Differential Geometry IV
Problem Set 7

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(1) Let $A \in \mathrm{M}_n(\mathbf{R})$ be symmetric and positiv definite. Define $f \in \mathcal{S}(\mathbf{R}^n, \mathbf{C})$ by

$$f(x) \coloneqq e^{-\pi\langle Ax, x\rangle}$$

Prove that its Fourier transform $\hat{f}$ satisfies

$$\hat{f}(\xi) = \det A^{-1/2} e^{-\pi\langle A^{-1}\xi, \xi\rangle}!$$

(2) Set $\mathfrak{H} \coloneqq \{\tau \in \mathbf{C} : \operatorname{Im}\tau > 0\}$. Let $\Lambda \subset \mathbf{R}^n$ be a lattice. Define $\theta_\Lambda \colon \mathfrak{H} \to \mathbf{C}$ by

$$\theta_\Lambda(\tau) \coloneqq \sum_{\lambda \in \Lambda} e^{\pi i \tau |\lambda|^2}.$$

Define $r_\Lambda \colon [0, \infty) \to \mathbf{N}_0$ by

$$r_\Lambda(m) \coloneqq \#\{\lambda \in \Lambda : |\lambda|^2 = m\}.$$

The theta function has the Fourier expansion

$$\theta_\Lambda(\tau) = \sum_{m \in [0,\infty)} r_\lambda(m) q^m \quad \text{with} \quad q \coloneqq e^{\pi i \tau}.$$

The numbers $m_0 \coloneqq \{m \in (0, \infty) : r_\lambda(m) \neq 0\}$ and $r_0 \coloneqq r_\Lambda(m_0)$ are particular interest. They are the square of the length of the shortest non-zero $\lambda \in \Lambda$ and the number of shortest elements respectively. For $\Lambda = \mathbf{Z}^n$, $r_\Lambda(k)$ is the number of ways in which k can be written as a sum of n squares.

The **dual lattice** $\Lambda^* \subset \mathbf{R}^n$ is defined by

$$\Lambda^* \coloneqq \{\mu \in \mathbf{R}^n : \langle \mu, \lambda\rangle \in \mathbf{Z} \text{ for every } \lambda \in \Lambda\}.$$

Prove that

$$\theta_{\Lambda^*}(-1/\tau) = \left(\frac{\tau}{i}\right)^{n/2} \operatorname{vol}(\mathbf{R}^n/\Lambda)\theta_\Lambda(\tau)!$$

Here the square root $\sqrt{\cdot}\colon -i\mathfrak{H} \to \mathbf{C}$ is such that $\sqrt{(0,\infty)} \subset (0,\infty)$. Moreover, $\mathrm{vol}(\mathbf{R}^n/\Lambda)$ is computed with respect to the Riemannian metric induced by the Euclidean metric.

[Hint: Choose a basis $(e_1,\ldots,e_n)$ of Λ. Define the positive definite matrix $A = (a_{ij}) \in \mathrm{M}_n(\mathbf{R})$ by $a_{ij} := \langle e_i, e_j\rangle$. Define $f \in \mathscr{S}(\mathbf{R}^n)$ by

$$f(x) := e^{-\pi t\left|\sum_{a=1}^n x_a e_a\right|^2} = e^{-\pi\langle tAx,x\rangle}$$

Compute $\hat{f}$ and apply. Apply the Poisson summation formula (and rearrange what needs to be rearranged.) This should prove the formula for $\tau = it$ with $t > 0$. Apply unique continuation.]

(3) Let Λ be such that $\Lambda^* = \Lambda$ (**selfdual**) and $|\Lambda|^2 \subset 2\mathbf{Z}$ for every $\lambda \in \Lambda$ (**even**). Prove that n must be a multiple of 8.

[Hint: The modular group $\mathrm{SL}_2(\mathbf{Z})$ acts on $\mathfrak{H}$ by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau := \frac{a\tau+b}{c\tau+d}.$$

Moreover, it is generated by

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since Λ is even, θ_Λ is invariant under T. Since selfdual and even,

$$\theta_\Lambda(S\tau) = \left(\frac{\tau}{i}\right)^{n/2}\theta_\Lambda(\tau).$$

If $8 \nmid n$, then (possibly after passing to $\Lambda^{\oplus 2}$ or $\Lambda^{\oplus 4}$) $n = 4 \bmod 8$. Consider $\omega_\Lambda := \theta_\Lambda(\mathrm{d}z)^{n/4}$. Compute the action of ST on ω_Λ. Prove that $(ST)^3 = \mathbf{1} \in \mathrm{SL}_2(\mathbf{Z})$.]

Remark. The above (almost) shows that θ_Λ is a modular form of weight $n/2$. The space of modular forms of weight 4 and 8 is 1–dimensional (and spanned by the Eisenstein series, whose Fourier expansion is known). This allows one to determine r_Λ in these cases. ♣

(4) Fourier transform of $\Delta f = g$ is

$$4\pi^2|\xi|^2\hat{f} = \hat{g}.$$

Therefore,

$$\hat{f} = \left(4\pi^2|\xi|^2\right)^{-1}\hat{g}.$$

(The fact that $\left(4\pi^2|\xi|^2\right)^{-1}$ is singular at the origin causes a headache, but let's not be deterred.) If

$$G(x) := \mathscr{F}^{-1}\left(\left(4\pi^2|\xi|^2\right)^{-1}\right),$$

then (formally) $f = G * g$. The function G is the **Green kernel (of Δ)**. Compute a formula for G!

(Hint: $1/a = \int_0^\infty e^{-ta}\mathrm{d}t$.)

(5) An analogous discussion for $(1 + \Delta) f = g$ leads to the **Bessel kernel**

$$G_2 \coloneqq \mathscr{F}^{-1}\big(1 + \left(4\pi^2 |\xi|^2\right)^{-1}\big).$$

Compute a formula for G_2!