MTH 993 Spring 2018: Spin Geometry

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These notes are not set in stone. Please, help improve them. If you find any typos or mistakes, let me know. If there are any examples or results you are particularly fond of and you think they should be in the notes, let me know and I will add them. If you have any other suggestions for improvement, please, also let me know.

Exercises. There are exercises in these notes. Some of them are formally stated as such, but many are parts of proofs left for you to fill in. Please, do these exercises (or at the very least attempt to do them). They are an important part of you learning this material.

Conventions.

- $\mathbf{N} = \{1, 2, \ldots\}$, $\mathbf{N}_0 = \{0\} \cup \mathbf{N}$.
- We write $X^{\diamond n}$ for $\underbrace{X \diamond \cdots \diamond X}_{n}$ for $\diamond = \times, \oplus, \otimes, \wedge, \ldots$. If $\diamond$ is absolutely clear from the context, we might omit it; but usually only write X^n if X is a number, e.g., 2^3.
- If R is ring, then $M_n(R)$ denotes the set of $n \times n$–matrices with entries in R. $M_n(R)$ acts on $R^{\oplus n}$ on the left (right) by matrix-multiplication.
- k is field of characteristic not equal to two. Usually $k = \mathbf{R}$ or $k = \mathbf{C}$.

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Contents

Why Clifford algebras?

The starting point of Spin Geometry is the following question.

Question 0.1. On $\mathbf{R}^{n+1}$ can we write the wave operator

$$\square = \partial_t^2 - \sum_{i=1}^{n} \partial_{x_i}^2$$

as a square

$$\square = \not{D}^2?$$

Remark 0.2. Dirac [Dir28] came across this question when trying to find a relativistic theory of the electron.

The premise of this course is that this question and its analogue for the Laplace operator $\Delta = -\sum_{i=1}^{n} \partial_{x_i}^2$, anything that helps answer this question, and anything that arises from studying this question is inherently interesting.

If $n = 0$, the answer is obviously yes. If $n = 1$, we make the ansatz

$$D\!\!\!/ = \gamma_0 \partial_t + \gamma_1 \partial_x$$

with γ_0 and γ_1 constant. The equation

$$\square = D\!\!\!/^2$$

amounts to

$$\gamma_0^2 = 1, \quad \gamma_0\gamma_1 + \gamma_1\gamma_0 = 0, \quad \text{and} \quad \gamma_1^2 = -1. \tag{0.3}$$

Exercise 0.4. (0.3) has no solution with $\gamma_0, \gamma_1 \in \mathbf{R}$ (or even $\mathbf{C}$).

However, (0.3) does have a solution in $\mathrm{M}_2(\mathbf{R})$:

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In fact, it is not terribly difficult to find matrices γ_i *by hand* such that

$$\left(\gamma_0 \partial_t - \sum_{i=1}^{n} \gamma_i \partial_{x_i} \right)^2 = \square.$$

The theory of Clifford algebras answers the question: given a symmetric matrix $(q_{ij}) \in \mathbf{R}$, how does one find universal matrices γ_i such that

$$\gamma_i\gamma_j + \gamma_j\gamma_i = q_{ij}?$$

1 Multilinear algebra

The first part of the class will be concerned with Clifford algebras and their representation theory. This part is rather algebraic in nature. Maybe, more algebraic than one would expect for a geometry class. In order to warm up, we will review some constructions from multi-linear algebra, in particular, the tensor algebra, the alternating algebra and the symmetric algebra of a vector space V over a field k.

Let k be a field. Throughout this section, all vector spaces are taken to be vector spaces over this field. All of the following can be vastly generalized.

1.1 The tensor product

Definition 1.1. Let $V_1, \ldots, V_r$ and W be vector spaces. A map $M \colon V_1 \times \cdots \times V_r \to W$ is called **multi-linear** if for each $i = 1, \ldots, r$ and each $(v_1, \ldots, \ldots, v_r) \in V_1 \times \cdots \times V_r$ the map $V_i \to W$ defined by

$$v \mapsto M(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_r)$$

is linear.

Denote by $\mathrm{Mult}(V_1, \ldots, V_r; W)$ the vector space of multi-linear maps from $V_1 \times \cdots \times V_r$ to W

Proposition 1.2 (Construction and universal property of the tensor product). *Let $V_1, \ldots, V_r$ be vector spaces.*

1. *Denote by $k\langle V_1 \times \cdots \times V_r \rangle$ the free vector space generated by the set $V_1 \times \cdots \times V_r$. Let R be the linear subspace spanned by elements of the form*

$$(v_1, \ldots, v_{i-1}, v_i + \lambda v_i', v_{i-1}, \ldots, v_r) \\ - (v_1, \ldots, v_{i-1}, v_i, v_{i-1}, \ldots, v_r) - \lambda(v_1, \ldots, v_{i-1}, v_i', v_{i-1}, \ldots, v_r).$$

   Set

$$V_1 \otimes \cdots \otimes V_r := k\langle V_1 \times \cdots \times V_r \rangle / R. \tag{1.3}$$

   The map $\mu \colon V_1 \times \cdots \times V_r \longrightarrow V_1 \otimes \cdots \otimes V_r$ defined by

$$\mu(v_1, \ldots, v_r) := [(v_1, \ldots, v_r)]. \tag{1.4}$$

   is multi-linear.

2. *The pair $(V_1 \otimes \cdots \otimes V_r, \mu)$ satisfies the following universal property. For any vector space W, the map*

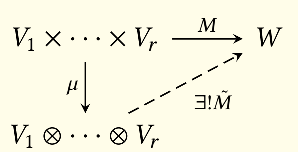
$$\mathrm{Hom}(V_1 \otimes \cdots \otimes V_r, W) \longrightarrow \mathrm{Mult}(V_1, \ldots, V_r; W), \quad \tilde{M} \mapsto \tilde{M} \circ \mu \tag{1.5}$$

   is bijective. In other words, if $M \colon V_1 \times \cdots \times V_r \longrightarrow W$ is a multi-linear map, then there exists a unique linear map $\tilde{M} \colon V_1 \otimes \cdots \otimes V_r \longrightarrow W$ such that

$$M = \tilde{M} \circ \mu.$$

Proof. Exercise. *Hint:* Use the universal property of the quotient vector space and the free vector space. ☐

Remark 1.6. Proposition 1.2 is often expressed by the following diagram:



Definition 1.7. The pair $(V_1 \otimes \cdots \otimes V_r, \mu)$ is called the **tensor product** of $V_1, \ldots, V_r$. We write

$$v_1 \otimes \cdots \otimes v_r := \mu(v_1, \ldots, v_r).$$

Remark 1.8. Almost everything about the tensor product can be proved using Proposition 1.2.

Proposition 1.9. *Let $V_1, \ldots, V_r, W_1, \ldots, W_r$ be vector spaces and $A_i \colon V_i \longrightarrow W_i$ be linear maps. There exists a unique linear map $A_1 \otimes \cdots \otimes A_r \colon V_1 \otimes \cdots \otimes V_r \longrightarrow W_1 \otimes \cdots \otimes W_r$ such that*

$$(A_1 \otimes \cdots \otimes A_r)(v_1 \otimes \ldots \otimes v_r) = A_1 v_1 \otimes \ldots \otimes A_r v_r$$

Proof. Denote by $\mu_V \colon V_1 \times \cdots \times V_r \longrightarrow V_1 \otimes \cdots \otimes V_r$ and $\mu_W \colon W_1 \times \cdots \times W_r \longrightarrow W_1 \otimes \cdots \otimes W_r$ the multilinear maps (1.4). The desired property of $A_1 \otimes \cdots \otimes A_r$ is that

$$\mu_W \circ (A_1 \times \cdots \times A_r) = (A_1 \otimes \cdots \otimes A_r) \circ \mu_V.$$

It is trivial to verify that the left-hand side of this equation is multi-linear. The existence of a unique $A_1 \otimes \ldots \otimes A_r$ is thus guaranteed by (1.5) being a bijection. □

Proposition 1.10. *If $V_1, \ldots, V_r$ are finite-dimensional, then*

$$\dim V_1 \otimes \cdots \otimes V_r = \prod_{i=1}^{m} \dim V_i.$$

More precisely if $(e^i_1, \ldots, e^i_{\dim V_i})$ are bases for V_i, then

$$\left\{ e^1_{i_1} \otimes \cdots \otimes e^r_{i_r} : i_j \in \{1, \ldots, \dim V_j\} \right\}$$

is a basis for $V_1 \otimes \cdots \otimes V_r$.

Proof. You should write a full proof as an exercise.

To just prove the dimension formula you can proceed as follows. We will implicitly prove later that

$$V_1 \otimes \cdots \otimes V_r \cong V_1 \otimes (V_2 \otimes \cdots \otimes V_r)$$

Thus it suffices to prove the result for $r = 2$. For this case show that if $V = V' \oplus V''$, then

$$V \otimes W = (V' \otimes W) \oplus (V'' \otimes W).$$

Knowing this the dimension formula will follow by induction. You can make this proof more concrete using a basis and prove the full result. □

Exercise 1.11. Use Proposition 1.2 to construct a linear map $V^* \otimes W \longrightarrow \mathrm{Hom}(V, W)$. When is this map injective? When is this map surjective?

1.2 The tensor algebra

Given any vector space, the tensor product gives rise to a natural unital graded algebra.

Definition 1.12. A **grading** on a vector space V is direct sum decomposition

$$V = \bigoplus_{r \in \mathbb{N}_0} V^r.$$

We call V^r the **degree** r component of V. A vector space together with a grading is called a **graded vector space**.

Definition 1.13. An **algebra** is a vector space A together with a bilinear map $m\colon A \times A \to A$ satisfying associativity, that is,

$$m \circ (m \times \mathrm{id}_A) = m \circ (\mathrm{id}_A \times m). \tag{1.14}$$

We often write $x \cdot y$ or xy instead of $m(x, y)$. With this notation (1.14) becomes the familiar $x(yz) = (xy)z$.

Definition 1.15. A **unital algebra** is a algebra (A, m) together with an element $1_A \in A$ such that

$$m(1_A, \cdot) = m(\cdot, 1_A) = \mathrm{id}_A.$$

Definition 1.16. A **graded algebra** is an algebra (A, m) with grading such that

$$m(A^r, A^s) \subset A^{r+s}.$$

Example 1.17. Denote by $k[x]$ the set of polynomials in the variable x with coefficients in k. The usual multiplication rule of polynomials is associative and makes $k[x]$ into an algebra. The polynomial 1 is a unit for this multiplication. This algebra has a natural grading with the r−th graded component consisting of homogeneous polynomial of degree r.

Proposition 1.18. *Let $r, s, t \in \mathbf{N}$. Denote by $(V^{\otimes r}, \mu_r)$ the tensor product of r copies of V.*

1. *There exists unique bilinear map $m_{r,s}\colon V^{\otimes r} \times V^{\otimes s} \to V^{\otimes r+s}$ such that*

$$m_{r,s}((v_1 \otimes \cdots \otimes v_r), (v_{r+1} \otimes \cdots \otimes v_{r+s})) = v_1 \otimes \cdots \otimes v_r \otimes v_{r+1} \otimes \cdots \otimes v_{r+s}.$$

2. *The maps $m_{r,s}$ satisfy associativity*

$$m_{r+s,t} \circ (m_{r,s} \times \mathrm{id}_{V^{\otimes t}}) = m_{r,s+t} \circ (\mathrm{id}_{V^{\otimes r}} \times m_{s,t}).$$

Remark 1.19. We extend $m_{r,s}$ to $r = 0$ and or $s = 0$ as follows. Set $V^{\otimes 0} = k$ and denote by $m_{0,r}\colon k \times V^{\otimes r} \to V^{\otimes r}$ and $m_{r,0}\colon V^{\otimes r} \times k \to V^{\otimes r}$ the scalar multiplication. In particular,the element $1 \in k = V^{\otimes 0}$ is a unit:

$$m_{0,r}(1, \cdot) = m_{r,0}(\cdot, 1) = \mathrm{id}_{V^{\otimes r}}.$$

Proof. This is basically trivial, but let me give a detailed proof to make it look complicated.

Given any vector space W, the maps

$$\begin{aligned}&\mathrm{Mult}((V_1 \otimes \cdots \otimes V_r), (V_{r+1} \otimes \cdots \otimes V_{r+s}); W)\\ &\qquad\longrightarrow \mathrm{Mult}(V_1 \times \cdots \times V_{r+1} \times \cdots \times V_{r+s}, W), \quad \tilde{B} \mapsto \tilde{B} \circ (\mu_r \times \mu_s)\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Mult}((V_1 \otimes \cdots \otimes V_r), (V_{r+1} \otimes \cdots \otimes V_{r+s}), (V_{r+t+1} \otimes \cdots \otimes V_{r+s+t}); W)\\ &\qquad\longrightarrow \mathrm{Mult}(V_1 \times \cdots \times V_{r+s+t}, W), \quad \tilde{C} \mapsto \tilde{C} \circ (\mu_r \times \mu_s \times \mu_t)\end{aligned}$$

are bijections. (It is an exercise to prove this using that (1.5) is a bijection.)

The desired property for $m_{r,s}$ is that

$$m_{r,s} \circ (\mu_r \times \mu_r) = \mu_{r,s}.$$

By the above such a $m_{r,s}$ exists and is uniquely determined.

The associativity follows from the the fact that

$$(\mu_r \times \mu_s) \times \mu_t = \mu_r \times \mu_s \times \mu_s = \mu_r \times (\mu_s \times \mu_t)$$

with respect to the identification $(X \times Y) \times Z = X \times Y \times Z = X \times (Y \times Z)$, as well as

$$m_{r+s,t} \circ (m_{r,s} \times \mathrm{id}_{V^{\otimes t}}) \circ ((\mu_r \times \mu_s) \times \mu_t) = m_{r+s,t} \circ (\mu_{r+s} \times \mu_t) = \mu_{r+s+t}$$

and

$$m_{r,s+t} \circ (\mathrm{id}_{V^{\otimes r}} \times m_{s,t}) \circ (\mu_r \times (\mu_s \times \mu_t)) = m_{r,s+t} \circ (\mu_r \times \mu_{s+t}) = \mu_{r+s+t}. \qquad \square$$

Proposition 1.20. *Set*

$$TV := \bigoplus_{r=0}^{\infty} V^{\otimes r}.$$

Given $x \in TV$, denote by x_r the component of x in $V^{\otimes r}$. The map $m\colon TV \times TV \longrightarrow TV$ defined by

$$m(x, y) = \sum_{r,s \in \mathbb{N}_0} m_{r,s}(x_r, x_s).$$

makes TV into graded unital associative algebra with unit $1 \in k = V^{\otimes 0} \subset TV$ with r-th graded piece $V^{\otimes r} \subset TV$.

Proposition 1.21 (Universal property of the tensor algebra). *Denote by $i\colon V \longrightarrow TV$ the inclusion map $V = V^{\otimes 1} \subset TV$. If A is a k-algebra together with a linear map $j\colon V \longrightarrow A$, then there exists a unique algebra homomorphism $f\colon TV \longrightarrow A$ such that*

$$f \circ i = j.$$

Proof. This is a consequence of the fact that V generates TV and $i: V \to TV$ is injective. $\square$

Definition 1.22. We call TV the **tensor algebra** on V. We write $x \otimes y$ for $m(x, y)$.

Exercise 1.23. Let $x_1, \ldots, x_n$ be n symbols. Denote by $k\langle x_1, \ldots, x_n \rangle$ the free vector space generated by these symbols. Prove that $Tk\langle x_1, \ldots, x_n \rangle$ is the free k−algebra generated by $x_1, \ldots, x_n$.

Exercise 1.24. Given a pair of algebras A and B, construct an algebra structure on $A \otimes B$.

With respect to this algebra structure, establish an algebra isomorphism

$$k[x] \otimes k[y] \cong k[x, y].$$

1.3 The alternating tensor product

Definition 1.25. A multi-linear map $M: V^{\times r} \to W$ is called **alternating** if

$$M(v_1, \ldots, v_r) = 0$$

whenever there is an $i = 1, \ldots, r-1$ such that $v_i = v_{i+1}$. We write $\mathrm{Alt}^r(V, W)$ for the space of alternating multi-linear maps $V^r \to W$.

Remark 1.26. Over $k = \mathbf{R}$ (or whenever k is not of characteristic 2), alternating is the same as

$$M(v_1, \ldots, v_i, v_{i+1}, \ldots, v_r) = -M(v_1, \ldots, v_{i+1}, v_i, \ldots, v_r).$$

Number theorist and algebraists will be mad at you if you define alternating like this in general.

Proposition 1.27 (Construction and universal property of the alternating tensor product). *Let V be a vector space and $r \in \mathbf{N}$.*

1. *Denote by R the linear subspace of $V^{\otimes r}$ spanned by elements of the form*

$$v_1 \otimes \cdots \otimes v_r$$

with $v_i = v_{i+1}$ for some $i = 1, \dots, r-1$. Set

$$\Lambda^r V \coloneqq V^{\otimes r}/R.$$

The multilinear map $\alpha\colon\ V^{\times r} \to \Lambda^r V$ defined by

$$\alpha(v_1, \dots, v_r) \coloneqq [v_1 \otimes \cdots \otimes v_r]$$

is alternating.

2. *The pair $(\Lambda^r V, \alpha)$ satisfies the following universal property. For any vector space W, the map*

$$\mathrm{Hom}(\Lambda^r V, W) \to \mathrm{Alt}^r(V, W), \quad \tilde{M} \mapsto \tilde{M} \circ \alpha$$

is bijective. In other words, if $M\colon\ V^{\times r} \to W$ is an alternating multi-linear map, then there exists a unique linear map $\tilde{M}\colon\ \Lambda^r V \to W$ such that

$$M = \tilde{M} \circ \alpha.$$

Proof. Exercise. □

Remark 1.28. Proposition 1.27 is often expressed by the following diagram:

$$\begin{array}{ccc} V^{\times r} & \xrightarrow{M} & W \\ {\scriptstyle\alpha}\downarrow & \nearrow_{\exists!\tilde{M}} & \\ \Lambda^r V & & \end{array}.$$

Definition 1.29. The vector space $\Lambda^r V$ together with the multi-linear α is called the k-th **exterior tensor product** of V. We write

$$v_1 \wedge \cdots \wedge v_r \coloneqq \alpha(v_1, \cdots, v_r).$$

Remark 1.30. If $\sigma \in S_r$ is a permutation of $\{1, \dots, k\}$, then

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \mathrm{sign}(\sigma) v_1 \wedge \cdots \wedge v_r.$$

Note that if k has characteristic 2, then $+1 = -1 \in k$.

Proposition 1.31. *Let V, W be vector spaces and $A\colon V \to W$ be a linear map. There exists a unique linear map $\Lambda^r A\colon \Lambda^r V \to \Lambda^r W$ such that*

$$(\Lambda^r A)(v_1 \wedge \ldots \wedge v_r) = Av_1 \wedge \ldots \wedge Av_r$$

Proof. Exercise. □

Proposition 1.32. *If V has dimension $n < \infty$, then*

$$\dim \Lambda^r V = \binom{n}{r}.$$

More precisely if $(e_1, \ldots, e_{\dim V})$ is a basis for V, then

$$\{e_{i_1} \wedge \cdots \wedge e_{i_r} : 1 \leqslant i_1 < \cdots < i_r \leqslant \dim V\}$$

is a basis for $\Lambda^r V$.

Proof. We can assume that $V = k^{\oplus n}$ with its standard basis. If $r = 0$ or $n = 0$, then the result is obvious. If $n \geqslant 1$, then

$$\Lambda^r k^{\oplus n} \cong (k\langle e_1 \rangle \otimes \Lambda^{r-1} k^{\oplus n}) \oplus \Lambda^r k\langle e_2, \ldots, e_n \rangle.$$

(You should prove this.) Set $d(r, n) \coloneqq \dim \Lambda^r k^{\oplus n}$. The above isomorphism implies that

$$d(r, n) = d(r - 1, n) + d(r, n - 1).$$

From this the dimension formula follows. In fact, the above isomorphism gives a geometrization/categorification of the combinatorial identity

$$\binom{n}{r} = \binom{n}{r-1} + \binom{n-1}{r}.$$

□

Exercise 1.33. Denote by $\mathrm{M}_n(k)$ the set of $n \times n$–matrices over k. If $V = k^n$ and $A \in \mathrm{M}_n(k) = \mathrm{End}(k^n)$, then

$$\Lambda^n A = \det A.$$

(If your definition of $\det A$ is $\Lambda^n A$, then work out the formula for $\det A$ in terms of the matrix entries of A.)

Proposition 1.34. *Let $r \in \mathbf{N}_0$. Suppose the characteristic of k does not divide r. Denote by $\pi\colon V^{\otimes r} \to \Lambda^r V$ the canonical projection map. There is a unique linear map $i\colon \Lambda^r V \to V^{\otimes r}$ such that*

$$\pi \circ i = \mathrm{id}_{\Lambda^r V}$$

and

$$i(v_1 \wedge \cdots \wedge v_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \operatorname{sign}(\sigma) v_1 \otimes \cdots \otimes v_r.$$

In particular, $i\colon \Lambda^r V \to V^{\otimes r}$ is an injection.

Proof. Exercise. □

Remark 1.35. If $k = \mathbf{R}$ (or if k has characteristic 0), then $\Lambda^r V$ can be identified with a subspace of $V^{\otimes r}$, but in general it is a quotient.

1.4 The exterior algebra

Proposition 1.36.

1. *There exists unique bilinear map $m_{r,s}\colon \Lambda^r V \times \Lambda^r V \to \Lambda^{r+s} V$ such that*

$$m_{r,s}((v_1 \wedge \cdots \wedge v_r), (v_{r+1} \wedge \cdots \wedge v_{r+s})) = v_1 \wedge \cdots \wedge v_r \wedge v_{r+1} \wedge \cdots \wedge v_{r+s}.$$

2. *The maps $m_{r,s}$ satisfy associativity:*

$$m_{r+s,t} \circ (m_{r,s} \times \mathrm{id}_{\Lambda^t V}) = m_{r,s+t} \circ (\mathrm{id}_{\Lambda^r V} \times m_{s,t}).$$

3. *The maps $m_{r,s}$ satisfy graded commutativity:*

$$m_{r,s}(x, y) = (-1)^{rs} m_{s,r}(y, x).$$

Proof. Exercise, cf. Proposition 1.18. □

We extend $m_{r,s}$ to the case $r = 0$ and $s = 0$ as in the construction of the tensor algebra.

Proposition 1.37. *Set*

$$\Lambda V := \bigoplus_{r=0}^{\infty} \Lambda^r V.$$

Given $x \in \Lambda V$, denote by x_r the component of x in $\Lambda^r V$. The map $m\colon \Lambda V \times \Lambda V \to \Lambda V$ defined by

$$m(x, y) = \sum_{r,s \in \mathbb{N}_0} m_{r,s}(x_r, y_s).$$

makes ΛV into a unital graded commutative associative algebra with unit $1 \in k = V^{\otimes 0} \subset \Lambda V$ with r−th graded piece $\Lambda^r V$.

Definition 1.38. We call ΛV the **exterior algebra** on V. We write $x \wedge y$ for $m(x, y)$.

Corollary 1.39. *Define an alternating map $(V^*)^r \to \mathrm{Alt}^r(V, k)$ by*

$$(v_1^*, \ldots, v_r^*) \mapsto \Big((v_1, \ldots, v_r) \mapsto \det\big((v_i^*(v_j))_{i,j=1}^r\big)\Big).$$

There is a unique linear map $\Lambda^r V^* \to \mathrm{Alt}^r(V, k)$ such that the diagram

$$\begin{array}{ccc} (V^*)^r & \longrightarrow & \mathrm{Alt}^r(V,k) \\ \big\downarrow & \nearrow_{\exists!} & \\ \Lambda^r V^* & & \end{array}$$

commutes. This map is injective. If V is finite-dimensional, this map is an isomorphism.

Remark 1.40. The map $\Lambda^r V^* \to \mathrm{Alt}^r(V, k)$ also make the following diagram commute:

$$\begin{array}{ccc} \Lambda^r V^* & \longrightarrow & \mathrm{Alt}^r(V,k) \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle \subset} \\ (V^*)^{\otimes r} & \longrightarrow & \mathrm{Mult}(V^r, k) \end{array}$$

Proposition 1.41. *There exists a unique linear map* $V \otimes \Lambda^r V^* \longrightarrow \Lambda^{k-1} V^*$

$$v \otimes \alpha \mapsto i(v)\alpha$$

such that

$$v \otimes (v_1^* \wedge \cdots \wedge v_r^*) \mapsto i(v) \sum_{i=1}^{r} (-1)^{i+1} v_i^*(v) \cdot v_1^* \wedge \cdots v_{i-1}^* \wedge v_{i+1}^* \cdots \wedge v_r^*$$

Thinking of α *and* $i(v)\alpha$ *as elements of* $\mathrm{Alt}^r(V,k)$ *and* $\mathrm{Alt}^{k-1}(V,k)$ *respectively, we have*

$$(i(v)\alpha)(v_1, \ldots, v_{r-1}) = \alpha(v, v_1, \ldots, v_{r-1}).$$

Definition 1.42. The map $i(v)$ called **contraction** with v.

1.5 The symmetric tensor product and the symmetric algebra

Definition 1.43. A multi-linear map $M\colon V^{\times r} \longrightarrow W$ is called **symmetric** if

$$M(v_1, \ldots, v_i, v_{i+1}, \ldots, v_r) = M(v_1, \ldots, v_{i+1}, v_i, \ldots, v_r).$$

for all $i = 1, \ldots, r-1$. We write $\mathrm{Sym}^r(V, W)$ for the space of symmetric multi-linear maps $V^r \longrightarrow W$.

Exercise 1.44. Work out the analogue of the discussion in Section 1.3 and Section 1.4. In particular, construct the **symmetric tensor product** $S^r V$ and the **symmetric algebra** SV. The symmetric algebra is a unital commutative graded algebra, that is, it commutative on the nose not graded commutative.

2 Quadratic Spaces

Definition 2.1. Let k be a field. Let V be a vector space. A **quadratic form** on V is a map $q\colon V \longrightarrow V$ of the form

$$q(v) = b(v, v)$$

for a bilinear map $b\colon V \times V \longrightarrow k$. We call (V, q) a **quadratic space**.

Remark 2.2. This concept generalizes to commutative rings R, but for our purposes working over fields k is enough. In fact, we could assume $k = \mathbf{R}$ or $k = \mathbf{C}$, but for the time being we will work with a general field for the fun of it.

2.1 Relation with symmetric bilinear forms

Example 2.3. If $(a_{ij}) \in \mathrm{M}_n(k)$, then $q\colon k^{\oplus n} \to k$ defined by

$$q(x_1, \dots, x_n) \coloneqq \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

is a quadratic form.

Example 2.4. Let $k = \mathbf{Z}/2\mathbf{Z}$. The matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

both give rise to the same quadratic form $q = 0$.

Exercise 2.5. Let $k = \mathbf{Z}/2\mathbf{Z}$. Show that the quadratic form $q\colon k^{\oplus 2} \to k$ defined by

$$q(x_1, x_2) \coloneqq x_1 x_2$$

cannot be represented by a *symmetric* matrix in $\mathrm{M}_2(k)$.

Remark 2.6. If k is not of characteristic 2, then

$$b(v, w) \coloneqq \frac{1}{2}(q(v + w) - q(v) - q(w))$$

is a *symmetric* bilinear form inducing q.

If k does not have characteristic 2, then quadratic forms are equivalent to symmetric bilinear forms. If k does have characteristic 2, then quadratic forms might not be representable by symmetric bilinear forms and if they are representable by symmetric bilinear forms, the representatives might be non-unique.

Because of this we will assume from now on that k has charactersitic not equal to two.

2.2 Isometries

Definition 2.7. Let (V_1, q_1), (V_2, q_2) be quadratic spaces. A linear map $f\colon V_1 \to V_2$ is called an **isometry** if f is invertible and

$$q_1(v) = q_2(f(v))$$

for all $v \in V$.

Remark 2.8. One can contemplate the more general notion of just a linear map satisfying $q_1(v) = q_2(f(v))$. Maybe one should call these maps quadratic, but quadratic linear map is an oxymoron.

Remark 2.9. Quadratic spaces and isometries form a category.

Exercise 2.10. If k does not have characteristic two and b_1 and b_2 are symmetric bilinear forms on V_1 and V_2 with associated quadratic forms q_1 and q_2, then $f\colon (V_1, q_1) \to (V_2, q_2)$ is an isometry if and only if

$$b_1(v_1, v_2) = b_2(f(v_1), f(v_2)).$$

Definition 2.11. Let (V, q) be a quadratic space. The **orthogonal group** associated with (V, q) is the group

$$\mathrm{O}(V, q) := \{f\colon V \to V : f \text{ is an isometry}\}.$$

Exercise 2.12. Read the mathoverflow post on "On the determination of a quadratic form from its isotropy group".

Definition 2.13. Let (V, q) be a quadratic space. The **special orthogonal group** associated with (V, q) the group

$$\mathrm{SO}(V, q) := \{f \in \mathrm{O}(V, q) : \det f = 1\}.$$

2.3 The Cartan–Dieudonné Theorem

Definition 2.14. Let (V, q) be a quadratic space over field of characteristic not equal to 2. Denote by b the symmetric bilinear map associated with q. We say that q is **non-degenerate** if the map $V \to V^*$ defined by

$$v \mapsto b(v, \cdot)$$

is an isomorphism.

Definition 2.15. Let (V, q) be a quadratic space. We say that $v \in V$ is **isotropic** if $q(v) = 0$ and **anisotropic** if $q(v) \neq 0$.

Exercise 2.16. Let (V, q) be a quadratic space over field of characteristic not equal to 2. Denote by b the symmetric bilinear map associated with q. If $v \in V$ is anisotropic, then the map $r_v\colon V \to V$ defined by

$$r_v(w) := w - 2\frac{b(v, w)}{q(v)}v$$

is an isometry of (V, q).

Definition 2.17. We call r_v the **reflection** in v.

Theorem 2.18 (Cartan–Dieudonné). *If q is a non-degenerate quadratic form on a vector space V, then any element of $O(V, q)$ can be written as the composition of at most n reflections.*

Proof. See Pete Clark's lecture notes. ◻

2.4 Classification of real and complex quadratic forms

We will be mostly interested in quadratic forms over $k = \mathbf{R}$ and $k = \mathbf{C}$. For these quadratic forms are classified as follows.

Theorem 2.19 (Sylvester's Law of Inertia). *Suppose k is a Euclidean field,*[a] *e.g., $k = \mathbf{R}$. Let (V, q) be a quadratic space of dimension n. There are unique numbers $n_+, n_-, n_0 \in \mathbf{N}_0$ such that (V, q) is isometric to $(1)^{\oplus n_+} \oplus (-1)^{n_-} \oplus (0)^{n_0}$.*

[a] A field k is called **Euclidean** if it is ordered and every $x \geqslant 0$ admits a square root.

Exercise 2.20. Prove Theorem 2.19.

Definition 2.21. The **signature** of q is the number

$$\sigma(q) \coloneqq n_+ - n_-$$

and the **nullity** of q is the number n_0.

Theorem 2.22. *Suppose k is algebraically closed, e.g., $k = \mathbf{C}$. If (V, q) is a quadratic space of dimension n, then there is a unique number $p \in \mathbf{N}_0$ such that (V, q) is isometric to $(1)^{\oplus p} \oplus (0)^{n-p}$*

3 Clifford algebras

The classical reference for the material in this section is Atiyah, Bott, and Shapiro [ABS64].

3.1 Construction and universal property of Clifford algebras

If V is vector space, then we denote by

$$TV = \bigoplus_{r=0}^{\infty} V^{\otimes r}$$

the tensor algebra over V.

Proposition 3.1 (Construction of universal property of the Clifford algebra). *Let (V, q) be a quadratic space.*

1. *Denote by I_q the ideal in TV generated by elements for the form*

$$v \otimes v - q(v).$$

Set

$$\mathrm{C}\ell(V, q) \coloneqq TV/I_q. \tag{3.2}$$

The obvious linear map[a] $\gamma\colon V \longrightarrow \mathrm{C}\ell(V, q)$ *satisfies*

$$\gamma(v)^2 = q(v). \tag{3.3}$$

2. *If A is an algebra together with a linear map $\delta\colon V \longrightarrow A$ such that*

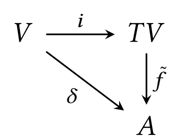
$$\delta(v)^2 = q(v),$$

then there exists a unique algebra homomorphism $f\colon \mathrm{C}\ell(V, q) \longrightarrow A$ such that

$$f(\gamma(v)) = \delta(v).$$

[a] In case you disagree that there is a canonical obvious map, $\gamma = \pi \circ i$ where $i\colon V = V^{\otimes 1} \subset TV$ and $\pi\colon TV \to \mathrm{C}\ell(V, q)$ is canonical projection.

Proof. By the universal property of TV, there exists a unique algebra homomorphism $\tilde{f}\colon TV \to A$ such that the diagram



commutes. Since $j(v)^2 = q(v)$, $\tilde{f}$ vanishes on the ideal I_q and thus factors through $\mathrm{C}\ell(V, q)$. This proves the existence of f. Since f extends to TV, it also proves the uniqueness by the universal property of TV. ◻

Definition 3.4. We call $\mathrm{C}\ell(V, q)$ together with γ the **Clifford algebra** associated with (V, q).

Remark 3.5. We have

$$\mathrm{C}\ell(V, 0) = \Lambda V.$$

Proposition 3.6. *The map γ is injective.*

Proof. Exercise. *Hint:* The map $\iota\colon V \to TV$ is obviously injective. You need to prove that

$$\operatorname{im}\iota \cap I_q = \{0\}.$$

Suppose that $v = xy$ for $x \in TV$ and $y = \sum w_i \otimes w_i - q(w_i)$ and derive that $v = 0$. □

Notation 3.7. Given that γ is injective, we will (from time to time) simply write v for $\gamma(v) \in \mathrm{C}\ell(V, q)$.

3.2 Automorphisms of $\mathrm{C}\ell(V, q)$

Exercise 3.8. Let $f\colon (V_1, q_1) \to (V_2, q_2)$ be an isometry. Prove that there is a unique algebra homomorphism $\mathrm{C}\ell(f)\colon \mathrm{C}\ell(V_1, q_1) \to \mathrm{C}\ell(V_2, q_2)$ such that

$$\mathrm{C}\ell(f) \circ i_{V_1} = i_{V_2} \circ f.$$

Prove that $\mathrm{C}\ell(f)$ is an algebra isomorphism.

Remark 3.9. The above makes $\mathrm{C}\ell$ into a functor from the category of quadratic spaces (with morphisms being isometries) to the category of algebras.

Corollary 3.10. $\mathrm{O}(V, q) \subset \mathrm{Aut}(\mathrm{C}\ell(V, q))$.

Definition 3.11. The map $V \to V, v \mapsto -v$ induces an involution $\alpha\colon \mathrm{C}\ell(V, q) \to \mathrm{C}\ell(V, q)$.

Definition 3.12. The anti-involution $v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto v_n \otimes \cdots \otimes v_2 \otimes v_1$ on TV preserves I_q and, hence, defines an anti-involution $\cdot^t\colon \mathrm{C}\ell(V, q) \to \mathrm{C}\ell(V, q)$ called **transposition**.

Definition 3.13. The anti-involution $\bar{\cdot} := \alpha \circ (\cdot)^t$ is called **conjugation**.

Definition 3.14. Let A be a unital algebra. An element $x \in A$ is called a **unit** if there exists an $x^{-1} \in A$ such that $xx^{-1} = x^{-1}x = 1_A$. We write $A^\times$ for the group of units in A.

Proposition 3.15. *If (V, q) is a non-degenerate quadratic space, then every automorphism arising from $\mathrm{O}(V, q)$ is of the form*

$$y \mapsto xy\alpha(x)^{-1}$$

for some $x \in \mathrm{C}\ell(V, q)^\times$.

Proof. By Theorem 2.18 it suffices to prove that if $v \in V$ is aniostropic, then the reflection r_v induces an automorphism of $\mathrm{C}\ell(V, q)$ of the asserted form. To this end observe that if v is anisotropic, then

$$v^{-1} = \frac{v}{q(v)} \in \mathrm{C}\ell(V, q)$$

and

$$-vwv^{-1} = -vw\frac{v}{q(v)} = v - 2\frac{b(v,w)}{q(v)}v = r_v w.$$

Since the automorphism of $C\ell(V,q)$ associated with r_v is determined by its action on V, it follows that it must be equal to $y \mapsto vy\alpha(v)^{-1}$. □

3.3 Real and complex Clifford algebras

One can develop the theory of Clifford algebras and study their structure over arbitrary k. We will be particularly (or rather: exclusively) interested in $k = \mathbf{R}$ and $k = \mathbf{C}$. In light of the previous exercises and Theorem 2.19 and Theorem 2.22, this means we care about the following Clifford algebras.

Definition 3.16. Given $r, s \in \mathbf{N}_0$, we define

$$C\ell_{r,s} \coloneqq C\ell(\mathbf{R}^{\oplus r+s}, q_{r,s}) \quad \text{with} \quad q_{r,s} = \operatorname{diag}(\underbrace{1,\dots,1}_{r}, \underbrace{-1,\dots,-1}_{s})$$

and

$$\mathbf{C\ell}_r \coloneqq C\ell(\mathbf{C}^{\oplus r}, q_r) \quad \text{with} \quad q_r = \operatorname{diag}(\underbrace{1,\dots,1}_{r}).$$

Our first major result in this lecture course will be the precise determination of what $C\ell_{r,s}$ and $\mathbf{C\ell}_r$ are. The first step is to work out what these algebras are if r, s are rather small.

Proposition 3.17. *If $V = k^{\oplus n}$ and $q = \operatorname{diag}(q_1,\dots,q_n)$, the there exists a unique algebra isomorphism*

$$C\ell(V,q) \cong k\langle x_1,\dots,x_n\rangle/\tilde{I}_q$$

with $\tilde{I}_q$ generated by $x_j^2 - q_j$ and $x_i x_j + x_j x_i$ such that $f\gamma = \tilde{\gamma}$ with $\tilde{\gamma}\colon k^{\oplus n} \to k\langle x_1,\dots,x_n\rangle/\tilde{I}_q$ defined by

$$\tilde{\gamma}(a_1,\dots,a_n) = a_1x_1 + \cdots + a_nx_n.$$

Proof. The linear map $\tilde{\gamma}$ satisfies

$$\tilde{\gamma}(a_1,\dots,a_n)^2 = \sum_{i=1}^n a_i^2 q_i.$$

Moreover, if A is an algebra and $\delta\colon k^{\oplus n} \to A$ is a linear map such that

$$\delta(a)^2 = q,$$

then $f\colon k\langle x_1,\dots,x_n\rangle/\tilde{I}_q \to A$ defined by

$$f(a_1x_1+\cdots+a_nx_n+b) := \sum_{i=1}^{n} a_i\delta(e_i)+b$$

is the unique algebra homomorphism satisfying

$$f\tilde{\gamma}=\delta.$$

This means that $k\langle x_1,\dots,x_n\rangle/\tilde{I}_q$ satisfies the universal property of $\mathrm{C}\ell(V,q)$. The existence and uniqueness of f follows immediately. □

Alternative proof. Identify $Tk^{\oplus n}=k\langle x_1,\dots,x_n\rangle$ and observe that $I_q=\tilde{I}_q$. □

Corollary 3.18. *Suppose $n=1$, that is, $\mathrm{C}\ell(k,q)=k[x]/(x^2-q)$. If $q=0$, then $\mathrm{C}\ell(k,q)$ is the ring of dual numbers over k. If there is a non-zero square root $\sqrt{q}\in k$, then $\mathrm{C}\ell(k,q)\cong k\oplus k$ via*

$$a+bx\mapsto(a+\sqrt{q}b,a-\sqrt{q}b).$$

Otherwise, $\mathrm{C}\ell(V,q)$ is the quadratic field extension $k(\sqrt{q})$ of k.

Remark 3.19. Suppose $n=2$, that is,

$$\mathrm{C}\ell(k^{\oplus 2},\mathrm{diag}(q_1,q_2))=k\langle x_1,x_2\rangle/(x_1^2-q_1,x_2^2-q_2,x_1x_2+x_2x_1).$$

This is isomorphic to the **quaternion algebra** $(q_1,q_2)_k$, that is, $\mathrm{C}\ell(V,q)=k\langle 1,i,j,k\rangle$ with

$$i^2=q_1,\quad j^2=q_2,\quad ij=k\quad\text{and}\quad ji=-k.$$

If $k=\mathbf{R}$ and $q_1=q_2=-1$, then this gives the Hamilton's quaternions $\mathbf{H}$.

Remark 3.20. We have $(1,1)_k\cong\mathrm{M}_2(k)$ with

$$i=\begin{pmatrix}1&\\&-1\end{pmatrix},\quad j=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\text{and}\quad ij=-ji=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

and $(1,-1)_k\cong\mathrm{M}_2(k)$ with

$$i=\begin{pmatrix}1&\\&-1\end{pmatrix},\quad j=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad\text{and}\quad ij=-ji=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

Knowing all of this it is easy to work out the real and complex Clifford algebras in dimension one and two.

Corollary 3.21. *We have*

$$\mathrm{C}\ell_{1,0} = \mathbf{R} \oplus \mathbf{R}, \quad \mathrm{C}\ell_{0,1} = \mathbf{C}, \quad \mathrm{C}\ell_{2,0} = \mathrm{M}_2(\mathbf{R}), \quad \mathrm{C}\ell_{1,1} = \mathrm{M}_2(\mathbf{R}), \quad \textit{and} \quad \mathrm{C}\ell_{0,2} = \mathbf{H}$$

and

$$\mathbf{C}\boldsymbol{\ell}_1 = \mathbf{C} \oplus \mathbf{C} \quad \textit{and} \quad \mathbf{C}\boldsymbol{\ell}_2 = \mathrm{M}_2(\mathbf{C}).$$

3.4 Digression: filtrations and gradings

Definition 3.22. Let V be a vector space. A **filtration** on V is a subspace $F^r V \subset V$ for every $r \in \mathbf{N}_0$ such that

$$F^r V \subset F^{r+1} V$$

for all $r \in \mathbf{N}_0$ and

$$V = \bigcup_{r \in \mathbf{N}_0} F^r V.$$

A vector space together with a filtration is called a **filtered vector space**.

