|^2_{L^2},$$

the above implies

$$\|F_A^{\mathrm{tw},-}\|^2_{L^2} - \|F_A^{\mathrm{tw},+}\|^2_{L^2} \leqslant \pi^2(3\sigma(X) + 2\chi(X)).$$

Together with the a priori bound on Φ this implies that

$$\|F_{A_L}\|_{L^2}^2 \leqslant c.$$

Therefore,

$$\int c_1(L) \wedge *c_1(L) \leqslant c(X).$$

This restriction is only satisfied by finitely many L. ■

103 The topology of $\mathscr{B}^*(\mathfrak{w})$

To understand the topology of $\mathscr{B}^*(\mathfrak{w})$ the crucial insight is to use the map

$$\deg\colon W^{s+1,2}\mathscr{G}(\mathfrak{w}) \to [X, \mathrm{U}(1)] = \mathrm{H}^1(X, \mathbf{Z}) \to \mathrm{H}^1_{\mathrm{dR}}(X)$$

defined by

$$\deg(u) \coloneqq [u^{-1}\mathrm{d}u] = [u^*\mu_{\mathrm{U}(1)}].$$

Remark 103.1. For every $[\alpha] \in \mathrm{im}(\mathrm{H}^1(X,\mathbf{Z}) \to \mathrm{H}^1_{\mathrm{dR}}(X))$ there is a unique harmonic $u\colon X \to \mathrm{U}(1)$ with $[\alpha] = [u^*\mu_{\mathrm{U}(1)}]$. ♣

Fix a base-point $A_0 \in \mathscr{A}(\mathfrak{w})$. The action of $W^{s+1,2}\mathscr{G}(\mathfrak{w})$ is given by

$$u^*(A_0 + a) = a + u^{-1}\mathrm{d}u = a + u^*\mu_{\mathrm{U}(1)}.$$

The stabiliser of the action of $W^{s+1,2}\mathscr{G}(\mathfrak{w})$ is $\mathrm{U}(1) < W^{s+1,2}\mathscr{G}(\mathfrak{w})$. The inclusion

$$\mathrm{U}(1) \hookrightarrow W^{s+1,2}\mathscr{G}(\mathfrak{w})$$

is split by the evaluation map $\mathrm{ev}_{x_0}\colon W^{s+1,2}\mathscr{G}(\mathfrak{w}) \to \mathrm{U}(1)$ for every $x_0 \in X$. The kernel

$$W^{s+1,2}\mathscr{G}(\mathfrak{w}; x_0) \coloneqq \ker \mathrm{ev}_{x_0}$$

is the **based gauge group**. We have

$$W^{s+1,2}\mathscr{G}(\mathfrak{w}) \cong W^{s+1,2}\mathscr{G}(\mathfrak{w}; x_0) \times \mathrm{U}(1).$$

Therefore, $W^{s,2}\mathscr{A}(\mathfrak{w})/W^{s+1,2}\mathscr{G}(\mathfrak{w}) = W^{s,2}\mathscr{A}(\mathfrak{w})/W^{s+1,2}\mathscr{G}(\mathfrak{w}; x_0)$ is homotopy equivalent to the torus

$$\frac{\mathrm{H}^1(X,\mathbf{R})}{\mathrm{H}^1(X,\mathbf{Z})} \cong T^{b_1(X)}.$$

$W^{s,2}\Gamma(W^+)\backslash\{0\}$ is homotopy equivalent to the unit sphere in the separable Hilbert space ℓ^2. Therefore, by Kuiper's theorem it is contractible. Consequently, $\big(W^{s,2}\Gamma(W^+)\backslash\{0\}\big)/\mathrm{U}(1)$ is homotopy equivalent to

$$B\mathrm{U}(1) = \mathbf{C}P^\infty.$$

Putting everything together, $\mathscr{B}^*(\mathfrak{w})$ is a fibre bundle over a space homotopy equivalent to $T^{b_1(X)}$ with fibres homotopy equivalent to $\mathbf{C}P^\infty$. Therefore, $\mathrm{H}^\bullet(\mathscr{B}^*(\mathfrak{w}))$ is generated by $\Lambda^\bullet\mathbf{Z}^{b_1(X)}$ (from the torus) and $\mathbf{Z}[x]$ with x of degree 2 from the $\mathbf{C}P^\infty$.

In Seiberg–Witten theory commonly only the cohomology from $\mathbf{C}P^\infty$ is used. The corresponding "generator" has the explicit description as

$$\mu := c_1(P) \in \mathrm{H}^2(\mathscr{B}^*(\mathfrak{w}))$$

with

$$P := \mathscr{A}(\mathfrak{w}) \times \big(\Gamma(W^+)\backslash\{0\}\big)/\mathscr{G}(\mathfrak{w}, x_0).$$

104 The Seiberg–Witten invariant

Let X be a closed oriented 4–manifold. Suppose that an orientation of $\mathrm{H}^1_{\mathrm{dR}}(X) \oplus (\mathrm{H}^+_{\mathrm{dR}}(X) \oplus \mathrm{H}^0_{\mathrm{dR}}(X))[1]$ has been chosen. The **Seiberg–Witten invariant** is the map

$$\mathrm{SW}\colon \{\mathfrak{w} \text{ spin}^{\mathrm{U}(1)}\text{–structure on } X\} \to \mathbf{Z}$$

defined by

$$\mathrm{SW}(\mathfrak{w}) := \begin{cases} \langle [\mathscr{M}(\mathfrak{w})], \mu^{d(\mathfrak{w})/2}\rangle & \text{if } d(\mathfrak{w}) = 0 \bmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

Lecture 22

The purpose of this lecture is to explain how to define the determinant line bundle (hopefully, without messing up the signs).

105 The determinant of a cochain complex

Definition 105.1. A **graded line** is a pair $L = (\underline{L}, \deg_L)$ consisting of a 1−dimensional $\mathbf{R}$ vector space $\underline{L}$ and an integer $\deg_L \in \mathbf{Z}$, the **degree** of L. Denote by $\mathscr{L}$ the category whose objects are graded lines and morphisms are

$$\hom(L, M) := \begin{cases} \mathrm{Iso}(\underline{L}, \underline{M}) & \text{if } \deg_L = \deg_M \\ \varnothing & \text{otherwise.} \end{cases} \qquad \bullet$$

The purpose of this category is to be the recipient of determinant functors. Denote by $\mathscr{V}^\times$ the category whose objects are finite dimensional vector spaces and morphisms are $\hom(V_0, V_1) := \mathrm{Iso}(V_0, V_1)$.

Definition 105.2. The **determinant** is the functor $\det\colon \mathscr{V}^\times \to \mathscr{L}$ defined by

$$\det V := (\Lambda^{\dim V} V, \dim V)$$

and

$$\det f := \Lambda^{\dim V} f. \qquad \bullet$$

$\mathscr{V}^\times$ is a symmetric monoidal category with the product given by the direct sum $\oplus$, unit 0, and the obvious associator, braiding, and unitors. $\mathscr{L}$ also can be given the structure of a symmetric monoidal category.

Definition 105.3. Equip $\mathscr{L}$ with the structure of a a symmetric monoidal category as follows:

(1) The tensor product $\otimes\colon \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ defined by

$$L \otimes M := (\underline{L} \otimes \underline{M}, \deg_L + \deg_M).$$

(2) The associator $\alpha_{K,L,M}\colon (K \otimes L) \otimes M \to K \otimes (L \otimes M)$ is defined by

$$\alpha\big((k \otimes \ell) \otimes m\big) := k \otimes (\ell \otimes m).$$

(3) The braiding $\beta\colon L \otimes M \to M \otimes L$ is defined by

$$\beta(\ell \otimes m) := (-1)^{\deg_L \deg_M} m \otimes \ell.$$

(4) The unit is

$$\mathbf{1} := (\mathbf{R}, 0).$$

(5) The left unitor $\lambda_L\colon\ \mathbf{1}\otimes L \to L$ and the right unitor $\rho\colon\ L\otimes\mathbf{1}\to L$ are defined by

$$\lambda(1\otimes\ell) \coloneqq \ell \quad\text{and}\quad \rho(\ell\otimes 1) \coloneqq \ell. \qquad\bullet$$

The proof of the following is straight-forward but somewhat time-consuming.

Proposition 105.4. *The functor* $\det\colon\ \mathscr{V}^\times \to \mathscr{L}$ *together with* $\mathrm{id}\colon\ \mathbf{1} = \det\mathbf{0}$ *and* $\phi_{V,W}\colon\ \det(V)\otimes\det(W)\to\det(V\oplus W)$ *defined by*

$$\phi_{V,W}(\nu\otimes\mu) \coloneqq \nu\wedge\mu$$

is a symmetric monoidal functor. ■

Remark 105.5. The sign in the braiding β is crucial for the above. In particular, the category of (ungraded) lines is not a suitable recipient of det—whenever the symmetric monoidal structure of $\mathscr{V}^\times$ plays any role. ♣

Remark 105.6. In the following—to keep a modicum of sanity–the maps using the associator α are swept under the rug. By Mac Lane's coherence theorem this does not cause any trouble. ♣

The determinant interacts well with exact sequences.

