# Differential Geometry IV
# Problem Set 1

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2022-04-07

**Due**: 2022-04-27

(1) Let $n \in \mathbf{N}$. The ansatz $\not{D} = \sum_{i=1}^{n} \gamma_i \partial_{x_i}$ for the Dirac operator leads to system of algebraic equations

$$\gamma_i^2 = -1 \quad \text{and} \quad \gamma_i \gamma_j + \gamma_j \gamma_i = 0.$$

This admits no solutions in $\mathbf{R}$ but it can be solved in complex matrices.

Find a solution for $n = 1$ and $n = 2$.

(2) Let $V$ be a vector space with $\dim V < \infty$. Let $b \in \mathrm{Hom}(S^2 V, \mathbf{R})$ by a symmetric bilinear form. Suppose that $b$ is non-degenerate (that is: the induced map $V \to V^*, v \mapsto b(v, \cdot)$ is an isomorphism). Suppose that $W \subset V$ is isotropic (that is: the restriction of $b|_W \in \mathrm{Hom}(S^2 W, \mathbf{R})$ vanishes) and $\dim W = \frac{1}{2} \dim V$.

Prove that the signature $\sigma(b)$ of $b$ vanishes. (*Hint:* Use a basis of $W$ and $b$ to construct a basis of $V$ in which $b$ becomes particularly simple but not diagonal, then diagonalise.)

(3) Let $X$ be a closed oriented manifold of dimension $4k$. Prove that there is an intersection form $b_X \in \mathrm{Hom}(S^2 \mathrm{H}^{2k}_{\mathrm{dR}}(X), \mathbf{R})$ satisfying $b_X([\alpha], [\beta]) = \int_X \alpha \wedge \beta$.

In particular, it makes sense to define $\sigma(X) \coloneqq \sigma(b_X)$.

(4) Prove that $\sigma(X \times Y) = \sigma(X)\sigma(Y)$.

(5) Let $Y$ be a compact oriented manifold with boundary $\partial Y = X$. Suppose that $\dim X = 4k$. Set $W \coloneqq \mathrm{im}(\mathrm{H}^{2k}_{\mathrm{dR}}(Y) \to \mathrm{H}^{2k}_{\mathrm{dR}}(X))$.

Prove that $W$ is isotropic.

Prove that $\dim W = \frac{1}{2} b^{2k}(X)$. (*Hint:* Try to assemble the long exact sequence of relative homology, Poincaré duality, and Poincaré–Lefschetz duality into a commutative diagram.)

Prove that $\sigma(X) = 0$.

(6) Let $X$ be a smooth manifold with $\dim X = 2n$. Define $\varepsilon \in \mathrm{End}(\Lambda T^* X \otimes \mathbf{C})$ by

$$\varepsilon \alpha \coloneqq i^{k(k-1)+n} * \alpha \quad \text{with} \quad k \coloneqq \deg \alpha.$$

Prove that:

(a) $\varepsilon^2 = \mathbf{1}$

(b) $\mathrm{d} + \mathrm{d}^*$ and $\varepsilon$ anti-commute.

(7) Recall the Chern–Weil homomorphism (for example, from Differential Geometry III). Prove that the first $L$ polynomial is $L_1 = \frac{1}{3}p_1$.