Every graded vector space V has a canonical filtration given by

$$F^r V = V^{\leqslant r} := \bigoplus_{s \leqslant r} V^s.$$

Definition 3.23. Given a filtered vector space V, the **associated graded vector space** $\mathrm{gr}\, V$ is

$$\mathrm{gr}\, V := \bigoplus_{r=0}^{\infty} \mathrm{gr}^r\, V \quad \text{with} \quad \mathrm{gr}^r\, V := F^r V / F^{r-1} V.$$

Here we use the convention $F^{-1} V = \{0\}$.

Exercise 3.24. If V is a graded vector space, then the associated graded vector space $\mathrm{gr}\, V$ of V with the canonical filtration is isomorphic to V.

Exercise 3.25. If V is filtered vector space, then there is a canonical linear map $i \colon V \to \mathrm{gr}\, V$. The map i is injective and, hence, an isomorphism if V is finite-dimensional.

Definition 3.26. Let (A, m) be a k–algebra. A **filtration** in A is a filtration on the underlying vector space such that

$$m(F^r A, F^s A) \subset F^{r+s} A.$$

Exercise 3.27. If A is a filtered algebra, then $\mathrm{gr}\, A$ inherits the structure of graded algebra.

Definition 3.28. If A is a filtered algebra, then $\mathrm{gr}\, A$ is called the **associated graded algebra**.

Exercise 3.29. Let A be a filtered algebra and let I be an ideal in A. Given $r \in \mathbf{N}_0$, define

$$F^r(A/I) := (F^r A)/(I \cap F^r A).$$

This defines a filtration on A/I.

3.5 The filtration on the Clifford algebra

Corollary 3.30. *$\mathrm{C}\ell(V,q)$ is a filtered algebra.*

Proposition 3.31. *The linear maps*

$$\delta\colon V^{\otimes r} \longrightarrow \mathrm{gr}^r\, \mathrm{C}\ell(V,q) = F^r \mathrm{C}\ell(V,q)/F^{r-1}\mathrm{C}\ell(V,q)$$

factor through $\Lambda^r V$ and induce an isomorphism of algebras

$$\delta\colon \Lambda V \cong \mathrm{gr}\, \mathrm{C}\ell(V,q).$$

Proof. The linear map $\delta\colon V^{\otimes r} \longrightarrow \mathrm{gr}^r\, \mathrm{C}\ell(V,q)$ is surjective. Since

$$v_1 \cdots v_{i-1} v v v_{i+2} \cdots v_r = q(v) v_1 \cdots v_{i-1} v_{i+2} \cdots v_r \in F^{r-1}\mathrm{C}\ell(V,q),$$

the kernel δ contains I_0. Consequently, δ factors through $\Lambda^r V$. We will prove that $\ker \delta = I_0$ and thus the map $\Lambda^r V \longrightarrow \mathrm{gr}^r\, \mathrm{C}\ell(V,q)$ is injective. The kernel of the map $\varepsilon\colon V^{\otimes r} \longrightarrow F^r\mathrm{C}\ell(V,q)$ is $I_q \cap V^{\otimes r}$. Therefore the kernel of δ is

$$\bigoplus_{r=0}^{\infty} \varepsilon^{-1}(F^{r-1}\mathrm{C}\ell(V,q)) = \bigoplus_{r=0}^{\infty} V^{\otimes r} \cap (I_q + TV^{\leqslant r-1}) = I_0. \qquad \square$$

Corollary 3.32. $\dim_k \mathrm{C}\ell(V,q) = 2^{\dim_k V}$.

Exercise 3.33. Suppose k has characteristic zero. Denote by $i\colon \Lambda V \longrightarrow TV$ the map from Proposition 1.34 Prove that the map

$$\Lambda V \xrightarrow{i} TV \longrightarrow \mathrm{C}\ell(V,q)$$

is an isomorphism.

This means that *as a vector space* we can identify $\mathrm{C}\ell(V,q)$ with ΛV but this a non-standard multiplication which does not preserve the grading but only the filtration.

Exercise 3.34. Work out how to write the multiplication induced on ΛV via the vector space isomorphism $\Lambda V \cong \mathrm{C}\ell(V,q)$.

3.6 The $\mathbf{Z}_2$ grading on the Clifford algebra

Definition 3.35. A $\mathbf{Z}_2$ **grading** on a vector space V is direct sum decomposition

$$V = V^0 \oplus V^1.$$

We call V^r the **degree** r component of V. A vector space together with a grading is called a $\mathbf{Z}_2$ **graded vector space** or a **super vector space**.

Definition 3.36. A $\mathbf{Z}_2$ **graded algebra** (or **super algebra**) is an algebra (A, m) together with a $\mathbf{Z}_2$ grading such that

$$m(A^r, A^s) \subset A^{r+s}$$

for al $r, s \in \{0, 1\} = \mathbf{Z}_2$.

Definition 3.37. Let A be $\mathbf{Z}_2$ graded algebra and let $I \subset A$ be an ideal. We say I is **homogeneous** if for all $x \in I$ we have $x_0 \in I$ and $x_1 \in I$.

Exercise 3.38. If I is an homogeneous ideal in a $\mathbf{Z}_2$ graded algebra A, then A/I is $\mathbf{Z}/2$ graded.

Definition 3.39. We define a $\mathbf{Z}_2$–grading on TV by declaring that

$$TV^0 := \bigoplus_{r \in \mathbf{N}_0} V^{2r} \quad \text{and} \quad TV^1 := \bigoplus_{r \in \mathbf{N}_0} V^{2r+1}.$$

Since I_q is homogeneous with respect to this $\mathbf{Z}_2$ grading, $\mathrm{C}\ell(V, q)$ inherits a canonical $\mathbf{Z}_2$ grading.

3.7 Clifford algebra of direct sums

The following shows that the $\mathbf{Z}_2$ grading on $\mathrm{C}\ell(V, q)$ is, in principle, very useful because it allows us to determine $\mathrm{C}\ell(V_1 \oplus V_2, q_1 \oplus q_2)$ in terms of $\mathrm{C}\ell(V_1, q_1)$ and $\mathrm{C}\ell(V_2, q_2)$.

Definition 3.40. Let A, B be two $\mathbf{Z}_2$ graded algebras. The $\mathbf{Z}_2$ **graded tensor product** is $\mathbf{Z}_2$ graded algebra $A \hat{\otimes} B$ with underlying vector space $A \otimes B$, grading

$$(A \hat{\otimes} B)^0 = A^0 \hat{\otimes} B^0 \oplus A^1 \hat{\otimes} B^1 \quad \text{and} \quad (A \hat{\otimes} B)^1 = A^0 \hat{\otimes} B^1 \oplus A^1 \hat{\otimes} B^0,$$

and multiplication

$$m(a_1 \hat{\otimes} b_1, a_2 \hat{\otimes} b_2) = (-1)^{\deg a_2 \cdot \deg b_1} (a_1 a_2) \hat{\otimes} (b_1 b_2).$$

Proposition 3.41. *The linear map* $\gamma_\oplus\colon V_1 \oplus V_2 \longrightarrow \mathrm{C}\ell(V_1, q_1) \hat{\otimes} \mathrm{C}\ell(V_2, q_2)$ *defined by*

$$\gamma_\oplus(v \oplus w) = \gamma_1(v) \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_2(w)$$

satisfies

$$\gamma_\oplus(v \oplus w)^2 = q_1(v) + q_2(w).$$

Given an algebra A *together with a linear map* $\delta\colon V_1 \oplus V_2 \longrightarrow A$ *such that*

$$\delta(v \oplus w)^2 = q_1(v) + q_2(w),$$

there exists a unique algebra homomorphism $f\colon \mathrm{C}\ell(V_1, q_1) \hat{\otimes} \mathrm{C}\ell(V_2, q_2) \longrightarrow A$ *such that*

$$\delta = f \circ \gamma_\oplus.$$

In particular, there is a canonical isomorphism

$$\mathrm{C}\ell(V_1 \oplus V_1, q_1 \oplus q_2) \cong \mathrm{C}\ell(V_1, q_1) \hat{\otimes} \mathrm{C}\ell(V_2, q_2).$$

Proof. We have

$$\begin{aligned}\gamma_\oplus(v \oplus w)^2 &= (\gamma_1(v) \hat{\otimes} 1 + 1 \hat{\otimes} \gamma_2(w))^2 \\ &= \gamma_1(v)^2 \hat{\otimes} 1 + \gamma_1(v) \hat{\otimes} \gamma_2(w) - \gamma_1(v) \hat{\otimes} \gamma_2(w) + 1 \hat{\otimes} \gamma_w(w))^2 \\ &= q_1(v) + q_2(w).\end{aligned}$$

Remark 3.42. The minus sign in the above computation comes from the sign in the definition of $\hat{\otimes}$. This sign is crucial.

Let A be an algebra and $\delta\colon V_1 \oplus V_2 \longrightarrow A$ such that

$$\delta(v \oplus w) = q_1(v) + q_2(w).$$

By the universal property of $\mathrm{C}\ell(V_1, q_1)$ and $\mathrm{C}\ell(V_2, q_2)$ there are unique algebra homomorphisms $f\colon \mathrm{C}\ell(V, q) \longrightarrow A$ and $g\colon \mathrm{C}\ell(W, p) \longrightarrow A$ such that

$$\delta(v \oplus w) = f \circ \gamma_1(v) + g \circ \gamma_2(w) = (f \hat{\otimes} g) \circ \gamma_\oplus(v \oplus w).$$

The universal property of the tensor product shows that $f \hat{\otimes} g$ is the unique linear map with this property.

This means that $(\mathrm{C}\ell(V_1, q_1) \hat{\otimes} \mathrm{C}\ell(V_2, q_2), \gamma_\oplus)$ satisfies the universal property of Clifford algebra and hence is isomorphic to it through a canonical isomorphism. □

Proposition 3.41 and our determination of $\mathrm{C}\ell_{1,0}$ and $\mathrm{C}\ell_{0,1}$ seems to be the answer to our question: Which algebra is $\mathrm{C}\ell_{r,s}$? After all, it yields an isomorphism

$$\mathrm{C}\ell_{r,s} \cong \underbrace{\mathrm{C}\ell_{1,0} \hat{\otimes} \cdots \hat{\otimes} \mathrm{C}\ell_{1,0}}_{r} \hat{\otimes} \underbrace{\mathrm{C}\ell_{0,1} \hat{\otimes} \cdots \hat{\otimes} \mathrm{C}\ell_{0,1}}_{s}.$$

It turns out, however, that it is not that easy to work out what the right hand side is.

Exercise 3.43. We have $C\ell_{0,1} = \mathbf{R} \oplus \mathbf{R}$ with

$$C\ell^0_{1,0} = \mathbf{R}(1,1) \quad \text{and} \quad C\ell^1_{1,0} = \mathbf{R}(1,-1).$$

Therefore,

$$C\ell_{2,0} = \langle (1,1) \mathbin{\hat{\otimes}} (1,1), (1,1) \mathbin{\hat{\otimes}} (1,-1), (1,-1) \mathbin{\hat{\otimes}} (1,1), (1,-1) \mathbin{\hat{\otimes}} (1,-1) \rangle.$$

Work out the multiplication table for the above generators and find an explicit isomorphism to $M_2(\mathbf{R})$.

3.8 Digression: What does it mean to determine an algebra?

One of our major goals in the first part of this lecture course is to determine what $C\ell_{r,s}$ is. Although it is not strictly necessary for the rest of the course, we should pause here and ask "what does that even mean?" We gave the definition of $C\ell_{r,s}$. Isn't that enough? Also, Proposition 3.41 as well as our computation of $C\ell_{1,0}$ and $C\ell_{0,1}$ allows us to write $C\ell_{r,s}$ in terms of simple pieces. Why bother any more? Largely, the reason for studying algebras is to understand their representations. A good answer to "what algebra is A?" should allow us to immediately understand the representations of A. Wedderburn's Structure Theorem tells us that this is possible—in principle.

Definition 3.44. Let A be an algebra and let V be a vector space. A **representation** of A on V is an algebra homomorphism $\rho\colon A \to \operatorname{End}(V)$.

It is customary to make ρ implicit and call V the representation of A and write xv for $\rho(x)v$.

Definition 3.45. A representation $\rho\colon A \to \operatorname{End}(V)$ is called **irreducible** if $V \neq \{0\}$ and for every $W \subset V$ satisfying

$$\rho(x)W \subset W$$

for all $x \in A$ we have either $W = \{0\}$ or $W = V$.

Exercise 3.46. Let V be a vector space and let $\rho\colon \Lambda V \to \operatorname{End}(W)$ be an irreducible representation of ΛV. Show that $\dim W = 1$ and that every $x \in \Lambda^{\geqslant 1} V$ acts trivially on W.

Definition 3.47. Let A be a finite dimensional algebra. The **Jacobson radical** of A is

$$J(A) := \{x \in A : \rho(x) = 0 \text{ for all irreducible representations } \rho\}.$$

Remark 3.48. The previous exercise shows that $J(\Lambda V) = \Lambda^{\geqslant 1} V$.

Exercise 3.49. Prove that $J(A)$ is an ideal of A.

Definition 3.50. A finite dimensional algebra A is called **semisimple** if $J(A) = 0$.

Definition 3.51. Let $\rho\colon A \to \mathrm{End}(V)$ be an irreducible representation. The **commuting algebra** of ρ is the subalgebra

$$\mathrm{End}_A(V) = \{x \in \mathrm{End}_k(V) : [x, \rho(y)] = 0 \text{ for all } y \in A\}.$$

Lemma 3.52 (Schur's Lemma). *Let V and W be irreducible representation of A. If $f \in \mathrm{Hom}_A(V, W) \subset \mathrm{Hom}_k(V, W)$, that is, $f\colon V \to W$ is k–linear and*

$$f(xv) = xf(v)$$

for all $x \in A$, then either $f = 0$ or f is invertible.

Corollary 3.53. *If V is an irreducible representation, then $\mathrm{End}_A(V)$ is a division algebra over k, that is, every non-zero $x \in \mathrm{End}_A(V)$ is invertible.*

Theorem 3.54 (Frobenius). *If D is a division algebra over $\mathbf{R}$, then D is isomorphic to either $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$.*

Proposition 3.55. *If k is an algebraically closed field, e.g., $k = \mathbf{C}$, then any division algebra over k is isomorphic to k.*

Theorem 3.56 (Wedderburn's Structure Theorem). *Let A be a finite dimensional algebra.*

1. *A has only finitely many irreducible representations $V_1, \ldots, V_n$ and each V_i is finite dimensional.*
2. *Denote by D_i the commuting algebra of V_i. We have*

$$A/J(A) \cong \prod_{i=1}^{n} \mathrm{End}_{D_i}(V_i).$$

Proof. You can find a proof in Igusa's lectures notes. □

3.9 Digression: Representation theory of finite groups

Theorem 3.56 together with the following result largely clarify the representation theory of finite groups.

Theorem 3.57 (Maschke's Theorem). *If the characteristic of k does not divide $|G|$, then $k[G]$ is semisimple.*

Proof. Suppose $k[G] \to \mathrm{End}(V)$ is a representation and $W \subset V$ is an invariant subspace. Denote by $\pi\colon V \to V$ a projection on W, that is, $\operatorname{im}\pi = W$ and $\pi|_W = \mathrm{id}_W$. Averaging over G, we can assume that π is G–invariant. Set $W^\perp := \ker\pi$. This is an invariant subspace.

This means that we can decompose $k[G]$ into irreducible representations $k[G] = \bigoplus V_i$. Every non-zero $x \in k[G]$ acts non-trivially on $k[G]$ and thus on at least one of the irreducible representations V_i. Consequently, $J(k[G]) = \{0\}$. □

3.10 Determination of $\mathrm{C}\ell_{r,s}$

Exercise 3.58. Let k be a field, let D be a k–algebra, and let $n, m \in \mathbf{N}_0$. We have

$$\mathrm{M}_n(k) \otimes \mathrm{M}_m(k) \cong \mathrm{M}_{nm}(k) \quad \text{and} \quad \mathrm{M}_n(k) \otimes_k D \cong \mathrm{M}_n(D).$$

Theorem 3.59. *For $r, s \in \{0, \dots, 7\}$, $\mathrm{C}\ell_{r,s}$ is as in Table 1. Moreover, we have*

$$\mathrm{C}\ell_{r+8,s} \cong \mathrm{C}\ell_{r,s} \otimes \mathrm{M}_{16}(\mathbf{R}) \quad \textit{and} \quad \mathrm{C}\ell_{r,s+8} \cong \mathrm{C}\ell_{r,s} \otimes \mathrm{M}_{16}(\mathbf{R}).$$

The proof of this result relies on the following observation.

Proposition 3.60. *Let (V, q) be a quadratic space. We have*

$$\mathrm{C}\ell\left((V,q) \oplus (\pm 1)^{\oplus 2}\right) \cong \mathrm{C}\ell(V, -q) \otimes \mathrm{C}\ell\left((\pm 1)^{\oplus 2}\right) \quad \textit{and}$$
$$\mathrm{C}\ell\left((V,q) \oplus (1) \oplus (-1)\right) \cong \mathrm{C}\ell(V, q) \otimes \mathrm{C}\ell\left((1) \oplus (-1)\right).$$

Proof. Denote by (e_1, e_2) the standard basis of $k^{\oplus 2}$. Define $\gamma\colon V \oplus k^{\oplus 2} \to \mathrm{C}\ell(V, -q) \otimes \mathrm{C}\ell\left((\pm 1)^{\oplus 2}\right)$ by

$$\gamma(v, x, y) := v \otimes e_1 e_2 + 1 \otimes x e_1 + 1 \otimes y e_2.$$

Since

$$\gamma(v, x, y)^2 = -q(v) \pm x^2 \pm y^2,$$

γ induces an algebra homomorphism $\mathrm{C}\ell\left((V,q) \oplus (\pm 1)^{\oplus 2}\right) \to \mathrm{C}\ell(V, -q) \otimes \mathrm{C}\ell\left((\pm 1)^{\oplus 2}\right)$. This map is surjective because it maps onto a set of generators. For dimension reasons it also injective and, hence, an algebra isomorphism.

The second isomorphism is constructed by the same argument. □

Corollary 3.61. *For $r, s \in \mathbf{N}_0$, we have*

$$\mathrm{C}\ell_{r,s} \otimes \mathrm{M}_2(\mathbf{R}) \cong \mathrm{C}\ell_{s+2,r},$$
$$\mathrm{C}\ell_{r,s} \otimes \mathbf{H} \cong \mathrm{C}\ell_{s,r+2}, \quad \textit{and}$$
$$\mathrm{C}\ell_{r,s} \otimes \mathrm{M}_2(\mathbf{R}) \cong \mathrm{C}\ell_{r+1,s+1}.$$

	$r=0$	1	2	3	4	5	6	7
$s=0$	$\mathbf{R}$	$\mathbf{R}^{\oplus 2}$	$M_2(\mathbf{R})$	$M_2(\mathbf{C})$	$M_2(\mathbf{H})$	$M_2(\mathbf{H})^{\oplus 2}$	$M_4(\mathbf{H})$	$M_8(\mathbf{C})$
1	$\mathbf{C}$	$M_2(\mathbf{R})$	$M_2(\mathbf{R})^{\oplus 2}$	$M_4(\mathbf{R})$	$M_4(\mathbf{C})$	$M_4(\mathbf{H})$	$M_4(\mathbf{H})^{\oplus 2}$	$M_8(\mathbf{H})$
2	$\mathbf{H}$	$M_2(\mathbf{C})$	$M_4(\mathbf{R})$	$M_4(\mathbf{R})^{\oplus 2}$	$M_8(\mathbf{R})$	$M_8(\mathbf{C})$	$M_8(\mathbf{H})$	$M_8(\mathbf{H})^{\oplus 2}$
3	$\mathbf{H}^{\oplus 2}$	$M_2(\mathbf{H})$	$M_4(\mathbf{C})$	$M_8(\mathbf{R})$	$M_8(\mathbf{R})^{\oplus 2}$	$M_{16}(\mathbf{R})$	$M_{16}(\mathbf{C})$	$M_{16}(\mathbf{H})$
4	$M_2(\mathbf{H})$	$M_2(\mathbf{H})^{\oplus 2}$	$M_4(\mathbf{H})$	$M_4(\mathbf{C})$	$M_{16}(\mathbf{R})$	$M_{16}(\mathbf{R})^{\oplus 2}$	$M_{32}(\mathbf{R})$	$M_{32}(\mathbf{C})$
5	$M_4(\mathbf{C})$	$M_4(\mathbf{H})$	$M_4(\mathbf{H})^{\oplus 2}$	$M_8(\mathbf{H})$	$M_8(\mathbf{C})$	$M_{32}(\mathbf{R})$	$M_{32}(\mathbf{R})^{\oplus 2}$	$M_{64}(\mathbf{R})$
6	$M_8(\mathbf{R})$	$M_8(\mathbf{C})$	$M_8(\mathbf{H})$	$M_8(\mathbf{H})^{\oplus 2}$	$M_{16}(\mathbf{H})$	$M_{16}(\mathbf{C})$	$M_{64}(\mathbf{R})$	$M_{64}(\mathbf{R})^{\oplus 2}$
7	$M_8(\mathbf{R})^{\oplus 2}$	$M_{16}(\mathbf{R})$	$M_{16}(\mathbf{C})$	$M_{16}(\mathbf{H})$	$M_{16}(\mathbf{H})^{\oplus 2}$	$M_{32}(\mathbf{H})$	$M_{32}(\mathbf{C})$	$M_{128}(\mathbf{R})$

Table 1: $C\ell_{r,s}$

Proposition 3.62. *We have*

$$\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C} \oplus \mathbf{C}, \quad \mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \cong \mathrm{M}_2(\mathbf{C}), \quad \text{and} \quad \mathbf{H} \otimes_{\mathbf{R}} \mathbf{H} \cong \mathrm{M}_4(\mathbf{R})$$

Proof. The the isomorphism $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \to \mathbf{C} \otimes \mathbf{C}$ is given by

$$z \otimes w \mapsto zw \oplus z\bar{w}.$$

Identifying $\mathbf{H} = \mathbf{C} \oplus \mathbf{C}j = \mathbf{C}^2$, $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H}$ acts on $\mathbf{C}^2$ via

$$(z \otimes q) \cdot v = zv\bar{q}.$$

This action is $\mathbf{C}$–linear. A computation shows that the resulting map $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \to \mathrm{End}_{\mathbf{C}}(\mathbf{C}^2) \cong \mathrm{M}_2(\mathbf{C})$ is an isomorphism.

Identifying $\mathbf{H} = \mathbf{R}^4$, $\mathbf{H} \otimes_{\mathbf{R}} \mathbf{H}$ acts on $\mathbf{R}^4$ via

$$(p \otimes q) \cdot v = pv\bar{q}.$$

This action is $\mathbf{C}$–linear. A computation shows that the resulting map $\mathbf{H} \otimes_{\mathbf{R}} \mathbf{H} \to \mathrm{End}_{\mathbf{R}}(\mathbf{R}^4) \cong \mathrm{M}_4(\mathbf{R})$ is an isomorphism. □

Proof of Theorem 3.59. The proof now proceeds as follows:

1. Determine $C\ell_{3,0}$, $C\ell_{4,0}$ using $C\ell_{0,s} \otimes \mathrm{M}_2(\mathbf{R}) \cong C\ell_{s+2,0}$.
2. Determine $C\ell_{0,3}$,...,$C\ell_{0,6}$ using $C\ell_{r,0} \otimes \mathbf{H} \cong C\ell_{0,r+2}$
3. Determine $C\ell_{5,0}$, $C\ell_{7,0}$ using $C\ell_{0,s} \otimes \mathrm{M}_2(\mathbf{R}) \cong C\ell_{s+2,0}$
4. Determine $C\ell_{0,7}$ using $C\ell_{r,0} \otimes \mathbf{H} \cong C\ell_{0,r+2}$
5. Determine the rest of Table 1 using $C\ell_{r,s} \otimes \mathrm{M}_2(\mathbf{R}) \cong C\ell_{r+1,s+1}$.

The establish the periodicity in r and s, observe that

$$C\ell_{r+8,s} \cong C\ell_{s,r+6} \otimes \mathrm{M}_2(\mathbf{R}) \cong C\ell_{r+4,s} \otimes \mathrm{M}_2(\mathbf{H}) \cong C\ell_{s,r+2} \otimes \mathrm{M}_4(\mathbf{H}) \cong C\ell_{r,s} \otimes \mathrm{M}_{16}(\mathbf{R})$$

and, similarly,

$$C\ell_{s,r+8} \cong C\ell_{s,r} \otimes \mathrm{M}_{16}(\mathbf{R}).$$

□

Proposition 3.63. *Denote by $\nu_{r,s}$ the number of irreducible representation of $C\ell_{r,s}$. Denote by $D_{r,s}$ the commuting algebra of an irreducible representation of $C\ell_{r,s}$. Denote by $d_{r,s}$ the dimension of an irreducible representation of $C\ell_{r,s}$ over $D_{r,s}$. We have*

$$\nu_{r,s} = \begin{cases} 2 & \text{if } r-s = 1 \mod 4 \\ 1 & \text{if } r-s \neq 1 \mod 4, \end{cases}$$

$$D_{r,s} = \begin{cases} \mathbf{R} & \text{if } r-s = 0,1,2 \mod 8 \\ \mathbf{C} & \text{if } r-s = 3 \mod 4 \\ \mathbf{H} & \text{if } r-s = 4,5,6 \mod 8, \end{cases} \quad \text{and}$$

$$d_{r,s} = \frac{2^{r+s}}{\nu_{r,s} \cdot \dim_{\mathbf{R}} D_{r,s}}.$$

Proof. This is a direct consequence of Theorem 3.59 □

3.11 Determination of $\mathbf{C\ell}_r$

Proposition 3.64. *For $r \in \mathbf{N}_0$, we have*

$$\mathbf{C\ell}_r \cong \begin{cases} \mathrm{M}_{2^{r/2}}(\mathbf{C}) & \text{if } r \text{ is even} \\ \mathrm{M}_{2^{(r-1)/2}}(\mathbf{C})^{\oplus 2} & \text{if } r \text{ is odd.} \end{cases}$$

Exercise 3.65. Prove that $C\ell_{r,s} \otimes \mathbf{C} = \mathbf{C\ell}_{r+s}$.

Exercise 3.66. Derive Proposition 3.64 from Theorem 3.59.

3.12 Digression: determining $C\ell_{r,s}$ via the representation theory of finite groups

Here is an alternative strategy for determining $C\ell_{r,s}$. Denote by $e_1, \ldots, e_{r+s}$ the standard orthonormal basis of $(\mathbf{R}^{r+s}, q_{r,s})$. Let $G_{r,s}$ be the finite subgroup of $C\ell_{r,s}^{\times}$ of elements of the form

$$\pm e_{i_1} \cdots e_{i_n}.$$

Show that a $C\ell_{r,s}$ representation is equivalent to a $G_{r,s}$ representation in which $-1 \in G_{r,s}$ acts as -1. Determine the irreducible representations of $G_{r,s}$ using the representation theory of finite groups. Prove that $C\ell_{r,s}$ is semi-simple. Use Theorem 3.56 to determine $C\ell_{r,s}$.

This is strategy is carried out in [Roe98] for $\mathbf{C\ell}_r$ where it is attributed to J.F. Adams.

3.13 Chirality

The following assumes $r + s > 0$. From our determination of $C\ell_{r,s}$ and $\mathbf{C\ell}_r$ we know that these algebras decompose into the direct sum of two algebras if $r - s = 1 \mod 4$ and $r = 1 \mod 2$ respectively. The following explains where this splitting comes from and how to distinguish the summands in the splitting.

Definition 3.67. Fix an orientation on $\mathbf{R}^{\oplus r+s}$ and denote by $e_1, \dots, e_{r+s}$ a positive orthonormal basis for $q_{r,s}$ that is

$$b_{r,s}(e_i, e_j) = \pm\delta_{ij}.$$

The **volume element** is

$$\omega \coloneqq e_1 \cdots e_{r+s} \in \mathrm{C}\ell_{r,s}.$$

Proposition 3.68. *The volume element ω is central in $\mathrm{C}\ell(V, q)$ (that is: $\omega x = x\omega$ for all $x \in \mathrm{C}\ell(V, q)$) if and only if $r + s = 1 \mod 2$ and it satisfies $\omega^2 = 1$ if and only if $r - s \in \{0, 1\} \mod 4$.*

Proof. We have

$$\begin{aligned} e_1 \cdots e_{r+s} \cdot e_1 \cdots e_{r+s} &= (-1)^{r+s-1} e_1^2 e_2 \cdots e_{r+s} \cdot e_2 \cdots e_{r+s} \\ &= (-1)^{\frac{(r+s)(r+s-1)}{2}} e_1^2 e_2^2 \cdots e_{r+s}^2 \\ &= (-1)^{\frac{(r+s)(r+s-1)}{2} + s} \end{aligned}$$

and

$$v\omega = (-1)^{r+s-1}\omega v$$

for all $v \in \mathrm{C}\ell(V, q)$. Consequently, ω is central and satisfies $\omega^2 = 1$ if and only if

$$r - s = (r + s)^2 \mod 4 \quad \text{and} \quad r + s = 1 \mod 2$$

respectively. This implies the assertion by checking the possible values of $r - s \mod 4$. □

Remark 3.69. The volume element ω is central and $\omega^2 = 1$ if and only if $r - s = 1 \mod 4$. The volume element ω is not central and $\omega^2 = 1$ if and only if $r - s = 0 \mod 4$ and $r + s > 0$.

Proposition 3.70. *Suppose that $r - s = 1 \mod 4$.*

1. *The linear maps $\pi_\pm \colon \mathrm{C}\ell_{r,s} \to \mathrm{C}\ell_{r,s}$ defined by*

$$\pi_\pm(x) \coloneqq \frac{1}{2}(1 \pm \omega)x$$

are algebra homomorphisms and satisfy

$$\pi_\pm^2 = \pi_\pm, \quad \pi_+\pi_- = \pi_-\pi_+ = 0, \quad \text{and} \quad \pi_+ + \pi_- = \mathrm{id}.$$

Consequently,

$$\mathrm{C}\ell_{r,s}^\pm \coloneqq \ker(\pi_\mp) = \mathrm{im}(\pi_\pm)$$

are subalgebras and

$$\mathrm{C}\ell_{r,s} = \mathrm{C}\ell_{r,s}^+ \oplus \mathrm{C}\ell_{r,s}^-.$$

2. *$\mathrm{C}\ell_{r,s}^\pm$ are isomorphic via the automorphism $\alpha \in \mathrm{Aut}(\mathrm{C}\ell_{r,s})$.*

3. *$\mathrm{C}\ell_{r,s}$ has two irreducible representations S^+ and S^-. The volume element ω acts as $\pm\mathrm{id}_{S^\pm}$ on $S^\pm$. $\mathrm{C}\ell_{r,s}^\mp$ acts trivially on $S^\pm$.*

Proof. The assertions about $\pi_\pm$ follow immediately from the fact that ω is central and that $\omega^2 = 1$. The fact that $\mathrm{C}\ell_{r,s}^\pm$ are isomorphic via α follows from the fact that ω is odd and thus $\alpha(\omega) = -\omega$ and, consequently, $\pi_\pm\alpha = \alpha\pi_\mp$. The assertion about the irreducible representations is left as an exercise. □

Definition 3.71. We call $\mathrm{C}\ell_{r,s}^\pm$ the **positive chirality** and **negative chirality** summand of $\mathrm{C}\ell_{r,s}$ respectively. We call $S^\pm$ the **positive chirality** and **negative chirality** irreducible representation of $\mathrm{C}\ell_{r,s}$.

These algebras are the two summands appearing in $\mathrm{C}\ell_{r,s}$ if $r - s = 1 \mod 4$. Reversing the orientation on $\mathbf{R}^{r+s}$ reverses the labels on the summands.

Proposition 3.72. *Suppose that $r - s = 0 \mod 4$ and $r + s > 0$.*

1. *If S is a representation of $\mathrm{C}\ell_{r,s}$, then there is a decomposition*

$$S = S^+ \oplus S^-$$

into the ± 1–eigenspaces of ω.

2. *If $v \in \mathbf{R}^{r+s}$ with $q_{r,s}(v) \neq 0$, then the action of v induces isomorphisms $S^\pm \to S^\mp$.*

3. *$S^\pm$ are representations of $\mathrm{C}\ell_{r,s}^0$.*

Proof. Exercise. □

Definition 3.73. We call $S^{\pm}$ the **positive chirality** and **negative chirality** summand of S respectively.

The discussion for $\mathbf{C}\boldsymbol{\ell}_r$ is very similar, except that one uses the **complex volume element** $\omega_{\mathbf{C}}$ defined by

$$\omega_{\mathbf{C}} := i^{\lfloor \frac{r+1}{2} \rfloor} e_1 \cdots e_r \in \mathbf{C}\boldsymbol{\ell}_r$$

for a positive orthonormal basis $e_1, \ldots, e_r$. I leave it to you actually carry out the discussion.

4 Pin and Spin Groups

Throughout, we assume that (V, q) is non-degenerate.

Definition 4.1. The **twisted adjoint representation** is the map $\widetilde{\mathrm{Ad}}\colon \mathrm{C}\ell(V,q)^{\times} \to \mathrm{GL}(\mathrm{C}\ell(V,q))$ defined by

$$\widetilde{\mathrm{Ad}}(x)y := xy\alpha(x)^{-1}.$$

4.1 The Clifford group

Before introducing the Pin and Spin groups it will be helpful to consider a slightly larger group.

Definition 4.2. Let (V, q) be a non-degenerate quadratic space. The **Clifford group** of (V, q) is the group

$$\Gamma(V,q) := \left\{ x \in \mathrm{C}\ell(V,q)^{\times} : \widetilde{\mathrm{Ad}}(x)(V) \subset V \right\}.$$

The **special Clifford group** of (V, q) is the group

$$S\Gamma(V,q) := \Gamma(V,q) \cap \mathrm{C}\ell(V,q)^0.$$

Proposition 4.3.

1. *If $x \in \Gamma(V,q)$, then $\widetilde{\mathrm{Ad}}(x) \in \mathrm{O}(V,q) \subset \mathrm{GL}(V)$.*
2. *The group homomorphism $\widetilde{\mathrm{Ad}}\colon \Gamma(V,q) \to \mathrm{O}(V,q)$ is surjective and its kernel is $k^{\times}$; that is, we have an exact sequence*

$$0 \to k^{\times} \to \Gamma(V,q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(V,q) \to 0.$$

Corollary 4.4. *Suppose (V, q) is non-degenerate. $\Gamma(V, q)$ is the subgroup of $\mathrm{C}\ell(V,q)^{\times}$ generated by $k^{\times}$ and anisotropic vectors $v \in V$.*

Proposition 4.5. *Define* $N\colon \mathrm{C}\ell(V,q) \to \mathrm{C}\ell(V,q)$ *by*

$$N(x) \coloneqq \bar{x}x.$$

1. *Given* $x \in \Gamma(V,q)$,
$$N(x) \in k^\times \subset \mathrm{C}\ell(V,q).$$
2. *If* $v \in V\backslash\{0\} \subset \Gamma(V,q)$, *then* $N(v) = -q(v)$.
3. *The map* $N\colon \Gamma(V,q) \to k^\times$ *is a group homomorphism.*

Definition 4.6. The group homomorphism $N\colon \Gamma(V,q) \to k^\times$ is called the **Clifford norm**.

Definition 4.7. Since $N(k^\times) \subset (k^\times)^2$, the Clifford norm induces a group homomorphism

$$N\colon \mathrm{O}(V,q) \to k^\times/(k^\times)^2$$

called the **spinor norm**.

Proof that $\ker \widetilde{\mathrm{Ad}} \cap \Gamma(V,q) = k^\times$. Suppose $x \in \Gamma(V,q)$ and $\widetilde{\mathrm{Ad}}(x) = \mathrm{id}_V$. Denote by x_0 the even part of x and by x_1 its odd part. We have

$$vx_0 = x_0v \quad \text{and} \quad -vx_1 = x_1v. \tag{4.8}$$

In an orthogonal basis $e_1,\dots,e_n$ of v we can write $x_0 = y_0 + e_1y_1$ with y_0 and y_1 only involving $e_2,\dots,e_n$. Since x_0 is even, y_0 must be even and y_1 must be odd. Applying (4.8) with $v = e_1$ yields

$$\begin{aligned} e_1y_0 + e_1^2y_1 &= y_0e_1 + e_1y_1e_1 \\ &= e_1y_0 - e_1^2y_1. \end{aligned}$$

Consequently, $e_1^2y_1 = q(e_1)y_1 = 0$ and thus $y_1 = 0$. This means that x_0 does not actually involve e_1. Arguing inductively it follows that x_0 does not involve any e_i and thus $x_0 \in k$.

We now write $x_1 = y_1 + e_1y_0$ with y_0 even and y_1 odd and neither involving e_1. From (4.8) with $v = e_1$ it follows that

$$\begin{aligned} -e_1y_1 - e_1^2y_0 &= y_1e_1 + e_1y_0e_1 \\ &= -e_1y_1 + e_1^2y_0 \end{aligned}$$

and thus $y_0 = 1$. This proves that x_1 does not involve e_1. It follows inductively that x_1 does not involve any e_i and thus $x_1 \in k$. In fact, $x_1 = 0$ because it is odd. Consequently, $x \in k$. Since x is invertible, we must have $x \in k^\times$. □

Proof of Proposition 4.5. We prove (1). Given $x \in \Gamma(V, q)$, we have

$$(\widetilde{\mathrm{Ad}}(x)v)^t = \widetilde{\mathrm{Ad}}(x)v$$

and, therefore,

$$\begin{aligned}\widetilde{\mathrm{Ad}}(N(x))v &= \bar{x}(xv\alpha(x)^{-1})(x^t)^{-1} \\ &= \bar{x}(xv\alpha(x)^{-1})^t(x^t)^{-1} \\ &= \bar{x}\bar{x}^{-1}vx^t(x^t)^{-1} = v\end{aligned}$$

for all $v \in V$. Therefore, $N(x) \in \ker \widetilde{\mathrm{Ad}} = k^\times$.

The assertion (2) is trivial.

We prove (3). If $x, y \in \Gamma(V, q)$, then

$$N(xy) = \bar{y}\bar{x}xy = \bar{y}N(x)y = N(x)\bar{y}y = N(x)N(y).$$

□

Proof of Proposition 4.3. We prove (1). Given $x \in \Gamma(V, q)$ and $v \in V$ with $q(v) \neq 0$, since $N(\alpha(x)) = N(x)$, we have

$$N(\widetilde{\mathrm{Ad}}(x)v) = N(xv\alpha(x)^{-1}) = N(x)N(x)^{-1}N(v) = N(v).$$

Consequently,

$$q(\widetilde{\mathrm{Ad}}(x)v) = q(v)$$

for all non-zero v; hence $\widetilde{\mathrm{Ad}}(x) \in \mathrm{O}(V, q)$.

The fact that $\ker \widetilde{\mathrm{Ad}} = k^\times$ was already proved and we also already proved that $\widetilde{\mathrm{Ad}}$ maps onto $\mathrm{O}(V, q)$. □

Exercise 4.9. Prove that $x \in \Gamma(V, q)$ implies $\bar{x} \in \Gamma(V, q)$.

4.2 Spin(V, q) and Pin(V, q)

Definition 4.10. The **pin group** associated with (V, q) is the group

$$\mathrm{Pin}(V, q) := \ker N \subset \Gamma(V, q).$$

The **spin group** associated with (V, q) is the group

$$\mathrm{Spin}(V, q) := \mathrm{Pin}(V, q) \cap S\Gamma(V, q).$$

The following is a consequence of Corollary 4.4.

Corollary 4.11. *The pin and spin group can be described explicitly as follows*

$$\mathrm{Pin}(V,q) = \left\{ \lambda v_1 \cdots v_n \in \mathrm{C}\ell(V,q)^\times : \lambda^2 \prod_i q(v_i) = (-1)^n \right\} \quad \textit{and}$$
$$\mathrm{Spin}(V,q) = \left\{ \lambda v_1 \cdots v_{2n} \in \mathrm{C}\ell(V,q)^\times : \lambda^2 \prod_i q(v_i) = 1 \right\}.$$

Here $\lambda \in k^\times$ and $v_i \in V$ are anisotropic vectors.

Remark 4.12. In [LM89] $\mathrm{Pin}(V,q)$ and $\mathrm{Spin}(V,q)$ are defined differently and their definitions do indeed give rise to different groups. However, for $k = \mathbf{R}$ and positive and negative definite forms our spin groups will be identical to those defined in [LM89]. In as much as definitions can be wrong, I think their definition is wrong.

There are exact sequences

$$0 \to \{\pm 1\} \to \mathrm{Pin}(V,q) \to \mathrm{O}(V,q) \xrightarrow{N} k^\times/(k^\times)^2$$

and

$$0 \to \{\pm 1\} \to \mathrm{Spin}(V,q) \to \mathrm{SO}(V,q) \xrightarrow{N} k^\times/(k^\times)^2.$$

Example 4.13. Suppose $q = q_{r,s}$ on $\mathbf{R}^{r+s}$. We have $\mathbf{R}^\times/(\mathbf{R}^\times)^2 \cong \{\pm 1\}$. The spinor norm of a reflection $r_v \in \mathrm{O}(V,q)$ in an anisotropic vector v is $-\operatorname{sign}(q(v))$. If $s = 0$, that is, q is positive definite, then $N = (-1)^{\det}$. Therefore, $\mathrm{Spin}_{r,0} = \mathrm{Pin}_{r,0}$! If $r = 0$, that is, q is negative definite, then $N = 1$.