Proposition 105.7. *For every short exact sequence*

$$0\to V' \xrightarrow{i} V \xrightarrow{p} V'' \to 0$$

of finite dimensional vector spaces there is an unique isomorphism

$$\psi_{i,p}\colon\ \det V'\otimes\det V''\to\det V$$

such that:

(1) *For the split exact sequence*

$$0\to V' \xrightarrow{\iota} V'\oplus V'' \xrightarrow{\pi} V''\to 0$$

with $\iota(v') \coloneqq (v',0)$ *and* $\pi(v',v'') \coloneqq v''$

$$\psi_{\iota,\pi} = \phi_{V',V''}.$$

(2) *If*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \xrightarrow{i} & V & \xrightarrow{p} & V'' & \longrightarrow & 0\\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & W' & \xrightarrow{j} & W & \xrightarrow{q} & W'' & \longrightarrow & 0. \end{array}$$

is a commutative diagram of finite dimensional vector spaces with exact rows and f, f', f'' *isomorphisms, then*

$$\begin{array}{ccc} \det V'\otimes\det V'' & \xrightarrow{\psi_{i,p}} & \det V\\ {\scriptstyle \det f'\otimes\det f''}\downarrow & & \downarrow{\scriptstyle\det f}\\ \det W'\otimes\det W'' & \xrightarrow[\psi_{j,q}]{} & \det W \end{array}$$

commutes.

Moreover:

(3) *If*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U' & \xrightarrow{i} & U & \xrightarrow{p} & U'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & V' & \xrightarrow{i} & V & \xrightarrow{p} & V'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & W' & \xrightarrow{j} & W & \xrightarrow{q} & W'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is a commutative diagram with exact rows and columns, then the diagram commutes.

Proof. Let $0 \to V' \xrightarrow{i} V \xrightarrow{p} V'' \to 0$ be an exact sequence. If p^R is a right inverse of p, then $(i\ p^R)\colon\ V' \oplus V'' \to V$ is an isomorphism and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V' & \xrightarrow{\iota} & V' \oplus V'' & \xrightarrow{\pi} & V'' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle (i\ p^R)} & & \| & & \\
0 & \longrightarrow & V' & \xrightarrow{i} & V & \xrightarrow{p} & V'' & \longrightarrow & 0
\end{array}$$

is as in (2). Define $\psi_{i,p}\colon\ \det V' \otimes \det V'' \to \det V$ by

$$\psi_{i,p} \coloneqq \det(i\ p^R) \circ \phi_{V',V''}.$$

This does not depend on the choice of p^R because if p_1^R, p_2^R are right inverses of p, then

$$\det\big((i\ p_2^R)^{-1}(i\ p_1^R)\big) = \det\begin{pmatrix} \mathbf{1} & * \\ 0 & \mathbf{1} \end{pmatrix} = 1.$$

Evidenlty, (1) holds A moment's thought shows that this construction satisfies (2). Obviously, this construction is unique. ■

The construction of the determinant of $\mathbf{Z}/2\mathbf{Z}$–graded vector space requires the following additional structure.

Definition 105.8. Define the functor $\cdot^\vee\colon\ \mathscr{L} \to \mathscr{L}$ and the natural isomorphism $\mathrm{ev}_L\colon\ L^\vee \otimes L \to \mathbf{1}$ by

$$L^\vee = (\underline{L}^\vee, \deg_{L^\vee}) \coloneqq (\underline{L}^*, -\deg_L) \quad \text{and} \quad f^\vee \coloneqq (f^{-1})^*$$

and

$$\mathrm{ev}_L(\ell^* \otimes \ell) \coloneqq (-1)^{\binom{\deg_L}{2}} \ell^*(\ell).$$ •

Remark 105.9.

(1) If for every graded line L a choice of a graded line $L^\vee$ together with an isomorphism $\mathrm{ev}_L\colon\ L^\vee \otimes L \longrightarrow \mathbf{1}$ has been made, then this induces a functor $\cdot^\vee$ by requiring that the diagram

$$\begin{array}{ccc} L^\vee \otimes L & \xrightarrow{\mathrm{ev}_M} & \mathbf{1} \\ {\scriptstyle f^\vee\otimes f}\downarrow & & \| \\ M^\vee \otimes M & \xrightarrow[\mathrm{ev}_L]{} & \mathbf{1} \end{array}$$

commutes. Any of those could be used alternatively in the upcoming constructing.

(2) Since ev_L induces an isomorphism $\underline{L}^\vee \cong \underline{L}^*$, Definition 105.8 appears natural—except possibly for the sign in ev_L.

(3) $\cdot^\vee$ and ev determine a unique natural isomorphism $\iota_L\colon\ L \longrightarrow L^{\vee\vee}$ such that the diagram

$$\begin{array}{ccccc} L \otimes L^\vee & \xrightarrow{\beta_{L,L^\vee}} & L^\vee \otimes L & \xrightarrow{\mathrm{ev}_L} & \mathbf{1} \\ {\scriptstyle \iota_L\otimes\mathrm{id}_{L^\vee}}\downarrow & & & & \| \\ L^{\vee\vee} \otimes L^\vee & & \xrightarrow[\mathrm{ev}_{L^\vee}]{} & & \mathbf{1} \end{array}$$

commutes. A brief computation shows that that ι_L is the canonical isomorphism $L \cong L^{**}$, $\ell \mapsto (\ell^* \mapsto \ell^*(\ell))$, with the choices made in Definition 105.8.

(4) $\cdot^\vee$ and ev determine a unique natural isomorphism $\kappa_{L,M}\colon\ L^\vee \otimes M^\vee \longrightarrow (L \otimes M)^\vee$ such that the diagram

$$\begin{array}{ccc} (L^\vee \otimes M^\vee) \otimes (L \otimes M) & \xrightarrow{\mathbf{1}\otimes\beta_{M^\vee,L}\otimes\mathbf{1}} & (L^\vee \otimes L) \otimes (M^\vee \otimes M) \\ & & \downarrow{\scriptstyle \mathrm{ev}_L\otimes\mathrm{ev}_M} \\ {\scriptstyle \kappa_{L,M}\otimes\mathbf{1}}\downarrow & & \mathbf{1} \otimes \mathbf{1} \\ & & \downarrow \\ (L \otimes M)^\vee \otimes (L \otimes M) & \xrightarrow[\mathrm{ev}_{L\otimes M}]{} & \mathbf{1} \end{array}$$

commutes. A computation shows that

$$\kappa_{L,M}(\ell^* \otimes m^*)(\ell \otimes m) = \ell^*(\ell) \cdot m^*(m)$$

with the choices made in Definition 105.8; that is: $\kappa_{L,M}$ is the canonical isomorphism $L^* \otimes M^* \cong (L \otimes M)^*$. ♣

Definition 105.10. Denote by $\mathscr{V}^\times_{\mathbf{Z}/2\mathbf{Z}}$ the category of $\mathbf{Z}/2\mathbf{Z}$–graded vector spaces with isomorphism only. Define the **determinant** functor $\det\colon\ \mathscr{V}^\times_{\mathbf{Z}/2\mathbf{Z}} \longrightarrow \mathscr{L}$ by

$$\det V := \det V^0 \otimes (\det V^1)^\vee \quad \text{and} \quad \det f := \det f^0 \otimes (\det f^1)^\vee. \qquad \bullet$$

The shift functor $[1]$ on $\mathscr{V}^{\times}_{\mathbf{Z}/2\mathbf{Z}}$ is compatible with det in the sense that the braiding β, ι, and κ induce a natural isomorphism

$$\delta := (\iota \otimes \mathbf{1}) \circ \beta \circ \kappa^{-1} \colon \det(\cdot[1]) \Rightarrow (\det \cdot)^{\vee}.$$

Proposition 105.11. $\det\colon \mathscr{V}^{\times}_{\mathbf{Z}/2\mathbf{Z}} \to \mathscr{L}$ *together with* $\varepsilon\colon \beta \circ \mathrm{ev}_{\mathbf{1}}^{-1}\colon \mathbf{1} \to \mathbf{1} \otimes \mathbf{1}^{\vee} = \det \mathbf{0}$ *and* $\phi_{V,W}\colon \det V \otimes \det W \to \det(V \oplus W)$ *defined by as the compositon*

$$\begin{array}{c} \det V \otimes \det W = \det V^0 \otimes (\det V^1)^{\vee} \otimes \det W^0 \otimes (\det W^1)^{\vee} \\ \downarrow{\scriptstyle 1\otimes\beta\otimes 1} \\ \det V^0 \otimes \det W^0 \otimes (\det V^1)^{\vee} \otimes (\det W^1)^{\vee} \\ \downarrow{\scriptstyle 1\otimes\kappa} \\ \det V^0 \otimes \det W^0 \otimes \big(\det V^1 \otimes \det W^1\big)^{\vee} \\ \downarrow{\scriptstyle \phi\otimes\phi} \\ \det(V^0 \oplus W^0) \otimes (\det(V^1 \oplus W^1))^{\vee} = \det(V \oplus W). \end{array}$$

is a symmetric monoidal functor. ∎

Proposition 105.12. *Proposition 105.7 holds for* $\det\colon \mathscr{V}^{\times}_{\mathbf{Z}/2\mathbf{Z}} \to \mathscr{L}$. ∎

Remark 105.13. If $f\colon V \to W$ is an isomorphism, then

$$\det f = \psi_{f,0} \circ (\mathbf{1} \otimes \varepsilon) \circ \rho^{-1}. \quad ♣$$

Denote by $\mathscr{C}^{\times}_{\mathbf{Z}/2\mathbf{Z}}$ the category whose objects are finite dimensional $\mathbf{Z}/2\mathbf{Z}$–graded differential vector space $(A, \mathrm{d}\colon A \to A[1])$ and whose morphisms are quasi-isomorphisms. $\mathscr{V}^{\times}_{\mathbf{Z}/2\mathbf{Z}}$ includes into $\mathscr{C}^{\times}_{\mathbf{Z}/2\mathbf{Z}}$ via $A \mapsto (A, 0)$.[4] The following extends det to $\mathscr{C}^{\times}_{\mathbf{Z}/2\mathbf{Z}}$.

