Differential Geometry IV
Problem Set 8

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(1) This exercise is concerned with (a version of the) trace theorem.

(a) Define $\iota\colon \mathbf{R}^{n-1} \to \mathbf{R}^n$ by $\iota(x) := (0, x)$. Define the **restriction map** $\mathrm{res}\colon \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}(\mathbf{R}^{n-1})$ by

$$\mathrm{res}(f) := f \circ \iota.$$

Define the **integration map** $I\colon \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}(\mathbf{R}^{n-1})$ by

$$(If)(x) := \int_{\mathbf{R}} f(w, x)\,\mathrm{d}x.$$

Prove that for every $f \in \mathscr{S}(\mathbf{R}^n)$

$$\widehat{\mathrm{res}(f)} = I\hat{f}$$

(b) Let $s > \frac{1}{2}$. Prove that there is a constant $c = c(s) > 0$ such that for every $R \geqslant 0$

$$\int_{\mathbf{R}} \frac{\left(1 + R^2\right)^{s-\frac{1}{2}}}{(1 + \eta^2 + R^2)^s}\,\mathrm{d}\eta \leqslant c.$$

(c) Prove the following.

Theorem 0.1 (Trace Theorem). *Let $s > \frac{1}{2}$. The restriction map* $\mathrm{res}\colon \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}(\mathbf{R}^{n-1})$ *extends to bounded operator* $\mathrm{res}\colon W^{s,2}(\mathbf{R}^n) \to W^{s-\frac{1}{2},2}(\mathbf{R}^{n-1})$.

(d) Assume $s > \frac{1}{2}$. Is $\mathrm{res}\colon W^{s,2}(\mathbf{R}^n) \to W^{s-\frac{1}{2},2}(\mathbf{R}^{n-1})$ surjective?

(e) Does the trace theorem hold for $s = \frac{1}{2}$?

(2) Let $p \in \mathrm{Hom}(S^k(\mathbf{R}^n)^*, \mathrm{Hom}(V, W))$. Consider the formal differential operator $D := p(\partial)\colon \mathbf{R}[x_1, \ldots, x_n] \otimes V \to \mathbf{R}[x_1, \ldots, x_n] \otimes W$. Suppose that p is elliptic. Prove that D is is surjective.

(3) Compute the symbols of $\mathrm{d} + \mathrm{d}^*\colon \Omega(X) \to \Omega(X)$, $\Delta\colon \Omega(X) \to \Omega(X)$, and of a Dirac operator.

(4) This exercise is concerned with Ehrling's Lemma and a simple applications to Sobolev interpolation.

(a) Prove the following.

Lemma 0.2 (Ehrling's Lemma). *Let X, Y, Z be Banach spaces. Let $K\colon X \to Y$ be a compact operator. Let $I\colon Y \to Z$ be an injective operator. For every $\varepsilon > 0$ there is a constant $c(\varepsilon) > 0$ such that for every $x \in X$*

$$\|Kx\|_Y \leqslant \varepsilon\|x\|_X + c(\varepsilon)\|IKx\|_Z.$$

(b) The Rellich theorem asserts that for every bounded open subset $\Omega \subset \mathbf{R}^n$ the inclusion $W^{k,2}(\Omega) \subset W^{\ell,2}(\Omega)$ is compact if $k > \ell$. We only need this for the **cube** $Q := (0,1)^n$. Prove that for every $\varepsilon > 0$ there is a constant $c_0(\varepsilon) > 0$ such that for every $f \in W^{2,2}(Q)$

$$\|f\|_{W^{1,2}} \leqslant \varepsilon\|f\|_{W^{2,2}} + c_0(\varepsilon)\|f\|_{L^2}.$$

(c) Prove that for every $\varepsilon > 0$ there is a constant $c_1(\varepsilon) > 0$ such that for every $f \in W^{2,2}(\mathbf{R}^n)$

$$\|\nabla f\|_{L^2} \leqslant \|\nabla^2 f\|_{L^2} + c_1(\varepsilon)\|f\|_{L^2}.$$

(d) By considering $f_r(x) := f(x)$ prove that for every $f \in W^{2,2}(\mathbf{R}^n)$ and $r > 0$

$$\|\nabla f\|_{L^2} \leqslant r\|\nabla^2 f\|_{L^2} + c_2 r^{-1}\|f\|_{L^2}.$$

(This holds although Rellich's theorem does not apply with $\Omega = \mathbf{R}^n$.

(e) Prove that for every $f \in W^{2,2}(\mathbf{R}^n)$ and $r > 0$

$$\|\nabla f\|_{L^2} \leqslant c_3\|\nabla^2 f\|_{L^2}^{1/2}\|f\|_{L^2}^{1/2}.$$

(These multiplicative forms of the the interpolation inequality are usually the most powerful versions.)

(f) Prove that $c_0(\varepsilon)$ can be taken to be $c_4\varepsilon^{-1}$.