Differential Geometry IV
Problem Set 3

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Due: 2022-05-11

(1) The purpose of this exercise is to go through the computation of $\mathbf{C}\ell_r$ in [Roe98, p. 59]. Denote by $e_1, \ldots, e_r$ the standard orthonormal basis of $\langle 1 \rangle^{\perp r}$. Consider the subgroup $G_r < \mathbf{C}\ell_r^\times$ consisting of elements of the form

$$\pm e_1^{i_1} \cdots e_r^{i_r}$$

with $i_j \in \{0, 1\}$. Denote the element $-1 \in G_r$ by ν. Denote by $\omega = e_1 \cdots e_r \in \mathbf{C}\ell_r$ the volume element.

(a) Prove the restriction bijection between $\mathbf{C}\ell_r$–modules and representations of G_r in which ν acts as $-\mathbf{1}$.

(b) Prove that there are precisely 2^r irreducible representations of G_r in which ν acts as $+\mathbf{1}$.
Hint: Since $\nu \in Z(G_r)$ and $\nu^2 = \mathbf{1}$, it acts as $\pm\mathbf{1}$ for any irreducible representation of G_r. The representations on which it acts as $+\mathbf{1}$ are actually representations of $G_r/\langle \nu \rangle$. What can you say about the latter group?

(c) Prove that the center $Z(G_r)$ of G_r is

$$Z(G_r) = \begin{cases} \{\mathbf{1}, \nu\} & \text{if } r \text{ is even} \\ \{\mathbf{1}, \nu, \omega, \nu\omega\} & \text{if } r \text{ is odd.} \end{cases}$$

(d) Let $g \in G_r$. Prove that the conjugacy class of (g) is $\{g\}$ if $g \in Z(G_r)$ and $\{g, \nu g\}$ otherwise.

(e) Prove that the number of conjugacy classes of elements of G_r is

$$\begin{cases} 2^r + 1 & \text{if } r \text{ is even} \\ 2^r + 2 & \text{if } r \text{ is odd.} \end{cases}$$

(f) Prove that G_r has one (two) irreducible representation in which ν acts as $-\mathbf{1}$ if r is even (odd).
Denote these by Δ and $\Delta^\pm$ respectively.

(g) Suppose that r is even. Prove that $\dim \Delta = 2^{r/2}$ and derive that $\mathbf{C}\boldsymbol{\ell}_r \cong \mathrm{M}_{r/2}(\mathbf{C})$.

(h) Suppose that r is odd. Prove that $\dim \Delta^{\pm} = 2^{\lfloor r/2 \rfloor}$ and derive that $\mathbf{C}\boldsymbol{\ell}_r \cong \mathrm{M}_{\lfloor r/2 \rfloor}(\mathbf{C}) \oplus \mathrm{M}_{\lfloor r/2 \rfloor}(\mathbf{C})$.

(2) Let q be a non-degenerate quadratic form. Prove that $\mathrm{C}\ell(q)$ is supercentral.

(3) Let q be a non-degenerate quadratic form. Prove that $\mathrm{C}\ell(q)$ is supersimple.

References

[Roe98] J. Roe. *Elliptic operators, topology and asymptotic methods.* Second. Pitman Research Notes in Mathematics Series 395. Longman, Harlow, 1998, pp. ii+209. MR: 1670907 (cit. on p. 1)