Definition 105.14.

(1) Let (A, d) be a $\mathbf{Z}/2\mathbf{Z}$–graded complex. Set

$$\det(A, \mathrm{d}) := \det A = \det A^0 \otimes (\det A^1)^{\vee}.$$

(2) Let (A, d) be a $\mathbf{Z}/2\mathbf{Z}$–graded complex. The short exact sequences

$$0 \to \operatorname{im} d \xrightarrow{i} \ker d \xrightarrow{p} \mathrm{H}(A, d) \to 0 \quad \text{and} \quad 0 \to \ker d \xrightarrow{j} A \xrightarrow{d} \operatorname{im} d[1] \to 0$$

induce the isomorphisms

$$\psi_{i,p}\colon \det \operatorname{im} d \otimes \det \mathrm{H}(A, d) \to \det \ker d$$

[4]Forgetting the differential is not a functor! This is because quasi-isomorphisms are not isomorphism.

and

$$\psi_{j,d}\colon\ \det\ker d\otimes\det\operatorname{im} d[1]\to\det A.$$

Define $\eta_{(A,d)}\colon\ \det \mathrm{H}(A,d)\to\det A$ as the composition

$$\begin{array}{c}\det \mathrm{H}(A,d)\\ \Big\downarrow{\scriptstyle(\mathrm{ev}^{-1}\otimes\mathbf{1})\circ\lambda^{-1}}\\ (\det\operatorname{im} d)^\vee\otimes[\det\operatorname{im} d\otimes\det\mathrm{H}(A,d)]\\ \Big\downarrow{\scriptstyle\delta^{-1}\otimes\psi_{i,p}}\\ \det\operatorname{im} d[1]\otimes\det\ker d\\ \Big\downarrow{\scriptstyle\psi_{j,d}\circ\beta}\\ \det A.\end{array}$$

(3) Let (A,d_A), (B,d_B) be $\mathbf{Z}/2\mathbf{Z}$–graded complexes. Let $f\colon\ (A,d_A)\to(B,d_B)$ be a quasi-isomorphism. Set

$$\det f\coloneqq\eta_{(B,d_B)}\circ\det\mathrm{H}(f)\circ\eta^{-1}_{(A,d_A)}\colon\ \det(A,d_A)\to\det(B,d_B).$$ •

Proposition 105.15. $\det\colon\ \mathscr{C}^\times_{\mathrm{Z}/2\mathrm{Z}}\to\mathscr{L}$ *is a functor and* $\eta\colon\ \det\circ\,\mathrm{H}\Rightarrow\det$ *is a natural transformation.* ■

Proposition 105.16. *If* $f\colon (A, d_A) \to (B, d_B)$ *is an isomorphism, then* $\det f\colon \det A \to \det B$ *and* $\det f\colon \det(A, d_A) \to \det(B, d_B)$ *agree.*

Proof. The assertion holds if the diagram

$$
\begin{array}{ccc}
\det \mathrm{H}(A, d_A) & \xrightarrow{\det \mathrm{H}(f)} & \det \mathrm{H}(B, d_B) \\
\downarrow{\scriptstyle (\mathrm{ev}^{-1} \otimes \mathbf{1}) \circ \lambda^{-1}} & & \downarrow{\scriptstyle (\mathrm{ev}^{-1} \otimes \mathbf{1}) \circ \lambda^{-1}} \\
(\det \operatorname{im} d_A)^\vee \otimes (\det \operatorname{im} d_A) \otimes \det \mathrm{H}(A, d_A) & \xrightarrow{(\det f)^\vee \otimes \det f \otimes \det \mathrm{H}(f)} & (\det \operatorname{im} d_B)^\vee \otimes (\det \operatorname{im} d_B) \otimes \det \mathrm{H}(B, d_B) \\
\downarrow{\scriptstyle \delta^{-1} \otimes \psi_{i_A, p_A}} & & \downarrow{\scriptstyle \delta^{-1} \otimes \psi_{i_B, p_B}} \\
\det \operatorname{im} d_A[1] \otimes \det \ker d_A & \xrightarrow{\det f[1] \otimes \det f} & \det \operatorname{im} d_B[1] \otimes \det \ker d_B \\
\downarrow{\scriptstyle \psi_{j_A, d_A} \circ \beta} & & \downarrow{\scriptstyle \psi_{j_B, d_B} \circ \beta} \\
\det A & \xrightarrow{\det f} & \det B
\end{array}
$$

commutes. The top square evidently commutes. The middle and bottom square commute by Proposition 105.12 applied to

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{im} d_A & \xrightarrow{i_A} & \ker d_A & \xrightarrow{p_A} & \mathrm{H}(A, d_A) & \longrightarrow & 0 \\
 & & {\scriptstyle \cong}\downarrow{\scriptstyle f} & & {\scriptstyle \cong}\downarrow{\scriptstyle f} & & {\scriptstyle \cong}\downarrow{\scriptstyle \mathrm{H}(f)} & & \\
0 & \longrightarrow & \operatorname{im} d_B & \xrightarrow{i_B} & \ker d_B & \xrightarrow{p_B} & \mathrm{H}(B, d_B) & \longrightarrow & 0
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker d_A & \xrightarrow{j_A} & A & \xrightarrow{d_A} & \operatorname{im} d_A[1] & \longrightarrow & 0 \\
 & & {\scriptstyle \cong}\downarrow{\scriptstyle f} & & {\scriptstyle \cong}\downarrow{\scriptstyle f} & & {\scriptstyle \cong}\downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \ker d_B & \xrightarrow{j_B} & B & \xrightarrow{d_B} & \operatorname{im} d_B[1] & \longrightarrow & 0.
\end{array}
$$

■

Proposition 105.17. *If* $0 \to (A, d_A) \xrightarrow{\iota} (B, d_B) \xrightarrow{\pi} (I, d_I) \to 0$ *is a short exact sequence of* $\mathbf{Z}/2\mathbf{Z}$*–graded differential vector spaces and* $\mathrm{H}(I, d_I) = 0$*, then*

$$\begin{array}{ccc} \det A & \xrightarrow{(1\otimes\varepsilon)\circ\rho^{-1}} & \det A \otimes \det \mathrm{H}(I, d_I) \\ {\scriptstyle \det \pi}\big\downarrow & & \big\downarrow{\scriptstyle 1\otimes\eta_{(I,d_I)}} \\ \det B & \xleftarrow[\psi_{\iota,\pi}]{} & \det A \otimes \det I \end{array}$$

commutes.

Proof. `Sadly, I have not been able to typeset this proof reasonably.` Proposition 105.12 applied to

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker d_A & \xrightarrow{j_A} & A & \xrightarrow{d_A} & \mathrm{im}\, d_A[1] & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \iota'} & & \big\downarrow{\scriptstyle \iota} & & \big\downarrow{\scriptstyle \iota''[1]} & & \\ 0 & \longrightarrow & \ker d_B & \xrightarrow{j_B} & B & \xrightarrow{d_B} & \mathrm{im}\, d_B[1] & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi''[1]} & & \\ 0 & \longrightarrow & \ker d_I & \xrightarrow{j_I} & I & \xrightarrow{d_I} & \mathrm{im}\, d_I[1] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{im}\, d_A & \xrightarrow{i_A} & \ker d_A & \xrightarrow{p_A} & \mathrm{H}(A, d_A) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \iota''} & & \big\downarrow{\scriptstyle \iota'} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle \mathrm{H}(\iota)} & & \\ 0 & \longrightarrow & \mathrm{im}\, d_B & \xrightarrow{i_B} & \ker d_B & \xrightarrow{p_B} & \mathrm{H}(B, d_B) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \pi''} & & \big\downarrow{\scriptstyle \pi'} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{im}\, d_I & = & \ker d_I & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

implies that

$$\begin{array}{ccc} \det\ker d_A \otimes \det\mathrm{im}\, d_A[1] \otimes \det\ker d_I \otimes \det\mathrm{im}\, d_I[1] & \xrightarrow{\psi_{j_A,d_A}\otimes\psi_{j_I,d_I}} & \det A \otimes \det I \\ \big\downarrow{\scriptstyle (1\otimes\psi_{\iota''[1],\pi''[1]})\circ\beta} & & \\ \det\ker d_A \otimes \det\ker d_I \otimes \det\mathrm{im}\, d_B[1] & & \big\downarrow{\scriptstyle \psi_{\iota,\pi}} \\ \big\downarrow{\scriptstyle \psi_{\iota',\pi'}\otimes 1} & & \\ \det\ker d_B \otimes \det\mathrm{im}\, d_B[1] & \xrightarrow{\psi_{j_B,d_B}} & \det B \end{array} \tag{105.18}$$

and

(105.19)
$$\begin{CD}
(\det \operatorname{im} d_A \otimes \det \mathrm{H}(A, d_A)) \otimes \det \operatorname{im} d_I @>{\psi_{i_A,p_A} \otimes \mathbf{1}}>> \det \ker d_A \otimes \det \ker d_I \\
@V{(\psi_{\iota'',\pi''} \otimes \det \mathrm{H}(i)) \circ \beta}VV @VV{\psi_{\iota',\pi'}}V \\
\det \operatorname{im} d_B \otimes \det \mathrm{H}(B, d_B) @>{\psi_{i_B,p_B}}>> \det \ker d_B
\end{CD}$$

commute.