Definition 4.14. Given $r, s \in \mathbf{N}_0$, we define

$$\mathrm{Pin}_{r,s} \coloneqq \mathrm{Pin}(\mathbf{R}^{r+s}, q_{r,s}) \quad \text{and} \quad \mathrm{Spin}_{r,s} \coloneqq \mathrm{Spin}(\mathbf{R}^{r+s}, q_{r,s}).$$

We will later restrict to definite quadratic forms and use the following conventions.

Definition 4.15. We define

$$\mathrm{O}(n) \coloneqq \mathrm{O}_{0,n} \quad \text{and} \quad \mathrm{SO}(n) \coloneqq \mathrm{SO}_{0,n}.$$

as well as

$$\mathrm{Pin}(n) \coloneqq \mathrm{Pin}_{0,n} \quad \text{and} \quad \mathrm{Spin}(n) \coloneqq \mathrm{Spin}_{0,n}.$$

Of course, $\mathrm{O}(n) = \mathrm{O}_{n,0}$ but the choice of the negative sign makes certain identities come out cleaner.

4.3 Digression: The Lorentz Group

The Lorentz group $\mathrm{O}(1,3) = \mathrm{O}(\mathbf{R}^4, q_{1,3})$ is the group of matrices $A \in \mathrm{M}_4(\mathbf{R})$ such that

$$A^t Q A = Q \quad \text{with} \quad Q = \mathrm{diag}(1, -1, -1, -1)$$

Definition 4.16. A vector $v \in (\mathbf{R}^4, q = q_{1,3})$ is called **time-like**, **space-like**, or **light-like** if $q(v) > 0$, $q(v) < 0$, $q(v) = 0$ respectively. We say that $v = (v_0, v_1, v_2, v_3)$ is **positive** if $v_0 > 0$.

The set of all light-like vector forms the **light-cone**.

The complement of light-cone in $\mathbf{R}^4$ has 3 components: positive light-like vectors, negative light-like vectors, and space-like vectors. Any **Lorentz transformation** $A \in \mathrm{O}(1,3)$ preserves the light-cone, but it might interchange the positive and negative light-cone; e.g., the time-inversion

$$T = \mathrm{diag}(-1, 1, 1, 1).$$

A moment's thought shows that $A = (a_{ij})$ switches the positive and negative light-like directions if and only if $a_{00} < 0$.

Definition 4.17. A Lorentz transformation $A = (a_{ij}) \in \mathrm{O}(1,3)$ is called **orthochronous** if $a_{00} > 0$. The **orthochronous Lorentz group** is the group

$$\mathrm{O}^+(1,3) = \{A \in \mathrm{O}(1,3) : a_{00} > 0\}$$

and the **proper, orthochronous Lorentz group** or **restricted Lorentz group** is

$$\mathrm{SO}^+(1,3) = \mathrm{SO}(1,3) \cap \mathrm{O}^+(1,3).$$

Proposition 4.18. *The group* $\mathrm{SO}^+(1,3)$ *is a connected normal subgroup. The quotient* $\mathrm{O}(1,3)/\mathrm{SO}^+(1,3)$ *is isomorphic to the subgroup of* $\mathrm{O}(1,3)$ *generated by*

$$T = \mathrm{diag}(-1, 1, 1, 1) \quad \textit{and} \quad P = \mathrm{diag}(1, -1, -1, -1)$$

which is itself isomorphic to the Klein four group. In particular $\mathrm{O}(1,3)$ *has* 4 *connected components.*

Proof. This should be in any self-respecting book on Special Relativity. □

Let $v \in (\mathbf{R}^4, q_{1,3})$. If v is space-like, then $N(v) > 0$ and $\widetilde{\mathrm{Ad}}(v) \in P \cdot \mathrm{SO}^+(1,3)$. If v is time-like, then $N(v) < 0$ and $\widetilde{\mathrm{Ad}}(v) \in T \cdot \mathrm{SO}^+(1,3)$. This means that the image of $\mathrm{Pin}_{1,3}$ is $\{1, P\} \cdot \mathrm{SO}^+(1,3) = \mathrm{O}^+(1,3)$ while the image of $\mathrm{Spin}_{1,3}$ is $\mathrm{SO}^+(1,3)$.

4.4 $\mathrm{Pin}_{r,s}$ and $\mathrm{Spin}_{r,s}$

One can prove, more generally, that if $r, s \in \mathbf{N}_0$, then $\mathrm{O}_{r,s}$ has at most 4 connected components distinguished by the value of $\det \times N\colon \mathrm{O}_{r,s} \to \{\pm 1\} \times \{\pm 1\}$. More precisely,

$$\pi_0(\mathrm{O}_{r,s}) = \begin{cases} \mathbf{Z}_2 \times \mathbf{Z}_2 & \text{if } r, s > 0, \\ \mathbf{Z}_2 & \text{otherwise.} \end{cases}$$

Following the notation above we set

$$\mathrm{SO}^+_{r,s} \coloneqq (\det \times N)^{-1}(+1, +1) \quad \text{and} \quad \mathrm{O}^+_{r,s} \coloneqq N^{-1}(+1).$$

We have exact sequences

$$0 \to \mathbf{Z}_2 \to \mathrm{Pin}_{r,s} \to \mathrm{O}^+_{r,s} \to 0 \quad \text{and} \quad 0 \to \mathbf{Z}_2 \to \mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s} \to 0.$$

Corollary 4.19. $\mathrm{Spin}_{r,s}$ *is a Lie group.*

Proposition 4.20. *If $r, s \in \mathbf{N}$ but $(r, s) \neq (1, 1)$, then the covering maps $\mathrm{Pin}_{r,s} \to \mathrm{O}^+_{r,s}$ and $\mathrm{Spin}_{r,s} \to \mathrm{SO}^+_{r,s}$ are non-trivial on each connected component of the base.*

Proof. It suffices to prove that $+1$ and -1 are connected in $\mathrm{Spin}_{r,s}$. Fix an orthonormal set e_1, e_2 with $q(e_1) = q(e_2) = \pm 1$ and define a path $\gamma\colon [0, \pi/2] \to \mathrm{Spin}_{r,s}$ by

$$\gamma(t) = (e_1 \cos(t) + e_2 \sin(t))(e_1 \cos(t) - e_2 \sin(t)).$$

Since

$$\gamma(\pi/2) = e_1^2 = \pm 1 \quad \text{and} \quad \gamma(\pi/2) = -e_2^2 = \mp 1,$$

this completes the proof. □

4.5 Comparing $\mathfrak{spin}_{r,s}$ and $\mathfrak{so}_{r,s}$.

Definition 4.21. We denote by $\mathfrak{spin}_{r,s}$ the Lie algebra of $\mathrm{Spin}_{r,s}$.

Proposition 4.22. *With respect to the identification*

$$\Lambda^2 \mathbf{R}^n = \left\{ \tfrac{1}{2}(vw - wv) : v, w \in \mathbf{R}^n \right\} \subset C\ell_{0,n},$$

we have

$$\mathfrak{spin}_{r,s} = \Lambda^2 \mathbf{R}^n.$$

Proof. Fix an orthonormal basis $e_1, \dots, e_n$ of $\mathbf{R}^n$. For $i < j$, the curve

$$\begin{aligned}\gamma(t) &= (e_i \cos(t/2) + e_j \sin(t/2)) \cdot (-e_i \cos(t/2) + e_j \sin(t/2)) \\ &= \cos(t) + \sin(t) e_i e_j\end{aligned}$$

lies in $\mathrm{Spin}(n)$ and its tangent vector at $\gamma(0)$ is $e_i e_j$. This means that $\mathfrak{spin}_{r,s} \subset \Lambda^2 \mathbf{R}^n$. By dimension counting the inclusion must be an identity. □

There also is a natural isomorphism $\Lambda^2 \mathbf{R}^n \cong \mathfrak{so}_{r,s}$ given by

$$(v \wedge w)x = b_{r,s}(w, x)v - b_{r,s}(v, x)w.$$

Here $b_{r,s}$ is the symmetric bilinear form associated with $q_{r,s}$.

Remark 4.23. This isomorphism is simply raising one index using $b_{r,s}$.

Remark 4.24. For $\mathrm{SO}(n) = \mathrm{SO}_{0,n}$, we have

$$(v \wedge w)x = \langle v, x \rangle w - \langle w, x \rangle v$$

with respect to the *positive definite* inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{R}^n$.

Proposition 4.25. *The Lie algebra map* $\mathrm{Lie}(\widetilde{\mathrm{Ad}}) \colon \mathfrak{spin}_{r,s} \to \mathfrak{so}_{r,s}$ *is given by*

$$\mathrm{Lie}(\widetilde{\mathrm{Ad}}) e_i e_j = 2 e_i \wedge e_j$$

or, more invariantly, by

$$\mathrm{Lie}(\widetilde{\mathrm{Ad}})([v, w]) = 4 v \wedge w.$$

Proof. Consider the curve $\gamma(t) = \cos(t) + \sin(t) e_i e_j$ in $\mathrm{Spin}(n)$. We have $\dot{\gamma}(0) = e_i e_j$ and

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \widetilde{\mathrm{Ad}}(\gamma(t))v &= e_i e_j v - v e_i e_j \\ &= e_i e_j v + e_i v e_j - 2 b_{r,s}(v, e_i) e_j \\ &= 2 b_{r,s}(v, e_j) e_i - 2 b_{r,s}(v, e_i) e_j \\ &= 2(e_i \wedge e_j)v.\end{aligned}$$

□

4.6 Identifying $\mathrm{C}\ell(V, q)^0$

In light of the definition of $\mathrm{Spin}(V, q)$, it becomes a relevant question to ask: what is $\mathrm{C}\ell(V, q)^0$ and what are its representatons?

If $S = S^+ \oplus S^-$ is $\mathbf{Z}_2$ graded vector space, then $\mathrm{End}(S)$ is a $\mathbf{Z}_2$ graded algebra with

$$\begin{aligned}\mathrm{End}(S)^0 &= \mathrm{End}(S^+) \oplus \mathrm{End}(S^-) \quad \text{and} \\ \mathrm{End}(S)^1 &= \mathrm{Hom}(S^+, S^-) \oplus \mathrm{Hom}(S^-, S^+).\end{aligned}$$

Definition 4.26. Let A be a $\mathbf{Z}_2$ graded algebra. A graded representation of A is a $\mathbf{Z}_2$ graded vector space $S^+ \oplus S^-$ together with a graded algebra homomorphism $A \to \mathrm{End}(S)$.

The following allows us to explicitly determine $\mathrm{C}\ell^0_{r,s}$.

Proposition 4.27. *We have*

$$\mathrm{C}\ell(V \oplus k, q \oplus (-1))^0 \cong \mathrm{C}\ell(V, q) \quad \textit{and} \quad \mathrm{C}\ell(V \oplus k, q \oplus (1))^0 \cong \mathrm{C}\ell(V, -q)$$

In particular,

$$\mathrm{C}\ell^0_{r,s+1} \cong \mathrm{C}\ell_{r,s}, \quad \mathrm{C}\ell^0_{r+1,s} \cong \mathrm{C}\ell_{s,r}, \quad \textit{and} \quad \mathbf{C}\boldsymbol{\ell}^0_{r+1} \cong \mathbf{C}\boldsymbol{\ell}_r.$$

Proof. Set $e_0 := (0,1) \in V \oplus k$. Define $\gamma \colon V \to \mathrm{C}\ell(V \oplus k)^0$ by

$$\gamma(v) := e_0 v.$$

Since

$$\gamma(v) = e_0 v e_0 v = -e_0^2 v^2 = q(v),$$

γ induces an algebra homomorphism $\mathrm{C}\ell(V, q) \to \mathrm{C}\ell(V \oplus k, q \oplus (-1))^0$.

Exercise 4.28. Prove that the homomorphism is surjective by proving that $e_0 v$ generates $\mathrm{C}\ell(V \oplus k)^0$.

Multiplication with e_0 induces a vector space isomorphism $\mathrm{C}\ell(V \oplus k, q \oplus (-1))^0 \to \mathrm{C}\ell(V \oplus k, q \oplus (-1))^1$. Consequently, $\dim \mathrm{C}\ell(V \oplus k, q \oplus (-1))^0 = 2^{\dim V + 1}/2 = \dim \mathrm{C}\ell(V, q)$. It follows that the algebra homomorphism is also injective.

The second isomorphism follows by the same argument. □

Remark 4.29. Note that the isomorphism $\mathrm{C}\ell^0_{r+1,s} \cong \mathrm{C}\ell_{s,r}$ changes the order of r and s.

4.7 Representation theory of $\mathrm{Pin}(V, q)$ and $\mathrm{Spin}(V, q)$

We are interested in representation of $\mathrm{Pin}(V, q)$ and $\mathrm{Spin}(V, q)$ in which -1 acts non-trivially. Representations in which -1 acts trivial, actually are representations of

$$\Omega(V, q) := \mathrm{im}(\mathrm{Pin}(V, q) \to \mathrm{O}(V, q)) \quad \text{and}$$
$$S\Omega(V, q) := \mathrm{im}(\mathrm{Spin}(V, q) \to \mathrm{SO}(V, q))$$

and thus yield nothing new.

Proposition 4.30. *If* $\rho \colon \mathrm{C}\ell(V, q) \to \mathrm{End}(W)$ *is a representation of* $\mathrm{C}\ell(V, q)$*, then its restriction to* $\mathrm{Pin}(V, q)$ *is a representation of* $\mathrm{Pin}(V, q)$ *in which* -1 *acts as* $-\mathrm{id}_W$ *If* $N(\Gamma(V, q)) \subset (k^\times)^2$*, then every such representation of* $\mathrm{Pin}(V, q)$ *arises from this construction.*

Proof. The first part of each of the assertions is trivial. Suppose $N(\Gamma(V,q)) \subset (k^\times)^2$. If $v \in V$ is anisotropic that is $N(v) = -q(v) \neq 0$, then $v/\sqrt{N(v)} \in \mathrm{Pin}(V,q)$. In fact, $\mathrm{Pin}(V,q)$ is generated by ± 1 and vectors $v \in V$ with $N(v) = 1$. There is basis $(v_1, \ldots, v_n)$ of V with $b(v_i, v_j) = \delta_{ij}$. Let $\sigma\colon \mathrm{Pin}(V,q) \to \mathrm{GL}(W)$ be a representation with $\sigma(-1) = -\mathrm{id}_W$. Define $\delta\colon V \to \mathrm{End}(W)$ to be the unique linear map such that

$$\delta(v_i) = \sigma(v_i).$$

Since

$$\delta(v_i)^2 = \sigma(v_i)^2 = \sigma(-1)^2 = -\mathrm{id}_W,$$

δ extends to a representation $\rho\colon \mathrm{C}\ell(V,q) \to \mathrm{End}(W)$. By construction ρ extends σ. □

Using this one can introduce the notion of a **pinor representation**. In this course, we will not work with pinors, but only spinors. Thus, let us proceed to the construction of the spinor representation immediately.

Proposition 4.31. *Denote by S the restriction of an irreducible representation of $\mathrm{C}\ell_{r,s}$ to $\mathrm{Spin}_{r,s}$.*

1. *S is independent of the choice of irreducible representation of $\mathrm{C}\ell_{r,s}$.*
2. *If $r - s \in \{-1, -2\} \mod 8$, then S of $\mathrm{Spin}_{r,s}$ decomposes into two equivalent $\mathrm{C}\ell^0_{r,s}$–irreducible representations $S = S' \oplus S'$. If $r - s \in \{-3, -5, -6, -7\} \mod 8$, then S of $\mathrm{Spin}_{r,s}$ is $\mathrm{C}\ell^0_{r,s}$–irreducible.*
3. *If $r - s = 0 \mod 4$, then S of $\mathrm{Spin}_{r,s}$ decomposes into two inequivalent $\mathrm{C}\ell^0_{r,s}$–irreducible representations $S = S^+ \oplus S^-$. $S^\pm$ is characterized by the volume element ω acting as $\pm\mathrm{id}_{S^\pm}$. Moreover, if $v \in V$, then $\gamma(v)S^\pm \subset S^\mp$.*

Remark 4.32. If $N(S\Gamma(V,q)) \subset (k^\times)^2$, then irreducible for $\mathrm{C}\ell^0_{r,s}$ implies irreducible for $\mathrm{Spin}_{r,s}$. This condition does not hold for indefinite quadratic forms (that is, when $r, s > 0$). In this case, I am not sure whether irreducible for $\mathrm{C}\ell^0_{r,s}$ implies irreducible for $\mathrm{Spin}_{r,s}$.

Definition 4.33. We call S in Proposition 4.31 the **spinor representation** of $\mathrm{Spin}_{r,s}$ and we call S^+ and S^- the **positive chirality spinor representation** and **negative chirality spinor representation**.

Proof of Proposition 4.31. We prove (1). If $r - s \neq 1 \mod 4$, there is a unique irreducible representation of $\mathrm{C}\ell_{r,s}$. For $r = s = 1 \mod 4$, there are two irreducible representations S^+ and S^- and the Clifford algebra decomposes as $\mathrm{C}\ell_{r,s} = \mathrm{C}\ell^+_{r,s} \oplus \mathrm{C}\ell^-_{r,s}$. The involution α interchanges $\mathrm{C}\ell^+_{r,s}$ and $\mathrm{C}\ell^-_{r,s}$ as well as S^+ and S^-. This means that if $(x,y) \in \mathrm{C}\ell^+_{r,s} \oplus \mathrm{C}\ell^-_{r,s}$, then $\alpha(x,y) = (\alpha y, \alpha x)$. Therefore,

$$\mathrm{C}\ell^0_{r,s} = \{(x, \alpha(x)) \in \mathrm{C}\ell^+_{r,s} \oplus \mathrm{C}\ell^-_{r,s}\}.$$

Since the two irreducible representations S^+ and S^- are related by α, they agree on $\mathrm{C}\ell^0_{r,s}$.

(2) follows by inspecting Table 1 and using Proposition 4.27. E.g., $C\ell_{0,1} = \mathbf{C}$ has the irreducible representation $\mathbf{C}$. Restricting to $C\ell^0_{0,0} = C\ell_{0,0} = \mathbf{R}$ this representation splits as $\mathbf{C} = \mathbf{R} \oplus i\mathbf{R} = \mathbf{R}^{\oplus 2}$. Similarly, $C\ell_{0,2} = \mathbf{H}$ has the irreducible representation $\mathbf{H}$. Restricting to $C\ell^0_{0,2} = C\ell_{0,1} = \mathbf{C}$ this representation splits as $\mathbf{H} = \mathbf{C} \oplus j\mathbf{C} = \mathbf{C}^{\oplus 2}$. However, the irreducible representation $\mathbf{H}$ of $C\ell_{0,3} = \mathbf{H}^{\oplus 2}$ stays irreducible upon restriction to $C\ell^0_{0,3} = C\ell_{0,2} = \mathbf{H}$.

(3) is immediate from Proposition 3.72. □

Proposition 4.34. *The spinor representations S and the representations S' are faithful.*

Proof. Exercise using Table 1. □

Proposition 4.35.

1. *The spinor representation S of $\mathrm{Spin}_{r,s}$ admits an inner product invariant under the action of $\mathrm{Spin}_{r,s}$. If $r = 0$ or $s = 0$, the inner product can be chosen to be positive definite. The action of $\mathbf{R}^{\oplus r+s}$ on S is skew-adjoint with respect to this inner product. If $r - s = 0 \mod 4$, then $S^+ \perp S^-$.*

2. *If $r - s = -1 \mod 8$, then S admits a complex structure I which is invariant under the action of $\mathrm{Spin}_{r,s}$ and orthogonal with respect to the Euclidean inner product. The action of $\mathbf{R}^{\oplus r+s}$ on S is $\mathbf{C}$–linear. The complex structure I does not preserve $S' \subset S$.*

3. *If $r - s = -2, \mod 8$, then S admits a quaternionic structure I, J, K which is invariant under the action of $\mathrm{Spin}_{r,s}$ and orthogonal with respect to the Euclidean inner product. The action of $\mathbf{R}^{\oplus r+s}$ on S is $\mathbf{H}$–linear. The complex structure I does preserve $S' \subset S$, but J and K do not.*

4. *If $r - s = -3, -4 \mod 8$, then S admits a quaternionic structure I, J, K which is invariant under the action of $\mathrm{Spin}_{r,s}$ and orthogonal with respect to the Euclidean inner product. The action of $\mathbf{R}^{\oplus r+s}$ on S is $\mathbf{H}$–linear.*

5. *If $r - s = -5 \mod 8$, then S admits a quaternionic structure I, J, K which is invariant under the action of $\mathrm{Spin}_{r,s}$ and orthogonal with respect to the Euclidean inner product; moreover, the action of $\mathbf{R}^{\oplus r+s}$ on S is $\mathbf{C}$–linear with respect to the complex structure I, but not with respect to J and K.*

Proof. If $r = 0$ or $s = 0$, then $\mathrm{Spin}_{r,s}$ is compact and the inner product can be constructed by averaging. In general, the construction is discussed in [Har90].

The rest is an exercise using Table 1. □

The preceding proposition, dimension counting, and some Lie algebra theory can be used to prove the following **accidental isomorphisms**.

Proposition 4.36.

$$\begin{aligned}
\mathrm{Spin}_{0,2} &= \mathrm{U}(1),\\
\mathrm{Spin}_{0,3} &= \mathrm{Sp}(1),\\
\mathrm{Spin}_{0,4} &= \mathrm{Sp}(1)\times\mathrm{Sp}(1),\\
\mathrm{Spin}_{0,5} &= \mathrm{Sp}(2),\\
\mathrm{Spin}_{0,6} &= \mathrm{SU}(4).
\end{aligned}$$

4.8 Pin^c and Spin^c

If there is one thing we have learned to so far is that the dependence of real Clifford algebras, spin representations, etc. on r and s is unpleasantly complicated. The whole story should be much simpler over the complex numbers. Our construction of $\mathrm{Pin}(V,q)$ and $\mathrm{Spin}(V,q)$ directly carries over to complex vector spaces, but we do not want to work complex vector spaces to start with. One way out is to complexify and pass to $V\otimes\mathbf{C}$, but then the pin and spin groups do not act on V any more. There is a modification of the definition of the Clifford group, and the pin and spin groups using the real structure (that is: the complex conjugation) on $V\otimes\mathbf{C}$. You can read about that in [ABS64, p. 9]. Here we will directly make the following definitions.

Definition 4.37. The pin^c and spin^c groups are defined as

$$\mathrm{Pin}^c(V,q) \coloneqq \mathrm{Pin}(V,q)\times_{\mathbf{Z}_2}\mathrm{U}(1) \quad\text{and}\quad \mathrm{Spin}^c(V,q) \coloneqq \mathrm{Spin}(V,q)\times_{\mathbf{Z}_2}\mathrm{U}(1)$$

We define

$$\mathrm{Pin}^c_s \coloneqq \mathrm{Pin}^c(\mathbf{R}^s, q_{0,s}) \quad\text{and}\quad \mathrm{Spin}^c_s \coloneqq \mathrm{Spin}^c(\mathbf{R}^s, q_{0,s}).$$

Identifying $\mathrm{U}(1)=\{z\in\mathbf{C} : |z|=1\}$, these groups naturally sit in $\mathrm{C}\ell(V\otimes\mathbf{C},q)$ via

$$\mathbf{C}\otimes\mathrm{C}\ell(V,q)\cong\mathrm{C}\ell(V\otimes\mathbf{C},q).$$

Definition 4.38. The **complex spinor representation** W of Spin^c_s is the restriction of an irreducible representation of $\mathbf{C}\boldsymbol{\ell}_s$. If $s=0 \mod 2$, then we can decompose $W=W^+\oplus W^-$ according to the action of the complex volume element. We call $W^\pm$ the positive/negative chirality complex spinor representation.

Remark 4.39. The complex spinor representations carries a Hermitian inner product which is Spin^c_s invariant.

Remark 4.40. Since $\mathrm{Spin}_{0,s}\subset\mathrm{Spin}^c_s$, we will also talk about the complex spinor representation of $\mathrm{Spin}_{0,s}$.

Remark 4.41. If G is a Lie group and with a choice of embedding $\mathbf{Z}_2 \subset Z(G)$, then one can also consider $\mathrm{Spin}^G_{r,s} = \mathrm{Spin}_{r,s} \times_{\mathbf{Z}_2} G$. This plays a role in the discussion of certain Seiberg–Witten equations.

Throughout, M is an *oriented* manifold of dimension $\dim M = n$ together with a *Riemannian* metric $g \in \Gamma(S^2T^*M)$. All of the theory developed in this chapter can be extended to the case of semi-Riemannian indefinite metrics. If you are used to working with semi-Riemannian manifold, you will probably have no trouble adjusting the following development to that case.

5 Clifford bundles

Definition 5.1. Let $\pi\colon E \to M$ be a vector bundle of rank r together with a Euclidean metric h. The **orthornormal frame bundle** $\mathrm{O}(E)$ is the principal $\mathrm{O}(r)$–bundle defined by

$$\mathrm{O}(E) \coloneqq \{(x, e_1, \ldots, e_r) \in M \times E^{\times r} : \pi(e_i) = x, h(e_i, e_j) = \delta_{ij}\}.$$

If E is oriented, then we also defined the **orthornormal frame bundle** $\mathrm{SO}(E)$ as the principal $\mathrm{SO}(r)$–bundle defined by

$$\mathrm{SO}(E) \coloneqq \{(x, e_1, \ldots, e_r) \in \mathrm{O}(E) : e_1 \wedge \cdots \wedge e_r > 0\}.$$

Exercise 5.2. Construct the obvious principal bundle structures on $\mathrm{O}(E)$ and $\mathrm{SO}(E)$.

Remark 5.3. If M is a manifold, then the Serre–Swan theorem identifies vector bundles E over M (or rather their spaces of sections $\Gamma(E)$) with finitely-generated projective modules over the ring $R \coloneqq C^\infty(M)$. The choice of an Euclidean metric is then simply a quadratic form on $\Gamma(E)$. One can construct $\mathrm{C}\ell(E)$ using a straight-forward extension of the theory developed earlier to quadratic forms on modules over rings (as opposed to just quadratic forms on vector spaces). I do not know if this is useful for anything.

Recall from Corollary 3.10, that $\mathrm{O}(n)$ acts on the Clifford algebra $\mathrm{C}\ell_{0,n}$.

Definition 5.4. If (E, h) is a Euclidean rank r vector bundle, then the **Clifford bundle** associated with (E, h) is the bundle

$$\mathrm{C}\ell(E) \coloneqq \mathrm{O}(E) \times_{\mathrm{O}(r)} \mathrm{C}\ell_{0,r}.$$

We denote by $\gamma\colon E \to \mathrm{C}\ell(E)$ the map induced by the inclusion $\mathbf{R}^{\oplus r} \to \mathrm{C}\ell_{0,r}$.

Remark 5.5. As *vector bundles*

$$\mathrm{C}\ell(E) = \Lambda E.$$

Clearly, the fibre $\mathrm{C}\ell(E)_x$ of $\mathrm{C}\ell(E)$ over $x \in M$ is the Clifford algebra $\mathrm{C}\ell(E_x, -h)$. All structures discussed on $\mathrm{C}\ell_{0,n}$ naturally carry over to $\mathrm{C}\ell(E)$. In particular, the involution α induces a bundle map $\alpha\colon \mathrm{C}\ell(E) \to \mathrm{C}\ell(E)$ and its $(+1)$– and (-1)–eigenspaces correspond to the even part $\mathrm{C}\ell(E)^0$ and $\mathrm{C}\ell(E)^1$. Similarly, if $n = 1 = -3 \mod 4$ and E is *oriented*, then $\mathrm{C}\ell(E)$ splits as $\mathrm{C}\ell(E) = \mathrm{C}\ell(E)^+ \oplus \mathrm{C}\ell(E)^-$.

Remark 5.6. If E is orientable, there is a splitting but without an choice of orientation we cannot label the summands. If E is not-orientable, the splitting exists locally, but globally it does not; there will be monodromies exchanging the summands in the splitting.

Definition 5.7. We denote the **Clifford bundle** associated to (TM, g) by $\mathrm{C}\ell(M)$.

Proposition 5.8. *There is a unique covariant derivative* $\nabla = \nabla_{\mathrm{C}\ell}\colon \Gamma(\mathrm{C}\ell(M)) \to \Omega^1(M, \mathrm{C}\ell(M))$ *on* $\mathrm{C}\ell(M)$ *such that for all* $v \in \Gamma(TM)$*, and* $x, y \in \Gamma(\mathrm{C}\ell(M))$ *we have*

$$\nabla_{\mathrm{C}\ell}\gamma(v) = \gamma(\nabla_{\mathrm{LC}}v) \quad \textit{and} \quad \nabla_{\mathrm{C}\ell}(xy) = (\nabla_{\mathrm{C}\ell}x)y + x(\nabla_{\mathrm{C}\ell}y).$$

6 Clifford module bundles

Definition 6.1. A **Clifford module bundle** over M is a vector bundle $\pi\colon S \to M$ together with a smooth map of algebra bundles $\mathrm{C}\ell(M) \to \mathrm{End}(S)$; that is, the map is smooth and for each $x \in M$ the induced map $\mathrm{C}\ell(M)_x \to \mathrm{End}(S_x)$ is an algebra homomorphism.

Definition 6.2. A **complex Clifford module bundle** over M is a complex vector bundle $\pi\colon S \to M$ together with a smooth map of algebra bundles $\mathrm{C}\ell(M) \to \mathrm{End}_{\mathbb{C}}(S)$.

Definition 6.3. If S is a Clifford module bundle, then the induced map $\gamma\colon TM \to \mathrm{End}(S)$ is called the **Clifford multiplication**.

Exercise 6.4. Prove that the Clifford multiplication satisfies

$$\gamma(v)^2 = -|v|^2\mathrm{id}_S.$$

Prove that the existence of such a Clifford multiplication is equivalent to the existence of a Clifford module structure.

Example 6.5. $\mathrm{C}\ell(M)$ is a Clifford module bundle.

Example 6.6. The bundle of exterior algebras

$$S := \Lambda TM = \bigoplus_{r=0}^{n} \Lambda^r TM$$

is a Clifford module bundle. To see this we need to define a Clifford multiplication $\gamma \colon TM \to \mathrm{End}(S)$. The map γ defined by

$$\gamma(v)(w_1 \wedge \cdots \wedge w_r) = v \wedge w_1 \wedge \cdots \wedge w_r - \sum_{s=1}^{r} (-1)^s \langle v, w_s \rangle w_1 \wedge \cdots \wedge \hat{w}_s \wedge \cdots \wedge w_r$$

satisfies

$$\begin{aligned}\gamma(v)\gamma(v)(w_1 \wedge \cdots \wedge w_r) &= -|v|^2 w_1 \wedge \cdots \wedge w_r \\ &\quad - \sum_{s=1}^{r} (-1)^{s+1} \langle v, w_s \rangle v \wedge w_1 \wedge \cdots \wedge \hat{w}_s \wedge \cdots \wedge w_r \\ &\quad - \sum_{s=1}^{r} (-1)^{s} \langle v, w_s \rangle v \wedge w_1 \wedge \cdots \wedge \hat{w}_s \wedge \cdots \wedge w_r \\ &= -|v|^2 w_1 \wedge \cdots \wedge w_r;\end{aligned}$$

hence, it is a Clifford multiplication.

The previous example has a natural $\mathbf{Z}_2$–grading given by

$$S^0 := \Lambda^{\mathrm{even}} TM \quad \text{and} \quad S^1 := \Lambda^{\mathrm{odd}} TM.$$

This makes it a **graded Clifford module bundle**.

Exercise 6.7 (Twisting Clifford modules). Suppose S is a Clifford module bundle and E is a vector bundle. Show that $S \otimes E$ also is a Clifford module bundle.

Proposition 6.8. *Suppose M is oriented.*

1. *If $n = 0 \mod 4$, then every Clifford module bundle S splits according to the action of the volume element $\omega = e_1 \cdots e_n \in C\ell(M)$ as*

$$S = S^+ \oplus S^-.$$

The Clifford multiplication exchanges S^+ and S^-.

2. *If $n = 0 \mod 2$, then every complex Clifford module bundle S splits according to the action of the complex volume element $\omega = i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdots e_n \in C\ell(M) \otimes \mathbf{C}$ as*

$$S = S^+ \oplus S^-.$$

The Clifford multiplication exchanges S^+ and S^-.

7 Dirac bundles and Dirac operators

Definition 7.1. A **Dirac bundle** is a Clifford module bundle S together with an inner product $\langle \cdot, \cdot \rangle$ and a covariant derivative $\nabla = \nabla_S \colon \Gamma(S) \to \Omega^1(M, S)$ satisfying

$$\langle \gamma(v)s, t \rangle + \langle s, \gamma(v)t \rangle = 0 \quad \text{and} \quad \mathrm{d}\langle s, t \rangle = \langle \nabla_S s, t \rangle + \langle s, \nabla_S t \rangle$$

as well as

$$\nabla_S(xs) = (\nabla_{C\ell} x)s + x(\nabla_S s).$$

A **complex Dirac bundle** is Dirac bundle where S is a complex vector bundle, the complex structure I is orthogonal with respect to $\langle \cdot, \cdot \rangle$, and ∇_S is complex linear.

Exercise 7.2. Show that if S is a (complex) Dirac bundle and E is a Euclidean (Hermitian) vector bundle with a compatible connection ∇_E, then $S \otimes E$ is a (complex) Dirac bundle.

Remark 7.3. The first identity above means that $\gamma(v)^* = -\gamma(v)$ with respect to $\langle \cdot, \cdot \rangle$. The second means that the map $C\ell(E) \to \mathrm{End}(S)$ is parallel with respect to $\nabla_{C\ell}$ and ∇_S.

Exercise 7.4. Let S be a Dirac bundle. Let $(e_1, \ldots, e_n)$ be a local orthonormal frame of TM and s a local section of S. The expression

$$Ds = \sum_{i=1}^{n} \gamma(e_i) \nabla_{S, e_i} s$$

does not depend on the choice of $(e_1, \ldots, e_n)$.

Proof. Since this is a crucial point, let me give the proof. If $(f_1, \ldots, f_n)$ is another local orthonormal frame, then there is an orthogonal matrix $A = (a_{ij})$ such that $f_i = Ae_i$. Therefore, using $\sum_i^n a_{ij} a_{ik} =$

$(A^t A)_{jk} = \delta_{jk}$, we have

$$\sum_{i=1}^{n} \gamma(f_i)\nabla_{S,f_i} = \sum_{i,j,k=1}^{n} a_{ij}a_{ik}\gamma(e_j)\nabla_{S,e_k} = \sum_{j,k=1}^{n} \delta_{jk}\gamma(e_j)\nabla_{S,e_k} = \sum_{i=1}^{n} \gamma(e_i)\nabla_{S,e_i}.$$

□

Remark 7.5. There is a slicker looking argument which says that this is just $\gamma(\nabla_S s)$. There is a secret identification $TM = T^*M$ in this argument; that is, one considers the Clifford multiplication as a map $T^*M \to \mathrm{End}(S)$. To justify this one uses the trace and proving yields the same definition of D involves the invariance of the trace, which is of course proved by the above argument. If one really wants to avoid the above computation, then one should define $C\ell(M) = C\ell(T^*M)$. This is probably "the right thing", but let us not bother with such details.

Definition 7.6. The **Dirac operator** associated with a Dirac bundle S is the differential operator $D\colon \Gamma(S) \to \Gamma(S)$ defined by

$$Ds := \sum_{i=1}^{n} \gamma(e_i)\nabla_{S,e_i}.$$

Example 7.7. $S = \Lambda TM$ with its natural Euclidean metric and covariant derivative is a Dirac bundle. The corresponding Dirac operator is

$$D = \mathrm{d} + \mathrm{d}^*\colon \Lambda TM \to \Lambda TM.$$

Proposition 7.8. *Suppose M is oriented. With respect to the splittings from Proposition 6.8 the following hold.*

1. *Let S be a Dirac bundle. If $n = 0 \mod 4$, then $D\colon \Gamma(S^+ \oplus S^-) \to \Gamma(S^+ \oplus S^-)$ decomposes as*

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

2. *Let S be a complex Dirac bundle. If $n = 0 \mod 2$, then $D\colon \Gamma(S^+ \oplus S^-) \to \Gamma(S^+ \oplus S^-)$ decomposes as*

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

Proposition 7.9. *We have*

$$\langle Ds,t\rangle - \langle s,Dt\rangle = \operatorname{div} V \quad \textit{with} \quad V := \sum_{i=1}^{n} \langle \gamma(e_i)s,t\rangle e_i$$

In particular, if M is a compact manifold with boundary and ν denotes outward pointing unit normal, then

$$\langle Ds,t\rangle_{L^2} - \langle s,Dt\rangle_{L^2} = \int_{\partial M} \langle \gamma(\nu)s,t\rangle;$$

and every $D\colon \Gamma(S) \to \Gamma(S)$ Dirac operator is formally self-adjoint, that is, if $s,t \in \Gamma(S)$ are compactly supported, then

$$\langle Ds,t\rangle_{L^2} = \langle s,Dt\rangle_{L^2}.$$

Proof. Let $(e_1,\dots,e_n)$ be a local orthonormal frame. We compute

$$\begin{aligned}\langle Ds,t\rangle &= \sum_{i=1}^{n} \langle \gamma(e_i)\nabla_{S,e_i}s,t\rangle \\ &= \sum_{i=1}^{n} \langle \nabla_{S,e_i}(\gamma(e_i)s),t\rangle - \langle \gamma(\nabla_{\mathrm{LC},e_i}e_i)s),t\rangle \\ &= \sum_{i=1}^{n} \langle s,\gamma(e_i)\nabla_{S,e_i}t\rangle + \partial_{e_i}\langle \gamma(e_i)s,t\rangle - \langle \gamma(\nabla_{\mathrm{LC},e_i}e_i)s,t\rangle.\end{aligned}$$

Since

$$\begin{aligned}\operatorname{div} V &= \sum_{i,j=1}^{n} \langle \nabla_{\mathrm{LC},e_i}\langle \gamma(e_j)s,t\rangle e_j, e_i\rangle \\ &= \sum_{i=1}^{n} \partial_{e_i}\langle \gamma(e_i)s,t\rangle + \langle \gamma(e_j)s,t\rangle\langle \nabla_{\mathrm{LC},e_i}e_j,e_i\rangle \\ &= \sum_{i=1}^{n} \partial_{e_i}\langle \gamma(e_i)s,t\rangle - \langle \gamma(e_j)s,t\rangle\langle e_j,\nabla_{\mathrm{LC},e_i}e_i\rangle \\ &= \sum_{i=1}^{n} \partial_{e_i}\langle \gamma(e_i)s,t\rangle - \langle \gamma(\nabla_{\mathrm{LC},e_i}e_i)s,t\rangle,\end{aligned}$$

the assertion follows. □

Exercise 7.10. If D is a Dirac operator on S, $s \in \Gamma(S)$ and $f \in C^\infty(M)$, then

$$D(fs) = \gamma(\nabla f)s + fDs.$$

8 Spin structures, spinor bundles, and the Atiyah–Singer operator

It is a natural question, whether a given manifold admits a Clifford module bundle with irreducible fibres. By Proposition 4.30 this question is tightly related to the existence of pin structures. Assuming the underlying manifold is oriented, this itself is essentially the same as the existence of spin structures. This is why we will directly go to spin structures and skip pin structures.

Definition 8.1. Let E be an oriented Euclidean vector bundle of rank r over M. A **spin structure** on E is a principal $\mathrm{Spin}(r)$–bundle $\mathfrak{s}$ over M together with an isomorphism

$$\mathfrak{s} \times_{\mathrm{Spin}(n)} \mathrm{SO}(n) \cong \mathrm{SO}(E).$$

Definition 8.2. A **spin structure** on a Riemannian manifold is a spin structure on TM. A **spin manifold** is a Riemannian manifold with a choice of spin structure.

Definition 8.3. If $\mathfrak{s}$ is a spin structure and S is the spinor representation, then we denote by S the associated bundle

$$\mathfrak{s} \times_{\mathrm{Spin}(n)} S.$$

We call S the **spinor bundle** associated with $\mathfrak{s}$. If $\mathfrak{s}$ is the spinor bundle of a spin manifold, we also write $\not{S}$ for the spinor bundle.

A section of $\not{S}$ is called a **spinor field** or, simply, **spinor**.

The structure of the spinor representation described Proposition 4.35 induces corresponding structures on S. In particular, S comes with an Euclidean metric with respect to which the Clifford multiplication $\gamma\colon E \to \mathrm{End}(S)$ is skew-adjoint. Suppose M is a spin manifold and S is the corresponding spinor bundle. What is missing in order to make S into a Dirac bundle is a covariant derivative compatible with the inner product and the Clifford multiplication. Before discussing this point in detail, we will address the existence question for spin structures.

8.1 Existence of spin structures

We begin by reviewing the axiomatic definition of the second Stiefel–Whitney class.

Definition 8.4. Let E be a real vector bundle of rank r over a topological space X. Let $k \in \mathbf{N}_0$. The **second Stiefel–Whitney class** is the unique class $w_2(E) \in H^2(X, \mathbf{Z}_2)$ such that:

1. If $E \to B\mathrm{SO}(r)$ is the universal bundle over $B\mathrm{SO}(r)$, then $w_2(E) \neq 0 \in H^2(B\mathrm{SO}(r), \mathbf{Z}_2) \cong \mathbf{Z}_2$.
2. If $f\colon X \to Y$ is continuous, then

$$w(f^*E) = f^*w(E).$$

Proposition 8.5.

1. *E admits a spin structure if and only if $w_2(E) = 0$.*

2. *If $w_2(E) = 0$, then the set of spin structures is a $H^1(M, \mathbf{Z}_2)$–torsor.*

Proof. The proof relies on the following observation.

