Differential Geometry IV
Problem Set 6

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2022-06-04

(1) Prove the Riemann–Lebesque-Lemma.

(2) Prove that $e^{-|x|^2}\sin(|x|^2)$ is not a Schwartz function.

(3) A smooth map $f \in C^\infty(\mathbf{R}^n, \mathbf{C})$ has **moderate growth** if for every $\alpha \in \mathbf{N}_0$ there is a $k = k(\alpha) \in \mathbf{N}_0$ such that

$$\sup \frac{|\partial^\alpha f(x)|}{(1+|x|^2)^k} < \infty.$$

Denote the ring of smooth functions of moderate growth by O_M^n. Observe, that $\mathscr{S}(\mathbf{R}^n, V)$ is an O_M^n–module. Prove the following:

Proposition 0.1. *Let $f\colon \mathbf{R}^n \to \mathbf{C}$ be measureable. If for every $\phi \in \mathscr{S}(\mathbf{R}^n, V)$ also $f\phi \in \mathscr{S}(\mathbf{R}^n, V)$, then $f \in O_M^n$ and the endomorphism of $f\cdot\colon \mathscr{S}(\mathbf{R}^n, V) \to \mathscr{S}(\mathbf{R}^n, V)$ is continuous, then $f \in O_M^n$.*

(4) Prove the following:

Theorem 0.2 (Poisson summation formula). *For every $f \in \mathscr{S}(\mathbf{R})$*

$$\sum_{k\in\mathbf{Z}} f(k) = \sum_{\xi\in\mathbf{Z}} \hat{f}(\xi).$$

(5) (a) The heat kernel $\tilde{h}_t$ on $\mathbf{R}$ satisfies

$$\tilde{h}_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

Derive a formula for the heat kernel h_t on $S^1 = \mathbf{R}/\mathbf{Z}$ from this.

(b) (Formally) derive a formula for h_t using Fourier series.

(c) Explain the relation between the two formulae for h_t.