The diagram

(105.20)
$$\begin{CD}
(\det \operatorname{im} d_A \otimes \det \mathrm{H}(A, d_A)) \otimes (\det \operatorname{im} d_I \otimes \det \operatorname{im} d_A[1] \otimes \det \operatorname{im} d_I[1]) @>{\psi_{i_A,p_A} \otimes \mathbf{1}}>> \det \ker d_A \otimes (\det \operatorname{im} d_I \otimes \det \operatorname{im} d_A[1] \otimes \det \operatorname{im} d_I[1]) \\
@V{\mathbf{1} \otimes \psi_{\iota''[1],\pi''[1]}}VV @VV{\mathbf{1} \otimes \psi_{\iota''[1],\pi''[1]}}V \\
(\det \operatorname{im} d_A \otimes \det \mathrm{H}(A, d_A)) \otimes \det \operatorname{im} d_I \otimes \det \operatorname{im} d_B[1] @>{\psi_{i_A,p_A} \otimes \mathbf{1}}>> \det \ker d_A \otimes \det \ker d_I \otimes \det \operatorname{im} d_B[1]
\end{CD}$$

obviously commutes. The diagram

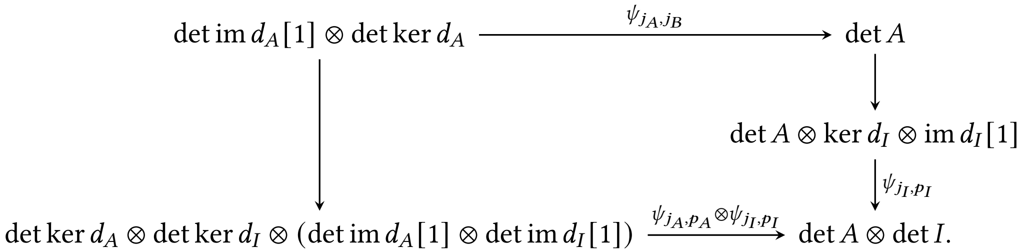
(105.21)
$$\begin{CD}
\det \operatorname{im} d_A[1] \otimes \det \operatorname{im} d_A \otimes \det \mathrm{H}(A, d_A) @>{\mathbf{1} \otimes \psi_{i_A,p_A}}>> \det \operatorname{im} d_A[1] \otimes \det \ker d_A \\
@V{\mathrm{ev}^{-1}}VV @VV{\mathrm{ev}^{-1}}V \\
(\det \operatorname{im} d_A \otimes \det \mathrm{H}(A, d_A)) \otimes \det \operatorname{im} d_I \otimes (\det \operatorname{im} d_A[1] \otimes \det \operatorname{im} d_I[1]) @>{\psi_{i_A,p_A} \otimes \mathbf{1}}>> \det \ker d_A \otimes \det \ker d_I \otimes (\det \operatorname{im} d_A[1] \otimes \det \operatorname{im} d_I[1])
\end{CD}$$

obviously commutes (because $\ker d_I = \operatorname{im} d_I$).

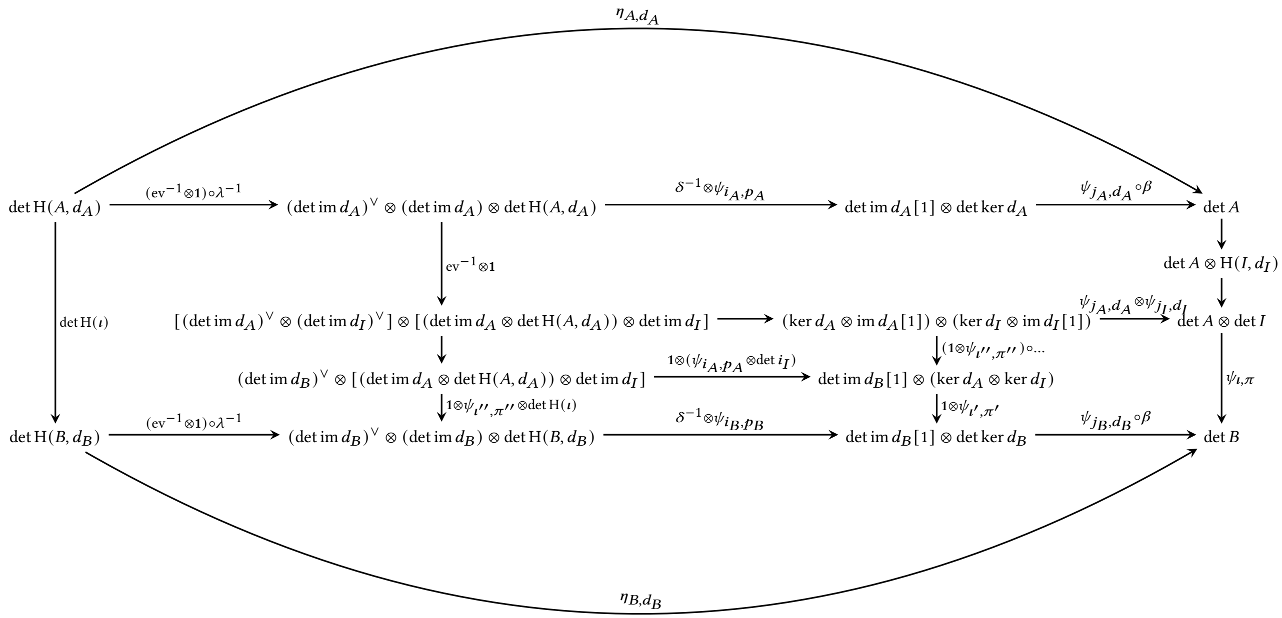
The diagram

(105.22)
$$\begin{CD}
\det \mathrm{H}(A, d_A) @>>> \det \operatorname{im} d_A[1] \otimes \det \operatorname{im} d_A \otimes \det \mathrm{H}(A, d_A) \\
@V{\mathrm{H}(\iota)}VV @VVV \\
@. (\det \operatorname{im} d_A \otimes \det \mathrm{H}(A, d_A)) \otimes \det \operatorname{im} d_I \otimes (\det \operatorname{im} d_A[1] \otimes \det \operatorname{im} d_I[1]) \\
@. @VVV \\
\det \mathrm{H}(B, d_B) @<<< \det \operatorname{im} d_B \otimes \det \mathrm{H}(B, d_B) \otimes \operatorname{im} d_B[1]
\end{CD}$$

commutes too an so does



These assemble into:



Since this diagram commutes the assertion follows. ■

106 The determinant line of a family of Fredholm operators

Let X, Y be Banach spaces.

$$\det := \coprod_{L\in\mathscr{F}(X,Y)} \det\ker L \otimes (\det\operatorname{coker} L)^\vee$$

This wants to be a vector bundle but it has to be given a topology first. Here is how to construct this topology. Let $\sigma\colon S \to Y$ be a linear map with $\dim S < \infty$. The operator

$$(L\ \sigma)\colon X \oplus S \to Y$$

is Fredholm. The Snake Lemma implies that the sequence

$$0 \to \ker L \xrightarrow{i_L} \ker(L\ \sigma) \xrightarrow{\mathrm{pr}_S} S \xrightarrow{\sigma} \operatorname{coker} L \to \operatorname{coker}(L\ \sigma) \to 0$$

is exact. Therefore, if $\operatorname{coker}(L\ \sigma) = 0$ and

$$C_{L,\sigma} := \big(\ker(L\ \sigma) \xrightarrow{\mathrm{pr}_S} S\big),$$

then i_L defines an isomorphism $i_L\colon \ker L \to \mathrm{H}^0(C_{L,\sigma})$ and σ induces an isomorphism $\sigma\colon \mathrm{H}^1(C_{L,\sigma}) \to \operatorname{coker} L$. Therefore

$$\det\ker L \otimes (\det\operatorname{coker} L)^\vee \xrightarrow{\det i_L \otimes (\det\sigma^{-1})^\vee} \det \mathrm{H}(C_{L,\sigma}) \xrightarrow{\upsilon} \det\ker(L\ \sigma) \otimes (\det S)^\vee$$

The subset

$$U_\sigma := \{L \in \mathscr{F}(X,Y) : \operatorname{im} L + \operatorname{im}\sigma = Y\}.$$

is open and

$$\coprod_{L\in U_\sigma} \ker(L\ \sigma) \to U_\sigma.$$

"is" a vector bundle. Therefore,

$$\coprod_{L\in U_\sigma} \det\ker(L\ \sigma) \otimes (\det S)^\vee \to U_\sigma$$

"is" too. Demanding that the above maps assemble into a vector bundle isomorphim defines the vector bundle structure on $\det|_{U_\sigma}$.