Proposition 8.6. *There is a bijection between the set of spin structures on E and the set of 2–sheeted covers of $\mathrm{SO}(E)$ such that the restriction to a fibre of $\mathrm{SO}(E)$ is a non-trivial cover.*

Proof. The isomorphism $\mathfrak{s} \times_{\mathrm{Spin}(n)} \mathrm{SO}(n) \cong \mathrm{SO}(E)$ defines a 2–sheeted covering map $\underline{\widetilde{\mathrm{Ad}}}$ via $\mathfrak{s} \to \mathfrak{s} \times \{1\} \to \mathfrak{s} \times_{\mathrm{Spin}(n)} \mathrm{SO}(n) \cong \mathrm{SO}(E)$; moreover, $\underline{\widetilde{\mathrm{Ad}}}$ satisfies $\underline{\widetilde{\mathrm{Ad}}}(xg) = \underline{\widetilde{\mathrm{Ad}}}(x)\,\widetilde{\mathrm{Ad}}(g)$. Conversely, any $\underline{\widetilde{\mathrm{Ad}}}$ gives rise to and isomorphism $\mathfrak{s} \times_{\mathrm{Spin}(n)} \mathrm{SO}(n) \cong \mathrm{SO}(E)$. The proposition now follows by observing that the cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is non-trivial. □

The set of 2–sheeted covers of $\mathrm{SO}(E)$ is identified with $H^1(\mathrm{SO}(E), \mathbf{Z}_2)$. The bijection is given by the monodromy. Associated to the fibration $\mathrm{SO}(E) \to M$ with fibre $\mathrm{SO}(n)$ there is an exact sequence

$$0 \to H^1(M, \mathbf{Z}_2) \xrightarrow{\pi^*} H^1(\mathrm{SO}(E), \mathbf{Z}_2) \xrightarrow{\mathrm{res}} H^1(\mathrm{SO}(n), \mathbf{Z}_2) \xrightarrow{\beta_E} H^2(M, \mathbf{Z}_2).$$

If $\xi \in H^1(\mathrm{SO}(E), \mathbf{Z}_2)$ is a 2–sheeted cover of $\mathrm{SO}(E)$, then $\mathrm{res}(\xi)$ is its restriction to $H^1(\mathrm{SO}(n), \mathbf{Z}_2) = \mathbf{Z}_2$. If ξ corresponds to a spin structure on E, then $\mathrm{res}(\xi)$ must be non-trivial, i.e., $\mathrm{res}(\xi) = [-1]$. Since the above sequence is exact, we must have $\beta_E([-1]) = 0$. Conversely, if $\beta_E([-1]) = 0$, then such an ξ exists. Moreover, the set of such ξ is $H^1(M, \mathbf{Z}_2)$–torsor.

It remains to identify $\beta_E([-1])$ with $w_2(E)$. Since $\beta_E([-1])$ is clearly natural, one only needs to verify by direct computation that $w_2(E) \neq 0$ for $E \to B\mathrm{SO}(r)$ the universal bundle over $B\mathrm{SO}(r)$. This completes the proof. □

Proof using Čech cohomology. For G equal to $\mathbf{Z}_2$, $\mathrm{Spin}(r)$, or $\mathrm{SO}(r)$, denote by $\underline{G}$ the sheaf of continuous maps to G. The exact sequence

$$0 \to \mathbf{Z}_2 \to \mathrm{Spin}(r) \to \mathrm{SO}(r) \to 0$$

of groups induces a corresponding exact sequence of sheaves. Since $\mathrm{SO}(r)$ is connected, $\check{H}^0(M, \underline{\mathrm{SO}(r)}) = \{0\}$. Hence, the above yields the following exact sequence of Čech cohomology groups:

$$0 \to \check{H}^1(M, \underline{\mathbf{Z}_2}) \to \check{H}^1(M, \underline{\mathrm{Spin}(r)}) \to \check{H}^1(M, \underline{\mathrm{SO}(r)}) \xrightarrow{\beta} \check{H}^2(M, \underline{\mathbf{Z}_2}).$$

E corresponds to an element in $\check{H}^1(M, \underline{\mathrm{SO}(r)})$, which we also denote by E. A spin structure on E corresponds to an element of $\check{H}^1(M, \underline{\mathrm{Spin}(r)})$ mapping to E. By exactness of the above sequence, the obstruction to the existence of such an element is precisely $\beta(E)$ and the set of such elements is an torsor over $\check{H}^1(M, \underline{\mathbf{Z}_2})$.

Since $\mathbf{Z}_2$ is discrete, $\underline{\mathbf{Z}_2}$ is the sheaf of locally constant sections of $\mathbf{Z}_2$. Therefore, $\check{H}^1(M, \underline{\mathbf{Z}_2}) = H^1(M, \mathbf{Z}_2)$.

It remains to identify $\beta(E)$ with $w_2(E)$. This follows from the naturality $\beta(E)$ and checking that $\beta(E)$ is non-trivial for the universal bundle over $B\mathrm{SO}(n)$. □

Remark 8.7. Recall, that E being orientable means is equivalent to $w_1(E) = 0$ is equivalent to f^*E being trivial for every continuous map $f\colon S^1 \to M$. E admitting a spin structure is equivalent to f^*E being trivial for all continuous map from any compact surface to M.

Theorem 8.8. *If M is an orientable 3-manifold, then $w_2(M) = 0$.*

8.2 Connections on spinor bundles

Proposition 8.9. *Given any metric covariant derivative ∇_E on E, there exists a unique metric covariant derivative on S such that*

$$\nabla_S(\gamma(v)s) = \gamma(\nabla_E v)s + \gamma(v)\nabla_S s.$$

Sketch of proof of Proposition 8.9. A metric covariant derivative on E is equivalent to a connection the principal bundle $\mathrm{SO}(E)$. The relation is as follows. Suppose $\mathscr{E} = (e_1, \dots, e_n)$ is a local section of $\mathrm{SO}(E)$. The covariant derivative on E induced by θ is defined by the rule

$$\nabla e_i = (\mathscr{E}^*\theta)(e_i).$$

A connection on $\mathrm{SO}(E)$ is encoded by a $\mathrm{SO}(n)$-equivariant 1-form

$$\theta \in \Omega^1(\mathrm{SO}(E), \mathfrak{so}(r))$$

which restricts to the Maurer–Cartan form on the fibres. The desired covariant derivative on S is equivalent to a connection the principal bundle $\mathfrak{s}$ such that the corresponding $\mathrm{Spin}(n)$-equivariant 1-form $\tilde{\theta} \in \Omega^1(\mathfrak{s}, \mathfrak{spin}(r))$ satisfies

$$\tilde{\theta} = \xi^*\theta$$

with $\xi\colon \mathfrak{s} \to \mathrm{SO}(E)$ denoting the covering map induced by $\mathfrak{s} \times_{\mathrm{Spin}(n)} \mathrm{SO}(n) \cong \mathrm{SO}(E)$ and under the identification $\mathfrak{spin}(r) = \Lambda^2\mathbf{R}^{\oplus r} = \mathfrak{so}(r)$. □

Remark 8.10. If the connection 1–form of ∇_E is given by

$$\theta_E = \sum_{i,j} \theta_{ij} e_i e^j$$

then the connection 1–form of ∇_S is given by

$$\theta_S = \frac{1}{4} \sum_{i,j} \theta_{ij} \cdot e_i \wedge e_j,$$

If we define ${F^E_{ijk}}^{\ell}$ by

$$F_E(e_i, e_j)e_k = \sum_{\ell} {F^E_{ijk}}^{\ell} e_\ell,$$

then F^S the curvature of ∇_S is given by

$$F_S = \frac{1}{4} \sum_{k,\ell} R^{\ell}_{ijk} \gamma(e^k)\gamma(e^\ell).$$

8.3 The Atiyah–Singer operator

Definition 8.11. If M is a spin manifold, then by the preceding discussion the spinor bundle $\not{S}$ naturally is a Dirac bundle. The associated Dirac operator $\not{D}$ is called the **Atiyah–Singer operator**.

Definition 8.12. A spinor $\Phi \in \Gamma(\not{S})$ is called **harmonic** if $\not{D}\Phi = 0$.

8.4 Universality of spinor bundles

Proposition 8.13. *Suppose M is a spin manifold and denote by $\not{S}$ its spinor bundle. Denote by D the commuting algebra for the spin representation of* $\mathrm{Spin}(\dim M)$. *Given any Dirac bundle S over M, there exists a unique Euclidean vector bundle (E, h) over M together with a metric connection such that*

$$S = \not{S} \otimes_D E$$

as Dirac bundles.

Proof. Take $E = \mathrm{Hom}_{C\ell(M)}(\not{S}, S)$. □

8.5 Spin^c structures

The condition to admit a spin structure is somewhat restrictive. One could be interested in a slightly weaker version.

Definition 8.14. Let E be an oriented Euclidean vector bundle of rank r over M. A **spin^c structure** on E is a principal $\mathrm{Spin}^c(r)$–bundle $\mathfrak{w}$ over M together with an isomorphism

$$\mathfrak{w} \times_{\mathrm{Spin}^c(n)} \mathrm{SO}(n) \cong \mathrm{SO}(E).$$

Definition 8.15. A **spin^c structure** on a Riemannian manifold is a spin structure on TM. A **spin^c manifold** is a Riemannian manifold with a choice of spin^c structure.

Definition 8.16. If $\mathfrak{w}$ is a spin^c structure and S is the complex spinor representation, then we denote by W the associated bundle

$$\mathfrak{w} \times_{\mathrm{Spin}^c(n)} W.$$

We call W the **spinor bundle** associated with $\mathfrak{w}$. Moreover, the **characteristic line bundle** associated with $\mathfrak{w}$ is the complex line bundle

$$L := \mathfrak{w} \times_{\mathrm{Spin}^c(n)} \mathbf{C}$$

associated with the representation in which $[x, z] \in \mathrm{Spin}^c(r) = \mathrm{Spin}(r) \times_{\mathbf{Z}_2} \mathbf{C}$ acts as z^2.

Definition 8.17. Denote by $\beta_2 \colon H^k(M, \mathbf{Z}_2) \to H^{k+1}(M, \mathbf{Z})$ the Bockstein homomorphism induced by the exact sequence $0 \to \mathbf{Z} \xrightarrow{2\times} \mathbf{Z} \to \mathbf{Z}_2 \to 0$. We define

$$W_{k+1}(E) := \beta_2 w_k(E).$$

Proposition 8.18.

1. *M admits a spin^c structure if and only if $w_2(M) \in \mathrm{im}(H^2(M, \mathbf{Z}) \to H^2(M, \mathbf{Z}_2)$ if and only if $W_3(M) = 0$.*
2. *If M admits a spin^c structure, then the set of spin^c structures is a torsor over $H^2(M, \mathbf{Z})$.*

Proof. For G a topological group, denote by $\underline{G}$ the sheaf of continuous maps to G. The exact sequence

$$0 \to \mathrm{U}(1) \to \mathrm{Spin}^c(r) \to \mathrm{SO}(r) \to 0$$

of groups induces a corresponding exact sequence of sheaves. Since $\mathrm{SO}(r)$ is connected, $\check{H}^0(M, \underline{\mathrm{SO}(r)}) = \{0\}$. Hence, the above yields the following exact sequence of Čech cohomology groups:

$$0 \to \check{H}^1(M, \underline{\mathrm{U}(1)}) \to \check{H}^1(M, \underline{\mathrm{Spin}^c(r)}) \to \check{H}^1(M, \underline{\mathrm{SO}(r)}) \xrightarrow{\beta} \check{H}^2(M, \underline{\mathrm{U}(1)}).$$

E corresponds to an element in $\check{H}^1(M, \underline{\mathrm{SO}(r)})$, which we also denote by E. A spin^c structure on E corresponds to an element of $\check{H}^1(M, \underline{\mathrm{Spin}^c(r)})$ mapping to E. By exactness of the above sequence, the obstruction to the existence of such an element is precisely $\beta(E)$ and the set of such elements is an torsor over $\check{H}^1(M, \underline{\mathrm{U}(1)})$.

The exact sequence

$$0 \to \mathbf{Z} \xrightarrow{2\pi i \times} i\mathbf{R} \xrightarrow{\exp} \mathrm{U}(1) \to 0,$$

gives rise to an exact sequence

$$\check{H}^1(M, \underline{i\mathbf{R}}) \to \check{H}^1(M, \underline{\mathrm{U}(1)}) \to \check{H}^2(M, \underline{\mathbf{Z}}) \to \check{H}^2(M, \underline{i\mathbf{R}}).$$

Since $\underline{i\mathbf{R}}$ is soft, $\check{H}^k(M, \underline{i\mathbf{R}})$ for all $k > 0$. Therefore,

$$\check{H}^k(M, \mathrm{U}(1)) = \check{H}^{k+1}(M, \underline{\mathbf{Z}}) = H^{k+1}(M, \mathbf{Z})$$

for all $k > 0$.

It remains to identify $\beta(E) \in \check{H}^2(M, \mathrm{U}(1)) = \check{H}^3(M, \mathbf{Z})$ as $W_3(E)$. There is a commutative diagram

$$\begin{array}{ccccc}
 & & \check{H}^1(M, \underline{\mathrm{Spin}(r)}) & & \\
 & & \downarrow & & \\
\check{H}^1(M, \underline{\mathrm{Spin}^c(r)}) & \longrightarrow & \check{H}^1(M, \underline{\mathrm{SO}(r)}) & \longrightarrow & \check{H}^2(M, \underline{\mathrm{U}(1)}) \\
 & & \downarrow & & \downarrow{=} \\
 & & \check{H}^2(M, \underline{\mathbf{Z}_2}) & \longrightarrow & \check{H}^2(M, \underline{\mathrm{U}(1)}).
\end{array}$$

Given this and the fact that $w_2(E)$ is the image of E under the map $\check{H}^1(M, \underline{\mathrm{SO}(r)}) \to \check{H}^2(M, \underline{\mathbf{Z}_2})$, we only need to prove that

$$H^2(M, \mathbf{Z}_2) = \check{H}^2(M, \underline{\mathbf{Z}_2}) \to \check{H}^2(M, \underline{\mathrm{U}(1)}) = H^3(M, \mathbf{Z})$$

agrees with β_2. This can be proved by a diagram chase. Specifically, one considers the exact sequence

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow \mathrm{U}(1) \xrightarrow{(-)^2} \mathrm{U}(1) \longrightarrow 1$$

and proves that the diagram

$$\begin{array}{ccccc}
H^k(M, \mathbf{Z}_2) & \longrightarrow & \check{H}^k(M, \underline{U(1)}) & \xrightarrow{(-)^2} & \check{H}^k(M, \underline{U(1)}) \\
\downarrow & & \downarrow{\cong} & & \downarrow{\cong} \\
H^k(M, \mathbf{Z}_2) & \longrightarrow & H^{k+1}(M, \mathbf{Z}) & \xrightarrow{2\times} & H^{k+1}(M, \mathbf{Z}).
\end{array}$$

commutes. □

Remark 8.19. The obstruction to admitting a spin^c–structure is that $w_2(E)$ lifts to an integral class. This holds for the tangent bundle of any orientable 4–manifold.

Remark 8.20. It is not uncommon to see the characteristic line bundle called the **determinant bundle**. The reason for that is that if M is an spin 4–manifold, then

$$L = \Lambda^2 W^+ = \Lambda^2 W^-.$$

Similary, if M is an spin 3–manifold, then

$$L = \Lambda^2 W.$$

If one is not talking exclusively about 3– or 4–manifold, one should not call L the determinant line bundle.

Proposition 8.21. *Given a metric covariant derivative on E and a metric covariant derivative on L, there exits a unique covariant derivative on S which makes the Clifford multiplication parallel and which induces the given covariant derivative on L.*

Remark 8.22. The fact that the spin connection depends on the choice of a connection on L, is important in the formulation of the classical Seiberg–Witten equation.

9 Weitzenböck formulae

Definition 9.1. Given a Dirac bundle S, we denote by $F_S \in \Omega^2(M, \mathfrak{so}(S))$ the curvature of ∇_S. Define $\mathscr{F}_S \in \Gamma(\mathrm{End}(S))$ by

$$\mathscr{F}_S = \frac{1}{2} \sum_{i,j=1}^{n} \gamma(e_i)\gamma(e_j)F_S(e_i, e_j).$$

Proposition 9.2 (Weitzenböck formula for Dirac bundles).

$$D^2 = \nabla_S^* \nabla_S + \mathscr{F}_S.$$

Proof. We pick a local orthonormal frame $(e_1, \ldots, e_n)$ around a point $x \in M$ such that at x we have $\nabla e_i = 0$. At the point $x \in M$, we compute

$$\begin{aligned} \sum_{i,j=1}^{n} \gamma(e_i)\nabla_{S,e_i}\gamma(e_j)\nabla_{S,e_j} &= \sum_{i,j=1}^{n} \gamma(e_i)\gamma(e_j)\nabla_{S,e_i}\nabla_{S,e_j} \\ &= -\sum_{i=1}^{n} \nabla_{S,e_i}\nabla_{S,e_i} + \sum_{i<j}^{n} \gamma(e_i)\gamma(e_j)[\nabla_{S,e_i}, \nabla_{S,e_j}] \\ &= \nabla_S^*\nabla_S + \sum_{i<j}^{n} \gamma(e_i)\gamma(e_j)F_s(e_i, e_j). \end{aligned}$$

□

Proposition 9.3 (Bochner). *If M is compact and $\mathscr{F}_S$ is non-negative definite (that is: $\langle \mathscr{F}_S\Phi, \Phi\rangle \geqslant 0$), then $D\Phi = 0$ implies $\nabla_S\Phi = 0$. Moreover, $\mathscr{F}_S$ is positive definite somewhere, then $\Phi = 0$.*

Proof. If $D\Phi = 0$, then we have

$$\int_M |\nabla\Phi|^2 + \langle \mathscr{F}_S\Phi, \Phi\rangle = 0.$$

□

The usefulness of Proposition 9.2 and Proposition 9.3 crucially depends on being able to understand what $\mathscr{F}_S$ is. In the following we will try to better understand F_S and, hence, $\mathscr{F}_S$.

Definition 9.4. Let S be a Dirac bundle. Define $R_S \in \Omega^2(M, \mathfrak{so}(S))$ by

$$R_S(v,w) := \frac{1}{4}\sum_{i,j=1}^n \gamma(e_i)\gamma(e_j)\langle R(v,w)e_i, e_j\rangle.$$

Proposition 9.5. *Let S be a Dirac bundle. Denote by $F_S \in \Omega^2(M, \mathrm{End}(S))$ the curvature of ∇_S. There is an $F_S^{\mathrm{tw}} \in \Omega^2(M, \mathfrak{so}(S))$ which commutes with Clifford multiplication such that*

$$F_S = R_S + F_S^{\mathrm{tw}}.$$

Proof. This result follows from the following two propositions.

Proposition 9.6. *Let S be a Dirac bundle. Denote by $F_S \in \Omega^2(M, \mathrm{End}(S))$ the curvature of ∇_S. We have*

$$[F_S(u,v), \gamma(w)] = \gamma(R(u,v)w).$$

Proof. We pick a local orthonormal frame $(e_1, \dots, e_n)$ around a point $x \in M$ such that at x we have $\nabla e_i = 0$. At the point $x \in M$, we compute

$$\begin{aligned}
[F_S(e_i,e_j), \gamma(e_k)] &= [[\nabla_{S,e_i}, \nabla_{S,e_j}], \gamma(e_k)] \\
&= [\nabla_{S,e_i}, [\nabla_{S,e_j}, \gamma(e_k)]] - [\nabla_{S,e_j}, [\nabla_{S,e_i}, \gamma(e_k)]] \\
&= [\nabla_{S,e_i}, \gamma(\nabla_{e_j}e_k)] - [\nabla_{e_j}, \gamma(\nabla_{e_i}e_k)] \\
&= \gamma(\nabla_{e_i}\nabla_{e_j}e_k) - \gamma(\nabla_{e_j}\nabla_{e_i}e_k) \\
&= \gamma(R(e_i,e_j)e_k).
\end{aligned}$$

□

Proposition 9.7. *Let S be a Dirac bundle. We have*

$$[R_S(u,v), \gamma(w)] = \gamma(R(u,v)w).$$

Proof. We pick a local orthonormal frame $(e_1, \dots, e_n)$ and compute

$$
\begin{aligned}
[R_S(e_k, e_\ell), \gamma(e_m)] &= \frac{1}{4} \sum_{i,j=1}^{n} \langle R(e_k, e_\ell)e_i, e_j\rangle [\gamma(e_i)\gamma(e_j), \gamma(e_m)] \\
&= \frac{1}{4} \sum_{i,j=1}^{n} \langle R(e_k, e_\ell)e_i, e_j\rangle (\gamma(e_i)\gamma(e_j)\gamma(e_m) - \gamma(e_m)\gamma(e_i)\gamma(e_j)).
\end{aligned}
$$

We have

$$
\gamma(e_i)\gamma(e_j)\gamma(e_m) - \gamma(e_m)\gamma(e_i)\gamma(e_j) = \begin{cases} 0 & \text{if } i = j, \\ 0 & \text{if } i, j, m \text{ are pairwise distinct,} \\ 2\gamma(e_j) & \text{if } i \neq j \text{ and } i = m, \\ -2\gamma(e_i) & \text{if } i \neq j \text{ and } j = m. \end{cases}
$$

Therefore,

$$
\begin{aligned}
[R_S(e_k, e_\ell), \gamma(e_m)] &= \frac{1}{2} \sum_{j=1}^{n} \langle R_S(e_k, e_\ell)e_m, e_j\rangle \gamma(e_j) - \frac{1}{2} \sum_{i=1}^{n} \langle R_S(e_k, e_\ell)e_i, e_m\rangle \gamma(e_i) \\
&= \sum_{j=1}^{n} \langle R_S(e_k, e_\ell)e_m, e_j\rangle \gamma(e_j).
\end{aligned}
$$

□

Given the above, simply define

$$
F_S^{\text{tw}} := F_S - R_S.
$$

□

Definition 9.8. The **Ricci curvature** of g is

$$
\operatorname{Ric}(v, w) := \sum_{i=1}^{n} \langle R(e_i, v)w, e_i\rangle
$$

and the **scalar curvature** of g is

$$
\operatorname{scal}_g := \sum_{i=1}^{n} \operatorname{Ric}(e_i, e_i).
$$

Exercise 9.9. Prove that $\operatorname{Ric}(v, w) = \operatorname{Ric}(w, v)$.

Proposition 9.10 (Weitzenböck formula for Dirac Bundles, II). *With*

$$
\mathscr{F}_S^{\text{tw}} = \frac{1}{2} \sum_{i,j=1}^{n} \gamma(e_i)\gamma(e_j)F_S^{\text{tw}}(e_i, e_j)
$$

and with scal_g *denoting the scalar curvature of* g*, we have*

$$
D^2 = \nabla_S^* \nabla_S + \frac{1}{4}\operatorname{scal}_g + \mathscr{F}_S^{\text{tw}}.
$$

Remark 9.11. Why is this better than Proposition 9.2? We know that $\mathscr{F}_S^{\mathrm{tw}}$ is $\mathrm{C}\ell(M)$–linear this strongly restricts what $\mathscr{F}_S^{\mathrm{tw}}$ could possibly be and, sometimes, makes easy to work out what it actually is.

The proof relies on the following computation. On first sight the computation looks off-putting, but the result of the computation is of fundamental importance and will be used repeatedly later.

Proposition 9.12. *We have*

$$\sum_{i,j,\ell=1}^{n} \gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k,e_\ell)e_i,e_j\rangle = -2\sum_{i=1}^{n}\gamma(e_i)\mathrm{Ric}(e_k,e_i)$$

and

$$\sum_{i,j,k,\ell=1}^{n} \gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k,e_\ell)e_i,e_j\rangle = 2\mathrm{scal}_g.$$

Proof. The first identity implies the second directly.

If i, j, ℓ are pairwise distinct, then

$$\gamma(e_\ell)\gamma(e_i)\gamma(e_j) = \gamma(e_i)\gamma(e_j)\gamma(e_\ell) = \gamma(e_j)\gamma(e_\ell)\gamma(e_i).$$

By the algebraic Bianchi identity

$$\langle R(e_k,e_\ell)e_i,e_j\rangle + \langle R(e_k,e_i)e_j,e_\ell\rangle + \langle R(e_k,e_j)e_\ell,e_i\rangle = 0.$$

Thus the sum of terms with i, j, ℓ pairwise distinct appearing the the left-hand side vanishes. The terms with $i = j$ vanish because $R(e_k,e_\ell)$ is skew-symmetric.

If $i \neq j = \ell$, then

$$\gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k,e_\ell)e_i,e_j\rangle = \gamma(e_i)\langle R(e_k,e_j)e_i,e_j\rangle = -\gamma(e_i)\langle R(e_j,e_k)e_i,e_j\rangle.$$

The sum of these expressions contributes

$$-\sum_{i=1}^{n}\gamma(e_i)\mathrm{Ric}(e_k,e_i)$$

to the left-hand side. If $j \neq i = \ell$, then

$$\gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k,e_\ell)e_i,e_j\rangle = -\gamma(e_j)\langle R(e_i,e_k)e_j,e_i\rangle$$

The sum of these expressions also contributes

$$-\sum_{i=1}^{n}\gamma(e_i)\mathrm{Ric}(e_k,e_i)$$

to the left-hand side. □

Proof of Proposition 9.10. Given an orthonormal frame $(e_1, \dots, e_n)$, by the previous proposition we have

$$\frac{1}{2}\sum_{k,\ell=1}^{n} \gamma(e_k)\gamma(e_\ell)R_S(e_k, e_\ell) = \frac{1}{8}\sum_{i,j,k,\ell=1}^{n} \gamma(e_k)\gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k, e_\ell)e_i, e_j\rangle$$
$$= \frac{1}{4}\mathrm{scal}_g. \qquad \square$$

Proposition 9.13. *If $S = \$$ is the spinor bundle, then*

$$F_S = R_S.$$

In particular,

$$D^2 = \nabla_S^*\nabla_S + \frac{1}{4}\mathrm{scal}_g.$$

Therefore, if $\mathrm{scal}_g \geqslant 0$, then every harmonic spinor is parallel; if scal_g is positive somewhere, then harmonic spinors must vanish.

Proof. The twisting curvature F_S^{tw} is 2–form with values in skew-symmetric endomorphisms of $\$$ which commute with the Clifford multiplication. Since $\$$ arises from an irreducible representation, by Schur's Lemma an endomorphism of $\$$ commuting with the Clifford multiplication must be a scalar. A skew-symmetric scalar vanishes. This shows that $F_S^{\mathrm{tw}} = 0$. $\square$

Alternative proof. One can proof directly that $F_S = R_S$ using Proposition 4.25. $\square$

Exercise 9.14. If $S = W$ is a complex spinor bundle, associated to a spin^c–structure prove that $F_S^{\mathrm{tw}} \in \Omega^2(M, i\mathbf{R})$. Identify F_S^{tw} in terms of the curvature of the connection on the characteristic line bundle L. More precisely, prove that $F_S^{\mathrm{tw}} = \frac{1}{2}F_A$ where F_A denotes the curvature of the connection on L.

10 Parallel spinors and Ricci flat metrics

Proposition 10.1 (cf. Hitchin [Hit74, Theorem 1.2]). *Let M be a spin manifold. If there exists a non-zero spinor $\Phi \in \Gamma(\$)$ such that*

$$\nabla\Phi = 0,$$

then M is Ricci flat.

Remark 10.2. This is well-known among physicists, because non-zero parallel spinor are closely related to super symmetry.

Proof. Since Ric is a symmetric tensor, we can chose a local orthonormal frame and functions $\lambda_1, \dots, \lambda_n$ such that

$$\mathrm{Ric}(e_i, e_j) = \lambda_i\delta_{ij}.$$

If Φ is parallel, then in particular $R_S\Phi = 0$. By Definition 9.4 and Proposition 9.12, this means that

$$\begin{aligned}
0 &= \sum_{\ell=1}^{n} \gamma(e_\ell) R_S(e_k, e_\ell)\Phi \\
&= \frac{1}{4} \sum_{i,j,\ell=1}^{n} \gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k, e_\ell)e_i, e_j\rangle\Phi \\
&= -\frac{1}{2} \sum_{i=1}^{n} \gamma(e_i)\mathrm{Ric}(e_k, e_i)\Phi \\
&= -\frac{1}{2}\lambda_k \gamma(e_k)\Phi.
\end{aligned}$$

It follows that $\lambda_1 = \cdots = \lambda_n = 0$ and therefore $\mathrm{Ric} = 0$. □

All known Ricci flat manifold have special holonomy, that is, $\mathrm{Hol}(g)$ is a strict subgroup of $\mathrm{SO}(n)$. It is a famous open question whether there are any compact Ricci-flat manifolds with $\mathrm{Hol}(g) = \mathrm{SO}(n)$. If M admits a parallel spinor, then it is impossible that $\mathrm{Hol}(g) = \mathrm{SO}(n)$, because the holonomy group of the spin bundle must reduce to a subgroup $\mathrm{Spin}(n-1) \subset \mathrm{Spin}(n)$. The possible holonomy groups have been classified by Berger [Ber55]. The following theorem clarifies the relation between parallel spinors and special holonomy.

Theorem 10.3 (Wang [Wan89]). *Let M be a complete, simply connected, irreducible spin manifold of dimension n. Set $d := \dim\ker \not{D}$. If M is not flat, then one of the following holds:*

1. $n = 2m$, $\mathrm{Hol}(g) = \mathrm{SU}(m)$ *(that is: M is Calabi–Yau,) and* $d = 2$.
2. $n = 4m$, $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *(that is: M is hyperkähler), and* $d = m + 1$.
3. $n = 7$, $\mathrm{Hol}(g) = G_2$, *and* $d = 1$.
4. $n = 8$, $\mathrm{Hol}(g) = \mathrm{Spin}(7)$, *and* $d = 1$.

Remark 10.4 (Friedrich [Fri00, Chatper 3, Exercise 4]). For $c > 0$, the metric

$$g = \frac{x_1}{x_1 + c}(\mathrm{d}x_1)^2 + x_1^2(\mathrm{d}x_2)^2 + x_1\sin(x_2)^2(\mathrm{d}x_3)^2 + \frac{x_1 + c}{x_1}(\mathrm{d}x_4)^2$$

is Ricci flat, but does not admit a non-trivial parallel spinor.

11 Spin structures and spin^c structures on Kähler manifolds

Proposition 11.1 (Atiyah, Bott, and Shapiro [ABS64, pp. 10, 13, 14]).

1. *The map* $\rho\colon \mathrm{U}(n) \to \mathrm{SO}(2n)$ *does not lift to* $\mathrm{Spin}(2n)$.
2. *The map* $\rho \times \det\colon \mathrm{U}(n) \to \mathrm{SO}(2n) \times \mathrm{U}(1)$ *lifts to* $\mathrm{Spin}^c(n)$*; that is,*

$$\begin{array}{ccc} & & \mathrm{Spin}^c(2n) \\ & \nearrow & \big\downarrow \widetilde{\mathrm{Ad}} \times (-)^2 \\ \mathrm{U}(n) & \xrightarrow[\rho\times\det]{} & \mathrm{SO}(2n) \times \mathrm{U}(1). \end{array}$$

3. *The complex spinor representation can be identified with* $\Lambda_{\mathbf{C}}(\mathbf{C}^n)^*$ *such that the lift* $\mathrm{U}(n) \to \mathrm{Spin}^c(n)$ *makes the following diagram commutative:*

$$\begin{array}{ccc} \mathrm{U}(n) & \longrightarrow & \mathrm{Spin}^c(n) \\ \big\downarrow & & \big\downarrow \\ \mathrm{End}_{\mathbf{C}}(\mathbf{C}^n) & \xrightarrow{(-)^*\circ\Lambda} & \mathrm{End}_{\mathbf{C}}(\Lambda_{\mathbf{C}}(\mathbf{C}^n)^*). \end{array}$$

   The Clifford multiplication on $\Lambda_{\mathbf{C}}(\mathbf{C}^n)^*$ *is given by*

$$\gamma(v)\alpha = v^* \wedge \alpha - i(v)\alpha.$$

Proof. (1) is a consequence of the fact that $\pi_1(\rho)\colon \pi_1(\mathrm{U}(n)) = \mathbf{Z} \to \pi_1(\mathrm{SO}(2n)) = \mathbf{Z}_2$ is surjective, while the image of the map $\pi_1(\widetilde{\mathrm{Ad}})\colon \pi_1(\mathrm{Spin}(2n)) \to \pi_1(\mathrm{SO}(2n))$ is trivial.

(2) is proved by constructing the lift explicitly. Given $f \in \mathrm{U}(n)$, chose a unitary basis $(e_1, \dots, e_n)$ of $\mathbf{C}^n$ in which f is diagonal; that is: $f = \mathrm{diag}(e^{i\alpha_1}, \cdots, e^{i\alpha_n})$. An orthonormal basis of the $2n$–dimensional real Euclidean space $\mathbf{C}^n$ is given by $(e_1, ie_1, \dots, e_n, ie_n)$. Define $\tilde f \in \mathrm{Spin}^c(2n)$ by

$$\tilde f \coloneqq \prod_{j=1}^{n} \left[\big(\cos(\alpha_j/2) + \sin(\alpha_j/2) e_j(ie_j)\big) \times e^{\frac{i}{2}\alpha_j} \right].$$

Observe that $\alpha_j \in \mathbf{R}/2\pi\mathbf{Z}$, so $\alpha_j/2 \in \mathbf{R}/\pi\mathbf{Z}$. Consequently, the both factors individually are only defined up to a sign. Their product, however, is well-defined. Clearly, $\left(\prod_{j=1}^n e^{\frac{i}{2}\alpha_j}\right)^2 = \det(f)$. The fact that $\rho(f) = \widetilde{\mathrm{Ad}}(\tilde f)$ follows from following observation.

Proposition 11.2. *Let* (e_1, e_2) *be an orthonormal basis of* $\mathbf{R}^2$ *and let* $\alpha \in \mathbf{R}$. *We have*

$$\widetilde{\mathrm{Ad}}\left(\cos(\alpha/2) + \sin(\alpha/2)e_1e_2\right) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Proof. Since

$$\alpha\left(\cos(\alpha/2)+\sin(\alpha/2)e_1e_2\right)^{-1}=\cos(\alpha/2)-\sin(\alpha/2)e_1e_2,$$

we have

$$\begin{aligned}\widetilde{\mathrm{Ad}}\left(\cos(\alpha/2)+\sin(\alpha/2)e_1e_2\right)e_i &= \left(\cos(\alpha/2)+\sin(\alpha/2)e_1e_2\right)^2 e_i\\ &= \big(\cos(\alpha/2)^2-\sin(\alpha/2)^2\\ &\qquad +2\cos(\alpha/2)\sin(\alpha/2)e_1e_2\big)e_i\\ &= \left(\cos(\alpha)+\sin(\alpha)e_1e_2\right)e_i.\end{aligned}$$

From this the assertion follows directly. □

The formula for the Clifford multiplication defines how $\mathrm{Spin}^c(2n)$ acts on $\Lambda_{\mathbf{C}}(\mathbf{C}^n)^*$. Proving (3) is a matter of a calculation using the explicit formula for the lift constructed above. □

Remark 11.3. Recall that if V is a real vector space with a complex structure I, then we decompose

$$V\otimes_{\mathbf{R}}\mathbf{C}=V^{0,1}\oplus V^{1,0}$$

with

$$V^{1,0}:=\{v\in V\otimes\mathbf{C}:Iv=iv\}\quad\text{and}\quad V^{0,1}:=\{v\in V\otimes\mathbf{C}:Iv=-iv\}.$$

Given $v\in V$, we denote by $v^{0,1}$ and $v^{1,0}$ its projections to $V^{0,1}$ and $V^{1,0}$ respectively; more precisely,

$$v^{1,0}:=\frac{1}{2}(v-iIv)\quad\text{and}\quad v^{0,1}:=\frac{1}{2}(v+iIv).$$

If V has Hermitian metric, then with respect to the induced metric on $V\otimes_{\mathbf{R}}\mathbf{C}$, we have

$$|v^{1,0}|^2=\frac{1}{4}(|v|^2+|Iv|^2)=\frac{1}{2}|v|^2\quad\text{and}\quad|v^{0,1}|^2=\frac{1}{2}|v|^2.$$

Consequently, $v\mapsto\sqrt{2}v^{0,1}$ is an isometry.

Definition 11.4. If M is a complex manifold, then its **canonical bundle** is

$$\mathscr{K}_M=\Lambda^n_{\mathbf{C}}T^{1,0}M^*$$

and the **anti-canonical bundle** is $\mathscr{K}_M^*$.

Remark 11.5. If M is a Kähler manifold with volume form vol, then there is a pairing $(\Lambda^nT^{1,0}M^*)\otimes(\Lambda^nT^{0,1}M^*)\to\mathbf{C}$ given by

$$\alpha\otimes\beta\mapsto\frac{\alpha\wedge\beta}{\mathrm{vol}}.$$

In particular,

$$\mathscr{K}_M^*\cong\Lambda^nT^{0,1}M^*\cong\Lambda^nT^{1,0}M.$$

Proposition 11.6. *Suppose M is a Kähler manifold.*

1. *For any Hermitian line bundle L, there is a unique spin^c structure $\mathfrak{w}$ on M whose complex spinor bundle is*
$$W = \bigoplus_{k=0}^{n} \Lambda^k T^{0,1}M^* \otimes L$$
whose characteristic line bundle is $L^{\otimes 2} \otimes_{\mathbf{C}} \mathscr{K}_M^*$. We have
$$W^+ = \bigoplus_{k=0}^{\frac{n}{2}} \Lambda^{2k} T^{0,1}M^* \otimes L \quad \text{and} \quad W^- = \bigoplus_{k=0}^{\frac{(n-1)}{2}} \Lambda^{2k+1} T^{0,1}M^* \otimes L$$

2. *The Clifford multiplication on W is given by*
$$\gamma(v)\alpha = \sqrt{2}(v^{0,1})^* \wedge \alpha - \sqrt{2}i(v^{0,1})\alpha.$$

3. *If A is a Hermitian connection on L, then the corresponding connection on W induced by the Levi–Civita connection on $\Lambda^k(T^*M)^{0,1}$ is compatible with the Clifford multiplication.*

4. *If A induces a holomorphic structure $\bar{\partial}_{\mathscr{L}}$ on L (that is: $F_A^{0,2} = 0$), then*
$$D = \sqrt{2}(\bar{\partial}_{\mathscr{L}} + \bar{\partial}_{\mathscr{L}}^*) \colon \Omega^{0,\bullet}(M, \mathscr{L}) \to \Omega^{0,\bullet}(M, \mathscr{L}).$$
In particular, if M is compact, then the space of positive and negative harmonic spinors can be identified with the cohomology groups
$$\bigoplus_{k=0}^{\lfloor n/2 \rfloor} H^{2k}(M, \mathscr{L}) \quad \text{and} \quad \bigoplus_{k=0}^{\lfloor (n-1)/2 \rfloor} H^{2k+1}(M, \mathscr{L}).$$

Proof. If M is a Kähler manifold, then the structure group of TM is canonically reduced from $\mathrm{SO}(2n)$ to $\mathrm{U}(n)$. It follows from Proposition 11.1, that any Kähler manifold has a canonical spin^c structure; moreover, the complex spinor bundle is given $\bigoplus_{k=0}^{n} \Lambda^k(T^*M)^{0,1}$ and the Clifford multiplication is as asserted. It is computation to verify that the characteristic line bundle of the canonical spin^c structure is given by $\mathscr{K}_M^*$. Taking into account that the set of spin^c structures is a torsor over the group of Hermitian line bundles, the above proves (1) and (2). (3) is obvious and the first half of (4) follows by a direct computation. The second half of (4) follows by Hodge theory. □

Proposition 11.7. *A spin^c structure $\mathfrak{w}$ arises from a spin structure if and only if its characteristic line bundle is trivial. The set of spin structures inducing a fixed spin^c structure is a torsor over* $\ker(H^1(M, \mathbf{Z}_2) \to H^2(M, \mathbf{Z})$ *(that is: the group of Euclidean line bundles with trivial complexification).*

Proof. $\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbf{Z}_2} \mathrm{U}(1)$ and we have an exact sequence

$$0 \to \mathrm{Spin}(n) \to \mathrm{Spin}^c(n) \to \mathrm{U}(1) \to 0.$$

Since characteristic line bundle is associated to the representation $\mathrm{Spin}^c(n) \to \mathrm{U}(1)$, its triviality is precisely the obstruction to lifting a spin^c structure to a spin structure. This proves the first part. The second part follows by observing that any two spin structures differ by a Euclidean line bundle $\mathfrak{l}$, while any two spin^c structures differ by a Hermitian line bundle. ◻

Remark 11.8. Serre duality asserts that for a holomorphic vector bundle $\mathscr{E}$ over a compact complex manifold,

$$H^k(M, \mathscr{E}) \cong H^{n-k}(M, \mathscr{E}^* \otimes \mathscr{K}_M)^*.$$

In terms of the Dolbeault resolution, this duality is induced on chain-level by the pairing

$$\begin{aligned}(\Lambda^k T^{0,1}M^* \otimes \mathscr{E}) \otimes (\Lambda^{n-k} T^{0,1}M^* \otimes \mathscr{E}^* \otimes \mathscr{K}_M) &\cong \mathscr{K}_M^* \otimes \mathscr{K}_M \otimes \mathscr{E} \otimes \mathscr{E}^* \\ &\to \Lambda^{2n} T^*M \otimes \mathbf{C} \\ &\to \mathbf{C}.\end{aligned}$$

This pairing induces an isomorphism

$$\Lambda^k T^{0,1}M^* \otimes \mathscr{E} \cong (\Lambda^{n-k} T^{0,1}M^* \otimes \mathscr{E}^* \otimes \mathscr{K}_M)^*.$$

Using the Hermitian inner product on $\mathscr{K}_M$, we obtain an *anti-linear* isomorphism

$$\sigma\colon \Lambda^k T^{0,1}M^* \otimes \mathscr{E} \cong \Lambda^{n-k} T^{0,1}M^* \otimes \mathscr{E}^* \otimes \mathscr{K}_M.$$

In particular, if $\mathscr{L}$ is a square root of $\mathscr{K}_M$ (that is: $\mathscr{L}^{\otimes 2} \cong \mathscr{K}_M$), then

$$\sigma\colon \Lambda^k T^{0,1}M^* \otimes \mathscr{L} \cong \Lambda^{n-k} T^{0,1}M^* \otimes \mathscr{L}.$$

Proposition 11.9. *Let M be a Kähler manifold.*

1. *M admits a spin structure if and only if there is a complex line bundle L satisfying $L^{\oplus 2} \cong \mathcal{K}_M$.*
2. *Suppose M is compact. There is a bijective correspondence between the set of spin structures on M and the set of isomorphism classes of holomorphic line bundles $\mathcal{L}$ satisfying $\mathcal{L}^{\otimes 2} \cong \mathcal{K}_M$. (Each such $\mathcal{L}$ inherits a Hermitian metric from $\mathcal{K}_M$.)*
3. *Suppose M is compact. Suppose that $\mathcal{L}$ is a square root of $\mathcal{K}_M$ and W denotes the associated complex spinor bundle.*

   (a) *If $\dim_{\mathbf{C}} M = 1 \mod 4$, then*

   $$\$ = W \quad \textit{and} \quad \not{D} = \sqrt{2}(\bar\partial + \bar\partial^*).$$

   The is a complex structure J on $\$$ which commutes with Clifford multiplication and anti-commutes with the complex structure i.