It remains to prove that if $\sigma\colon S \to Y$ and $\tau\colon T \to Y$ are given, then the trivialisations overlap continuously on $U_\sigma \cap U_\tau$. That is one needs to show that the map $\star$ in the commutative diagram

$$\begin{array}{ccccc}
\det\ker L \otimes (\det\operatorname{coker} L)^\vee & \longrightarrow & \det\mathrm{H}(C_{L,\sigma}) & \xrightarrow{\eta} & \det\ker(L\ \sigma) \otimes (\det S)^\vee \\
\| & & & & \downarrow{\scriptstyle\star} \\
\det\ker L \otimes (\det\operatorname{coker} L)^\vee & \longrightarrow & \det\mathrm{H}(C_{L,\tau}) & \xrightarrow{\eta} & \det\ker(L\ \tau) \otimes (\det T)^\vee.
\end{array}$$

varies continuously in $U_\sigma \cap U_\tau$. It suffices to prove this assuming that there is a surjective map $p\colon\ S \to T$ such that

$$\begin{array}{ccc} S & \xrightarrow{p} & T \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} \\ Y & = & T \end{array}$$

commutes. In this case $\mathbf{1} \oplus p$ and p define a surjective quasi-isomorphism

$$\mathbf{p}\colon\ C_{L,\sigma} \to C_{L,\tau}$$

with

$$\ker \mathbf{p} = (\ker p \xrightarrow{\mathrm{id}} \ker p).$$

By definition $\star$ is $\det \mathbf{p}$. By Proposition 105.17 can be written in terms of $\det \ker \mathbf{p} \to \mathbf{1}$ and thus varies continuously in $U_\sigma \cap U_\tau$.

Lecture 23

107 Witten's vanishing theorem

Theorem 107.1. *Let be X is a closed manifold with $b^+(X) \geqslant 2$. If X carries a metric with positive scalar curvature, then* $\mathrm{SW} = 0$.

Proof. This is immediate from the Weitzenböck formula which forces $\mathscr{M}(\mathfrak{w}, \eta) = 0$ for sufficiently small η. ■

108 The simple type conjecture

Conjecture 108.1. *Let be X is a closed manifold with $b^+(X) \geqslant 2$. Let $\mathfrak{w}$ be a spin$^{\mathrm{U}(1)}$–structure on X. If $d(\mathfrak{w}) > 0$, then* $\mathrm{SW}(\mathfrak{w}) = 0$. □

Remark 108.2.

(1) There is no known counter-example.

(2) $\mathrm{SW}(\mathfrak{w}) = 0$ does not mean that $\mathscr{M}(\mathfrak{w}, \eta) = \varnothing$. In fact, it is possible to produce non-empty moduli spaces by gluing constructions even if $d > 0$. ♣

109 Charge conjugation symmetry

The map $c\colon \mathrm{Spin}^{\mathrm{U}(1)}(n) \to \mathrm{Spin}^{\mathrm{U}(1)}(n)$ defined by

$$c[g, z] := [g, \bar{z}]$$

is a group homomorphism. If $\mathfrak{w}$ is a spin$^{\mathrm{U}(1)}$–structure, then so is the **charge conjugate**

$$\tilde{\mathfrak{w}} := \mathfrak{w} \times_c \mathrm{Spin}^{\mathrm{U}(1)}(n).$$

This construction defines an involution on the set of isomorphism classes of spin$^{\mathrm{U}(1)}$–structures.

It is clear that the characteristic line bundle L of $\mathfrak{w}$ and $\tilde{L}$ of $\tilde{\mathfrak{w}}$ are related by $\tilde{L} \cong \bar{L}$ Moreover, the complex spinor bundles $S^{\pm}$ and $\tilde{S}^{\pm}$ associated with $\mathfrak{w}$ and $\tilde{\mathfrak{w}}$, and the Dirac operators, are (also) related by a complex anti-linear isomorphism. This sets up a bijection of Seiberg–Witten moduli spaces

$$\mathscr{M}(\mathfrak{w}, \eta) \cong \mathscr{M}(\tilde{\mathfrak{w}}, -\eta).$$

It is a good exercise in orientations to prove the following:

Proposition 109.1.

$$\mathrm{SW}(\mathfrak{w}) = (-1)^{\frac{\chi(X)+\sigma(X)}{4}} \mathrm{SW}(\tilde{\mathfrak{w}}).$$

110 $b^+ = 1$ and wall-crossing

Let X be a closed oriented 4–manifold $b^+(X) = 1$. Fix a Riemannian metric g on X. Suppose that $\mathrm{H}^1_{\mathrm{dR}}(X) \oplus (\mathrm{H}^+_{\mathrm{dR}}(X) \oplus \mathrm{H}^0_{\mathrm{dR}}(X))[1]$ has been oriented and, indeed, $\mathrm{H}^+_{\mathrm{dR}}(X)$ has been oriented as well.

Let $\mathfrak{w}$ be a spin$^{\mathrm{U}(1)}$–structure on X and $\eta \in \Omega^+(X, i\mathbf{R})$. The map $\mathrm{H}^+_{\mathrm{dR}}(X) = \mathscr{H}^+(X) \to \mathbf{R}$ defined by

$$\alpha \mapsto \langle i\eta, \alpha\rangle_{L^2} - \pi\langle \mathrm{c}_1(L) \cup [i\alpha], [X]\rangle.$$

If the map vanishes, then $\varepsilon = \varepsilon(\eta, \mathfrak{w}) := 0$ (this is the case when reducibles exists). Otherwise, if it is orientation preserving (reversing), set $\varepsilon := +1$ ($\varepsilon := -1$). The function ε decomposes $\Omega^+(X, i\mathbf{R})$ into two chambers $\{\pm\varepsilon > 0\}$ and a wall $\{\varepsilon = 0\}$. The arguments used to define the bordism class $[\mathscr{M}(\mathfrak{w})]$ if $b^+(X)$ can be used to define

$$[\mathscr{M}(\mathfrak{w})^+] := [\mathscr{M}(\mathfrak{w}, \eta)] \quad \text{with} \quad \pm\,\varepsilon(\eta) > 0.$$

This gives the Seiberg–Witten invariants

$$\mathrm{SW}^\pm(\mathfrak{w}) \in \mathbf{Z}.$$

Theorem 110.1 (Wall-crossing formula for the Seiberg–Witten invariant). *If $b_1(X) = 0$, then*

$$\mathrm{SW}^+(\mathfrak{w}) - \mathrm{SW}^-(\mathfrak{w}) = 1.$$

Remark 110.2. The above is true regardless of the orientation of $\mathrm{H}^+(X)$ because flipping it exchanges + and − but also flips the sign of the invariant. ♣

Remark 110.3. There also is a wall-crossing formula if $b_1(X) \neq 0$ ♣

Example 110.4. Consider $\mathbf{C}P^2$. Since $\mathbf{C}P^2$ is Kähler, it carries a distinguished spin$^{\mathrm{U}(1)}$–structure $\mathfrak{w}_0$ and every other spin$^{\mathrm{U}(1)}$–structure $\mathfrak{w}_k$ is obtained by twisting with $\mathscr{O}_{\mathbf{C}P^2}(k)$ for some $k \in \mathbf{Z}$. Since $K_{\mathbf{C}P^2} = \mathscr{O}_{\mathbf{C}P^3}(-3)$, the characteristic line bundle L_k associated with $\mathfrak{w}_k$ is

$$L_k \cong \mathscr{O}_{\mathbf{C}P^3}(2k+3).$$

Therefore,

$$d(\mathfrak{w}_k) = \frac{1}{4}\big((2k+3)^2 - 9\big) = k(k+3).$$

For $k \in \{-1, -2\}$ this is negative and, therefore,

$$\mathrm{SW}^\pm(\mathfrak{w}_k) = 0 \quad \text{for} \quad k \in \{-1, -2\}.$$

The Fubini–Study metric on $\mathbf{C}P^2$ has positive scalar curvature and, therefore, $\mathscr{M}(\mathfrak{w}_k, \eta) = \varnothing$ for sufficiently small η. Since

$$\varepsilon(0, \mathfrak{w}_k) = \mathrm{sign}(-(2k+3)),$$

it follows that

$$\mathrm{SW}^+(\mathfrak{w}_k) = 0 \quad \text{for} \quad k \leqslant -2 \quad \text{and} \quad \mathrm{SW}^+(\mathfrak{w}_k) = 0 \quad \text{for} \quad k \geqslant -1.$$

The wall-crossing formula determines the remaining values of $\mathrm{SW}^\pm(\mathfrak{w}_k)$ to be ± 1. ♠

Remark 110.5. As similar discussion applies to $\mathbf{C}P^1 \times \mathbf{C}P^1$. ♣

Lecture 23

111 The Seiberg–Witten energy identity

Proposition 111.1. *Let* $(X, \mathfrak{w})$ *be a closed spin*$^{\mathrm{U}(1)}$ 4*–manifold. For every* $(A, \Phi) \in \mathscr{A}(\mathfrak{w}) \times \Gamma(W^+)$

$$E(A,\Phi) := \int_X |\nabla_A \Phi|^2 + \frac{1}{4}\mathrm{scal}_g|\Phi|^2 + \frac{1}{2}|\mu(\Phi) + \eta|^2 + \frac{1}{4}|F_A^{\mathrm{tw}}|^2$$

satisfies

$$E(A,\Phi) = \int_X |D_A\Phi|^2 + \frac{1}{2}|F_A^+ - \mu(\Phi) - \eta|^2 + \langle F_A^{\mathrm{tw},+}, \eta\rangle - \frac{1}{4}\pi^2 \mathrm{c}_1(L_{\mathfrak{w}})^2.$$

Proof. By the Weitzenböck formula,

$$\int_X |D_A\Phi|^2 = \int_X |\nabla_A\Phi|^2 + \frac{1}{4}\mathrm{scal}_g|\Phi|^2 + \langle F_A^{\mathrm{tw},+}, \mu(\Phi)\rangle.$$

Therefore,

$$E(A,\Phi) = \int_X |D_A\Phi|^2 + \frac{1}{2}|F_A^{\mathrm{tw},+} - \mu(\Phi) - \eta|^2 + \frac{1}{4}|F_A^{\mathrm{tw}}|^2 - \frac{1}{2}|F_A^{\mathrm{tw},+}|^2 + \langle F_A^{\mathrm{tw},+}, \eta\rangle$$

The assertion follows because

$$\int_X |F_A^{\mathrm{tw},-}|^2 - |F_A^{\mathrm{tw},+}|^2 = \int_X F_A^{\mathrm{tw}} \wedge F_A^{\mathrm{tw}} = \left(\frac{\pi}{i}\mathrm{c}_1(L_{\mathfrak{w}})\right)^2 = -\pi^2\mathrm{c}_1(L_{\mathfrak{w}})^2$$

■

112 Seiberg–Witten invariants of Kähler surfaces

Let X be a connected closed Kähler surface. X carries a canonical spin$^{\mathrm{U}(1)}$–structure $\mathfrak{w}_{\mathbf{C}}$ whose characteristic line bundle is K_X^*. Every other spin$^{\mathrm{U}(1)}$–structure $\mathfrak{w}_L$ is obtained by twisting by a Hermitian line bundle L. For $\mathfrak{w}_L$:

$$S^+ = \Lambda^{0,0}T^*X \otimes_{\mathbf{C}} L \oplus \Lambda^{0,2}T^*X_{\mathbf{C}}L \quad \text{and} \quad S^- = \Lambda^{0,1}T^*X_{\mathbf{C}}L$$

and

$$\gamma(v)\alpha = \sqrt{2}[(v^\flat)^{0,1} \wedge \alpha - i_{v^{0,1}}\alpha]$$

and

$$D_A = \sqrt{2}(\bar\partial_A + \bar\partial_A^*).$$

with $A \in \mathscr{A}(L)$.

To understand the Seiberg–Witten equation in this context we also need to compute μ. Recall that on a Kähler surface $\Lambda^+T^*X \otimes \mathbf{C} = \langle\omega\rangle \oplus \Lambda^{2,0}T^*X \oplus \Lambda^{0,2}T^*X$. In particular, $\mu \in \Omega^+(X, i\mathbf{R})$ is determined by $\Lambda\mu$ (with Λ denoting the dual Lefschetz operator) and $\mu^{0,2}$. A computation shows that for $(\alpha, \beta) \in \Gamma(L) \oplus \Omega^{0,2}(X, L)$

$$i\Lambda\mu(\alpha, \beta) = \frac{1}{4}(|\beta|^2 - |\alpha|^2) \quad \text{and} \quad \mu(\alpha, \beta)^{0,2} = \frac{1}{2}\bar{\alpha}\beta.$$

Therefore, the η-perturbed Seiberg–Witten equation becomes the following equation for $A \in \mathscr{A}(L)$ and $(\alpha, \beta) \in \Gamma(L) \oplus \Omega^{0,2}(X, L)$:

$$\begin{aligned} \bar{\partial}_A\alpha + \bar{\partial}_A^*\beta &= 0 \\ i\Lambda(-F_{K_X} + F_A) &= \frac{1}{4}(|\beta|^2 - |\alpha|^2) + i\Lambda\eta \\ F_A^{0,2} &= \frac{1}{2}\bar{\alpha}\beta + \eta^{0,2} \end{aligned} \tag{112.1}$$

One often restricts to perturbations with $\eta^{0,2} = 0$.

Proposition 112.2. *Suppose that $\eta^{0,2} = 0$. Set*

$$t := -\frac{i}{2\pi}\int_X \eta \wedge \omega.$$

If (A, α, β) be a solution of (112.1), then:

1. $F_A^{0,2} = 0$*; that is: $\mathscr{L} = (L, \bar{\partial}_A)$ is a holomorphic line bundle.*
2. *If $\deg L \geqslant \frac{1}{2}\deg K_X + t$, then $\alpha = 0$, $\bar{\partial}_A^*\beta = 0$.*
3. *If $\deg L \leqslant \frac{1}{2}\deg K_X + t$, then $\beta = 0$, $\bar{\partial}_A\alpha = 0$.*

Proof. Applying $\bar{\partial}_A$ to the first equation in (112.1) yields

$$F_A^{0,2}\alpha + \bar{\partial}_A\bar{\partial}_A^*\beta = 0.$$

Using the last equation in (112.1) this becomes

$$\frac{1}{2}|\alpha|^2\beta + \bar{\partial}_A\bar{\partial}_A^*\beta = 0.$$

Taking the L^2 inner product with β yields

$$\frac{1}{2}\int_X |\alpha|^2|\beta|^2 + |\bar{\partial}_A^*\beta| = 0.$$

By unique continuation it follows that either α vanishes (and $\bar{\partial}_A^*\beta = 0$) or β vanishes. In particular, $F_A^{0,2}$.

Since $\frac{iF_A}{\pi}$ represents the first Chern class of the characteristic line bundle and the latter is $L^2 \otimes K_X^*$,

$$\deg(L^2 \otimes K_X^*) = \int_X \frac{iF_A}{\pi} \wedge \omega = \frac{1}{\pi}\int_X i\Lambda F_A \mathrm{vol} = \frac{1}{4\pi}\int_X |\beta|^2 - |\alpha|^2 \mathrm{vol} - 2t.$$

This determine whether α or β vanishes. ∎

Remark 112.3. Here is another argument for based on the energy identity; that is:

$$E(A,\Phi) := \int_X |\nabla_A\Phi|^2 + \frac{1}{4}\mathrm{scal}_g|\Phi|^2 + \frac{1}{2}|\mu(\Phi)+\eta|^2 + \frac{1}{4}|F_A^{\mathrm{tw}}|^2$$

satisfies

$$E(A,\Phi) = \int_X |D_A\Phi|^2 + \frac{1}{2}|F_A^+ - \mu(\Phi) - \eta|^2 + \langle F_A^{\mathrm{tw}}, \eta\rangle + \frac{1}{4}\pi^2 \mathrm{c}_1(L_{\mathfrak{w}})^2.$$

The map $(A,\alpha,\beta)\mapsto(A,\alpha,-\beta)$ does not affect $E(A,\Phi)$ (because $\eta^{0,2}=0$. Therefore, it maps solutions to solutions. Hence:

$$-\frac{1}{2}\bar{\alpha}\beta = F_A^{0,2} = \frac{1}{2}\bar{\alpha}\beta = 0.$$

♣

Remark 112.4.

(1) By the charge conjugation symmetry/Serre duality it is enough to consider the case $\deg L \geqslant \frac{1}{2}\deg K_X$ with $t=0$.

(2) If $\deg L = \frac{1}{2}\deg K_X$ and $t=0$, then the above exhibits $\mathscr{M}(\mathfrak{w}_L)$ as the moduli space of ASD instantons on L.

(3) In any case, by passing to $t \gg 1$ one can always reduce to the case $\beta=0$.

♣

Proposition 112.5. *If $\mathscr{L}=(L,\bar\partial)$ is a holomorphic line bundle with $\deg L < \frac{1}{2}\deg K_X + t$ and $\alpha \in H^0(X,\mathscr{L})\backslash\{0\}$, then L admits a Hermitian metric H such that the Chern connection $A=A(\bar\partial,H)$ satisfies*

$$i\Lambda F_A = -\frac{1}{4}|\alpha|^2 - \pi t$$

Proof. Choose a Hermitian metric H_0 on L. Denote the corresponding Chern connection by A_0. Any other Hermitian metric is of the form $H=H_0e^{2f}$. A computation shows that

$$i\Lambda F_A = i\Lambda F_{A_0} + \Delta f.$$

Therefore, the task at hand is to find f such that

$$\Delta f + ae^{2f} + b = 0 \quad \text{with} \quad a := \frac{1}{4}H_0(\alpha,\alpha) \quad \text{and} \quad b := i\Lambda(F_{A_0}-\eta).$$

This is a Kazdan–Warner equation. It can be solved because $\int_X b < 0$ and $a \neq 0$. (The solution theory of this equation also appears in an analytic proof of the metric uniformisation theorem.) ∎

The above (together with some thinking about gauge equivalence) exhibits a bijection between $\mathscr{M}(\mathfrak{w}_L)$ and pairs $[\mathscr{L},\alpha]$ consisting of a holomorphic line bundle (with underlying complex line bundle L) and a non-zero holomorphic section $\alpha \in \mathbf{P}\mathrm{H}^0(X,\mathscr{L})$ up to scaling. This is essentially the same data the effective divisor $D := Z(\alpha)$.

Finally, we discuss the actual invariants. We employ the following orientation convention. $\mathrm{H}^1(X)$ is oriented so that the projection onto $\mathrm{H}^{0,1}(X,\mathbf{C})$ is orientation preserving if the latter is given the complex orientation. $\mathrm{H}^+(X) = \langle\omega\rangle \oplus \mathrm{H}^{0,2}(X,\mathbf{C})$ is oriented so that ω is positive and $\mathrm{H}^{0,2}(X,\mathbf{C})$ is given the complex orientation.

Proposition 112.6 (Witten [Wit94, §4]). *If* $b^+(X) \geqslant 2$ *and if* $\mathrm{SW}(\mathfrak{w}_L) \neq 0$*, then* $0 \leqslant \deg L \leqslant \deg K_X$.