   (b) *If $\dim_{\mathbf{C}} M = 2 \mod 4$, then*

   $$\$^\pm = W^\pm \quad \textit{and} \quad \not{D}^\pm = \sqrt{2}(\bar\partial + \bar\partial^*);$$

   moreover, there is a complex structure J on $\$^\pm$ which commutes with Clifford multiplication and anti-commutes with the complex structure i.

   (c) *If $\dim_{\mathbf{C}} M = 3 \mod 4$, then is a real structure on W which respect to which $\sqrt{2}(\bar\partial + \bar\partial^*)$ is real. With respect to this real structure we have*

   $$\$ = \operatorname{Re} W \quad \textit{and} \quad \not{D} = \sqrt{2}(\bar\partial + \bar\partial^*).$$

   (d) *If $\dim_{\mathbf{C}} M = 4 \mod 4$, then is a real structure on $W^\pm$ With respect to this real structure we have*

   $$\$^\pm = \operatorname{Re} W^\pm \quad \textit{and} \quad \not{D} = \sqrt{2}(\bar\partial + \bar\partial^*).$$

Proof. (1) follows from Proposition 11.7.

(2) Denote $\mathcal{O}^\times$ the sheaf of nowhere vanishing holomorphic functions on M. There is a short exact sequence of sheaves

$$1 \to \mathbf{Z}_2 \to \mathcal{O}^\times \xrightarrow{x \mapsto x^2} \mathcal{O}^\times \to 1.$$

The corresponding long exact sequence in cohomology reads as follows:

$$H^0(M, \mathcal{O}^\times) \to H^0(M, \mathcal{O}^\times) \to H^1(M, \mathbf{Z}_2) \xrightarrow{\alpha} H^1(M, \mathcal{O}^\times) \to H^1(M, \mathcal{O}^\times) \xrightarrow{\beta} H^2(M, \mathbf{Z}_2)$$

The map α is injective, because the map $\mathbf{C}^\times = H^0(M, \mathcal{O}^\times) \to H^0(M, \mathcal{O}^\times) = \mathbf{C}^\times$ is surjective. Recall, that $H^1(M, \mathcal{O}^\times)$ classifies a holomorphic line bundles. A holomorphic line bundle $\mathcal{L}$ has a square root if and only if $\beta([\mathscr{L}]) = (c_1(L) \mod 2) = 0$. If $\beta([\mathscr{L}]) = 0$, then by the above the set of square roots is a torsor over $H^1(M, \mathbf{Z}_2)$.

For the proof of (3), using 1, one first analyzes the relationship between the spinor representation S and the complex spinor representation W in dimension n and determines the following:

1. If $n = 2 \mod 8$, then $S = W$ and W has a complex anti-linear complex structure J. $S = \mathbf{H}$, $W = W^+ \oplus W^- = \mathbf{C} \oplus \mathbf{C}$.

2. If $n = 4 \mod 8$, then $S^\pm = W^\pm$ and $W^\pm$ have a complex anti-linear complex structure J.

3. If $n = 6 \mod 8$, then there is a real structure on W and $S = \operatorname{Re} W$. This real structure does not respect the splitting $W = W^+ \oplus W^-$. Clifford multiplication is real with respect to this real structure.

4. If $n = 8 \mod 8$, then there is a real structure on $W^\pm$ and $S^\pm = \operatorname{Re} W^\pm$. Clifford multiplication is real with respect to this real structure.

□

12 Dirac operators on symmetric spaces

12.1 A brief review of symmetric spaces

Suppose G is a compact Lie group and K is a closed subgroup. Set

$$M := G/K.$$

Tautologically, $\pi\colon G \to M$ is a principal K–bundle. M can be made into a Riemannian manifold via the following construction. A Riemannian manifold obtained by this construction is called a **symmetric space**.

Definition 12.1. Let G be a Lie group. Set $L_g h := gh$ and $R_g h := hg$. The **Maurer–Cartan** form of G is the unique differential form $\mu_G \in \Omega^1(G, \mathfrak{g})$ such that

$$\mu_G(\mathrm{d}L_g(\xi)) = \xi.$$

Exercise 12.2. The Maurer–Cartan form satisfies

$$R_g^*\mu = \operatorname{Ad}(g^{-1})\mu.$$

Proposition 12.3. *Set*

$$\mathfrak{k} := \mathrm{Lie}(K) \quad \textit{and} \quad \mathfrak{g} := \mathrm{Lie}(G).$$

Let $\mathfrak{m}$ be a complement of $\mathfrak{k} \subset \mathfrak{g}$ such that for all $k \in K$

$$\mathrm{Ad}(k)\mathfrak{m} \subset \mathfrak{m}$$

and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$.

1. *The 1–form $\theta \in \Omega^1(G, \mathfrak{k})$ defined by*
$$\theta := \pi_{\mathfrak{k}}\mu_G$$
is a connection 1–form on the principal K–bundle $G \to M$.

2. *The curvature tensor of θ is given by*
$$\Omega = -\frac{1}{2}[\pi_{\mathfrak{m}}\mu_G \wedge \pi_{\mathfrak{m}}\mu_G].$$

3. *There is unique K–equivariant vector bundle isomorpism $TM \cong G \times_K \mathfrak{m}$ which agrees with the canonical identification $T_{[1]}M = \mathfrak{m}$ at $[1] \in G/K$.*

4. *Suppose $\langle \cdot, \cdot \rangle$ is an $\mathrm{Ad}(K)$–invariant Euclidean inner product on $\mathfrak{m}$. By slight abuse of notation also use $\langle \cdot, \cdot \rangle$ to denote the induced Riemannian metric on TM. The connection on TM induced by θ is the Levi-Civita connection.*

Proof. (1) Given $\xi \in \mathfrak{k}$ and $g \in G$, we have

$$\theta(g\xi) = \pi_{\mathfrak{k}}\mu_G(\mathrm{d}L_g\xi) = \pi_{\mathfrak{k}}(\xi) = \xi.$$

Moreover, if $k \in K$, then

$$R_k^*\theta = \pi_{\mathfrak{k}}R_k^*\mu_g = \pi_{\mathfrak{k}}\,\mathrm{Ad}(k)^{-1}\mu_g = \mathrm{Ad}(k)^{-1}\pi_{\mathfrak{k}}\mu_g = \mathrm{Ad}(k)^{-1}\theta.$$

This proves that θ is a connection 1–form.

(2) The curvature of θ is

$$\begin{aligned}\Omega &= \mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta]\\ &= \pi_{\mathfrak{k}}\mathrm{d}\mu_G + \frac{1}{2}\pi_{\mathfrak{k}}[\theta \wedge \theta].\end{aligned}$$

Since $\mathfrak{m}$ is $\mathrm{Ad}(K)$–invariant, we have $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. Since, moreover, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$, we have

$$\begin{aligned}\pi_{\mathfrak{k}}[\mu_G \wedge \mu_G] &= [\pi_{\mathfrak{k}}\mu_G \wedge \pi_{\mathfrak{k}}\mu_G] + [\pi_{\mathfrak{m}}\mu_G \wedge \pi_{\mathfrak{m}}\mu_G]\\ &= [\theta \wedge \theta] + [\pi_{\mathfrak{m}}\mu_G \wedge \pi_{\mathfrak{m}}\mu_G].\end{aligned}$$

It follows that

$$\begin{aligned}\Omega &= \mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta] \\ &= \pi_{\mathfrak{k}}(\mathrm{d}\mu_G + \frac{1}{2}[\mu_G \wedge \mu_G]) - \frac{1}{2}[\pi_{\mathfrak{m}}\mu_G \wedge \pi_{\mathfrak{m}}\mu_G].\end{aligned}$$

Since μ_G satisfies the Maurer–Cartan equation

$$\mathrm{d}\mu_G + \frac{1}{2}[\mu_G \wedge \mu_G] = 0,$$

the curvature Ω is given by the asserted formula.

(3) is obvious.

(4) It is clear that the connection induced by θ is a metric connection. It is an exercise to show that this connection is also torsion-free and, hence, agrees with the Levi-Civita connection. □

12.2 Homogeneous spin structures

Definition 12.4. Assume the situation of Proposition 12.3. A **homogeneous spin structure** on $M = G/K$ is a homomorphism $\widetilde{\mathrm{Ad}}\colon K \to \mathrm{Spin}(\mathfrak{m})$ such that the following diagram commutes

$$\begin{array}{ccc} & & \mathrm{Spin}(\mathfrak{m}) \\ & \overset{\widetilde{\mathrm{Ad}}}{\nearrow} & \downarrow \\ K & \xrightarrow[\mathrm{Ad}]{} & \mathrm{SO}(\mathfrak{m}). \end{array}$$

Given a homogeneous spin structure,

$$\mathfrak{s} \coloneqq G \times_K \mathrm{Spin}(n)$$

defines a spin structure in the usual sense on M. If $\mathrm{Spin}(\mathfrak{m}) \to \mathrm{GL}(S)$ denotes the spinor representation, then the spinor bundle of $\mathfrak{s}$ is given by

$$\slashed{S} \coloneqq G \times_K S.$$

The connection induced by θ yields the spin connection.

A spinor $\psi \in \Gamma(\slashed{S})$ can be identified with a K–equivariant map

$$\psi\colon G \to S \quad \text{with} \quad \psi(gk) = \widetilde{\mathrm{Ad}}(k^{-1})\psi(g).$$

The Clifford multiplication by $v \in T_xM \cong \mathfrak{m}$ is given simply by the Clifford multiplication of $\mathfrak{m}$ on S. The derivative $\nabla\psi \in \Omega^1(M, \slashed{S})$ can be identified with the K–equivariant 1–form on G with values in S defined by

$$(\nabla\psi)(\xi) = (\mathrm{d}\psi)(\xi) + \widetilde{\mathrm{ad}}(\theta(\xi))\psi$$

with

$$\widetilde{\mathrm{ad}} = \mathrm{Lie}(\widetilde{\mathrm{Ad}}).$$

Therefore, if $(e_1, \dots, e_m)$ is an orthonormal basis for $\mathfrak{m}$, the Dirac operator is given by

$$\not{D}\psi = \sum_{i=1}^{m} \gamma(e_i)\mathscr{L}_{e_i}\psi.$$

12.3 The Weitzenböck formula for symmetric spaces

Suppose that $\langle\cdot,\cdot\rangle$, in fact, arises from G–invariant inner product on $\mathfrak{g}$; e.g., G is semi-simple and $\langle\cdot,\cdot\rangle$ is the negative of the Killing form.

Definition 12.5. The **Casimir operator** of G is the differential operator $\Omega_G \colon C^\infty(G) \to C^\infty(G)$ defined by

$$\Omega_G \coloneqq -\sum_{i=1}^{n} \mathscr{L}_{e_i}\mathscr{L}_{e_i}$$

for some orthonormal basis $(e_1, \dots, e_n)$ of $\mathfrak{g}$.

Proposition 12.6. *We have*

$$\not{D}^2 = \Omega_G + \frac{1}{8}\mathrm{scal}.$$

Sketch of proof. Since $[e_i, e_j] \in \mathfrak{k}$ and, for $\xi \in \mathfrak{k}$, $\mathscr{L}_\xi\psi = -\widetilde{\mathrm{ad}}(\xi)$, we have

$$\begin{aligned}
\not{D}^2\psi &= \sum_{i,j=1}^{m} \gamma(e_i)\gamma(e_j)\mathscr{L}_{e_i}\mathscr{L}_{e_j}\psi \\
&= -\sum_{i=1}^{m} \mathscr{L}_{e_i}^2\psi + \frac{1}{2}\sum_{i,j=1}^{m} \gamma(e_i)\gamma(e_j)\mathscr{L}_{[e_i,e_j]}\psi \\
&= -\sum_{i=1}^{m} \mathscr{L}_{e_i}^2\psi - \frac{1}{2}\sum_{i,j=1}^{m} \gamma(e_i)\gamma(e_j)\widetilde{\mathrm{ad}}([e_i, e_j]).
\end{aligned}$$

Let $(f_1, \dots, f_k)$ be an orthonormal basis of $\mathfrak{k}$. The above formula can then be written as

$$\not{D}^2 = \Omega_G + \sum_{j=1}^{k} \widetilde{\mathrm{ad}}(f_j)\widetilde{\mathrm{ad}}(f_j) - \frac{1}{2}\sum_{i,j=1}^{m} \gamma(e_i)\gamma(e_j)\widetilde{\mathrm{ad}}([e_i, e_j])\psi.$$

A computation identifies the sum of the last two terms with $\frac{1}{8}\mathrm{scal}$. □

Here is why the above is useful. The scalar curvature scal of a symmetric space is constant. $L^2\Gamma(\$)$ is acted upon by G and can be decomposed into irreducible representations

$$L^2\Gamma(\$) = \bigoplus_{\lambda\in\Lambda} V_\lambda.$$

On an irreducible representation, the Casimir operator acts as a constant $c(\lambda)$. Consequently, the spectrum of $D\!\!\!/^2$ is given by

$$\operatorname{spec}(D\!\!\!/^2) = \left\{c(\lambda) + \frac{1}{8}\operatorname{scal} : \lambda \in \Lambda\right\}.$$

This can (in principle) be used to compute the spectrum of $D\!\!\!/^2$ using representation theory.

Example 12.7 (Toy example). Consider the circle $S^1 = \mathbf{R}/2\pi\mathbf{Z}$. It has two spin structures. For one of them, the spinor bundle is the trivial bundle $\$ = \underline{\mathbf{C}}$ and the Dirac operator is simply $D\!\!\!/ = i\partial_t$. Consequently,

$$\operatorname{spec} D\!\!\!/ = \mathbf{Z}$$

with eigenspinors given by $\psi_k(t) = e^{ikt}$.

We can think of S^1 as the symmetric space $\mathrm{U}(1)/\{e\}$. Since $\mathrm{Spin}(1) = \{\pm 1\}$ and $\mathrm{U}(1)$ is connected, there is a unique homogeneous spin structure on S^1. This is the spin structure considered above. The irreducible representation of $\mathrm{U}(1)$ are parametrized by $\mathbf{Z}$: given $k \in \mathbf{Z}$, $\mathrm{U}(1) \to \mathrm{GL}(\mathbf{C})$, $z \mapsto z^k$ is irreducible. Each of these representations appear with multiplicity one in $L^2\Gamma(\$)$ (by Fourier theory). The Casimir operator on the representation parametrized by $k \in \mathbf{Z}$ takes value k^2. Consequently, the above discussion tells us that

$$\operatorname{spec} D\!\!\!/^2 = \{k^2 : k \in \mathbf{Z}\}.$$

Of course, this derivation is the same as direct derivation in the previous paragraph.

Remark 12.8. This method has been used by Sulanke to determine the spectrum of $D\!\!\!/$ on $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ in her PhD thesis. A simpler way to determine the spectrum of $D\!\!\!/$ on S^n was found by Bär [Bär96]. In fact, Bär's method also determines an explicit eigenbasis with respect to $D\!\!\!/$.

13 Killing Spinors

Definition 13.1. Let M be a spin manifold. A **Killing spinor** is a spinor $\psi \in \Gamma(\$)$ satisfying

$$\nabla_v\psi - \mu\gamma(v)\psi = 0$$

for some constant $\mu \in \mathbf{R}$ and all $v \in TM$. We call μ the **Killing number** of ψ.

Proposition 13.2. *A Killing spinor with Killing number μ is an eigenspinor with eigenvalue $-n\mu$.*

13.1 Friedrich's lower bound for the first eigenvalue of $\not{D}$

As far as I know, the origin of the study of Killing spinors is the following result.

Theorem 13.3 (Friedrich [Fri80]). *Let M be a compact spin manifold with non-negative but non-vanishing scalar curvature. Denote by λ^+ and λ^- the smallest positive and negative eigenvalues of $\not{D}$ respectively. With*

$$\mathrm{scal}_0 := \min \mathrm{scal},$$

we have

$$(\lambda^\pm)^2 \geqslant \frac{n}{4(n-1)}\mathrm{scal}_0.$$

If equality holds, then M admits a non-trivial Killing spinor with Killing number $+\frac{n}{4(n-1)}\mathrm{scal}$ or $-\frac{n}{4(n-1)}\mathrm{scal}$.

Remark 13.4. The obvious lower bound on $\lambda^\pm$ arising from Proposition 9.13 is $\lambda^\pm \geqslant \frac{1}{4}\mathrm{scal}_0$.

The proof is based on an important trick. The basic idea is that if $f \in C^\infty(M, \mathbf{R})$, then there is a Weitzenböck formula for $\not{D} + f$ which give sharper bounds that Proposition 9.13. More generally, one can replace f with a suitable endomorphism of $\not{S}$.

Definition 13.5. Given $f \in C^\infty(M, \mathbf{R})$, define the covariant derivative ${}^f\nabla$ on $\not{S}$ by

$${}^f\nabla_v\Phi := \nabla\Phi - f\gamma(v)\Phi.$$

Remark 13.6. ${}^f\nabla_v\Phi$ is a metric covariant derivative.

Proposition 13.7. *We have*

$$(\not{D} + f)^2 = {}^f\nabla^{*f}\nabla + \frac{1}{4}\mathrm{scal} + (1-n)f^2$$

Proof. Since

$$\not{D}(f\phi) = \gamma(\nabla f) + f\not{D}(\phi),$$

we have

$$(\not{D} + f)^2 = \not{D}^2 + 2f\not{D} + \gamma(\nabla)f + f^2.$$

By Proposition 9.13, we have

$$(\not{D} + f)^2 = \nabla^*\nabla + 2f\not{D} + \gamma(\nabla f) + f^2 + \frac{1}{4}\mathrm{scal}.$$

We have

$$
\begin{aligned}
{}^f\nabla^{*f}\nabla &= -\sum_{i=1}^{n} {}^f\nabla_{e_i}{}^f\nabla_{e_i} \\
&= -\sum_{i=1}^{n} (\nabla_{e_i} - f\gamma(e_i))(\nabla_{e_i} - f\gamma(e_i)) \\
&= -\sum_{i=1}^{n} \left(\nabla_{e_i}^2 - f^2 - \gamma_{e_i}\nabla_{e_i}f - 2f\gamma_{e_i}\nabla_{e_i}\right) \\
&= \nabla^*\nabla + nf^2 + \gamma(\nabla f) + 2f\not{D},
\end{aligned}
$$

which can be rewritten as

$$\nabla^*\nabla = {}^f\nabla^{*f}\nabla - nf^2 - \gamma(\nabla f) - 2f\not{D}.$$

This proves the asserted identity. □

Corollary 13.8. *If ψ is compactly supported, then*

$$\int_M \langle(\not{D} + f)^2\psi, \psi\rangle = \int_M \left(\frac{1}{4}\mathrm{scal} + (1-n)f^2\right)|\psi|^2 + |{}^f\nabla\psi|^2.$$

Proof of Theorem 13.3. Suppose λ is an eigenvalue of $\not{D}$ and ψ is an eigenspinor for λ. Using Corollary 13.8 with $f = \mu$ a constant, we obtain

$$0 = \int_M \left(\frac{1}{4}\mathrm{scal} + (1-n)\mu^2 - (\lambda+\mu)^2\right)|\psi|^2 + |{}^f\nabla\psi|^2.$$

Consequently,

$$\frac{1}{4}\mathrm{scal}_0 \leqslant (\lambda+\mu)^2 + (n-1)\mu^2.$$

The minimum of the right-hand side is $\frac{n-1}{n}\lambda^2$; it is achieved at $\mu = -\lambda/n$. This implies the bound. □

13.2 Killing spinors and Einstein metrics

Proposition 13.9. *If M admits a non-trivial Killing spinor with Killing number μ, then M is Einstein with Einstein constant $4(n-1)(\mu/n)^2$.*

Proof. The curvature of ${}^f\nabla$ is given by

$$
\begin{aligned}
{}^fF_{k\ell} &= [\nabla_k - f\gamma_k, \nabla_\ell - f\gamma_\ell] \\
&= R^S_{k\ell} - \partial_k f\gamma_k\gamma_\ell + \partial_\ell f\gamma_\ell\gamma_k + f^2[\gamma_k, \gamma_\ell].
\end{aligned}
$$

Arguing as in the proof of Proposition 10.1 with $f = -\lambda/n$, we have

$$\begin{aligned} 0 &= \sum_{\ell=1}^{n} \gamma(e_\ell)^f F(e_k, e_\ell)\psi \\ &= \frac{1}{4}\sum_{i,j,\ell=1}^{n} \gamma(e_\ell)\gamma(e_i)\gamma(e_j)\langle R(e_k, e_\ell)e_i, e_j\rangle\psi + \sum_{\ell=1}^{n}(\lambda/n)^2\gamma(e_\ell)[\gamma(e_k), \gamma(e_\ell)]\psi \\ &= -\frac{1}{2}\sum_{i=1}^{n}\gamma(e_i)\mathrm{Ric}(e_k, e_i)\psi + 2(n-1)(\lambda/n)^2\gamma(e_k)\psi \\ &= \left(-\frac{1}{2}\lambda_k + 2(n-1)(\lambda/n)^2\right)\gamma(e_k)\psi. \end{aligned}$$

It follows that

$$\mathrm{Ric} = 4(n-1)(\lambda/n)^2.$$ □

13.3 The spectrum of the Atiyah–Singer operator on S^n

Theorem 13.10. *Let $n \geqslant 3$. On S^n, we have*

$$\mathrm{spec}(D\!\!\!/) = \{\pm(n/2 + k) : k \in \mathbf{N}_0\}.$$

The multiplicity of $\lambda_{\pm,k} = \pm(n/2+k)$ is

$$\mathrm{rk}\,\$ \cdot \binom{k+n-1}{k}.$$

Proof. The following argument goes back to Bär [Bär96].

Proposition 13.11. *Let $n \geqslant 3$. The spinor bundle $\$$ of S^n can be trivialized by Killing spinors with Killing number $+1/2$ and also by Killing spinors with Killing number $-1/2$.*

Proof. Consider the covariant derivative ${}^{\pm 1/2}\nabla$ defined by ${}^{\pm 1/2}\nabla_v\psi = \nabla_v\psi \mp 1/2\gamma(v)\psi$. A computation shows that the curvature of ${}^{\pm 1/2}\nabla$ vanishes. Since S^n is simply-connected, it follows that $\$$ admits a trivialization by $\tilde{\nabla}^{\pm}$–parallel spinors. □

Proposition 13.12. *We have*

$$(D\!\!\!/ \pm 1/2)^2 = {}^{\pm 1/2}\nabla^{*\,\pm 1/2}\nabla + \frac{1}{4}(n-1)^2.$$

Proof. This is Proposition 13.7. □

Pick Killing spinors $(\psi_1^\pm, \dots, \psi_m^\pm)$ with Killing number $\pm 1/2$ forming a basis for $\$$ point-wise. Here $m = \operatorname{rk} \$$. Let (f_k) be a complete L^2 orthonormal basis of eigenfunctions for Δ on S^n. Denote by λ_k the eigenvalue corresponding to f_k.

Proposition 13.13. *We have*

$$\operatorname{spec}(\Delta_{S^n}) = \{k(n+k-1) : k \in \mathbf{N}_0\}.$$

The eigenvalue $\lambda_k = k(n+k-1)$ has multiplicity

$$m_k = \binom{n+k-1}{k} \frac{n+2k-1}{n+k-1}.$$

Clearly $(f_i \psi_j^\pm)$ forms an L^2 orthonormal basis of $L^2\Gamma(\$)$. Since $\psi_j^\pm$ is ${}^{\pm 1/2}\nabla$–parallel we have,

$$(\not{D} \pm 1/2)^2 (f_i \psi_j^\pm) = \left(\lambda_i + \frac{1}{4}(n-1)^2\right) f_i \psi_j^\pm.$$

Therefore, $(f_i \psi_j^\pm)$ is an eigenbasis for $(\not{D} \pm 1/2)^2$. Using Proposition 13.13, we can compute the spectrum of $(\not{D} \pm 1/2)^2$.

Corollary 13.14. *We have*

$$\operatorname{spec}\left((\not{D} \pm 1/2)^2\right) = \left\{k(n+k-1) + (n-1)^2/4 : k \in \mathbf{N}_0\right\}.$$

The eigenvalue $\lambda_k = k(n+k-1) + (n-1)^2/4$ has multiplicity

$$m(\lambda_k) = \binom{n+k-1}{k} \frac{n+2k-1}{n+k-1} \cdot \operatorname{rk} \$.$$

Proposition 13.15. *If $A^2 x = \lambda^2 x$, then $x^\pm = \pm\lambda x + Ax$ satisfy*

$$Ax^\pm = \pm \lambda x^\pm.$$

We have $\sqrt{k(n+k-1) + (n-1)^2/4} = k + \frac{n-1}{2}$. For $\varepsilon = \pm 1$, define

$$\psi_{k\ell}^{\varepsilon\pm} := (\not{D} \pm 1/2)(f_k \psi_\ell^\pm) + \varepsilon(k + (n-1)/2)(f_k \psi_\ell^\pm).$$

A brief computation shows that

$$\psi_{k\ell}^{\varepsilon\pm} = \varepsilon(\pm(1-n)/2 + \varepsilon(k + (n-1)/2))(f_k \psi_\ell^\pm) + \gamma(\nabla f_k)\psi_\ell^\pm.$$

Except $\psi_{0,\ell}^{++}$ and $\psi_{0,\ell}^{--}$ these spinors are non-vanishing. It follows that

$$\operatorname{spec}(\not{D} \pm 1/2) \subset \{\varepsilon(k + (n-1)/2) : \varepsilon = \pm 1, k \in \mathbf{N}_0\} \backslash \{\pm(n-1)/2)\}.$$

This implies the claim about $\operatorname{spec}(\not{D})$. For the computation of the multiplicities we refer the reader to [Bär96, Lemma 5]. □

14 Dependence of Atiyah–Singer operator on the Riemannian metric

The following goes back to the work of Bourguignon and Gauduchon [BG92]. The conformal invariance was already noted by Hitchin [Hit74, Section 1.4].

14.1 Comparing spin structures with respect to different metrics

Let (M, g) be a Riemannian manifold. Let $\mathrm{SO}(M, g)$ denote its orientated frame-bundle. Let $\mathfrak{s}$ be a spin-structure on M.

Proposition 14.1. *If $\tilde{g}$ is a different metric on M, then there exists a unique section h of g–self-adjoint endomorphisms of TM such that $\tilde{g} = ge^{2h}$; that is,*

$$\tilde{g}(v, w) = g(e^{2h}v, w) = g(e^h v, e^h w).$$

In other words, $e^h\colon (TM, \tilde{g}) \to (TM, g)$ is an isometry. This means it induces an isomorphism of $\mathrm{SO}(n)$–bundles $b\colon \mathrm{SO}(M, \tilde{g}) \to \mathrm{SO}(M, g)$.

Proof. This is basic linear algebra. □

Proposition 14.2. *Let $\mathfrak{s}$ be a spin structure for (M, g). There is a unique spin structure $\tilde{\mathfrak{s}}$ for $(M, \tilde{g})$ such that e^h lifts to an isomorphism of $\mathrm{Spin}(n)$–bundles $e^h\colon \tilde{\mathfrak{s}} \to \mathfrak{s}$.*

We have

$$e^{-h}(\gamma(v)\Psi) = \tilde{\gamma}(e^{-h}v)e^{-h}\Psi.$$

Proof. Let $g_t = ge^{2th}$. Then we have isomorphism b_t. These can be lifted to isomorphism of spin structures $e^h\colon \mathfrak{s} \to \tilde{\mathfrak{s}}$. This isometrically identifies the spinor bundles with respect to the different metrics and these isomorphism are also compatible with the Clifford multiplication. □

Remark 14.3. It should be pointed out as a warning that the above construction depends on the choice of path. In particular, it might not behave as one might expect with respect to concatenations.

Given this, we can compare Dirac operators with respect to different metrics.

Definition 14.4. In the situation above, we set

$$\widetilde{\not{D}}^{g,h} := e^h \not{D}^{ge^{2h}} e^{-h}.$$

Having chosen a reference metric g, the above allows us to view the Dirac operators for other metrics as an operator on the spinor bundle with respect to g. This makes it possible to compare Dirac operator with respect to different metrics.

14.2 Conformal invariance of the Dirac operator

Proposition 14.5. *Let g be a Riemannian metric and let $f \in C^\infty(M)$. Then*

$$\widetilde{\not{D}}^{g,f} = e^{-\frac{n+1}{2}f} \not{D}_g e^{\frac{n-1}{2}f}$$

Proposition 14.6. *Let $(e_1, \dots, e_n)$ be a local orthonormal frame with respect to g and denote by Γ the Christoffel symbols of ∇^g, that is,*

$$\nabla_{e_i} e_j = \partial_{e_i} + \Gamma_{ij}^k e_k.$$

Denote by $\tilde{e}_i = e^{-f} e_i$ the corresponding local orthonormal frame with respect to $\tilde{g} = ge^{2f}$. The Christoffel symbols $\tilde{\Gamma}$ of $\nabla^{\tilde{g}}$ are given by

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k - \delta_{ik} \cdot \partial_j f + \delta_{ij} \cdot \partial_k f.$$

Proof. Exercise. □

Corollary 14.7. *In the above situation, the spin connections $\not{\nabla}^g$ and $\not{\nabla}^{\tilde{g}}$ are related by*

$$e^{-f} \not{\nabla}^{\tilde{g}}_{\tilde{e}_i} e^f = e^{-f} \left(\not{\nabla}^g_{e_i} + \frac{1}{4} (\partial_j f) [\gamma_j, \gamma_i] \right).$$

Proof of Proposition 14.5. We have

$$\begin{aligned}
e^f \not{D}^{\tilde{g}} e^{-f} &= \sum_{i=1}^n e^f \tilde{\gamma}(\tilde{e}_i) \not{\nabla}^{\tilde{g}}_{\tilde{e}_i} e^{-f} \\
&= \sum_{i=1}^n \gamma(e_i) e^f \not{\nabla}^{\tilde{g}}_{\tilde{e}_i} e^{-f} \\
&= \sum_{i=1}^n \gamma(e_i) e^{-f} \left(\not{\nabla}^g_{e_i} + \frac{1}{4} (\partial_j f) [\gamma_j, \gamma_i] \right) \\
&= e^{-f} \left(\not{D}^g + \frac{n-1}{2} \gamma(\nabla f) \right) \\
&= e^{-\frac{n+1}{2}f} \not{D}_g e^{\frac{n-1}{2}f}.
\end{aligned}$$

□

Corollary 14.8. *The dimension of the space of harmonic spinors is a conformal invariant.*

14.3 Variation of the spin connections

In order to make use of $\widetilde{\not{D}}$ we need to understand how the spin connections with respect to g and $\tilde{g}$ are related. Denote by $\not{\nabla}^g$ the connection on the spinor bundle with respect to g.

Proposition 14.9. *Let*

$$\tilde{\not{\nabla}}^{g,h} = e^h \not{\nabla}^{ge^{2h}} e^{-h} = \not{\nabla}^g + \not{a}_h.$$

We have

$$\not{a}_h(\cdot) = -\frac{1}{4}\sum_{i,j}\Big(g(\cdot, (\nabla^g_{e_j} h)e_i) - g(\cdot, (\nabla^g_{e_i} h)e_j)\Big)\gamma(e_i)\gamma(e_j) + O(h^2)$$

The proof follows immediately from the following observation regarding the Levi-Civita connection.

Proposition 14.10. *Denote by $\nabla^{\tilde{g}}$ the Levi-Civita connection for $\tilde{g}$. Set*

$$\tilde{\nabla}^{g,h} = e^h \nabla^{\tilde{g}} e^{-h}.$$

Define a_h by

$$\tilde{\nabla}^{g,h} = \nabla^g + a_h$$

Write $a_h = \hat{a}(h) + O(h^2)$. We have

$$g(\hat{a}_h(u)v, w) = g(u, (\nabla^g_v h)w) - g(u, (\nabla^g_w h)v).$$

Remark 14.11. If (e_i) is a orthonormal basis for T_xM, then

$$\hat{a}_h = \sum_{i,j} (\hat{a}_h)^i_j e_i e^j$$

with

$$(\hat{a}_h)^i_j(e_k) = \langle (\hat{a}_h)(e_k)e_j, e_i\rangle = g(e_k, (\nabla^g_{e_j} h)e_i) - g(e_k, (\nabla^g_{e_i} h)e_j).$$

Proof. This is essentially proved by taking the derivative of the usual formula for the Levi-Civita connection. The following computation makes this look more complicated than it should be, but it also derives an explicit formula $\nabla^{\tilde{g}}$ in terms of ∇^g and h. (We highly recommend to skip this computation.)

Recall that $\exp(-y)\mathrm{d}_x \exp(y) = \Upsilon_x y$ with

$$\Upsilon_x := \frac{e^{\mathrm{ad}_x} - \mathrm{id}}{\mathrm{ad}_x}.$$

We have

$$\begin{aligned}
\tilde{g}(\nabla^{\tilde{g}}_u v, w) &= \frac{1}{2}\big(\mathscr{L}_u \tilde{g}(v, w) + \mathscr{L}_v \tilde{g}(w, u) - \mathscr{L}_w \tilde{g}(u, v) \\
&\quad + \tilde{g}([u, v], w) - \tilde{g}([v, w], u) + \tilde{g}([w, u], v)\big) \\
&= \frac{1}{2}\big(\mathscr{L}_u g(e^h v, e^h w) + \mathscr{L}_v g(e^h w, e^h u) - \mathscr{L}_w g(e^h u, e^h v)
\end{aligned}$$

$$
\begin{aligned}
&+ g(e^h[u,v], e^h w) - g(e^h[v,w], e^h u) + g(e^h[w,u], e^h v)\Big) \\
= \frac{1}{2}\Big(&\tilde{g}((\Upsilon(-h)\nabla_u h)v, w) + \tilde{g}(v, (\Upsilon(-h)\nabla_u h)w) \\
&+ \tilde{g}((\Upsilon(-h)\nabla_v h)w, u) + \tilde{g}(w, (\Upsilon(-h)\nabla_v h)u) \\
&- \tilde{g}((\Upsilon(-h)\nabla_w h)u, v) - \tilde{g}(u, (\Upsilon(-h)\nabla_w h)v)\Big) \\
&+ \frac{1}{2}\big(g(e^h\nabla_u v, e^h w) + g(e^h v, e^h\nabla_u w) \\
&+ g(e^h\nabla_v w, e^h u) + g(e^h w, e^h\nabla_v u) \\
&- g(e^h\nabla_w u, e^h v) - g(e^h w, e^h\nabla_w v) \\
&+ g(e^h[u,v], e^h w) - g(e^h[v,w], e^h u) + g(e^h[w,u], e^h v)\big) \\
= \frac{1}{2}\Big(&\tilde{g}((\Upsilon(-h)\nabla_u h)v, w) + \tilde{g}(v, (\Upsilon(-h)\nabla_u h)w) \\
&+ \tilde{g}((\Upsilon(-h)\nabla_v h)w, u) + \tilde{g}(w, (\Upsilon(-h)\nabla_v h)u) \\
&- \tilde{g}((\Upsilon(-h)\nabla_w h)u, v) - \tilde{g}(u, (\Upsilon(-h)\nabla_w h)v) \\
&+ \tilde{g}(\nabla_u v, w)
\end{aligned}
$$

Note that if x, y are self-adjoint, then $(\Upsilon(x)y)^* = \Upsilon(-x)y$. Set

$$\Theta(x) := \frac{1}{2}(\Upsilon(x) + \Upsilon(-x)).$$

Thus a defined by $\nabla^{\tilde{g}} = \nabla^g + a$ is characterized by

$$\tilde{g}(a(u)v, w) = \tilde{g}((\Theta(h)\nabla^g_u h)v, w) + \tilde{g}((\Theta(h)\nabla^g_v h)w, u) - \tilde{g}((\Theta(h)\nabla^g_w h)u, v).$$

Therefore, $\tilde{a}$, defined by

$$e^h\nabla^{\tilde{g}}e^{-h} = \nabla^g + \tilde{a}$$

satisfies

$$
\begin{aligned}
\tilde{g}(\tilde{a}(u)v, w) = +\tilde{g}(-(\Upsilon(-h)\nabla^g h)v, w) &+ \tilde{g}((e^{\mathrm{ad}_h}\Theta(h)\nabla^g_u h)v, w) \\
&+ \tilde{g}((e^{\mathrm{ad}\,h}\Theta(h)\nabla^g_v h)w, u) - \tilde{g}((e^{\mathrm{ad}_h}\Theta(h)\nabla^g_w h)u, v)
\end{aligned}
$$

because

$$\begin{aligned}
\tilde{g}(e^h \nabla^{\tilde{g}}_u e^{-h} v, w) &= \tilde{g}(e^h \nabla^g_u e^{-h} v, w) + \tilde{g}(e^h a(u) e^{-h} v, w) \\
&= \tilde{g}(\nabla^g_u v, w) + \tilde{g}(-(\Upsilon(-h)\nabla^g h)v, w) \\
&\quad + \tilde{g}(e^h(\Theta(h)\nabla^g_u h)e^{-h} v, w) + \tilde{g}(e^h(\Theta(h)\nabla^g_v h)e^{-h} w, u) \\
&\quad - \tilde{g}(e^h(\Theta(h)\nabla^g_w h)e^{-h} u, v) \\
&= \tilde{g}(\nabla^g_u v, w) \\
&\quad + \tilde{g}(-(\Upsilon(-h)\nabla^g h)v, w) + \tilde{g}((e^{\mathrm{ad}_h}\Theta(h)\nabla^g_u h)v, w) \\
&\quad + \tilde{g}((e^{\mathrm{ad}\, h}\Theta(h)\nabla^g_v h)w, u) - \tilde{g}((e^{\mathrm{ad}_h}\Theta(h)\nabla^g_w h)u, v). \qquad \square
\end{aligned}$$

14.4 Variation of the Dirac operator

Proposition 14.12. *Set* $\tilde{D\!\!\!/}^{g,h} = e^h D\!\!\!/^{ge^{2h}} e^{-h}$. *At* $h = 0$, *we have*

$$\mathrm{d}D\!\!\!/^{g,h}(\hat{h}) = -\sum_i \gamma(e_i)\nabla^g_{\hat{h}e_i} + \frac{1}{2}\gamma(\nabla^* h + \nabla\,\mathrm{tr}(h)).$$

Proof. Let (e_i) be an orthonormal basis of (TM, g). Then $(\tilde{e}_i) = (e^{-h} e_i)$ is an orthonormal basis of $(TM, \tilde{g})$ and

$$\begin{aligned}
e^h(D\!\!\!/^{\tilde{g}}(e^{-h}\Psi)) &= \sum_{i=1}^n e^h\left(\tilde{\gamma}(\tilde{e}_i)\nabla^{\tilde{g}}_{\tilde{e}_i} e^{-h}\Psi\right) \\
&= \sum_{i=1}^n \gamma(e_i) e^h \nabla^{\tilde{g}}_{\tilde{e}_i} e^{-h}\Psi \\
&= \sum_{i=1}^n \gamma(e_i)\tilde{\nabla}^h_{e^{-h}e_i}\Psi \\
&= \sum_{i=1}^n \gamma(e_i)\left(\nabla^g_{e^{-h}e_i} + a\!\!\!/^h(e^{-h}e_i)\right)\Psi. \qquad \square
\end{aligned}$$

Definition 14.13. The **stress-energy tensor** of ψ is

$$T_\psi(v, w) := \langle \gamma(v)\nabla_w\psi + \gamma(w)\nabla_v\psi, \psi\rangle$$

Corollary 14.14. *We have*

$$\langle \psi, \mathrm{d}D\!\!\!/^{g,h}(\hat{h})\psi\rangle = -\frac{1}{2}\langle T_\psi(v, w), \hat{h}\rangle. \tag{14.15}$$

Proof. We have

$$
\begin{aligned}
e^h(\not{D}^{ge^{2h}}(e^{-h}\Psi)) = \not{D}^g\Psi \\
&- \sum_i \gamma(e_i)\nabla^g_{he_i} \\
&- \frac{1}{4}\sum_{i,j,k}\Big(\langle e_i, (\nabla^g_{e_k}h)e_j\rangle - \langle e_i, (\nabla^g_{e_j}h)e_k\rangle\Big)\gamma(e_i)\gamma(e_j)\gamma(e_k) \\
&+ O(h^2).
\end{aligned}
$$

At a point where $\nabla_{e_i}e_j = 0$, we need to compute

$$
\sum_{ijk}(\nabla_k h_{ji} - \nabla_j h_{ki})\gamma_i\gamma_j\gamma_k.
$$

We can split this sum into five contributions depending on the incidence of the indices: $i = j = k$, $i \neq j = k$, $i = j \neq k$, $k = i \neq j$, $i \neq j \neq k$. Only the third and the fourth sum contribute and we get

$$
\begin{aligned}
\sum_{ijk}(\nabla_k h_{ji} - \nabla_j h_{ki})\gamma_i\gamma_j\gamma_k &= -\sum_{i\neq k}(\nabla_k h_{ii} - \nabla_i h_{ki})\gamma_k + \sum_{i\neq j}(\nabla_i h_{ji} - \nabla_j h_{ii})\gamma_k \\
&= +2\sum_{i,j}(\nabla_i h_{ij} - \nabla_j \operatorname{tr}(h))\gamma_k \\
&= -2\gamma(\nabla^* h + \nabla \operatorname{tr}(h)).
\end{aligned}
$$

□

15 L^2 elliptic theory for Dirac operators

Dirac operators naturally are defined as differential operators acting on smooth sections $\Gamma(S)$ a Dirac bundle. The space $\Gamma(S)$, naturally, is a Fréchet space. Unfortunately, functional analysis for Fréchet spaces is very delicate. It will be easier for us to work with Hilbert spaces of $W^{k,2}$ sections of S.