Proof. If $\mathrm{SW}(\mathfrak{w}_L) \neq 0$, then the Seiberg–Witten moduli space must be non-empty for $t \gg 1$. Therefore, $\mathrm{H}^0(X, L) \neq 0$; hence, $\deg L \geqslant 0$. Similarly, the Seiberg–Witten moduli space must be non empty for $t \gg 1$. Therefore, $\mathrm{H}^1(X, L) \neq 0$. By Serre duality, $\mathrm{H}^0(X, K_X \otimes L^*)$; hence $\deg L \leqslant \deg K_X$. ∎

Theorem 112.7. *If* $b^+(X) \geqslant 2$*, then*

$$\mathrm{SW}(\mathfrak{w}_{\mathbf{C}}) = \mathbf{1} \quad \textit{and} \quad \mathrm{SW}(\mathfrak{w}_{K_X}) = (-1)^{\frac{\sigma(X)+\chi(X)}{4}}.$$

Proof. By the charge conjugation symmetry it suffices to prove that $\mathrm{SW}(\mathfrak{w}_{\mathbf{C}}) = \mathbf{1}$. Indeed, the following argument proves that $\mathscr{M}(\mathfrak{w}_{\mathbf{C}}, \eta)$ consists of precisely one point, which is unobstructed, and positively oriented.

Take the perturbation to be

$$\eta = -\frac{1}{2}F^+_{K_X} + \frac{it}{2}\omega$$

with $t > 0$. If $[A, \alpha, \beta] \in \mathscr{M}(\mathfrak{w}_{\mathbf{C}}, \eta)$, then $\beta = 0$ (by the above proposition). Moreover, A induces a holomorphic structure $\bar\partial_A$ on the trivial bundle $\underline{\mathbf{C}}$. Therefore, the section α must be nowhere vanishing. That is α is an isomorphism of $(\underline{\mathbf{C}}, \bar\partial_A)$ with the trivial holomorphic bundle $(\underline{\mathbf{C}}, \bar\partial)$. Finally,

$$\frac{1}{4}|\alpha|^2 - t = 0.$$

Therefore, $[A, \alpha, \beta]$ is gauge equivalent to $[A_0, \sqrt{2t}, 0]$ with A_0 denoting the trivial connection on $\underline{\mathbf{C}}$.

Observe that

$$d(\mathfrak{w}_C) = \frac{1}{4}\big(c_1(K_X^*)^2 - 2\chi(X) - 3\sigma(X)\big).$$

Therefore, one expects $\mathscr{M}(\mathfrak{w}_{\mathbf{C}}, \eta)$ to be zero-dimensional. The linearisation of the Seiberg–Witten map at $(A_0, \sqrt{2t}, 0)$ is

$$T\mathfrak{sw} : (a, \hat\alpha, \hat\beta) \mapsto \begin{pmatrix} \sqrt{2}(\bar\partial\hat\alpha + \bar\partial^*\hat\beta) \\ i\Lambda \mathrm{d}a \\ \bar\partial a^{0,1} - \sqrt{t/2}\hat\beta \\ \mathrm{d}^* a \end{pmatrix} + \begin{pmatrix} \sqrt{2t}a^{0,1} \\ \frac{1}{2}\langle\sqrt{2t}, \hat\alpha\rangle \\ 0 \\ i\langle\sqrt{2t}, \hat\alpha\rangle. \end{pmatrix}$$

`The constants are almost certainly messed up.`

Suppose that $T\mathfrak{sw}(a, \hat\alpha, \hat\beta) = 0$. Applying $\bar\partial$ to the first equation and using the fourth equation gives

$$\bar\partial\bar\partial^*\beta + t/\sqrt{2}\beta = 0.$$

Since $t > 0$, this implies that $\hat\beta = 0$. Applying $\bar\partial^*$ to the first equation and using the third and fifth equation yields

$$\bar\partial^*\bar\partial\hat\alpha + 4t\hat\alpha = 0.$$

Therefore, $\hat{\alpha} = 0$. Considering the first equation again proves that $a = 0$. Since $d(\mathfrak{w}_{\mathbb{C}}) = 0$, it follows that $T\mathfrak{sw}$ is invertible.

I'm not proving that the orientation is indeed positive. (It suspect that the $T\mathfrak{sw}$, with the correct constants, is complex linear and the homotopy to the model operator is through complex linear operators.) ■

Lecture 24

113 The simple type conjecture for Kähler surfaces

Theorem 113.1 (Witten). *The simple type conjecture holds for Kähler surfaces.*

Proof. We need to prove that if $\mathrm{SW}(\mathfrak{w}_L) \neq 0$, then

$$d(\mathfrak{w}_L) = \frac{1}{4}(\mathrm{c}_1(K_X^* \otimes L^2)^2 - 2\chi(X) - 3\sigma(X)) = (\mathrm{c}_1(L) - \mathrm{c}_1(K_X))\mathrm{c}_1(L) \leqslant 0.$$

The above discussion already proves that L admits a holomorphic structure $\bar{\partial}$. Therefore, $\mathrm{c}_1(L)$ is of type $(1,1)$.

Since $b^+(X) \geqslant 2$, K_x admits a non-zero section. Let $\bar{\eta} \in \mathrm{H}^0(X, K_X) = \mathscr{H}^{2,0}(X)$ and take η to be our perturbation. The energy identity becomes that

$$E(A,\Phi) := \int_X |\nabla_A \Phi|^2 + \frac{1}{4}\mathrm{scal}_g |\Phi|^2 + \frac{1}{2}|\mu(\Phi) + \eta|^2 + \frac{1}{4}|F_A^{\mathrm{tw}}|^2$$

satisfies

$$E(A,\Phi) = \int_X |D_A \Phi|^2 + \frac{1}{2}|F_A^+ - \mu(\Phi) - \eta|^2 + \frac{1}{4}\pi^2 \mathrm{c}_1(L_{\mathfrak{w}})^2.$$

This is invariant under $(A, \alpha, \beta, \eta) \mapsto (A, \alpha, -\beta, -\eta)$. Therefore,

$$-(\bar{\alpha}\beta - \eta) = F_A^{0,2} = \bar{\alpha}\beta - \eta = 0.$$

Moreover, $\bar{\partial}_A \alpha = 0$ and $\bar{\partial}_A^* \beta = 0$.

Therefore, $\mathscr{L} = (L, \bar{\partial})$ is holomorphic, $\alpha \in \mathrm{H}^0(X, \mathscr{L})$, $\bar{\beta} \in \mathrm{H}^0(X, K_X \otimes \mathscr{L}^*)$ and

$$\eta = \alpha\bar{\beta}.$$

Decompose K_X into irreducible effective divisors as follows

$$\mathrm{c}_1(K_X) = \sum_i r_i [C_i].$$

The above shows that

$$\mathrm{c}_1(L) = \sum_i s_i [C_i].$$

with $0 \leqslant s_i \leqslant r_i$.

Therefore, to show that

$$d(\mathfrak{w}) = -(\mathrm{c}_1(K_X) - \mathrm{c}_1(L))\mathrm{c}_1(L) \leqslant 0$$

it suffices to prove that

$$(\mathrm{c}_1(K_X) - \mathrm{c}_1(L))[C_i] \geqslant 0.$$

By positivity of intersections,

$$(c_1(K_X) - c_1(L))[C_i] \geqslant (r_i - s_i)[C_i][C_i].$$

If $[C_i][C_i] \geqslant 0$, then this is sufficient.

By the adjunction formula,

$$2p_g(C_i) - 2 = [C_i][C_i] + c_1(K_X)[C_i].$$

Therefore,

$$-[C_i][C_i] - 2 \leqslant c_1(K_X)[C_i].$$

If $[C_i][C_i] < 0$ and $s_i \geqslant 1$, then

$$c_1(L)[C_i] \leqslant s_i[C_i]^2 \leqslant -[C_i][C_i] - 2 \leqslant c_1(K_X)[C_i].$$

This also proves the required statement. ■

114 Enriques–Kodaira classification

Definition 114.1. Let X be a complex surface. The **Kodaira dimension** is

$$\kappa(X) := \limsup_{n\to\infty} \frac{\log \dim H^0(X, K_X^n)}{\log n}.$$

X is **minimal** if it contains no holomorphic sphere E with $E \cdot E = -1$. X is ruled **ruled** if it is as $\mathbf{C}P^1$–bundle over a Riemann surface. X is **elliptic** it it has a holomorphic map $f\colon X \to \mathbf{C}P^1$ with generic fibre a torus and finitely many singular fibres. •

Remark 114.2. By Castelnuovo's contraction theorem, every X can be blow-down until it is minimal; that is: every sphere E with $E \cdot E = -1$ can be removed by an inverse blow-up procedure and finitely many such steps suffice to achieve minimality. ♣

Theorem 114.3 (Enriques–Kodaira classification). *Let X be a minimal Kähler surface.* $\kappa(X) \in \{-\infty, 0, 1, 2\}$ *and:*

(1) $\kappa(X) = -\infty$ *if and only if* X *is* $\mathbf{C}P^2$ *or ruled.*

(2) $\kappa(X) = 0$ *if and only if* X *is finitely covered by* T^4 *or* $K3$. *(*$c_1(K_X)$ *is torsion)*

(3) $\kappa(X) = 1$ *if and only if* X *is elliptic;* $c_1(K_X)$ *is not torsion and* $c_1(K_X)^2 = 0$.

(4) $\kappa(X) = 2$ *then* $c_1(K_X)^2 > 0$. ***(general type)***

If X is a minimal surface wit $\kappa(X) = 2$, then K_X is **nef**; that is: $K_X \cdot D \geqslant 0$ for every effective divisor D.

115 Diffeomorphism invariance of $\pm K_X$ for minimal surfaces of general type

Here is a neat application of Seiberg–Witten theory to complex algebraic geometry. It requires a tiny bit of preparation.

Theorem 115.1 (Hodge index theorem). *Let X be a Kähler surface. The intersection form on $\mathrm{H}^{1,1}(X, \mathbf{R})$ has signature $(1, b^{1,1} - 1)$.*

Theorem 115.2. *If X is a minimal Kähler surface of general type, then $\mathrm{SW}(\mathfrak{w}_L) \neq 0$ if and only if $L = \underline{\mathbf{C}}$ or $L = K_X$.*

Proof. If $\mathrm{SW}(\mathfrak{w}_L) \neq 0$, then there is a holomorphic structure $\bar\partial$ on L such that $\mathscr{L} = (L, \bar\partial)$ has a section and, therefore, determines an effective divisor.

By the simple type conjecture of Kähler surfaces, $\mathrm{c}_1(L)^2 = \mathrm{c}_1(L) \cdot \mathrm{c}_1(K_X)$. By hypothesis,

$$\mathrm{c}_1(L)^2 = \mathrm{c}_1(L) \cdot \mathrm{c}_1(K_X) \geqslant 0.$$

By charge conjugation symmetry/Serre duality, $\deg K_X \geqslant \deg L \geqslant 0$:

$$\mathrm{c}_1(K_X)^2 + \mathrm{c}_1(L)^2 - 2\mathrm{c}_1(K_x)\mathrm{c}_1(L) = \mathrm{c}_1(K_X)^2 - \mathrm{c}_1(L) \cdot \mathrm{c}_1(K_X) = \mathrm{c}_1(K_X)^2 - \mathrm{c}_1(L)^2 \geqslant 0.$$

Without loss of generality, $\deg L > 0$—otherwise pass to $K_X \otimes L^*$. Therefore, there is a $t \in (0, 1]$ such that $\deg K_X = t \deg L$

By the Hodge index theorem,

$$\begin{aligned} 0 &\geqslant (\mathrm{c}_1(K_x) - t\mathrm{c}_1(L))^2 \\ &= \mathrm{c}_1(K_X)^2 - 2t\mathrm{c}_1(K_X)\mathrm{c}_1(L) + t^2\mathrm{c}_1(L)^2 \\ &= \mathrm{c}_1(K_X)^2 + (-2t + t^2)\mathrm{c}_1(L)^2. \end{aligned}$$

The minimum of $-2t+t^2$ is achieved at $t = 1$ and takes the value -1. Therefore, $\mathrm{c}_1(L)^2 \geqslant \mathrm{c}_1(K_X)^2$. Comparing with the above, it follows that $\mathrm{c}_1(L)^2 = \mathrm{c}_1(K_X)^2$. Therefore, $\deg K_x = \deg \mathrm{c}_1(L)$ and also $(\mathrm{c}_1(K_X) - \mathrm{c}_1(L))^2 = 0$. Therefore, $\mathrm{c}_1(K_X \otimes L^*)$ is is torsion. Since $K_X \otimes \mathscr{L}^*$ is effective, it must be trivial. Therefore, $L = K_x$. ■

Corollary 115.3. *$\pm\mathrm{c}_1(K_X)$ for minimal surfaces of general type is a diffeomorphism invariant.* ■

116 More facts about the Seiberg–Witten invariant

Here are some further important facts about Seiberg–Witten theory which one should have heard about.

Theorem 116.1 (Vanishing on connected sums). *If X_1, X_2 are oriented closed smooth 4–manifolds with $b^+(X_i) \geqslant 1$, then the Seiberg–Witten invariant of $X_1 \# X_2$ vanishes.*

Theorem 116.2 (Blow-up formula for the Seiberg–Witten invariant). *Let X, E be oriented closed smooth 4–manifolds with $b^+(X) \geqslant 2$, $b^1(E) = b^+(X) = 0$ (e.g.: $E = \overline{\mathbf{C}P}^2$). If $\mathfrak{w}$ is a spin$^{\mathrm{U}(1)}$–structure on X and $\mathfrak{e}$ is a spin$^{\mathrm{U}(1)}$–structure on E such that*

$$w \cdot w - 2\chi(X) - 3\sigma(X) + e \cdot e + b^2(E) \geqslant 0 \quad \text{with} \quad w := c_1(L_{\mathfrak{w}}) \quad \text{and} \quad e := c_1(L_{\mathfrak{e}}),$$

then

$$\mathrm{SW}(\mathfrak{w}\#\mathfrak{e}) = \mathrm{SW}(\mathfrak{w})$$

Theorem 116.3 (Kronheimer–Mrowka's adjunction formula). *If X is an oriented closed smooth 4–manifold with $b^+(X) \geqslant 2$, and $\Sigma \subset X$ is an oriented closed connected embedded surface with $[\Sigma] \in \mathrm{H}_2(X, \mathbf{Z})$ non-torsion and $\Sigma \cdot \Sigma \geqslant 0$, then*

$$2g(\Sigma) - 2 \geqslant \Sigma \cdot \Sigma + |\langle \mathrm{c}_1(L_{\mathfrak{w}}), [\Sigma] \rangle|$$

whenever

$$\mathrm{SW}(\mathfrak{w}) \neq 0.$$

One should also be aware of Taubes' work on SW = Gr [Tau99].

117 Donaldson's diagonalisation theorem via Seiberg–Witten theory

Donaldson's diagonalisation theorem admits a technically rather simple proof using Seiberg–Witten theory. This requires a bit of algebra.

Definition 117.1. Let $q\colon \mathbf{Z}^n \to \mathbf{Z}$ be an integral quadratic form. A **characteristic element** with respect to q is a $c \in \mathbf{Z}^n$ with

$$q(c, x) = q(x, x) \bmod 2$$

for every $x \in \mathbf{Z}^n$. •

Theorem 117.2 (van der Blij [MH73, (5.2) Lemma]). *Let $q\colon \mathbf{Z}^n \to \mathbf{Z}$ be an integral quadratic form. If c is characteristic, then*

$$q(c, c) = \sigma(q) \bmod 8.$$

Theorem 117.3 (Elkies). *If $q\colon \mathbf{Z}^n \to \mathbf{Z}$ is negative definite unimodular form which is not diagonalisable, then there is a characteristic element $c \in \mathbf{Z}^n$ with*

$$q(c, c) \geqslant 8 - n.$$

In light of this, the following evidently implies Donaldson's diagonalisation theorem.

Proposition 117.4 (Kronheimer). *Let X be a oriented closed smooth 4–manifold with negative definite intersection form Q. Every characteristic $\gamma \in \mathrm{H}_2(X, \mathbf{Z})$ satisfies*

$$|\gamma \cdot \gamma| \geqslant b^2(X).$$

Remark 117.5. The condition on b_1 is easy to remove. ♣

Sketch proof. (To simplify the argument we assume that $b_1 = 0$.) Suppose not; that is: there is a characteristic $\gamma \in \mathrm{H}_2(X, \mathbf{Z})$ with $|\gamma \cdot \gamma| < b_2(X)$. Since Q is negative definite and $|\gamma \cdot \gamma| - b_2(X)$ is divisible by 8,

$$\gamma \cdot \gamma = 8k - b_2(X) \quad \text{with} \quad k \geqslant 1.$$

Choose a $\mathrm{spin}^{\mathrm{U}(1)}$–structure $\mathfrak{w}$ with $\mathrm{c}_1(L_{\mathfrak{w}}) = \gamma$ (up to torsion). By construction,

$$\operatorname{index} D_A^+ = 2k \geqslant 2 \quad \text{and} \quad d(\mathfrak{w}) = 2k + b_1(X) + 1.$$

For a generic choice of η,

$$\mathscr{M}^*(\mathfrak{w}, \eta)$$

is cut-out transversely and thus smooth of dimension $d(\mathfrak{w})$.

$\mathscr{M}^*(\mathfrak{w}, \eta)$ is not compact, but $\mathscr{M}(\mathfrak{w}, \eta)$. If $[A, \Phi] \in \mathscr{M}(\mathfrak{w}, \eta) \backslash \mathscr{M}^*(\mathfrak{w}, \eta)$, then $F_A^+ = \eta$ and $\Phi = 0$. In fact, there is a unique such point $[A_\eta, 0]$. It can be arranged that $\operatorname{coker} D_{A_\eta}^+ = 0$.

The truncation

$$\mathscr{M}^\delta(\mathfrak{w}, \eta) := \{[A, \Phi] \in \mathscr{M}^*(\mathfrak{w}, \eta) : \|\Phi\|_{L^2} \geqslant \delta\}$$

is. The boundary

$$\partial \mathscr{M}^\delta(\mathfrak{w}, \eta) = \{[A, \Phi] \in \mathscr{M}^*(\mathfrak{w}, \eta) : \|\Phi\|_{L^2} = \delta\}$$

is cobordant to

$$\mathscr{R} := \{(A, \Phi) : F_A^+ = \eta, D_A \Phi = 0, \|\Phi\|_{L^2} = \delta\} / \mathscr{G}(\mathfrak{w}) \cong [A_\eta] \times \mathbf{P} \ker D_{A_\eta}^+.$$

The class μ used to define the Seiberg–Witten invariants satisfies

$$\int_{\mathscr{R}} \mu^{k-1} = 1.$$

This contradicts Stokes's theorem since $\mathscr{R}$ is a boundary. ■

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