15.1 $W^{k,2}$ sections

Definition 15.1. Let M be a compact, oriented, Riemannian manifold. Let E be a Hermitian or Euclidean vector bundle over M. Suppose ∇ is a metric covariant derivative on E. Let $k \in \mathbf{N}_0$. Given $s, t \in \Gamma(E)$, define

$$\langle s, t\rangle_{W^{k,2}} := \sum_{j=0}^{k} \int_M \langle \nabla^k s, \nabla^k t\rangle_{T^*M^{\otimes k}\otimes E} \operatorname{vol}_g \quad \text{and} \quad \|s\|_{W^{k,2}} := \sqrt{\langle s, s\rangle_{W^{k,2}}}.$$

We denote by $W^{k,2}\Gamma(E)$ the Hilbert space obtained as the completion of $\Gamma(E)$ with respect to the norm $\|\cdot\|_{W^{k,2}}$; that is,

$$W^{k,2}\Gamma(E) := \overline{\Gamma(E)}^{\|\cdot\|_{W^{k,2}}}.$$

Instead of $W^{k,2}$ we will simply write L^2.

If E is Hermitian, then $W^{k,2}\Gamma(E)$ is a complex Hilbert space; otherwise, it is a real Hilbert space.

Remark 15.2. Using measure theory and distribution theory, the space $W^{k,2}\Gamma(E)$ can be constructed directly, without going through the abstract machinery of completion.

Exercise 15.3. The norm $\|\cdot\|_{W^{k,2}}$ does depend on the choice of inner product h and covariant derivative ∇ on E. However, since M is compact, different choices lead to comparable norms; that is, for some constant $c > 0$

$$c^{-1}\|\cdot\|_{W^{k,2}_{\nabla,h}} \leqslant \|\cdot\|_{W^{k,2}_{\nabla',h'}} \leqslant c\|\cdot\|_{W^{k,2}_{\nabla,h}}.$$

Consequently, $W^{k,2}\Gamma(E)$, as a topological vector space, is independent of the choice of ∇ and h.

Proposition 15.4. *If $D\colon \Gamma(E) \to \Gamma(F)$ is a differential operator of order $\ell \in \mathbf{N}_0$, then it extends uniquely to a bounded linear operator $D\colon W^{k+\ell,2}\Gamma(E) \to W^{k,2}\Gamma(F)$ for all $k \in \mathbf{N}_0$.*

Proof. Exercise. First proof that this holds for $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ and 0–th order differential operators. □

There are two fundamental theorems about $W^{k,2}$ spaces.

Theorem 15.5 (Rellich–Kondrachov). *The embedding $W^{k+1,2}\Gamma(E) \to W^{k,2}\Gamma(E)$ is compact.*

Proof. It suffices to restrict to $k = 0$. We need to prove that if (s_i) is a sequence in $W^{1,2}\Gamma(E)$ with $\|s_i\|_{W^{1,2}} \leqslant 1$, then the corresponding sequence in $L^2\Gamma(E)$ has a convergent subsequence.

Step 1. *We can assume that $M = T^n$ and $E = \underline{\mathbf{C}}$.*

Using a partition of unity we can write

$$s_i = s_{i,1} + \ldots + s_{1,m}$$

with $s_{i,j}$ supported in a coordinate chart. It suffices to prove convergence of the $s_{i,j}$ for fixed j. Such a coordinate chart can be embedded into T^n. In a coordinate chart E is trivial and we can decompose $s_{i,j}$ into its components (and possibly complexify).

Step 2. *Preliminary steps using Fourier analysis.*

By Fourier analysis, we can write any $s \in W^{1,2}\Gamma(E)$ as

$$s(x) = \sum_{\alpha \in \mathbf{Z}^n} \hat{s}_\alpha e^{i\langle \alpha, x\rangle}$$

and by Parseval's Theorem we have

$$\|s\|_{W^{1,2}}^2 = \sum_{\alpha \in \mathbf{Z}^n} \left(1 + |\alpha|^2\right) |s_\alpha|^2.$$

Denote by s^N the following truncation of the Fourier series of s:

$$s^N(x) := \sum_{\substack{\alpha \in \mathbf{Z}^n \\ |\alpha| \leqslant N}} \hat{s}_\alpha e^{i\langle \alpha, x\rangle}.$$

According to the above and Parseval's Theorem, we have

$$\left\|s^N\right\|_{L^2}^2 \leqslant \|s\|_{W^{1,2}} \quad \text{and} \quad \left\|s - s^N\right\|_{L^2}^2 = \sum_{\substack{\alpha \in \mathbf{Z}^n \\ |\alpha| > N}} |s_\alpha|^2 \leqslant \frac{\|s\|_{W^{1,2}}}{1 + N^2}.$$

Step 3. *Completion of the proof.*

Since $\|s_i\|_{W^{1,2}} \leqslant 1$, the above means that, for each $N \in \mathbf{N}$,

$$\left\|s_i^N\right\|_{L^2} \leqslant 1 \quad \text{and} \quad \left\|s_i - s_i^N\right\|_{L^2} \leqslant 1/N.$$

For each N, (s_i^N) has is a bounded sequence in the finite dimensional space of smooth functions spanned by $e^{i\langle \alpha, x\rangle}$ with $\alpha \in \mathbf{Z}^n$ satisfying $|\alpha| \leqslant N$. A diagonal sequence argument finds shows that after passing to a subsequence, we can assume that s_i^N converges for each N. Since $\left\|s_i - s_i^N\right\|_{L^2} \leqslant 1/N$, it follows that s_i converges as well. □

Theorem 15.6 (Morrey–Sobolev embedding). *If $\ell - n/2 > 0$, then*

$$\|s\|_{C^k} \lesssim \|s\|_{W^{k+\ell,2}};$$

in particular, $W^{k+\ell,2}\Gamma(E) \to C^k\Gamma(E)$.

Proof. It suffices to prove this for $k = 0$. Since the estimate is local, we can assume that s is supported in a coordinate chart of radius 1 and work on $\mathbf{R}^n$. We need to estimate $|s|(0)$ in terms of $\|s\|_{W^{\ell,2}}$.

For $\hat{x} \in S^{n-1}$, by multiple applications of the fundamental theorem of calculus and rearranging integrals we have

$$\begin{aligned} s(0) &= -\int_0^1 \partial_r s(r\hat{x}) \mathrm{d}r \\ &= +\int_0^1 \int_{r_1}^1 \partial_{r_2}^2 s(r_2\hat{x})\, \mathrm{d}r_2 \mathrm{d}r_1 \\ &= \cdots \\ &= (-1)^\ell \int_0^1 \int_{r_1}^1 \cdots \int_{r_{\ell-1}}^1 \partial_{r_\ell}^\ell s(r_\ell \hat{x})\, \mathrm{d}r_\ell \cdots \mathrm{d}r_2 \mathrm{d}r_1 \\ &= (-1)^\ell \int_0^1 \int_0^{r_\ell} \int_0^{r_{\ell-1}} \cdots \int_0^{r_2} \partial_{r_\ell}^\ell s(r_\ell \hat{x})\, \mathrm{d}r_1 \mathrm{d}r_2 \cdots \mathrm{d}r_\ell \\ &= (-1)^\ell/(\ell-1)! \int_0^1 r^{\ell-1} \partial_r^\ell s(r\hat{x})\, \mathrm{d}r. \end{aligned}$$

Therefore, integrating over S^{n-1} we obtain

$$\begin{aligned} |s(0)| &\lesssim \int_{S^{n-1}} \int_0^1 r^{\ell-1} |\partial_r^\ell s|(r\hat{x})\, \mathrm{d}r \mathrm{d}\hat{x} \\ &= \int_{B_1} |x|^{\ell-n} |\nabla^\ell s| \,\mathrm{vol}_{\mathbf{R}^n} \\ &\leqslant \left(\int_{B_1} |\nabla^\ell s|^2 \right)^{1/2} \cdot \left(\int_{B_1} |x|^{2(\ell-n)} \right)^{1/2}. \end{aligned}$$

The second factor is integrable provided $2(\ell - n) > n$, that is, $\ell > n/2$. □

Remark 15.7. The above argument is due to Nirenberg [Nir59, p. 127]. The inclined reader will observe that it can be used to prove a considerably more general result.

There also is a Fourier theoretic argument, which the reader can find in [Roe98, Theorem 5.7].

15.2 Elliptic estimates

Proposition 15.8. *Let $k \in \mathbf{N}_0$. There is a constant $c > 0$ such that*

$$\|s\|_{W^{k+1,2}} \leqslant c \left(\|Ds\|_{W^{k,2}} + \|s\|_{L^2} \right).$$

Proof. By the Weitzenböck formula $D^2 = \nabla^*\nabla + \mathscr{F}$. Consequently,

$$\begin{aligned} \|s\|_{W^{1,2}}^2 &\leqslant \langle D^2 s, s\rangle_{L^2} + c\|s\|_{L^2}^2 \\ &\leqslant \|Ds\|_{L^2}^2 + c\|s\|_{L^2}^2. \end{aligned}$$

This proves the assertion for $k = 0$.

Exercise 15.9. Prove that $[D, \nabla]$ is a 0[th] order differential operator.

Given this, we have

$$\begin{aligned}\|\nabla s\|_{W^{1,2}}^2 &\leqslant \|D\nabla s\|_{L^2}^2 + c\|s\|_{L^2}^2 \\ &\leqslant \|Ds\|_{W^{1,2}}^2 + c\|s\|_{L^2}^2.\end{aligned}$$

This proves the assertion for $k = 1$. The assertion for arbitrary k follows by induction. □

Lemma 15.10. *Let X, Y, Z be Banach spaces. Let $D\colon X \to Y$ be a bounded linear operator. Let $K\colon X \to Z$ be a compact linear operator. If there is a constant $c > 0$ such that*

$$\|x\|_X \leqslant c\left(\|Dx\|_Y + \|Kx\|_Z\right),$$

then $\ker D$ *is finite-dimensional, and* $\operatorname{im} D$ *is closed.*

Corollary 15.11. $\ker\left(D\colon W^{k+1,2}\Gamma(S) \to W^{k,2}\Gamma(S)\right)$ *is finite-dimensional.*

15.3 Elliptic regularity

Exercise 15.12. Let X, Y be a Hilbert spaces and $X^* = \mathscr{L}(X, \mathbf{R})$ and Y^* their duals. Let $L\colon X \to Y$ be a bounded linear operator. Set

$$\|L\|_{\mathscr{L}(X,Y)} \coloneqq \sup\{\|Lx\| : \|x\| = 1\}.$$

Prove that

$$\|L^*\|_{\mathscr{L}(Y^*,X^*)} = \|L\|_{\mathscr{L}(X,Y)}.$$

Definition 15.13. Denote by $W^{-k,2}\Gamma(S)$ the dual space Hilbert space of $W^{k,2}\Gamma(S)$.

Remark 15.14. By Theorem 15.6, $\bigcup_{k\in\mathbf{Z}} W^{k,2}\Gamma(S)$ is the space of S–valued distributions. We denote this space by $\mathscr{D}'\Gamma(S)'$.

We can extend D to $D\colon W^{k+1,2}\Gamma(S) \to W^{k,2}\Gamma(S)$ for all $k \in \mathbf{Z}$. If $k \geqslant 0$, it is clear how to define D. For $k \geqslant 1$ and $\psi \in W^{-k+1,2}\Gamma(S)$, we define $D\psi \in W^{-k,2}\Gamma(S)$ by

$$\langle D\psi, \phi\rangle_{W^{-k,2},W^{k,2}} \coloneqq \langle \psi, D\phi\rangle_{W^{-k+1,2},W^{k-1,2}}.$$

For $\ell \geqslant k \geqslant 0$, $W^{\ell,2}\Gamma(S) \subset W^{k,2}\Gamma(S)$, we have

$$W^{-k,2}\Gamma(S) = (W^{k,2}\Gamma(S))^* \subset (W^{\ell,2}\Gamma(S))^* = W^{-\ell,2}\Gamma(S).$$

Therefore,

$$W^{\ell,2}\Gamma(S) \subset W^{k,2}\Gamma(S).$$

for all $\ell \geqslant k$. All the results proved above extend to $k \in \mathbf{Z}$, in particular, the elliptic estimates.

Proposition 15.15. *If $\psi \in \mathscr{D}'\Gamma(S)$ and $D\psi \in W^{k,2}\Gamma(S) \subset W^{-\infty,2}\Gamma(S)$, then $\psi \in W^{k+1,2}\Gamma(S)$.*

There are many ways of proving this result. A popular method is to use difference quotients, see [Eva10, Section 6.3]. We will use the a Friedrich's mollifier.

Definition 15.16. A **Friedrich's mollifier** for S is a family $(F_\varepsilon)_{\varepsilon\in(0,1)}$ of smoothing operators $W^{-\infty,2}\Gamma(S) \to \Gamma(S)$ with the following properties for each $k \in \mathbf{Z}$:

1. There is a constant $c_k > 0$ such that, for all $s \in L^2\Gamma(S)$,
$$\|F_\varepsilon \phi\|_{W^{k,2}} \leqslant c\|\phi\|_{W^{k,2}}.$$
2. For any first order differential operator B there is a constant $c_B > 0$ such that for all $s \in \Gamma(\Sigma)$ with
$$\|[B, F_\varepsilon]\phi\|_{W^{k,2}} \leqslant c\|\phi\|_{W^{k,2}}$$
3. For any $s \in \mathscr{D}'\Gamma(S)$ and $t \in \Gamma(S)$, we have
$$\langle F_\varepsilon \phi, \psi\rangle \to \langle \phi, \psi\rangle.$$

Exercise 15.17. Suppose $M = \mathbf{R}^n$ and S is trivial. Let $\phi \in C_c^\infty([0,\infty),[0,1])$ be a compactly supported function with $\int \phi = 0$. For $\varepsilon > 0$, define
$$(F_\varepsilon s)(x) \coloneqq \varepsilon^{-n}\int_{\mathbf{R}^n} \phi((x-y)/\varepsilon)s(y)\mathrm{d}y.$$
Prove that (F_ε) is a mollifier and use this construction to prove the existence of mollifiers in general.

Sketch of proof of Proposition 15.15. We will prove that, for all $k \in \mathbf{Z}$, if $\psi \in W^{k,2}\Gamma(S)$ and $D\psi \in W^{k,2}\Gamma(S)$, then $\psi \in W^{k+1,2}\Gamma(S)$.

The definition of Friedrich's mollifiers implies that there is a constant $c > 0$ independent of ε such that
$$\|F_\varepsilon s\|_{W^{k,2}} \leqslant \|s\|_{W^{k,2}} \quad \text{and} \quad \|[F_\varepsilon, D]s\|_{W^{k,2}} \leqslant \|s\|_{W^{k,2}}.$$

By elliptic estimates
$$\begin{aligned}\|F_\varepsilon\psi\|_{W^{k+1,2}} &\leqslant c\left(\|DF_\varepsilon\psi\|_{W^{k,2}} + \|F_\varepsilon\psi\|_{W^{k,2}}\right)\\ &= c\left(\|[D,F_\varepsilon]\psi\|_{W^{k,2}} + \|F_\varepsilon\psi\|_{W^{k,2}}\right)\\ &\leqslant c\|\psi\|_{W^{k,2}}.\end{aligned}$$

It follows that $F_\varepsilon\psi$ converges weakly in $W^{k+1,2}\Gamma(S)$. By definition of F_ε, the limit of $F_\varepsilon\psi$ in $\mathscr{D}'\Gamma(S)$ is ψ. Since these limits must agree, we have $\psi \in W^{k+1,2}\Gamma(S)$. □

Corollary 15.18. $\ker\left(D\colon W^{k+1,2}\Gamma(S) \to W^{k,2}\Gamma(S)\right) = \ker D$*; in particular, it is independent of k.*

16 The index of a Dirac operator

Let M be a spin (or spin^c) manifold of even dimension with complex spinor bundle $W = W^+ \oplus W^-$. Let S be a complex Dirac bundle. For some Hermitian vector bundle E we can write

$$S = W \otimes_{\mathbf{C}} E$$

and thus

$$S = S^+ \oplus S^- \quad \text{with} \quad S^\pm := W^\pm \otimes_{\mathbf{C}} E.$$

We call this the **canonical grading** on S.

The Dirac operator on S splits according to this grading as

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

with

$$D^\pm = (D^\mp)^*.$$

By the preceding discussion $\ker D^+$ and $\operatorname{coker} D^+ \cong \ker D^-$ are both finite dimensional.

Definition 16.1. The **index** of D with respect to the canonical grading is

$$\operatorname{index}(D) = \dim\ker D^+ - \dim\ker D^-.$$

It is a truly remarkable fact that $\operatorname{index}(D)$ can be computed in terms of the topology of the underlying manifold M and in terms of the topology of S. The proof of this fact and the index formula will occupy most of the rest of this course.

Remark 16.2. Let $V = V^+ \oplus V^-$ be a $\mathbf{Z}_2$ graded Euclidean (or Hermitian) vector space and let $T\colon V \longrightarrow V$ be linear map. Writing

$$T = \begin{pmatrix} T_+^+ & T_+^- \\ T_-^+ & T_-^- \end{pmatrix}.$$

The **super trace** of T is

$$\operatorname{str} T = \operatorname{tr} T_+^+ - \operatorname{tr} T_-^-.$$

Suppose $A\colon V \longrightarrow V$ is of degree 1 and self-adjoint, that is

$$A = \begin{pmatrix} 0 & A^- \\ A^+ & 0 \end{pmatrix}.$$

with $A^\pm = (A^\mp)^*$. We can compute

$$\operatorname{index} A = \dim\ker A^+ - \dim\ker A^-$$

as follows.

Write

$$A^2 = \begin{pmatrix} (A^+)^*A^+ & 0 \\ 0 & (A^-)^*A^- \end{pmatrix}.$$

Observe that

$$\lim_{t\to\infty} e^{-t(A^\pm)^*A^\pm} = \Pi_\pm$$

with $\Pi^\pm$ denoting the orthogonal projection to $\ker A^\pm$. Because of this

$$\operatorname{index} A = \lim_{t\to\infty} \operatorname{str}(e^{-tA^2}) = \operatorname{str}\begin{pmatrix} \Pi_+ & 0 \\ 0 & \Pi_- \end{pmatrix}.$$

We have

$$\partial_t \operatorname{str}(e^{-tA^2}) = \operatorname{str}(A^2 e^{-tA^2}) = \operatorname{str}([Ae^{-\frac{1}{2}tA^2}, Ae^{-\frac{1}{2}tA^2}]_s) = 0.$$

with $[\cdot,\cdot]_s$ denoting the $\mathbf{Z}_2$–graded commutator. Therefore,

$$\operatorname{index} A = \operatorname{str}(e^{-tA^2})$$

for any t. Now, it turns out that in the infinite dimensional setting for $A = D$ the same reason goes through (once one makes sense of e^{-tD^2}, the trace, etc.). Moreover, one can compute $\lim_{t\to 0}\operatorname{str}(e^{-tD^2})$ as a integral of certain characteristic classes.

17 Spectral theory of Dirac operators

Theorem 17.1. *There is a complete orthonormal basis $(\phi_n)_{n\in\mathbf{N}}$ of $L^2\Gamma(S)$, which consists of smooth sections of S, and sequence of real numbers $(\lambda_n)_{n\in\mathbf{N}}$ such that*

$$D\phi_n = \lambda_n\phi_n \quad \textit{and} \quad \lim_{n\to\infty}|\lambda_n| = +\infty.$$

The above are unique up to renumbering.

Definition 17.2. The set of λ_n is called the **spectrum** of D and is denoted by $\mathrm{spec}(D)$.

Remark 17.3. One can prove a similar result directly for D, but it turns out we only need the result for D^2. Indeed, the proof of the above result is somewhat simpler.

We require following well-known result from Hilbert space theory.

Theorem 17.4 (Spectral theorem for compact self-adjoint operators). *Let H be a Hilbert space. Let $T\colon H \to H$ be a compact self-adjoint operator. There exists a complete orthonormal basis $(x_n)_{n\in\mathbf{N}}$ and a sequence of real numbers $(\lambda_n)_{n\in\mathbf{N}}$ such that*

$$Tx_n = \lambda_n x_n \quad \textit{and} \quad \lim_{n\to\infty}\lambda_n = 0.$$

The above are unique up to renumbering.

Exercise 17.5. Prove Theorem 17.4.

Proof of Theorem 17.1.

Step 1. *$D^2 + c\colon W^{1,2} \to W^{-1,2}$ is invertible.*

It follows from the Weitzenböck formula that, for $c \gg 1$,

$$\|\phi\|_{W^{1,2}} \leqslant \sup\left\{\langle(D^2+c)\phi,\psi\rangle_{L^2} : \|\psi\|_{W^{1,2}} \leqslant 1\right\} = \|(D^2+c)\phi\|_{W^{-1,2}}. \lesssim \|\phi\|_{W^{1,2}}$$

Therefore, $D^2 + c$ has trivial kernel and closed image. It also follows that $D^2 + c$ is surjective. The above shows that the standard inner product on $W^{1,2}$ is equivalent to

$$\langle Ds, Dt\rangle_{L^2} + c\langle s,t\rangle_{L^2}.$$

Consequently, it follows from the Riesz representation theorem, that $D^2 + c$ is surjective.

Step 2. *Application of the spectral theorem to the resolvent.*

The resolvent $R\colon L^2 \longrightarrow L^2$ is defined as the composition

$$L^2 \longrightarrow W^{-1,2} \xrightarrow{(D^2+c)^{-1}} W^{1,2} \longrightarrow L^2.$$

It is compact and self-adjoint. Consequently, Theorem 17.4 yields a complete orthonormal system (ϕ_n) and a null-sequence (μ_n) of positive real numbers such that

$$R\phi_n = \mu_n \phi_n.$$

Step 3. *Completion of the proof.*

The above can be rewritten as

$$D^2\phi_n = \lambda_n^2 \phi_n \quad \text{with} \quad \lambda_n^2 = (1/\mu_n - c).$$

Since $D^2 = D^*D$, $\lambda_n \geqslant 0$. Since $\mu_n \longrightarrow 0$, $\lambda_n \longrightarrow +\infty$. By elliptic regularity ϕ_n is smooth.

Recall that: If $A^2x = \lambda^2 x$, then $x^{\pm} = \pm\lambda x + Ax$ satisfy

$$Ax^{\pm} = \pm\lambda x^{\pm}.$$

This means that the eigenspinors for D^2 determine the eigenspinors for D. The eigenspaces for different eigenvalues of D perpendicular. For the eigenspaces themselves, are spanned by smooth sections which can be renormalized to be orthonormal by Gram–Schmidt. □

18 Functional Calculus of Dirac Operators

Let D be a Dirac operator. For any $\phi \in L^2\Gamma(S)$, write

(18.1)
$$\phi = \sum_{\lambda \in \operatorname{spec}(D)} \phi_\lambda$$

with ϕ_λ denoting the L^2–orthogonal projection to the λ–eigenspace of D. We understand the right-hand side as a series in $L^2\Gamma(S)$.

Proposition 18.2. *A section $\phi \in L^2\Gamma(S)$ is smooth if and only if*

$$\|\phi_\lambda\|_{L^2} = O(|\lambda|^\ell)$$

for all $\ell \geqslant 0$.

Proof. The section ϕ_λ is an eigensection of D with eigenvalue λ. Therefore, by elliptic regularity we have

$$\|\phi_\lambda\|_{W^{k,2}} \lesssim_k \lambda^k \|\phi_\lambda\|_{L^2}.$$

It follows that the right-hand side of (18.1) converges in $W^{k,2}\Gamma(S)$. Since k is arbitrary, it follows that the right-hand side is smooth.

Conversely, if ϕ is smooth, then $D^k\phi \in L^2$ for all k. Therefore,

$$\sum_{\lambda \in \mathrm{spec}(D)} \lambda^k \phi_\lambda$$

is L^2 summable for all k. Hence,

$$\sum_{\lambda \in \mathrm{spec}(D)} \lambda^{2k} \|\phi_\lambda\|_{L^2} < \infty.$$

This implies the asserted decay. □

Proposition 18.3. *Set*

$$L^\infty(\mathbf{R}) \coloneqq \{f\colon\ \mathrm{spec}(D) \to \mathbf{R} : f \text{ is bounded}\}.$$

For every $f \in L^\infty(\mathbf{R})$, there is a constant $c > 0$ such that, for all $s \in L^2\Gamma(S)$, the series

$$f(D)\phi = \sum_{\lambda \in \mathrm{spec}(D)} f(\lambda)\phi_\lambda$$

converges in L^2 and its value satisfies

$$\|f(D)\phi\| \leqslant c\|\phi\|_{L^2}.$$

Proposition 18.4 (Bounded Functional Calculus).

1. *The map $f \mapsto f(D)$ is homomorphism $L^\infty(\mathbf{R}) \to \mathscr{L}(L^2\Gamma(S))$ of unital Banach algebras.*
2. *If $T \in \mathscr{L}(L^2\Gamma(S))$ commutes with D, then it also commutes with $f(D)$.*
3. *If ϕ is smooth, then $f(D)\phi$ is smooth.*
4. *If f has rapid decay, that is, it satisfies*

   $$|f(\lambda)| = O(|\lambda|^{-k})$$

   for all $k \geqslant 1$, then $f(D)\phi$ is smooth for any $\phi \in L^2\Gamma(S)$.

Proposition 18.5. *(1) is a simple computation.*

(2) follows from the fact that if T commutes with D then it must preserve the eigenspaces of D.

(3) and (4) are consequences of Proposition 18.2.

Definition 18.6. Let $\pi_i\colon M \times M \to M$ denote the projection onto the i−th factor. Set

$$S \boxtimes S^* := \pi_1^* S \otimes \pi_2^* S^*.$$

A **smooth kernel** is a section $k \in \Gamma(S \boxtimes S^*)$. To any smooth kernel we associate the operator $K \in \mathscr{L}(L^1\Gamma(S), L^\infty\Gamma(S))$ defined by

$$(K\phi)(x) := \int_M k(x,y)s(y).$$

We say that an operator admits a smooth kernel if it is of the form K for some smooth kernel k.

Exercise 18.7. Proof that for any $\phi \in L^1\Gamma(S)$, $K\phi$ is smooth.

Proposition 18.8. *If f has rapid decay, then there exists a smooth kernel k such that*

$$f(D) = K.$$

Proof. Fix an L^2 orthonormal eigenbasis (ϕ_n) of D with eigenvalues λ_n. We can write

$$\begin{aligned}(f(D)\psi)(x) &= \sum_{n\in\mathbf{N}} f(\lambda_n)\phi_n(x)\langle\phi_n,\psi\rangle_{L^2}\\ &= \sum_{n\in\mathbf{N}} f(\lambda_n)\phi_n(x)\int_M\langle\phi_n(y),\psi(y)\rangle\\ &= \int_M\sum_{n\in\mathbf{N}} f(\lambda_n)\phi_n(x)\langle\phi_n(y),\psi(y)\rangle.\end{aligned}$$

Since f is rapidly decaying,

$$k(x,y) := \sum_{n\in\mathbf{N}} f(\lambda_n)\phi_n(x)\langle\phi_n(y),\cdot\rangle.$$

converges and defines a smooth kernel. □

19 The heat kernel associated Dirac of operator

Proposition 19.1.

1. *Let $\phi_0 \in L^2\Gamma(S)$. There exists a unique $\phi \in \Gamma((0,\infty)\times M, S)$ such that*

$$(\partial_t + D^2)\phi = 0 \quad \textit{and} \quad \lim_{t\to 0}\|\phi(t,\cdot) - \phi_0\|_{L^2} = 0.$$

2. *If $\phi_0 \in \Gamma(S)$, then $\phi \in \Gamma(([0,\infty)\times M, S)$ and $\phi(0,\cdot) = \phi_0$.*

Proof. We first prove uniqueness. If ϕ satisfies $(\partial_t + D^2)\phi = 0$, then

$$\begin{aligned}\partial_t \|\phi_t\|_{L^2}^2 &= -2\langle \partial_t \phi_t, \phi_t\rangle_{L^2} \\ &= -2\|D\phi_t\|_{L^2}^2 \leqslant 0.\end{aligned}$$

This implies uniqueness because if ϕ and ψ satisfy the heat equation with initial condition, then $\delta = \phi - \psi$ satisfies the heat equation with initial condition 0 and thus vanishes.

We establish existence. Define

$$\phi(t, x) := (e^{-tD^2}\phi_0)(x).$$

Here e^{-tD^2} is obtained using bounded functional calculus for $f_t(\lambda) = e^{-t\lambda^2}$. Since bounded functional calculus is continuous, we have $\lim_{t\to 0}\|\phi(t,\cdot) - \phi_0\|_{L^2} = 0$.

For fixed $t > 0$, $\phi(t,\cdot)$ is smooth because f_t is rapidly decaying. It also depends smoothly on t, which can be seen as follows. Since $f \mapsto f(D)$ is a homomorphism of Banach algebras, we can take the limit of

$$\frac{\phi_t(x) - \phi_{t+\varepsilon}(x)}{\varepsilon} = \frac{e^{-tD^2} - e^{-(t+\varepsilon)D^2}}{\varepsilon}\phi_0$$

as $\varepsilon = 0$ and deduce that

$$\partial_t \phi_t = -D^2 e^{-tD^2}\phi_0 = -D^2\phi_t.$$

Repeated applications of this argument show that $\partial_t^k \phi_t = -D^{2k}\phi_t$. This proves ϕ is smooth and satisfies the heat equation.

If ϕ_0 is smooth, the above argument also works at $t = 0$. This establishes the second part of the proposition. ◻

Proposition 19.2. *There exists a unique $k_t \in C^1((0,\infty), C^2\Gamma(S \boxtimes S^*))$ such that for all $\phi \in \Gamma(S)$ the following holds:*

1. *$\Phi(t, x) := (K_t\phi)(x)$ satisfies the heat equation*

$$(\partial_t + D^2)\Phi = 0.$$

2. *For all $\phi \in \Gamma(S)$, $\lim_{t\to 0}\|K_t\phi - \phi\|_{L^\infty} = 0$.*

In fact, such a k_t is a smooth.

Definition 19.3. We call k_t the **heat kernel** associated of D.

Proof. The uniqueness of k_t follows from the uniqueness of the solution to the heat equation.

From the proof of Proposition 19.1 it is clear that $K_t = e^{-tD^2}$ and k_t is the kernel associated with K_t via Proposition 18.8. The argument used to establish smoothness of ϕ in t in the proof of Proposition 19.1 also proves that k is smooth in t. ◻

Proposition 19.4 (Duhamel's Principle). *Let $\psi \in C^0((0,\infty), C^2\Gamma(S))$.*

1. *There exists a unique $\phi \in C^1([0,\infty), C^2\Gamma(S))$ satisfying*

$$(\partial_t + D^2)\phi = \psi \quad \text{and} \quad \phi_0 = 0.$$

2. *It is given by*

$$\phi_t = \int_0^t e^{-(t-\tau)D^2}\psi_\tau \, \mathrm{d}\tau.$$

3. *For each $\ell \geqslant 0$, we have*

$$\|\phi_t\|_{W^{\ell,2}} \lesssim_\ell t \sup_{\tau\in[0,t]} \|\psi_t\|_{W^{\ell,2}}.$$

Proof. Uniqueness is a consequence of the uniqueness statement in Proposition 19.1. We have $\phi_0 = 0$ and

$$\begin{aligned}\partial_t\phi_t &= \psi_\tau + \int_0^t -D^2 e^{-(t-\tau)D^2}\psi_\tau \, \mathrm{d}\tau \\ &= \psi_\tau - D^2\phi_t.\end{aligned}$$

This ϕ_t solves the inhomogenous heat equation.

It remains to prove the estimate. We clearly, have

$$\|\phi_t\|_{W^{\ell,2}} \lesssim_\ell t \sup_{\tau\in[0,t]} \left\| e^{-(t-\tau)D^2}\psi_t \right\|_{W^{\ell,2}}.$$

Since $e^{-\sigma D^2} \colon L^2\Gamma(S) \to L^2\Gamma(S)$ is bounded independent of σ and by elliptic estimates, we have

$$\begin{aligned}\|e^{-\sigma D^2}\psi\|_{W^{\ell,2}} &\lesssim \|D^\ell e^{-\sigma D^2}\psi\|_{L^2} + \|e^{-\sigma D^2}\psi\|_{L^2} \\ &= \|e^{-\sigma D^2}D^\ell\psi\|_{L^2} + \|e^{-\sigma D^2}\psi\|_{L^2} \\ &\lesssim \|D^\ell\psi\|_{L^2} + \|\psi\|_{L^2} \\ &\lesssim \|\psi\|_{W^{\ell,2}}.\end{aligned}$$

This means that Since $e^{-\sigma D^2} \colon W^{\ell,2}\Gamma(S) \to W^{\ell,2}\Gamma(S)$ is bounded independent of σ. From this the asserted estimate follows. □

20 Asymptotic Expansion of the Heat Kernel

Definition 20.1. Set

$$k_t^0 := \frac{1}{(4\pi t)^{n/2}} e^{-d(\cdot,\cdot)^2/4t}.$$

The proof of the index theorem which will discuss is based on rather carefully understanding the heat kernel. The heat kernel for the Laplacian Δ on $\mathbf{R}^n$ is given by

$$k_t^{\mathbf{R}^n}(x,y) := \frac{1}{(4\pi t)^{n/2}} e^{-d_{\mathbf{R}^n}(x,y)^2/4t} \quad \text{with} \quad d_{\mathbf{R}^n}(x,y) = |x-y|.$$

Knowing this and the Weitzenböck formula Proposition 9.2

$$D^2 = \nabla^*\nabla + \mathscr{F}_S$$

one might guess that k_t is approximately

$$k_t^0 = \frac{1}{(4\pi t)^{n/2}} e^{-d(\cdot,\cdot)^2/4t}$$

where d denotes the Riemannian distance on M. It turns out that this is true. In fact, one can do better and find a precise asymptotic expansion of k_t at $t = 0$ with leading term k_t^0.

Definition 20.2. Let X be a Banach space. Let $f\colon (0,\infty) \to X$ be a function. If $a_i\colon (0,\infty) \to X$ $i \in \mathbf{N}_0$ are functions such that for all $n \in \mathbf{N}_0$ there is an $m_0 = m_0(n)$ such that for all $m \geqslant m_0$

$$\left\| f(t) - \sum_{i=1}^{m} a_i(t) \right\| \lesssim_{n,m} |t|^n \quad \text{for} \quad t \ll_{n,m} 1,$$

then we say that (a_i) are an **asymptotic expansion** of f at $t = 0$ and write

$$f(t) \sim \sum_{i=0}^{\infty} a_k(t) \quad \text{as} \quad t \to 0.$$

Remark 20.3. The definition of asymptotic expansion makes no reference to convergence of the series $\sum_{i=0}^{\infty} a_i(t)$. If $f\colon \mathbf{R} \to X$ is a smooth function, then its Taylor expansion at 0 is is an asymptotic expansion

$$f(t) \sim \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} t^i \quad \text{as} \quad t \to 0.$$

However, the right-hand side converges only if f is analytic near zero.

Remark 20.4. Asymptotic expansion are in no way unique!

Theorem 20.5. *Let M be a compact Riemannian manifold. Let S be a Clifford bundle over M with Dirac operator D. Denote by k_t the heat kernel of M.*

1. *There are $\Theta_i \in \Gamma(S \boxtimes S^*)$ such that*
$$k_t \sim \sum_{i=0}^{\infty} \frac{t^i}{(4\pi t)^{n/2}} e^{-d(\cdot,\cdot)^2/4t}\Theta_i \quad as \quad t \to 0.$$
is an asymptotic expansion at $t = 0$ of $k\colon (0,\infty) \to C^r\Gamma(S \boxtimes S^*)$. for all $r \in \mathbf{N}_0$.

2. *$\Theta_0(x,x) = \mathrm{id}_S$ and $\Theta_1(x,x) = \frac{1}{6}\mathrm{scal}_g(x) - \mathscr{F}_S(x)$ with $\mathscr{F}_S$ as in Proposition 9.2.*

3. *The section $x \mapsto \Theta_j(x,x)$ can be computed in terms of algebraic expressions involving the metric, connection coefficients, and their derivatives.*

The proof requires some preparation.

Definition 20.6. Let $m \in \mathbf{N}_0$. An **approximate heat kernel** of order m is a time-dependent kernel $\tilde{k}_t$ such that

1. $\Phi(t,x) := (\tilde{K}_t\phi)(x)$ satisfies the heat equation
$$(\partial_t + D^2)\tilde{k}_t = t^m r_t$$
with $r_t \in C^0([0,\infty), C^m\Gamma(S \boxtimes S^*))$.

2. For all $\phi \in \Gamma(S)$, $\lim_{t\to 0}\|\tilde{K}_t\phi - \phi\|_{L^\infty} = 0$.

Proposition 20.7. *Let k_t be the heat kernel of D. Let $m \in \mathbf{N}_0$. If $\tilde{k}_t$ is an approximate heat kernel of D to order $\tilde{m} \geqslant m + \dim M/2$, then*
$$k_t - \tilde{k}_t = t^m e_t$$
with $e_t \in C^0([0,\infty), C^m\Gamma(S \boxtimes S^*))$.

Proof. Write
$$(\partial_t + D_x^2)\tilde{k}_t(x,y) = t^{\tilde{m}} r_t(x,y)$$
With $r_t \in C^0([0,\infty), C^{\tilde{m}}\Gamma(S \boxtimes S^*))$. By Proposition 19.4, there is a unique q_t such that
$$(-\partial_t + D_x^2)q_t(x,y) = -t^{\tilde{m}} r_t(x,y) \quad \text{and} \quad q_0 = 0.$$
Moreover,
$$\|q_t\|_{W^{\tilde{m},2}} \lesssim t^{\tilde{m}+1}.$$
From the uniqueness of the heat kernel it follows that
$$k_t - \tilde{k}_t = q_t.$$
By Theorem 15.6, the desired estimate on $e_t = t^{-m}q_t$ follows. □

In light of this proposition, we need to find Θ_i such that for every m there exists an $n_0 = n_0(m)$ such that for $n \geqslant n_0$

$$\frac{1}{(4\pi t)^{n/2}} e^{-d(\cdot,\cdot)^2/4t} \sum_{i=0}^{n} t^i \Theta_i$$

is an approximate heat kernel to order m.

Let us first analyze to what what extend k_t^0 fails to be a heat kernel.

Proposition 20.8. *Let $y \in M$. Fix normal coordinates in a neighborhood U of y in M. Set*

$$g = \det(g_{ij}).$$

In U, we have

$$\nabla k_t^0(\cdot, y) = -\frac{k_t^0(\cdot, y)}{2t} r\partial_r \quad \text{and} \quad (\partial_t + \Delta) k_t^0(\cdot, y) = \frac{r k_t^0(\cdot, y)}{4gt} \partial_r g.$$

Proof. We have

$$\nabla k_t^0 = -\frac{k_t^0}{4t} \nabla d(\cdot, y)^2 = -\frac{k_t^0}{4t} \nabla r^2 = -\frac{k_t^0}{2t} r\partial_r.$$

This proves the first identity.

To prove the second identity, we compute

$$\partial_t k_t^0(\cdot, y) = \left(-\frac{n}{2t} + \frac{r^2}{4t^2}\right) k_t^0(\cdot, y)$$

and

$$\begin{aligned}
\Delta k_t^0(\cdot, y) &= \nabla^* \nabla k_t^0(\cdot, y) \\
&= \nabla^* \left(-\frac{k_t^0(\cdot, y)}{2t} r\partial_r\right) \\
&= \nabla^* \left(-\frac{k_t^0(\cdot, y)}{2t} r\partial_r\right) \\
&= \frac{r\partial_r k_t^0(\cdot, y)}{2t} - \frac{k_t^0(\cdot, y)}{2t} \nabla^*(r\partial_r) \\
&= -\frac{r^2}{4t^2} k_t^0(\cdot, y) - \frac{k_t^0(\cdot, y)}{2t} \left(-n - \frac{r}{2g}\partial_r g\right) \\
&= \left(-\frac{r^2}{4t^2} + \frac{n}{2t} + \frac{r}{4gt}\partial_r g\right) k_t^0(\cdot, y).
\end{aligned}$$

Here we used that

$$\nabla^*(r\partial_r) = -g^{-1/2} \sum_{i=1}^{n} \partial_i (g^{1/2} x_i) = -n - \frac{r}{2g}\partial_r g.$$

□

Proof of Theorem 20.5. Let U be a neighborhood of the diagonal $\{(x,x) \in M \times M : x \in M\} \subset M \times M$ such that if $(x,y) \in \bar{U}$, then $d(x,y)$ is less than $\varepsilon > 0$, which itself is less the half injectivity radius. Let $\chi \in C^\infty(M \times M, [0,1])$ supported in U and with $\chi(x,x) = 1$ for all $x \in M$. We will construct Θ_i of the form $\chi\tilde{\Theta}_i$ with $\tilde{\Theta}_i$ defined on $\bar{U}$.

Pick $y \in M$ and choose normal coordinates on $B_{2\varepsilon}(y)$. If $\tilde{\Theta}_y$ is a section of $S \otimes S_y^*$, then by Proposition 20.8 and Proposition 9.2

$$(\partial_t + D^2)(k_t^0(\cdot,y)\tilde{\Theta}_y) = k_0^t \left(\partial_t \tilde{\Theta}_y + D^2 \tilde{\Theta}_y + \frac{r}{4gt} \partial_r g \cdot \tilde{\Theta}_y + \frac{1}{t} \nabla_{r\partial_r} \tilde{\Theta}_y \right).$$

The last term arises from $\langle \nabla k_t^0, \nabla \tilde{\Theta}_y \rangle$.

We make the ansatz that $\tilde{\Theta}_y$ is a *formal power series* in t; that is:

$$\tilde{\Theta}_y = \sum_{i=0}^{\infty} t^i \tilde{\Theta}_{i,y}$$

with $\tilde{\Theta}_{i,y}$ smooth and independent of t. We set $\tilde{\Theta}_{-1,y} = 0$. The condition that this formal power series is such that

$$(\partial_t + D^2)(k_t^0(\cdot,y)\tilde{\Theta}_y) = 0$$

is simply that

$$\nabla_{r\partial_r} \tilde{\Theta}_{i,y} + \left(i + \frac{r}{4g} \partial_r g \right) \tilde{\Theta}_{i,y} = -D^2 \tilde{\Theta}_{i-1,y}$$

or, equivalently,

$$\nabla_{\partial_r} \left(r^i g^{1/4} \tilde{\Theta}_{i,y} \right) = -r^{i-1} g^{1/4} D^2 \tilde{\Theta}_{i-1,y}.$$

Fixing $\tilde{\Theta}_{0,y}(x) = \mathrm{id}_{S_y}$, the ODE for $\tilde{\Theta}_{0,y}$ has a unique solution. Recursively, we can solve the ODE's for $\tilde{\Theta}_{i,y}$ for $i \in \mathbf{N}_0$. At each stage $\tilde{\Theta}_{i,y}$ is determined uniquely up to constant multiple of a term which is of order r^{-i} near x. Since we require $\tilde{\Theta}_{i,y}$ to be smooth, this term must vanish. We define

$$\Theta_i(x,y) \coloneqq \chi(x,y)\tilde{\Theta}_{i,y}(x).$$

We need to see that for each $m \in \mathbf{N}_0$ there is an $N_0 = N_0(m)$ such that for $N \geqslant N_0$

$$k_t^N \coloneqq k_t^0 \sum_{i=0}^{N} t^i \Theta_i$$

is a an approximate heat kernel of order m.

Since $\Theta_i(x,x) = \mathrm{id}_{S_x}$ and $k_t^0(x,y) \to \delta_{x,y}$ as $t \to 0$, it follows that $k_t^n(x,y) \to \delta_{x,y}$ as $t \to 0$.

By construction of k_t^N we have

$$(\partial_t + D^2) k_t^N(\cdot,y) = t^N k_t^0(\cdot,y) e_t^N(\cdot,y)$$

where e_t^N is smooth. If for $N > 2m + n/2$, $t^N k_t^0 = O(t^m)$ in C^m. Thus k_t^N is an approximate heat kernel of order m. This completes the construction of the asymptotic expansion of k_t.

The assertion about the computability of $\Theta_i(x,x)$ should be clear from the construction.

By construction $\Theta_0(x,x) = \mathrm{id}_{S_x}$. It remains to compute $\Theta_1(x,x)$. We have

$$\tilde{\Theta}_{0,y} = g^{-1/4}$$

since this solves the ODE and is id_{S_y} at 0. By construction

$$\nabla_{\partial_r}\left(rg^{1/4}\tilde{\Theta}_{1,y}\right) = -D^2\tilde{\Theta}_{0,y}$$

From this it follows that

$$\tilde{\Theta}_{1,y}(y) = (-D^2\tilde{\Theta}_{0,y})(y).$$

By Proposition 9.2 the right-hand side is

$$-\Delta(g^{1/4}) - \mathscr{F}_S(y).$$

We have

$$g_{ij} = \delta_{ij} + \frac{1}{3}\sum_{k,\ell} R_{ik\ell j}x_k x_\ell + O(|x|^3)$$

and consequently

$$g^{-1/4} = \det(g_{ij})^{-1/4} = 1 - \frac{1}{12}\sum_{i,k,\ell} R_{ik\ell i}x_k x_\ell + O(|x|^3).$$

Thus

$$\Delta(g^{-1/4}) = -\frac{1}{12}\sum_{i,k,\ell} R_{ikki} = \frac{1}{6}\mathrm{scal}.$$

□

21 Trace-class operators

Proposition 21.1. *Let X and Y be two separable Hilbert spaces. Let (e_i) and (f_j) orthonormal bases of X and Y, respectively. Given a bounded linear operator $A\colon X \to Y$,*

$$\|A\|_{HS} = \sum_{i,j} |\langle Ae_i, e_j\rangle|^2 \in [0,\infty]$$

is independent of the choice of bases. Moreover,

$$\|A\|_{HS} = \|A^*\|_{HS}.$$

Proof. We have

$$\sum_i \|Ae_i\|^2 = \sum_{i,j} |\langle Ae_i, f_j\rangle|^2$$
$$= \sum_{i,j} |\langle e_i, A^* f_j\rangle|^2$$
$$= \sum_j \|A^* f_j\|^2.$$

The left-hand side manifestly is independent of (f_j) while the right-hand site is manifestly independent of (e_i). ◻

Definition 21.2. If $A \in \mathscr{L}(X, Y)$ satisfies $\|A\|_{HS} < \infty$, then A is called a **Hilbert–Schmidt operator** and $\|A\|_{HS}$ is called its **Hilbert–Schmidt norm**.

Proposition 21.3.

1. *The set of Hilbert–Schmidt operators is a Hilbert spaces with respect to the inner product*

$$\langle A, B\rangle_{HS} = \sum_{ij} \langle f_j, Ae_i\rangle \langle Be_i, f_j\rangle$$

2. *For $A \in \mathscr{L}(X, Y)$,*

$$\|A\|_{\mathscr{L}} \leqslant \|A\|_{HS}.$$

3. *Hilbert–Schmidt operators are compact.*
4. *If A is Hilbert–Schmidt and B is bounded, then AB and BA (whenever they are defined) are Hilbert–Schmidt.*

Definition 21.4. We say that $T \in \mathscr{L}(X)$ is of **trace-class** if there are Hilbert–Schmidt operators A and B such that

$$T = AB.$$

The **trace** of a trace-class operator is

$$\operatorname{tr}(T) = \langle A^*, B\rangle_{HS} = \sum_j \langle Te_j, e_j\rangle.$$

Proposition 21.5. *If T is self-adjoint and of trace-class, then* $\operatorname{tr} T$ *is the sum of the eigenvalues of T.*

Proposition 21.6. *If T is of trace-class and B is bounded or T and B are both Hilbert–Schmidt, then BT and TB are of trace-class and*

$$\operatorname{tr}(TB) = \operatorname{tr}(BT).$$

Proposition 21.7. *If k is a continuous kernel, then K is Hilbert–Schmidt and*

$$\|K\|_{HS}^2 = \int_{M\times M} |k|^2.$$

Proof. Let (e_i) be an orthonormal basis for $L^2\Gamma(E)$. Then $(e_j \boxtimes e_i^*)$ is an orthonormal basis for $L^2\Gamma(E \boxtimes E^*)$. We have

$$\begin{aligned}
\|K\|_{HS}^2 &= \sum_{i,j} \left|\langle Ke_i, e_j\rangle_{L^2}\right|^2 \\
&= \sum_{i,j} \left|\int_{M\times M} \langle k(x,y)e_i(y), e_j(x)\rangle \mathrm{d}y\mathrm{d}x\right|^2 \\
&= \sum_{i,j} \left|\int_{M\times M} \langle k(x,y), e_j(x)e_i^*(y)\rangle \mathrm{d}y\mathrm{d}x\right|^2 \\
&= \sum_{i,j} \left|\int_{M\times M} \langle k(x,y), e_j(x)e_i^*(y)\rangle \mathrm{d}y\mathrm{d}x\right|^2 \\
&= \|k(x,y)\|_{L^2}^2 \\
&= \int_{M\times M} |k|^2. \qquad \square
\end{aligned}$$

Proposition 21.8. *If k is a smooth kernel, then K is of trace-class and*

$$\operatorname{tr} K = \int_M \operatorname{tr} k(x,x)\mathrm{d}x.$$

Proof. For $N \gg 1$, $(\Delta + 1)^{-N}$ has a continuous kernel g (and therefore is Hilbert–Schmidt). The operator

$$H = (\Delta + 1)^N K$$

is a smoothing operator which has some kernel h, which is self-adjoint. In terms of g and k, we have

$$k(x,y) = \int_M g(x,z)h(z,y)\mathrm{d}z.$$

Therefore, K is trace-class and

$$\begin{aligned}
\operatorname{tr} K &= \int_{M\times M} \langle g(x,z), h(z,x)\rangle \mathrm{d}z\mathrm{d}x \\
&= \int_{M\times M} \operatorname{tr}(g(x,z)h(z,x))\mathrm{d}z\mathrm{d}x \\
&= \int_M \operatorname{tr} k(x,x)\mathrm{d}x. \qquad \square
\end{aligned}$$

22 Digression: Weyl's Law

As an application of the asymptotic expansion of the heat kernel k_t we prove the following.

Definition 22.1. Denote by $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ the eigenvalues of D^2. Define $N\colon [0,\infty) \to \mathbf{N}_0$ by

$$N(\lambda) := \max\{k \in \mathbf{N}_0 : \lambda_k \leqslant \lambda\}.$$

Theorem 22.2 (Weyl). *We have*

$$N(\lambda) \sim \frac{\operatorname{rk} S \cdot \operatorname{vol}(M)}{(4\pi)^{n/2}\Gamma(n/2+1)}\lambda^{n/2} \quad \textit{as} \quad \lambda \to \infty.$$

and

$$\lambda_k \sim 4\pi \left(\frac{\operatorname{rk} S \cdot \operatorname{vol}(M)}{\Gamma(n/2+1)}\right)^{2/n} k^{2/n} \quad \textit{as} \quad k \to \infty.$$

Proof. It follows from Theorem 20.5 that

$$\lim_{t\to 0} t^{n/2} \sum_{k=0}^{\infty} e^{-t\lambda_k} = \frac{\operatorname{vol}(M)}{(4\pi)^{n/2}}.$$

This, in fact, applies the asserted statement about $N(\lambda)$ by the following result. □

Theorem 22.3 (Karamata). *If (λ_k) is an increasing sequence such that*

$$\lim_{t\to 0} t^{\alpha} \sum_{k=0}^{\infty} e^{-t\lambda_k} = A$$

then

$$N(\lambda) \sim A\lambda^{\alpha}/\Gamma(\alpha+1) \quad \textit{as} \quad \lambda \to \infty.$$

Proof. For any continuous function f on $[0,1]$, define

$$\phi_f(t) := \sum_{k=0}^{\infty} f(e^{-t\lambda_k})e^{-t\lambda_k}.$$

We have

$$\lim_{t\to 0} t^{\alpha}\phi_f(t) = \frac{A}{\Gamma(\alpha)} \int_0^{\infty} f(e^{-s})s^{\alpha-1}e^{-s}\,\mathrm{d}s.$$

Since and f can be approximated by polynomials and since everything is linear in f, it suffices to prove this for $f(x) = x^m$. The left-hand side is then

$$\lim_{t\to 0} t^{\alpha} \sum_{k=0}^{\infty} e^{-(m+1)t\lambda_k} = \lim_{t\to 0}(t/(k+1))^{\alpha} \sum_{k=0}^{\infty} e^{-t\lambda_k} = (k+1)^{-\alpha}A$$

while the right-hand side is

$$\frac{A}{\Gamma(\alpha)}\int_0^\infty e^{-(k+1)s}s^{\alpha-1}e^{-s}\,\mathrm{d}s = (k+1)^{-\alpha}A$$

(by a computation).

Now for $r \in [0,1)$, define f_r such that f_r vanishes on $[0, r/e]$, is affine on $[r/e, 1/e]$, and $f_r(x) = 1/x$ for $x \in [1/e, 1]$. We have

$$\phi_{f_r}(1/r\lambda) \leqslant N(\lambda) \leqslant \phi_{f_r}(1/\lambda).$$

Consequently,

$$\frac{Ar^\alpha}{\alpha\Gamma(\alpha)} \leqslant \liminf \lambda^{-\alpha}N(\lambda) \leqslant \limsup \lambda^{-\alpha}N(\lambda) \leqslant \frac{A^\alpha}{\alpha\Gamma(\alpha)}.$$

Taking the limit $r \to 1$ proves the result. □

23 Digression: Zeta functions

Proposition 23.1. *Let D be a Dirac operator and denote by $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$ the eigenvalues of D^2 counted with multiplicity. Suppose that zero is not an eigenvalue. For $\operatorname{Re} s > n/2$, the series*

$$\zeta_D(s) := \sum_{k=1}^\infty \lambda_k^{-s}.$$

converges. The series extends to a meromorphic function on all of $\mathbf{C}$ with poles contained in $n/2 - \mathbf{N}_0$. The function is holomorphic at 0 and its value is given by

$$\zeta_D(0) = \frac{1}{(4\pi)^{n/2}}\int_M \operatorname{tr}\Theta_{n/2}.$$

Definition 23.2. We call ζ_D the **zeta function** of D.

It will be useful to recall(?) some some properties of the Mellin transform.

Definition 23.3. Given $f \in C^\infty(0,\infty)$, its **Mellin transform** is defined as

$$M(f)(s) := \Gamma(s)^{-1}\int_0^\infty f(t)t^{s-1}\mathrm{d}t.$$

Proposition 23.4. *Let $f \in C^\infty(0,\infty)$ have an asymptotic expansion of the form*

$$f \sim \sum_{j=0}^{\infty} a_j t^{-n/2+j}$$

and suppose that $|f|(t) \lesssim e^{-ct}$. In this situation, the following hold:

1. *$M(f)$ converges for $\operatorname{Re} s > n/2$.*
2. *$M(f)$ has a meromorphic extension to $\mathbf{C}$ with poles contained in $n/2 - \mathbf{N}_0$.*
3. *$M(f)$ is holomorphic at 0 and*

$$M(f)(0) = \begin{cases} a_{n/2} & n \in 2\mathbf{Z} \\ 0 & n \in 2\mathbf{Z}+1. \end{cases}$$

Proof. We write

$$\Gamma(s)M(f)(s) = \int_0^1 f(t)t^{s-1}\mathrm{d}t + \int_1^\infty f(t)t^{s-1}\mathrm{d}t.$$

Since $f(t)$ has exponential decay, the second integral converges and defines a entire function. Using the asymptotic expansion, the first integral can be written as

$$\begin{aligned}\int_0^1 f(t)t^{s-1}\mathrm{d} &= \sum_{j=0}^{k} a_j \int_0^1 t^{-n/2+j+s-1}\mathrm{d}t + r(s) \\ &= \sum_{j=0}^{k} \frac{a_j}{s+j-n/2} + r(s).\end{aligned}$$

Here $r(s)$ arises as

$$r(s) = \int_0^1 O(t^{k/2-n/2+s-1})\mathrm{d}t.$$

It is holomorphic for $\operatorname{Re} s > n/2 - k/2$. The first term is meromorphic and has poles contained in $n/2 - \mathbf{N}_0$. This proves the first two assertions since $\Gamma(s)^{-1}$ is entire.

Since $\Gamma(s)^{-1} = s + O(s^2)$ we have

$$\begin{aligned}M(f)(0) &= \left.\left(\sum_{j=0}^{k} \frac{sa_j}{s+j-n/2} + sr(s)\right)\right|_{s=0} \\ &= \begin{cases} a_{n/2} & n \in 2\mathbf{Z} \\ 0 & n \in 2\mathbf{Z}+1. \end{cases}\end{aligned}$$

This completes the proof. □

Proof of Proposition 23.1. By Theorem 22.2,

$$\lambda_k \sim c(M)k^{2/n} \quad \text{as} \quad \lambda \to \infty.$$

Therefore, if $\operatorname{Re} s > n/2$, then

$$\sum_{k=1}^{\infty} \left|\lambda_k^{-s}\right| \lesssim_M 1 + \sum_{k=1}^{\infty} k^{2s/n} < \infty.$$

Consequently, ζ_D is summable provided $\operatorname{Re} s > n/2$.

Denote by $\Gamma(s)$ the gamma function. Recall that $\Gamma(s)$ has poles at $-\mathbf{N}_0 \subset \mathbf{C}$ and that $\Gamma(s)^{-1}$ is entire (that is: holomorphic on all of $\mathbf{C}$). We have

$$\lambda^{-s} = \Gamma(s)^{-1} \int_0^\infty e^{-t\lambda} t^{s-1} \mathrm{d}t$$

Using this we can write

$$\begin{aligned} \zeta_D(s) &= \sum_{k=1}^{\infty} \Gamma(s)^{-1} \int_0^\infty e^{-t\lambda_k} t^{s-1} \mathrm{d}t \\ &= \Gamma(s)^{-1} \int_0^\infty \operatorname{tr}(e^{-tD^2}) t^{s-1} \mathrm{d}t. \end{aligned}$$

That is, ζ_D is the **Mellin transform** of $t \mapsto \operatorname{tr}(e^{-tD^2})$. The result now follows from Proposition 23.4 and Theorem 20.5. ☐

Definition 23.5. If D^2 has trivial kernel, we define its **determinant** by

$$\det(D^2) := e^{-\zeta_D'(0)}.$$

We have,

$$\zeta_D(s) = \sum_{k=1}^{\infty} \exp(-s \log \lambda_k);$$

hence,

$$\zeta_D'(s) = \sum_{k=1}^{\infty} \log \lambda_k \exp(-s \log \lambda_k).$$

Formally, evaluating at $s = 0$, we obtain

$$\zeta_D'(0) \text{ "=" } \sum_{k=1}^{\infty} \log \lambda_k \quad \text{and} \quad e^{\zeta_D'(0)} \text{ "=" } \prod_{k=1}^{\infty} \lambda_k.$$

The expressions on the right-hand side actually cannot be defined. The expressions on the left-hand side are “regularizations” of those on the right-hand side.

24 From the asymptotic expansion of the heat kernel to the index theorem

Let S be a $\mathbf{Z}_2$ graded Dirac bundle, that is $S = S^+ \oplus S^-$ and D decomposes as

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

Recall that,

$$\operatorname{index}_{\mathbf{Z}_2} D = \operatorname{index} D^+ = \dim\ker D^+ - \dim\ker D^-.$$

Theorem 24.1 (McKean–Singer). *Let Θ_i be as in Theorem 20.5. If n is odd, then* $\operatorname{index}_{\mathbf{Z}_2} = 0$. *If n is even, then*

$$\operatorname{index}_{\mathbf{Z}_2} D = \frac{1}{(4\pi)^{n/2}} \int_M \operatorname{str} \Theta_{n/2}.$$

Proof. The limit as $t \longrightarrow \infty$ of e^{-tD^2} is the orthogonal projection onto $\ker D$. Hence,

$$\lim_{t\longrightarrow\infty} \operatorname{str} e^{-tD^2} = \operatorname{index}_{\mathbf{Z}_2} D.$$

In fact, denoting by $m_\pm$ the dimension of the $S^\pm$ component of the eigenspace for $\lambda \in \operatorname{spec}(D^2)$, we have

$$\begin{aligned} \operatorname{str} e^{-tD^2} &= \sum_{\lambda\in\operatorname{spec}(D^2)} e^{-t\lambda}(m_+(\lambda) - m_-(\lambda)) \\ &= \operatorname{index}_{\mathbf{Z}_2} D + \sum_{\lambda\in\operatorname{spec}(D^2)\setminus\{0\}} e^{-t\lambda}(m_+(\lambda) - m_-(\lambda)). \end{aligned}$$

If $\psi \in \Gamma(S^\pm)$ is an eigenspinor for D^2 with eigenvalue $\lambda \neq 0$, then

$$D^2 D\psi = \lambda D\psi.$$

This gives a map from the λ eigenspace in $\Gamma(S^\pm)$ to the one in $\Gamma(S^\mp)$. Since $\lambda \neq 0$, this map is invertible. It follows that $m_+(\lambda) = m_-(\lambda)$ for $\lambda \neq 0$. Therefore, the second term in the above expression vanishes. It follows that $\operatorname{str}(e^{-tD^2})$ is independent of t and always computes $\operatorname{index}_{\mathbf{Z}_2} D$.

Using Theorem 20.5, we have

$$\begin{aligned} \lim_{t\longrightarrow 0} \operatorname{str} e^{-tD^2} &= \lim_{t\longrightarrow 0} \int_M \operatorname{str} k_t(x,x)\mathrm{d}x \\ &= \lim_{t\longrightarrow 0} \frac{1}{(4\pi)^{n/2}} \sum_{i=0}^{\lceil n/2\rceil} t^{i-n/2} \int_M \operatorname{str} \Theta_i(x,x)\mathrm{d}x. \end{aligned}$$

Since the left-hand side is finite, we must have

$$\int_M \operatorname{str} \Theta_i(x,x)\mathrm{d}x = 0 \quad \text{for} \quad i < n/2.$$

If n is odd, the remaining term is the limit $O(t^{1/2})$ and vanishes. If n is even, we have

$$\operatorname{index}_{\mathbf{Z}_2} D = \lim_{t\to 0} \operatorname{str} e^{-tD^2} = \frac{1}{(4\pi)^{n/2}} \int_M \operatorname{str} \Theta_{n/2}(x,x)\mathrm{d}x.$$

□

The task at hand is now to compute

$$\operatorname{str} \Theta_{n/2}(x,x).$$

Results computing this term are called local index theorems. In Theorem 20.5 we computed that

$$\Theta_1(x,x) = \frac{1}{6}\operatorname{scal}_g(x) - \mathscr{F}_S(x).$$

Here is an application.

Theorem 24.2 (Gauss–Bonnet). *If Σ is a closed Riemann surface, then*

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma \operatorname{scal}_g.$$

Proof. Consider the Dirac bundle

$$S = \Lambda^\bullet T^*\Sigma$$

with the $\mathbf{Z}_2$ grading given by the parity of the degree; that is:

$$S^+ = \Lambda^0 T^*\Sigma \oplus \Lambda^2 T^*\Sigma \quad \text{and} \quad S^- = \Lambda^1 T^*\Sigma$$

The Dirac operator is given by

$$D = \mathrm{d} + \mathrm{d}^*.$$

The term $\mathscr{F}_S$ in the Weitzenböck formula

$$\nabla^*\nabla = (\mathrm{d}+\mathrm{d}^*)^2 + \mathscr{F}_S$$

vanishes on 0 and 2–forms, and on 1–forms is given by the Ricci curvature Ric_g. It follows that

$$\Theta_1(x,x) = \begin{pmatrix} \frac{1}{6}\operatorname{scal}_g(x)\mathrm{id}_{S^+} & 0 \\ 0 & \frac{1}{6}\operatorname{scal}_g(x)\mathrm{id}_{S^-} - \operatorname{Ric}_g(x). \end{pmatrix}$$

Hence,

$$\operatorname{str} \Theta_1(x,x) = \operatorname{tr} \operatorname{Ric}_g(x) = \operatorname{scal}_g(x).$$

This implies the result. □

25 The local index theorem

Theorem 25.1. *Let M be a spin manifold of even-dimension $2n$ and denote by $W = W^+ \oplus W^-$ the complex spinor bundle. Let E be a Hermitian vector bundle with a metric connection ∇_E. Set*

$$S := W \otimes_{\mathbb{C}} E \quad \text{and} \quad S^\pm := W^\pm \otimes E.$$

Denote by D the Dirac operator on S. We have

$$\operatorname{index}_{\mathbb{Z}_2} D = \int_M \hat{A}(\nabla_{TM})\operatorname{ch}(\nabla_E)$$

with

$$\hat{A}(\nabla) = \det \sqrt{\frac{F_\nabla/4\pi i}{\sinh(F_\nabla/4\pi i)}}$$

and

$$\operatorname{ch}(\nabla) = \operatorname{tr} e^{iF_\nabla/2\pi}.$$

Here we define the integral to vanish on the components of $\hat{A}(\nabla_{TM})\operatorname{ch}(\nabla_E)$ which are not of degree $\dim M$.

Remark 25.2. The expression

$$\det \sqrt{\frac{F_\nabla/4\pi i}{\sinh(F_\nabla/4\pi i)}}$$

can be understood as follows. Using $\det(A) = \exp \operatorname{tr} \log A$, we can write

$$\det \sqrt{\frac{F_\nabla/4\pi i}{\sinh(F_\nabla/4\pi i)}} = \exp\left(\frac{1}{2} \operatorname{tr} \log \frac{F_\nabla/4\pi i}{\sinh(F_\nabla/4\pi i)}\right).$$

This can be expanded as a power series:

$$1 - \frac{1}{12} \operatorname{tr}[(F_\nabla/4\pi i)^2] + \frac{1}{5760} \operatorname{tr}[(F_\nabla/2\pi i)^4] + \frac{1}{4608}[\operatorname{tr}(F_\nabla/2\pi i)^2]^2 + \cdots.$$

Here for a 2–form ω with values in $\operatorname{End}(V)$, $\omega^{\wedge k}$ is obtained by taking the k–th wedge power of the 2–form part of ω and composing the parts in $\operatorname{End}(V)$. In terms of the Pontrjagin forms, we have

$$\hat{A}(\nabla) = 1 - \frac{p_1(\nabla)}{24} + \frac{7p_1(\nabla)^2 - 4p_2(\nabla)}{5760} + \cdots.$$

We will prove the following stronger form:

Theorem 25.3. *In the situation of Theorem 25.1, we have*

$$\lim_{t\to 0} \operatorname{str} k_t(x,x)\mathrm{vol} = \left[\hat{A}(\nabla_{TM})\mathrm{ch}(\nabla_E)\right]_n(x).$$

The following proof of this result is due to Freed and based on a rescaling argument around x. Choose a small ball $B_r(0) \subset T_xM$ and identify $B_r(0)$ with $\exp(B_r(0)) \subset M$. (This means in particular that we identify 0 with x). Using radial parallel transport, for every $y \in B_r$, identify W_y with W_0 and E_y with E_0. We have

$$\begin{aligned} k_t(0,y) &\in \mathrm{Hom}(W_y \otimes E_y, W_0 \otimes E_0) \\ &= \mathrm{Hom}(W_y, W_0) \otimes_{\mathbf{C}} \mathrm{Hom}(E_y, E_0) \\ &= \mathrm{End}(W_0) \otimes \mathrm{End}(E_0) \\ &= (\mathrm{C}\ell(T_0M) \otimes_{\mathbf{R}} \mathbf{C}) \otimes \mathrm{End}(E_0). \end{aligned}$$

Proposition 25.4. *Let $c \in \mathbf{C}\boldsymbol{\ell}_{2n} \cong \mathrm{End}(W)$ with W denoting the irreducible representation of $\mathbf{C}\boldsymbol{\ell}_{2n}$. Write c as*

$$c = \sum_I c_I e_I.$$

with

$$e_{i_1\ldots i_k} = e_{i_1} \cdots e_{i_k}$$

and I ranging over all increasing multi-indices. With respect to the splitting $W = W^+ \oplus W^-$ induced by the complex volume form, we have

$$\operatorname{str} c = (-2i)^n c_{1\cdots 2n}.$$

Proof. The complex volume form is

$$\omega_{\mathbf{C}} = i^n e_1 \cdots e_{2n}.$$

It acts by $+1$ on W^+ and -1 on W^-. Consequently,

$$\operatorname{str}(c) = \operatorname{tr}(\omega_{\mathbf{C}} c) = \sum_I c_i \operatorname{tr}(\omega_{\mathbf{C}} e_I).$$

If $I = (1, \ldots, 2n)$, then

$$\operatorname{tr}(\omega_{\mathbf{C}} e_I) = i^n \operatorname{tr}(e_I e_I) = (-i)^n \operatorname{tr}(\mathrm{id}_W) = (-2i)^n.$$

If $I \neq (1, \ldots, 2n)$, then up to a constant $\omega_{\mathbf{C}} e_I = e_{I^c}$ with I^c denoting the complement of I in $(1, \ldots, 2n)$. The action of e_{I^c} on the basis elements of $\mathbf{C}\boldsymbol{\ell}_{2n}$ has no fixed points. Consequently,

$$\operatorname{tr}_{\mathbf{C}\boldsymbol{\ell}_{2n}} e_{I^c} = 0.$$

Since $\mathbf{C}\boldsymbol{\ell}_{2n} = W \otimes W^*$, we have

$$\operatorname{tr}_{\mathbf{C}\ell_{2n}} e_{I^c} = \dim W^* \cdot \operatorname{tr}_W e_{I^c}.$$

Thus $\operatorname{tr}_W e_{I^c}$ vanishes. ◻

This tells us that we need to extract the top coefficient of $k_t(0,y)$ in $\mathrm{C}\ell(T_0M) \otimes_{\mathbf{R}} \mathbf{C}$ to compute $\operatorname{str} k_t(0,0)$. Set

$$q_t = k_t(\cdot, 0).$$

We have an asymptotic expansion

$$q_t(0) \sim \frac{1}{(4\pi t)^n} \sum_{j=0}^{\infty} \sum_I t^j \Theta_{j,I} e_I$$

with $\Theta_{j,I} \in \operatorname{End}(W_0)$.

Proposition 25.5.

1. *$\Theta_{j,I} = 0$ if $|I| > 2j$.*
2. *$\operatorname{tr} \Theta_{n,1\dots 2n} = (2\pi i)^n \left[\hat{A}(\nabla_{TM})\operatorname{ch}(\nabla_E)\right]_n (x)$.*

Proof of Theorem 25.3. Using Proposition 25.4 and Proposition 25.5, we have

$$\lim_{t\to 0} \operatorname{str} k_t(x,x) = \frac{(-2i)^n}{(4\pi)^n} (2\pi i)^n \left[\hat{A}(\nabla_{TM})\operatorname{ch}(\nabla_E)\right]_n (x) = \left[\hat{A}(\nabla_{TM})\operatorname{ch}(\nabla_E)\right]_n (x).$$ ◻

The proof of Proposition 25.5 relies on a scaling argument.

Definition 25.6. Given $\varepsilon > 0$, define $U_\varepsilon \colon \mathbf{C}\boldsymbol{\ell}_{2n} \to \Lambda\mathbf{R}^n$ by

$$U_\varepsilon(e_{i_1} \cdots e_{i_k}) \coloneqq \varepsilon^{-k} e_{i_1} \wedge \ldots \wedge e_{i_k}.$$

Define the **rescaled heat kernel** $q_t^\varepsilon \colon B_{r/\varepsilon}(0) \to \Lambda\mathbf{R}^n \otimes \operatorname{End}(E_0)$ by

$$q_t^\varepsilon(y) \coloneqq \varepsilon^n U_\varepsilon k_{\varepsilon^2 t}(\varepsilon^2 y, 0).$$

The function $q_t^\varepsilon(0)$ has an asymptotic expansion

(25.7) $$q_t^\varepsilon(0) \sim \frac{1}{(4\pi)^n} \sum_{j=0}^{\infty} \sum_I \varepsilon^{2j-|I|} t^{j-n} \Theta_{j,I} e_I,$$

Proposition 25.8. *Define* $M_\varepsilon\colon B_{r/\varepsilon}(0) \longrightarrow B_r(0)$ *by*

$$M_\varepsilon(y) = \varepsilon y.$$

Set

$$S_\varepsilon := U_\varepsilon M_\varepsilon^* \quad \text{and} \quad P_\varepsilon := \varepsilon^2 S_\varepsilon D^2 S_\varepsilon^{-1}.$$

With the above notation we have

$$(\partial_t + P_\varepsilon)\, q_t^\varepsilon = 0 \quad \text{and} \quad \lim_{t\to 0} q_t^\varepsilon = \delta_0.$$

Proof. By definition

$$q_t^\varepsilon = \varepsilon^n S_\varepsilon q_{\varepsilon^2 t}.$$

We have

$$(\partial_t + D^2) q_t = 0 \quad \text{and} \quad \lim_{t\to 0} q_t \longrightarrow \delta_0.$$

This implies the proposition directly. □

Proposition 25.9. *We have*

$$\lim_{\varepsilon\to 0} P_\varepsilon = P_0 = -\sum_{k=1}^{2n}\left(\partial_k - \frac{1}{4}\sum_{\ell=1}^{2n} R_{k\ell}(0) y_\ell\right)^2 + F_E(0)$$

acting on $C^\infty(\mathbf{R}^{2n}, \Lambda(\mathbf{R}^{2n})^* \otimes E_0)$. *Here* $R_{k\ell}$ *is identified with a 2–form and acts by taking wedge product.* $F_E(0)$ *acts by taking the wedge product with the 2–form part and applying the component in* $\mathrm{End}(E_0)$.

The proof of Proposition 25.9 is a lengthy computation. We will defer it for a while.

Proposition 25.10. *The heat kernel for* P_0 *evaluated at* $(0,0)$ *is*

$$(4\pi t)^{-n/2} \det \sqrt{\left(\frac{tR/2}{\sinh(tR/2)}\right)} e^{-tF_E(0)}.$$

Proof. Set

$$Q := -\sum_{k=1}^{2n}\left(\partial_k - \frac{1}{4}\sum_{\ell} R_{k\ell}(0) y_\ell\right)^2 \quad \text{and} \quad F := F_E(0).$$

Since 2–forms commute, the operator Q commutes with F.

The heat kernel of P_0 is thus given by

$$e^{-tP_0} = e^{-tQ} e^{-tF}.$$

The expression e^{-tF} can be computed using the power-series expression for exp since $F^{n+1} = 0$, and thus the series is a finite sum.

If necessary, adjust the coordinates so that $R = (R_{k\ell})$ is block-diagonal:

$$R = \begin{pmatrix} 0 & -\omega_1 & & & \\ \omega_1 & 0 & & & \\ & & 0 & \omega_2 & \\ & & -\omega_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

Then Q can be written as

$$Q \coloneqq \sum_{k=1}^{n} Q_k \quad \text{with} \quad Q_k = -\left(\partial_{2k-1} + \frac{1}{4}\omega_k x_{2k}\right)^2 - \left(\partial_{2k} - \frac{1}{4}\omega_k x_{2k-1}\right)^2$$

These Q_k can be expanded as

$$Q_k = Q_k^0 + Q_k^1$$

with

$$Q_k^0 \coloneqq -(\partial_{2k-1}^2 + \partial_{2k}^2) - \frac{1}{16}\omega_k^2 (x_{2k-1}^2 + x_{2k}^2) \quad \text{and} \quad Q_k^1 \coloneqq \frac{1}{2}\omega_k (x_{2k-1}\partial_{2k} - x_{2k}\partial_{2k-1}).$$

Mehler's formula, discussed in Section 26, shows that the heat kernel of the operator $-\partial_x^2 + a^2x^2$ is given by

$$\frac{1}{\sqrt{4\pi t}}\left(\frac{2at}{\sinh 2at}\right)^{1/2} \exp\left(-\frac{x^2}{4t}\frac{2at}{\tanh 2at}\right).$$

Applying this with $a = \frac{i}{4}\omega_k$ exhibits the heat-kernel of Q_k^0 as

(25.11) $$h_{k,t}(x) \coloneqq (4\pi t)^{-1}\left(\frac{it\omega_k/2}{\sinh it\omega_k/2}\right)\exp\left(\frac{-x^2}{4t}\left(\frac{it\omega_k/2}{\tanh it\omega_k/2}\right)\right).$$

Since $Q_k^1 h_{k,t} = 0$.

$$(\partial_t + Q_k)h_{k,t} = 0 \quad \text{and} \quad \lim_{t\to 0} h_{k,t} = \delta_0.$$

Therefore,

$$\prod h_{k,t}(0) = (4\pi t)^{-n}\prod \frac{it\omega_k/2}{\sinh it\omega_k/2}.$$

To rewrite this in terms of R, observe the following. The power series

(25.12) $$f(x) = \frac{x/2}{\sinh(x/2)}$$

involves only even powers of x; hence, there is a power series g with

$$f(x) = g(x^2). \tag{25.13}$$

Evidently,

$$\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}^2 = \begin{pmatrix} -x^2 & 0 \\ 0 & -x^2 \end{pmatrix}. \tag{25.14}$$

Therefore,

$$f\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} = \begin{pmatrix} g(-x^2) & 0 \\ 0 & g(-x^2) \end{pmatrix} = \begin{pmatrix} f(ix) & 0 \\ 0 & f(ix) \end{pmatrix} \tag{25.15}$$

and

$$\det\sqrt{f\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}} = \sqrt{\det f\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}} = f(-ix). \tag{25.16}$$

Consequently,

$$\det\sqrt{\frac{tR/2}{\sinh(tR/2)}} = \prod \frac{it\omega_k/2}{\sinh it\omega_k/2}. \tag{25.17}$$

□

Proof of Proposition 25.5. Set

$$q_t^0 := (4\pi t)^{-n/2}\sqrt{\det\left(\frac{tR/2}{\sinh(tR/2)}\right)}e^{-tF}.$$

Using the Taylor expansion of the last two factors, we can write

$$q_t^0 = (4\pi)^{-n}\sum_j P_j(R/2, -F)t^{j-n}$$

with P_j homogeneous of degree j.

Since P_ε varies continuously ε, so do the associated heat kernels. The *homogeneous* asymptotic expansions of the heat kernel evaluated at $(0, 0)$ are unique and thus also vary continuously. Consequently,

$$\lim_{\varepsilon\to 0}(4\pi)^{-n}\sum_{j=0}^{\infty}\sum_I \varepsilon^{2j-|I|}t^{j-n}\Theta_{j,I}e_I = (4\pi)^{-n}\sum_j P_j(R/2, -F)t^{j-n}.$$

Hence,

$$\lim_{\varepsilon\to 0}\sum_I \varepsilon^{2j-|I|}\Theta_{j,I}e_I = P_j(R/2, -F).$$

It follows that for $|I| > 2j$, the coefficient $\Theta_{j,I}$ must vanish. We also obtain the formula

$$\sum_{|I|=2j} \Theta_{j,I} = P_j(R/2, -F).$$

Finally, using that P_n is homogeneous of degree n, we have

$$\begin{aligned}\Theta_{n,1\dots 2n} &= P_n(R/2, -F)\\ &= (2\pi i)^n P_n(R/4\pi i, iF/2\pi)\\ &= (2\pi i)^n \left[\hat{A}(\nabla_{TM})\mathrm{ch}(\nabla_E)\right]_n (x).\end{aligned}$$

□

Proof of Proposition 25.9. By the Weitzenböck formula

$$P_1 = \nabla^*\nabla + \frac{1}{4}\mathrm{scal}_g + F_E$$

Denote by Γ^k_{ij} the Christoffel symbols, that is,

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k.$$

Denote by a the local connection 1–form of ∇_E. We have

$$\begin{aligned}P_1 = -g^{k\ell}(y)&\left(\partial_k + \frac{1}{2}\Gamma^j_{ki}(y)e^*_i \wedge e^*_j + a(y, e_k)\right)\left(\partial_\ell + \frac{1}{2}\Gamma^j_{\ell i}(y)e^*_i \wedge e^*_j + a(y, e_\ell)\right)\\ &+ g^{k\ell}(y)\Gamma^m_{k\ell}(y)\left(\partial_m + \frac{1}{2}\Gamma^j_{mi}(y)e^*_i \wedge e^*_j + a(y, e_m)\right)\\ &+ \frac{1}{4}\mathrm{scal}_g(y) + F_E(y).\end{aligned}$$

The rescaling involved in passing from P_1 to P_ε means scaling y to εy (and correspondingly for derivatives and 1–forms), and scaling e^*_i

We have

$$\begin{aligned}P_\varepsilon = \varepsilon^2 g^{k\ell}(\varepsilon y)&\left(\varepsilon^{-1}\partial_k + \varepsilon^{-2}\frac{1}{2}\Gamma^j_{ki}(\varepsilon y)e^*_i \wedge e^*_j + a(\varepsilon y, e_k)\right)\\ &\cdot\left(\varepsilon^{-1}\partial_\ell + \varepsilon^{-2}\frac{1}{2}\Gamma^j_{\ell i}(y)e^*_i \wedge e^*_j + a(\varepsilon y, e_\ell)\right)\\ &+ \varepsilon^2 g^{k\ell}(y)\Gamma^m_{k\ell}(y)\left(\varepsilon^{-1}\partial_m + \frac{1}{2}\Gamma^j_{mi}(y)e^*_i \wedge e^*_j + a(\varepsilon y, e_k)\right)\\ &+ \frac{\varepsilon^2}{4}\mathrm{scal}_g(\psi y) + F_E(y).\end{aligned}$$

Using

$$\Gamma^b_{ka} = -\frac{1}{2}R^b_{k\ell a}y_\ell + O\left(|y|^2\right),$$

this becomes

$$P_\varepsilon = g^{k\ell}(0)\left(\partial_k - \frac{1}{4}R^j_{kmi}y_m e_i^* \wedge e_j^*\right)\left(\partial_\ell - \frac{1}{4}R^j_{\ell mi}y_m e_i^* \wedge e_j^*\right) + F_E(y) + O(\varepsilon).$$

Since $g^{k\ell}(0) = \delta^{k\ell}$, this proves the that P_ε has a limit and that the limit has the the asserted form. □

26 Mehler's formula

In the proof of the local index theorem we used (25.11). We could verify by hand that this is the desired heat kernel for the model operator, but it also can be derived from Mehler's formula. Since this formula also explains the appearance of the otherwise quite mysterious $\hat{A}$ genus, let us discuss it now.

Let A be a commutative **R**–algebra. Let $a \in A$ be a nilpotent element. (Or assume that power series can be evaluated on A for some other reason.) Consider the heat equation

$$\partial_t f - \partial_x^2 f + a^2 x^2 f = 0. \tag{26.1}$$

We want to find a solution $f_t(x)$ of this equation with initial condition

$$\lim_{t\to 0} f_t = \delta_0.$$

We make the ansatz

$$f_t(x) = \alpha(t)e^{-\frac{1}{2}x^2\beta(t)}.$$

Plugging this ansatz into (26.1), we obtain

$$\alpha'/\alpha - \frac{1}{2}x^2\beta' + \beta(t) - x^2\beta + a^2x^2 = 0$$

or, equivalently,

$$\log(\alpha)' = -\beta \quad \text{and} \quad \beta' = 2(a^2 - \beta^2).$$

Recall that

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \text{and} \quad \coth x = \frac{\cosh x}{\sinh x}.$$

A computation shows that

$$\coth' = 1 - \coth^2.$$

The ODE for β is solved by

$$\beta = a\coth(2at + C) = \frac{1}{2}\partial_t \log\sinh(2at + C)$$

Thus

$$\alpha(t) = D/\sqrt{\sinh(2at + C)}$$

Since $\sinh x = x + O(x^3)$, the initial condition holds for

$$C = 0 \quad \text{and} \quad D = \sqrt{a/2\pi}.$$

This show that the desired f_t is given by

$$f_t(x) = \frac{1}{\sqrt{4\pi t}} \left(\frac{2at}{\sinh 2at}\right)^{1/2} \exp\left(-\frac{x^2}{4t}\frac{2at}{\tanh 2at}\right). \tag{26.2}$$

27 Computation of the $\hat{A}$ genus

27.1 Review of Chern–Weil theory

Let G be a Lie group. Let $k = \mathbf{R}$ or $\mathbf{C}$ Let $k[[\mathfrak{g}]]^G$ be the algebra of G–invariant power series on the Lie algebra $\mathfrak{g}$. Any $p \in k[[\mathfrak{g}]]^G$ can be written as

$$p(X) = \sum_{k=0}^{\infty} p_k(\underbrace{X \otimes \cdots \otimes X}_{k \text{ times}})$$

where $p_k\colon (S^k\mathfrak{g})^G \to k$ is a linear map.

Let P be a principal G–bundle over M Let $F_A \in \Omega^2(M, \mathfrak{g}_P)$ be the curvature of a connection on P. We define

$$p(F_A) := \sum_{k=0}^{\infty} p_k(F_A \wedge \cdots \wedge F_A) \in \Omega^\bullet(M, k).$$

Here p_k acts as $S^k\mathfrak{g}_p \to k$. There are only finitely many summands.

Chern–Weil theory asserts that the cohomology class

$$[p(F_A)] \in H^\bullet(M, \mathbf{C})$$

depends only on P.

As we will see for computations it is useful to recall that if $\mathfrak{t}$ is a maximal torus and W is the Weyl group, then the inclusion

$$k[\mathfrak{t}]^W \subset k[\mathfrak{g}]^C$$

is an isomorphism.

27.2 Chern classes

Example 27.1. The **Chern classes** arise from

$$c(X) = \det\left(1 + \frac{iX}{2\pi}\right).$$

for $X \in \mathfrak{gl}_n(\mathbf{C})$. The total Chern class of a complex vector bundle of rank n is

$$c(E) = c_0(E) + \ldots + c_n(E) = [c(F_A)].$$

Example 27.2. The **Chern character** arises from

$$ch(X) = \operatorname{tr}\exp\left(\frac{iX}{2\pi}\right)$$

for $X \in \mathfrak{gl}_n(\mathbf{C})$. The total Chern character of a complex vector bundle is

$$ch(E) = [ch(F_A)].$$

For computations it is useful to observe that diagonalizable matricies are dense in $\mathfrak{u}(n)$; hence,

$$\mathbf{C}[[\mathfrak{u}(n)]]^G \cong \mathbf{C}[[x_1, \ldots, x_n]]^{S_n}.$$

The Chern classes correspond to

$$\prod_{j=1}^{n}(1 + ix_j/2\pi) = \sum_{k=0}^{n}\left(\frac{i}{2\pi}\right)^k \sum_{1\leqslant j_1<\cdots<i_k\leqslant k} x_{j_1}\cdots x_{j_k}.$$

The Chern character corresponds to

$$\sum_{k=0}^{\infty}\left(\frac{i}{2\pi}\right)^k \sum_{j=1}^{n}\frac{x_j^k}{k!}.$$

From the relation

$$\sum_{j=1}^{n}\frac{x_j^2}{2} = \frac{1}{2}\left(\sum_{j=1}^{n} x_j\right)^2 - \sum_{1\leqslant j_1<\cdots<i_2\leqslant n} x_{j_1}x_{j_2}.$$

we deduce

$$ch_2 = \frac{1}{2}c_1^2 - c_2.$$

27.3 Pontrjagin classes

Example 27.3. The **Pontrjagin class** of a real vector bundle is

$$p(V) = c_0(V \otimes \mathbf{C}) - c_2(V \otimes \mathbf{C}) + \cdots + (-1)^n c_{2n}(V \otimes \mathbf{C}).$$

There is a maximal torus $\mathfrak{t}$ in $\mathfrak{o}(2n)$ generated by block-diagonal matrices with blocks of the form

$$Y = \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}.$$

Over $\mathbf{C}$ this block can be diagonalized to

$$\begin{pmatrix} iy & 0 \\ 0 & -iy \end{pmatrix}.$$

It follows that

$$c_1(X) = 0 \quad \text{and} \quad c_2(X) = \left(\frac{i}{2\pi}\right)^2 (iy)(-iy) = -\frac{y^2}{4\pi^2}.$$

With respect to $\mathbf{R}[\mathfrak{o}(2n)]^{O(2n)} \cong \mathbf{R}[y_1, \ldots, y_n]^{S_n}$, the Pontrjagin classes correspond to

$$\prod_{j=1}^{n} \left(1 + y_j^2/4\pi^2\right) = \sum_{k=0}^{n} \left(\frac{1}{4\pi^2}\right)^k \sum_{1 \leqslant j_1 < \cdots < i_k \leqslant n} y_{j_1}^2 \cdots y_{j_k}^2.$$

27.4 Genera

The Chern character is an example of a **genus**.

Definition 27.4. Let $f \in \mathbf{C}[[x]]$. The **Chern f–genus** of a complex vector bundle E is defined characteristic class $p_f(E)$ associated to

$$c_f(X) = \det\left(f\left(\frac{iX}{2\pi}\right)\right).$$

Proposition 27.5.

1. *If L is a complex line bundle, then $c_f(L) = f(c_1(L))$.*
2. *If V_1 and V_2 are two complex vector bundles, then $c_f(V_1 \oplus V_2) = c_f(V_1) \cup c_f(V_2)$.*

Definition 27.6. Let $g \in \mathbf{C}[[x]]$ with $g(x) = 1 + \cdots$. Set $f(x) \coloneqq \sqrt{g(x^2)}$. The **Pontrjagin g genus** of a real vector bundle V is

$$p_g(V) = c_f(V \otimes \mathbf{C}).$$

Remark 27.7. Suppose E is a complex vector bundle. Then $E \otimes_{\mathbf{R}} \mathbf{C} \cong E \oplus E$. Consequently,

$$p_g(E) = c_f(E \oplus E) = c_f(E)^2.$$

The $\hat{A}$–genus arises as the Pontrjagin genus of

$$g(x) = \frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)}.$$

Let us discuss how to understand these genera using with respect to the isomorphism $\mathbf{R}[\mathfrak{o}(2n)]^{O(2n)} \cong \mathbf{R}[y_1, \dots, y_n]^{S_n}$. Consider the block

$$Y = \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}.$$

Over $\mathbf{C}$, this block is conjugate to

$$\begin{pmatrix} iy & 0 \\ 0 & -iy \end{pmatrix}.$$

Thus

$$\begin{aligned} p_g(Y) &= \det \begin{pmatrix} f(y/2\pi) & \\ & f(-y/2\pi) \end{pmatrix} \\ &= f(y/2\pi) f(-y/2\pi) \\ &= g\left(y^2/4\pi^2\right). \end{aligned}$$

Therefore, writing

$$g(x) = \sum_{k=0}^{\infty} a_k x^k$$

we have

$$p_g = \prod_{j=1}^{n} \left(\sum_{k=0}^{\infty} a_k \left(\frac{y_j^2}{4\pi^2} \right)^k \right).$$

27.5 Expressing $\hat{A}$ in terms of Pontrjagin classes

To understand the $\hat{A}$–genus, recall that

$$\frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)} = 1 - \frac{x}{24} + \frac{7x}{5760} + \dots.$$

Therefore,

$$\hat{A} = \prod_{j=1}^{n} \left(1 - \frac{1}{24} \frac{y_j^2}{4\pi^2} + \frac{7}{5760} \left(\frac{y_j^2}{4\pi^2} \right)^2 + \cdots \right)$$
$$= 1 - \frac{1}{24} \frac{1}{4\pi^2} \sum_{j=1}^{n} y_j^2 + \left(\frac{1}{4\pi^2} \right)^2 \left(\frac{7}{5670} \sum_{j=1}^{n} y_j^4 + \frac{1}{576} \sum_{1 \leqslant j_1 < j_2 \leqslant n} y_{j_1}^2 y_{j_2}^2 \right) + \cdots .$$

To express this in terms of Pontrjagin classes, recall that,

$$p_1 = \frac{1}{4\pi^2} \sum_{j=1}^{n} y_j^2 \quad \text{and} \quad p_2 = \left(\frac{1}{4\pi^2} \right)^2 \sum_{1 \leqslant j_1 < j_2 \leqslant n} y_{j_1}^2 y_{j_2}^2.$$

Therefore,

$$\left(\frac{1}{4\pi^2} \right)^2 \sum_{j=1}^{n} y_j^4 = p_1^2 - 2p_2.$$

It follows that

$$\hat{A} = 1 - \frac{1}{24} p_1 + \frac{1}{5760} \left(7p_1^2 - 14p_2 + 10p_2 \right) + \cdots$$
$$= 1 - \frac{1}{24} p_1 + \frac{7p_1^2 - 4p_2}{5760} + \cdots .$$

28 Hirzebruch–Riemann–Roch Theorem

Definition 28.1. Let E be a complex vector bundle over M. The **Todd class** $\mathrm{td}(E)$ of E is the characteristic Chern genus associated with the function

$$f(x) = \frac{x}{e^x - 1}.$$

That is, if F_A is the curvature of some connection on E, then

$$\mathrm{td}(E) = \left[\det \left(\frac{iF_A/2\pi}{e^{iF_A/2\pi} - 1} \right) \right].$$

Exercise 28.2. Prove that

$$\mathrm{td} = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \frac{-c_1^4 + 4c_1^2 c_2 + c_1 c_3 + 3c_2^2 - c_4}{720} + \cdots .$$

Theorem 28.3 (Hirzebruch–Riemann–Roch). *Let M be a compact spin Kähler manifold. Let $\mathscr{E} = (E, \bar\partial_E)$ be a holomorphic vector bundle together with a Hermitian metric h. We have*

$$\chi(\mathscr{E}) = \operatorname{index}\left(\bar\partial_E + \bar\partial_E^* \colon\ \Omega^{0,\mathrm{ev}}(M,\mathscr{E}) \to \Omega^{0,\mathrm{odd}}(M,\mathscr{E})\right) = \int_M \operatorname{td}(TM)\operatorname{ch}(E).$$

Remark 28.4. The spin condition can be dropped. This requires to discuss how Theorem 25.1 can be formulated on non-spin manifold.

Proof. The first identity is a consequence of Hodge theory. To compute the index we will use Theorem 25.1 and the discussion in Section 11.

For a spin Kähler manifold has the complex spinor bundle can be written as

$$W^+ = \Lambda^{0,\mathrm{ev}} T^{0,1}M^* \otimes \mathscr{K}_M^{-1/2} \quad \text{and} \quad W^- = \Lambda^{0,\mathrm{odd}} T^{0,1}M^* \otimes \mathscr{K}_M^{-1/2}$$

and the Dirac operator is given by

$$D = \sqrt{2}\left(\bar\partial + \bar\partial^*\right).$$

It follows from Theorem 25.1 that

$$\chi(\mathscr{E} \otimes \mathscr{K}_M^{-1/2}) = \int_M \hat{A}(M)\operatorname{ch}(E).$$

Consequently,

$$\chi(\mathscr{E}) = \int_M \hat{A}(M)\operatorname{ch}(K_M^{1/2} \otimes E) = \int_M \hat{A}(M)\sqrt{\operatorname{ch}(K_M)}\operatorname{ch}(E).$$

Recall that

$$TM \otimes \mathbf{C} = T^{1,0}M \oplus T^{0,1}M \quad \text{and} \quad \mathscr{K}_M = \Lambda^n_{\mathbf{C}} T^{1,0}M^*.$$

Write R for the Riemann curvature tensor on TM and $R_{\mathbf{C}}$ for the curvature on the complex vector bundle $T^{1,0}M$. We have

$$\sqrt{\operatorname{ch}}(K_M) = \sqrt{\exp(-\operatorname{tr} iR_{\mathbf{C}}/2\pi)} = \det(e^{-iR_{\mathbf{C}}/4\pi})$$

By Remark 27.7, we have

$$\sqrt{\det\left(\frac{iR/2\pi}{e^{iR/4\pi} - e^{-iR/4\pi}}\right)} = \det\left(\frac{iR_{\mathbf{C}}/2\pi}{e^{iR_{\mathbf{C}}/4\pi} - e^{-iR_{\mathbf{C}}/4\pi}}\right).$$

Therefore,

$$\begin{aligned}\hat{A}(M)\sqrt{\operatorname{ch}(K_M)} &= \left[\det\left(\frac{iR/2\pi}{e^{iR/4\pi} - e^{-iR/4\pi}}\right)\det(e^{-iR/4\pi})\right] \\ &= \left[\det\left(\frac{iR/2\pi}{e^{iR/2\pi} - 1}\right)\right] \\ &= \operatorname{td}(M).\end{aligned}$$

This proves the index formula. □

Theorem 28.5 (Riemann–Roch). *Let Σ be an compact Riemann surface and let $\mathscr{L}$ be a holomorphic vector bundle. We have*

$$\chi(\mathscr{L}) = \deg \mathscr{L} - g(\Sigma) + 1.$$

Proof. The Todd class of a Riemann surface is

$$\begin{aligned} \operatorname{td}(\Sigma) &= 1 + \frac{c_1(\Sigma)}{2}[\Sigma] \\ &= 1 + \frac{\chi(\Sigma)}{2}[\Sigma] \\ &= 1 + (1 - g(\Sigma))[\Sigma] \end{aligned}$$

and

$$\begin{aligned} \operatorname{ch}(\mathscr{L}) &= 1 + c_1(\mathscr{L}) \\ &= 1 + \deg(\mathscr{L})[\Sigma]. \end{aligned}$$

Consequently,

$$\begin{aligned} \chi(\mathscr{L}) &= \int_\Sigma \operatorname{td}(\Sigma)\operatorname{ch}(\Sigma) \\ &= \int_\Sigma (1 + (1 - g(\Sigma))[\Sigma])(1 + \deg(\mathscr{L})[\Sigma]) \\ &= \deg(\mathscr{L}) + g(\Sigma) - 1. \end{aligned}$$

□

29 Hirzebruch Signature Theorem

Proposition 29.1. *Let M be a compact oriented manifold of dimension $2n$.*

1. *The operator $\tau\colon \Omega^\bullet(M,\mathbf{C}) \to \Omega^\bullet(M,\mathbf{C})$ defined by*

$$\tau\omega \coloneqq i^{d(d-1)+n} * \omega \quad \textit{with} \quad \deg\omega = d.$$

   is an involution.

2. *Denote by $\Omega_\pm(M,\mathbf{C})$ the ± 1–eigenspace of τ. With respect to the splitting $\Omega^\bullet(M,\mathbf{C}) = \Omega_+(M,\mathbf{C}) \oplus \Omega_-(M,\mathbf{C})$, the operator $\mathrm{d}+\mathrm{d}^*\colon \Omega^\bullet(M,\mathbf{C}) \to \Omega^\bullet(M,\mathbf{C})$ is of the form*

$$\mathrm{d}+\mathrm{d}^* = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}.$$

   Equivalently,

$$(\mathrm{d}+\mathrm{d}^*)\tau + \tau(\mathrm{d}+\mathrm{d}^*) = 0.$$

3. *We have*

$$\operatorname{index} D = \begin{cases} 0 & \textit{if } n \textit{ is odd} \\ \sigma(M) & \textit{if } n \textit{ is even.} \end{cases}$$

   Here we denote by $\sigma(M)$ the signature of the intersection form Q on $H^{2k}(M,\mathbf{R})$.

4. *If M is spin and $W = W^+ \oplus W^-$ denotes the complex spinor bundle, then with respect to the isomorphism $\Lambda TM \otimes \mathbf{C} \cong W \otimes W$ induced by Clifford multiplication we have*

$$\Omega^\pm(M,\mathbf{C}) \cong \Gamma(W^\pm \otimes_{\mathbf{C}} W).$$

   and

$$D = \not{D}^+_W.$$

Definition 29.2. The operator $D\colon \Omega_+(M,\mathbf{C}) \to \Omega_-(M,\mathbf{C})$ is called the **signature operator**.

Proof of Proposition 29.1. (1) We have

$$** \omega = (-1)^{d(2n-d)}\omega \quad \text{with} \quad \deg\omega = d.$$

Consequently,

$$\begin{aligned}\tau^2\omega &= (-1)^{d(2n-d)} i^{d(d-1)+n} i^{(2n-d)(2n-d-1)+n}\omega \\ &= i^{2d(2n-d)+d(d-1)+n+(2n-d)(2n-d-1)+n}\omega.\end{aligned}$$

We have

$$\begin{aligned}&2d(2n-d)+d(d-1)+n+(2n-d)(2n-d-1)+n\\&\quad=(-2+1+1)d^2+(4-4)dn+4n^2+(-1+1)d+(1-2+1)n=4n^2.\end{aligned}$$

Therefore, $\tau^2=\mathrm{id}$.

(2) We have

$$\mathrm{d}^*\omega=(-1)^{2n(d-1)+1}*\mathrm{d}*\omega=-*\mathrm{d}*\omega.$$

Therefore,

$$\begin{aligned}(\mathrm{d}+\mathrm{d}^*)\tau&=i^{d(d-1)+n}\mathrm{d}*\omega-i^{d(d-1)+n}*\mathrm{d}**\omega\\&=i^{d^2-d+n}\mathrm{d}*\omega-(-1)^{d(2n-d)}i^{d^2-d+n}*\mathrm{d}\omega\\&=i^{d^2-d+n}\mathrm{d}*\omega-i^{2d(2n-d)+d^2-d+n}*\mathrm{d}\omega\\&=i^{d^2-d+n}\mathrm{d}*\omega-i^{-d^2-d+n}*\mathrm{d}\omega\end{aligned}$$

and

$$\begin{aligned}\tau(\mathrm{d}+\mathrm{d}^*)\omega&=-i^{(d-1)(d-2)+n}**\mathrm{d}*\omega+i^{(d+1)d+n}*\mathrm{d}\omega\\&=-i^{d^2-3d+2+n}(-1)^{(2n-d+1)(d-1)}\mathrm{d}*\omega+i^{d^2+d+n}*\mathrm{d}\omega\\&=-i^{d^2-3d+2+n+2(2n-d+1)(d-1)}\mathrm{d}*\omega+i^{d^2+d+n}*\mathrm{d}\omega\\&=-i^{-d^2+d+n}\mathrm{d}*\omega+i^{d^2+d+n}*\mathrm{d}\omega.\end{aligned}$$

Since

$$d^2\pm d=-d^2\mp d+2(d^2\pm d)\quad\text{and}\quad 2|(d^2\pm d),$$

it follows that

$$(\mathrm{d}+\mathrm{d}^*)\tau+\tau(\mathrm{d}+\mathrm{d}^*)=0.$$

(3) Since

$$(\mathrm{d}+\mathrm{d}^*)^2=\Delta,$$

we have

$$D^*D=\Delta\colon\ \Omega_+(M,\mathbf{C})\to\Omega_+(M,\mathbf{C})\quad\text{and}\quad DD^*=\Delta\colon\ \Omega_-(M,\mathbf{C})\to\Omega_-(M,\mathbf{C}).$$

It follows that the kernel of D consists of complex harmonic forms α satisfying $\tau\alpha=\alpha$ and that the kernel of D^* consists of complex harmonic forms α satisfying $\tau\alpha=-\alpha$. That is:

$$\begin{aligned}\ker D&=\mathscr{H}_+(M,\mathbf{C}):=\mathscr{H}(M,\mathbf{C})\cap\Omega_+(M,\mathbf{C})\quad\text{and}\\ \ker D^*&=\mathscr{H}_-(M,\mathbf{C}):=\mathscr{H}(M,\mathbf{C})\cap\Omega_-(M,\mathbf{C}).\end{aligned}$$

The maps $\Omega^0(M, \mathbf{C}) \oplus \cdots \oplus \Omega^{n-1}(M, \mathbf{C}) \oplus \Omega^n_\pm(M, \mathbf{C}) \to \Omega_\pm(M, \mathbf{C})$ defined by

$$(f, \ldots, \alpha, \beta) \mapsto (f \pm \tau f, \ldots, \alpha \pm \tau\alpha, \beta).$$

are isomorphisms. It follows that

$$\mathscr{H}_\pm(M, \mathbf{C}) \cong \mathscr{H}^0(M, \mathbf{C}) \oplus \cdots \oplus \mathscr{H}^{n-1}(M, \mathbf{C}) \oplus \mathscr{H}^n_\pm(M, \mathbf{C}).$$

Consequently,

$$\operatorname{index} D = \dim \mathscr{H}^n_+(M, \mathbf{C}) - \dim \mathscr{H}^n_-(M, \mathbf{C}).$$

If n is odd and $d = n$, then

$$*\tau = i^{n(n-1)+n} * = i^{n^2} *$$

and

$$\begin{aligned} \tau\bar{\alpha} &= i^{n^2} * \bar{\alpha} \\ &= -\overline{i^{n^2} * \alpha} \\ &= -\overline{\tau\alpha}. \end{aligned}$$

Consequently, $\bar{\cdot}\colon \mathscr{H}^n_+(M, \mathbf{C}) \to \mathscr{H}^n_-(M, \mathbf{C})$ is an (anti-linear) isomorphism. Hence, $\dim \mathscr{H}^n_+(M, \mathbf{C}) = \dim \mathscr{H}^n_-(M, \mathbf{C})$.

If $n = 2k$ and $d = n$, then

$$\tau = i^{2k(2k-1)+2k} * = *.$$

Consequently, $\mathscr{H}^n_\pm(M, \mathbf{C})$ consists of (anti-)self-dual harmonic forms. Thus, it follows from Hodge theory that

$$\operatorname{index} D = \sigma(M).$$

(4) Exercise. (Sadly, not fun.) □

Proposition 29.1 (4) allows us to compute $\sigma(M)$ using Theorem 25.1 if we can compute $\operatorname{ch}(W)$. Let me preempt the answer.

Definition 29.3. The **L genus** of a real vector bundle V is the Pocntrjagin genus associated with

$$L(y) = \frac{\sqrt{y}}{\tanh \sqrt{y}}$$

Exercise 29.4. We have

$$L = \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2) + \ldots.$$

Theorem 29.5 (Hirzebruch). *If M is a compact oriented manifold of dimension $4k$, then*

$$\sigma(M) = \int_M L(TM).$$

Remark 29.6. Historically, the Hirzebruch Signature Theorem was discovered before the Atiyah–Singer Index Theorem.

Sketch of proof of Theorem 29.5. We can assume that M is spin, because this is true locally and it suffices for the proof of Theorem 25.1.

Exercise 29.7. Prove that

$$\mathrm{ch}(W) = g(TM) \quad \text{with} \quad g(y) = 2\coth(\sqrt{y}/2)$$

Hint: The formula in Remark 8.10 asserts that if R^{ℓ}_{ijk} is the Riemann curvature tensor, then the spin curvature of W is given by

$$F_W(e_i, e_j) = \frac{1}{4}\sum_{a,b} R^b_{ija}\gamma(e^a)\gamma(e^b).$$

In a suitable basis we can assume that for every i and j

$$(R^{\ell}_{ijk}) = \begin{pmatrix} 0 & -y_1 & & & \\ y_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -y_{2k} \\ & & & y_{2k} & 0 \end{pmatrix}.$$

The corresponding spin curvature is then

$$F_W(e_i, e_j) = \frac{1}{2}\sum_{a=1}^{k} y_k \cdot \gamma(e_{2a-1})\gamma(e_{2a}).$$

From the exercise it follows that

$$\hat{A}(TM)\mathrm{ch}(W)$$

is equal to the Pontrjagin genus associated with

$$\begin{aligned} \hat{A}(y)g(y) &= \frac{\sqrt{y}/2}{\sinh(\sqrt{y}/2)} 2\coth(\sqrt{y}/2) \\ &= \frac{\sqrt{y}}{\tanh(\sqrt{y}/2)}. \end{aligned}$$

While this is not exactly L, it turns out that if V has rank r, then

$$L_r(V) = (\hat{A}(V)g(V))_r.$$

In particular,

$$\int_M \hat{A}(TM)g(TM) = \int_M (\hat{A}(TM)g(TM))_{4k} = \int_M L(TM)_{4k} = \int_M L(TM).$$

□

Remark 29.8. Hirzebruch's proof of Theorem 28.3 proceeded in a completely different way. One first proves that the signature is invariant under oriented cobordism. In fact, σ induces a ring homomrphism from the oriented cobordism ring Ω^{SO} to $\mathbf{Z}$. Thom proved that $\Omega^{SO} \otimes \mathbf{Q}$ is generated by the complex projective spaces $\mathbf{C}P^n$. Thus it suffices to prove Theorem 28.3 for $\mathbf{C}P^n$. Proceeding in this way requires to figure out the formula for L genus that one wants to verify. The original proof of the Atiyah–Singer Index Theorem, in fact, followed as similar (although much more involved) line of reasoning. While these approaches are very beautiful, they require a certain ingenuity in "guessing" what the right index formula might be. One of the key advantages of the of the heat kernel proof of the index theorem that we discussed in this class is that this approach automatically reveals the index formula (and one does not have to predict the answer).

Suppose now that M is a 4–manifold. In this case we have

$$\sigma(M) = \frac{1}{3}p_1(M).$$

In particular, $p_1(M)$ is divisible by 3. If M is spin, then we also have

$$\operatorname{index}(\not{D}\colon \Gamma(W^+) \to \Gamma(W^-)) = -\frac{1}{24}p_1(M) = -\frac{\sigma(M)}{8}.$$

It follows that $\sigma(M)$ is divisible by 8.

Theorem 29.9 (Rokhlin). *If M is a compact spin 4–manifold, then $\sigma(M)$ is divisible by 16.*

Proof. The original proof of this result requires what I would consider heavy lifting in algebraic topology.

Since we already have the Atiyah–Singer index theorem we can give the following simpler proof (due to Atiyah and Singer). Recall that the real spinor representation of $\mathrm{Spin}(4)$ is $\mathbf{H} \oplus \mathbf{H}$. The complex spinor representation is obtained by forgetting the quaternionic structure on $\mathbf{H}$ and identifying it with $\mathbf{C}^2$. From this it follows that the complex spinor bundles $W^\pm$ are obtained from the real spinor bundles by forgetting the quaternionic structure. That is, as complex vector bundles

$$\$^\pm = W^\pm.$$

Moreover, we have

$$\big(\not{D}\colon \Gamma(\$^+) \to \Gamma(\$^-)\big) = \big(\not{D}\colon \Gamma(W^+) \to \Gamma(W^-)\big)$$

as complex linear operators. Consequently,

$$\begin{aligned}\operatorname{index}_{\mathbf{C}}\left(\not{D}\colon \Gamma(W^+) \to \Gamma(W^-)\right) &= \operatorname{index}_{\mathbf{C}}\left(\not{D}\colon \Gamma(\$^+) \to \Gamma(\$^-)\right)\\ &= \frac{1}{2}\operatorname{index}_{\mathbf{R}}\left(\not{D}\colon \Gamma(\$^+) \to \Gamma(\$^-)\right).\end{aligned}$$

Now, the bundles $\$^\pm$ actually have quaternionic structures and $\not{D}$ is quaternionic linear. It follows that $\operatorname{index}_{\mathbf{R}}\left(\not{D}\colon \Gamma(\$^+) \to \Gamma(\$^-)\right)$ is divisible by 4. Consequently, $\operatorname{index}_{\mathbf{C}}\left(\not{D}\colon \Gamma(W^+) \to \Gamma(W^-)\right)$ is divisible by 2. □

Remark 29.10. The same argument shows that if M is a compact spin $(8k+4)$–manifold, then $\hat{A}_{4k}(M)$ is even.

Theorem 29.11 (Freedman). *There exists unique a compact simply connected* topological 4–*manifold with* $w_2(M) = 0$ *and intersection form*

$$E_8 = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}.$$

Since $\sigma(E_8) = 8$, it follows that M cannot admit a smooth structure.

Rohklin's theorem also leads to the following invariant of a spin 3–manifold.

Definition 29.12. Let $(N, \mathfrak{s})$ be spin 3–manifold. The **Rokhlin invariant** of $(M, \mathfrak{s})$ is defined as

$$\mu(N, \mathfrak{s}) = \sigma(M) \mod 16 \in \mathbf{Z}/16\mathbf{Z}.$$

where M is any compact spin 4–manifold with $\partial M = N$.

Invariants of this type still play an important role in geometry and topology. One of the recent applications of this idea is the ν–invariant of G_2–manifold due to Crowley and Nordström [CN12].

30 Example Index Computations

Example 30.1. Let M be a closed spin 4–manifold. Denote by $\not{D}^+ : \Gamma(W^+) \to \Gamma(W^-)$ the positive chirality complex Atiyah–Singer operator. By Theorem 25.1,

$$\operatorname{index} \not{D}^+ = \int_M \hat{A}(TM) = -\frac{1}{24} \int_M p_1(TM).$$

By the Theorem 29.5, we have

$$\sigma(M) = \frac{1}{3} \int_M p_1(TM).$$

Consequently,

$$\operatorname{index} \not{D}^+ = -\frac{\sigma(M)}{8}.$$

1. $H^2(S^4, \mathbf{R}) = 0$. Consequently, $\sigma(S^4) = 0$. It follows that

   $$\operatorname{index} \not{D}^+ = 0.$$

   In fact, we can see this by other means. The standard round metric g_0 on S^4 has positive scalar curvature. Therefore, it follows from the Weitzenböck formula that

   $$\ker \not{D}_{g_0} = 0.$$

   This means that

   $$\ker \not{D}^+_{g_0} = 0 \quad \text{and} \quad \operatorname{coker} \not{D}^+_{g_0} \cong \ker \not{D}^-_{g_0} = 0.$$

   Since the index is homotopy invariant and the space of metrics is convex and, hence, contractible, it follows that $\operatorname{index} \not{D}^+ = 0$ for every metric.

2. $\mathbf{C}P^2$ is not spin. One way to see this is to note that $\sigma(\mathbf{C}P^2) = 1$ and thus not divisible by 8.

3. Consider a smooth quartic Q in $\mathbf{C}P^3$. That is Q is the zero locus of a generic section $s \in H^0(\mathscr{O}_{\mathbf{C}P^3}(4))$. Along Q, we have

   $$0 \to \mathscr{T}_Q \to \mathscr{T}_{\mathbf{C}P^3}|_Q \to \mathscr{O}_{\mathbf{C}P^3}(4) \to 0$$

   Consequently,

   $$\mathscr{K}_{\mathbf{C}P^3}|_Q = \Lambda^3 \mathscr{T}^*_{\mathbf{C}P^3}|_Q = \Lambda^2 \mathscr{T}^*_Q \otimes \mathscr{O}_{\mathbf{C}P^3}(-4)|_Q = \mathscr{K}_Q \otimes \mathscr{O}_{\mathbf{C}P^3}(-4)|_Q.$$

   From the Euler sequence

   $$0 \to \mathscr{O}_{\mathbf{C}P^3} \to \mathscr{O}_{\mathbf{C}P^3}(1)^{\oplus 4} \to \mathscr{T}_{\mathbf{C}P^3} \to 0$$

   it follows that

   $$\mathscr{K}_{\mathbf{C}P^3} = \mathscr{O}_{\mathbf{C}P^3}(-4).$$

   Therefore,

   $$\mathscr{K}_Q = \mathscr{O}_Q.$$

   That is $\mathscr{K}_Q$ is holomorphically trivial. It follows that Q is spin.

   In fact, choosing the spin structure corresponding to $\sqrt{\mathscr{K}_Q} = \mathscr{O}_Q$, we have

   $$W^+ = \mathscr{O}_Q \oplus \Lambda^2 \mathscr{T}^*_Q \quad \text{and} \quad W^- = \mathscr{T}^*_Q.$$

   and

Example 30.2. Let M be a closed spin 4–manifold. Let be E a Hermitian vector bundle with a unitary connectio. Denote by $\not{D}_E^+\colon \Gamma(W^+ \otimes E) \to \Gamma(W^- \otimes E)$ the positive chirality complex Atiyah–Singer operator. By Theorem 25.1 and Theorem 29.5 we have

$$\begin{aligned}\operatorname{index} \not{D}^+ &= \int_M \hat{A}(TM)\operatorname{ch}(E) \\ &= -\frac{1}{24}\int_M p_1(TM) + \int_M \operatorname{ch}_2(E) \\ &= -\frac{1}{8}\sigma(M) + \frac{1}{2}\int_M c_1(E)^2 - 2c_2(E).\end{aligned}$$

Sadly(?), not every 4–manifold is spin. However, every oriented 4–manifold spin^c. This allows us to define an Atiyah–Singer operator $\not{D}^+$. The operator now depends on the choice of a connection on the characteristic line bundle L of the chosen spin^c–structure $\mathfrak{w}$. In dimension 4 the characteristic line bundle is

$$L = \Lambda^2 W^+ = \Lambda^2 W^-.$$

Any two choices of spin^c–structures and connections on L are related by tensoring W with a Hermitian line bundle ℓ with connection. This has the following effect on the characteristic line bundles:

$$\Lambda^2(W^+ \otimes \ell) = \Lambda^2 W^+ \otimes \ell^2.$$

Suppose M is actually spin. Fix a spin structure $\mathfrak{s}_0$. This induces a spin^c structure $\mathfrak{w}_0$ with trivial characteristic line bundle $L_0 = \Lambda^+ W_0^+$. Denote by $\not{D}_0^+$ the positive chirality Atiyah–Singer operator for $\mathfrak{s}_0$ or, equivalently, $\mathfrak{w}_0$. Let ℓ be a Hermitian line bundle with connection. Denote $\mathfrak{w}_\ell$ the spin^c structure obtained by twisting $\mathfrak{w}_0$ by ℓ. Denote by W_ℓ the corresponding complex spinor bundle and denote by $\not{D}_\ell^+$ the corresponding positive chirality Atiyah–Singer operator. We have

$$c_1(\Lambda^2 W_\ell^+) = c_1(\Lambda^2 W_0^+ \otimes \ell) = 2c_1(\ell).$$

From the above discussion it follows that

$$\begin{aligned}\operatorname{index} \not{D}_\ell^+ &= -\frac{1}{8}\sigma(M) + \frac{1}{2}\int_M c_1(\ell)^2 \\ &= -\frac{1}{8}\sigma(M) + \frac{1}{8}\int_M c_1(\Lambda^2 W_\ell^+)^2.\end{aligned}$$

In fact, this index formula is valid even if M is just spin^c. Our proof of the index formula is easily adapted to establish this. The point is that in the Weitzenböck formula an additional term arising from the curvature on L appears and this yields correction term of $\operatorname{ch}(L) = e^{c_1(L)}$. That is the Atiyah–Singer Index Theorem for a spin^c–manifold is

$$\operatorname{index} \not{D}^+ = \int_M \hat{A}(TM)\operatorname{ch}(L)$$

with L denoting the characteristic line bundle.

The above index formula plays an important role in **Seiberg–Witten theory**. It determines the (virtual) dimension of the moduli spaces in question.

Example 30.3. Let M be closed oriented 4–manifold. Let G be semi-simple Lie group. (Take $G = \mathrm{SU}(2)$ if you want.) Let P be principal G–bundle over M and denote by $\mathfrak{g}_P$ the associated adjoint bundle. That is

$$\mathfrak{g}_P = P \times_G \mathfrak{g}$$

with G acting on $\mathfrak{g}$ through the adjoint representation. Since G is semi-simple, the negative of the Killing form is a G–invariant inner product on $\mathfrak{g}$. This makes $\mathfrak{g}_P$ into Euclidean vector bundle. Suppose A is a **ASD instanton** on P; that is: a connection on P such that the self-dual part of its curvature vanishes

$$F_A^+ = 0 \quad \text{or, equivalently,} \quad * F_A = -F_A.$$

The deformation theory of A as an ASD instanton is controlled by the **Atiyah–Hitchin–Singer complex**:

$$0 \longrightarrow \Omega^0(M, \mathfrak{g}_P) \xrightarrow{\mathrm{d}_A} \Omega^1(M, \mathfrak{g}_P) \xrightarrow{\mathrm{d}_A^+} \Omega^2_+(M, \mathfrak{g}_P) \longrightarrow 0.$$

The virtual dimension

$$d(\mathfrak{g}_P)$$

of the moduli space of ASD instantons is thus the Euler characteristic of the cohomology of this complex. Equivalently it is minus the index of the operator

$$\delta_A = \begin{pmatrix} \mathrm{d}_A^+ \\ \mathrm{d}_A^* \end{pmatrix} : \Omega^1(M, \mathfrak{g}_P) \longrightarrow \Omega^0(M, \mathfrak{g}_P) \oplus \Omega^2_+(M, \mathfrak{g}_P).$$

This operator is the Dirac operator associated to the tensor product of $\mathfrak{g}_P$ with the Clifford module

$$S^+ = T^*M \quad \text{and} \quad S^- = \mathbf{R} \oplus \Lambda^2_+ T^*M$$

with Clifford muliplication given by

$$\gamma(v)\alpha = \alpha(v) + (v^* \wedge \alpha)^+.$$

Exercise 30.4. Prove this!

A computation verifies that if $W^\pm$ denote the positive and negative chirality complex spin repesentations

$$\begin{aligned} W^+ \otimes W^+ \oplus W^- \otimes W^+ = W \otimes W^+ &\cong (\mathbf{R} \oplus \Lambda^2_+ \mathbf{R}^4 \oplus TM) \otimes \mathbf{C} \\ &= (\mathbf{R} \oplus \Lambda^2_+ \mathbf{R}^4) \otimes \mathbf{C} \oplus TM \otimes \mathbf{C}. \end{aligned}$$

as graded Clifford modules. In particular, the complexification of δ_A agrees with the negative chirality Atiyah–Singer operator twisted by $W^+ \otimes \mathfrak{g}_P^{\mathbf{C}}$. By Theorem 25.1, we have

$$\begin{aligned} d(\mathfrak{g}_P) &= \operatorname{index} \delta_A \\ &= \operatorname{index} \not{D}^-_{W^+ \otimes \mathfrak{g}_P^{\mathbf{C}}} \\ &= -\operatorname{index} \not{D}^+_{W^+ \otimes \mathfrak{g}_P^{\mathbf{C}}} \\ &= -\int_M \hat{A}(TM)\mathrm{ch}(W^+)\mathrm{ch}(\mathfrak{g}_P^{\mathbf{C}}). \end{aligned}$$

One can evaluate this by working out $\mathrm{ch}(W^+)$, but we will proceed in this way. instead, we observe that $c_1(\mathfrak{g}_P^{\mathbf{C}}) = 0$, $\mathrm{ch}_0(\mathfrak{g}_P^{\mathbf{C}}) = \mathrm{rk}\,\mathfrak{g}_P = \dim \mathfrak{g}$, $\mathrm{ch}_0(W^+) = rkW^+ = 2$, and $\hat{A}_0(TM) = 1$ imply that

$$d(P) = -\int_M \hat{A}(TM)\mathrm{ch}(W^+)\mathrm{ch}(\mathfrak{g}_P^{\mathbf{C}})$$

31 The Atiyah–Patodi–Singer index theorem

Definition 31.1. Let (M, g) be a non-compact Riemannian manifold of dimension n. We say that (M, g) is **asymptotically cylindrical (ACyl)** if there exists a compact subset K, $\delta > 0$, a closed Riemannian manifold (N, g_∞), and a diffeomorphism $\phi\colon M\backslash K \to [1,\infty) \times N$ such that

$$|\nabla^k(\phi_* g - g_\infty)| = O(e^{-\delta\ell}) \quad \text{for all} \quad k \in \mathbf{N}_0$$

Here ℓ is the coordinate function on $[1, \infty)$.

The Atiyah–Singer Index Theorem does not apply to the Dirac operator on an ACyl Riemannian manifold. The extension of index theorem to this setting is (a special case) of the Atiyah–Patodi–Singer Index Theorem.

We shall first address the question when a Atiyah–Singer operator on an ACyl manifold is Fredholm.

Proposition 31.2. *Let N be a closed Riemannian manifold. Let $I \subset \mathbf{R}$ be an interval.*

1. *The product $I \times N$ is spin if and only if N is spin.*
2. *There is a canonical bijection between spin structure on N and spin structures on $I \times N$.*

Suppose that N is odd dimensional.

3. *Fix a spin structure on N and equip $I \times N$ with the corresponding spin structure. If W_N denotes the complex spinor bundle of N and $W_M^\pm$ denotes the complex spinor bundles of W, then there are isomorphisms*
$$W_M^\pm \cong \pi_N^* W_N.$$
4. *With respect to the above isomorphism, the complex Atiyah–Singer operator $\not{D}^+\colon \Gamma(W_M^+) \to \Gamma(W_M^-)$ is identified with*
$$\not{D}_W^+ = \partial_\ell + \not{D}_N.$$

Proof. The proof is left as an exercise. □

The above shows that the Atiayh–Singer operator on a spin ACyl manifold (of even dimension) is **asymptotically translation-invariant**. (I will spare you and not give a definition of what precisely it means to be asymptotically translation-invariant. Surely, you can figure out the definition yourself.)

Proposition 31.3. *Let M be an ACyl manifold with asymptotic cross-section N. Let $D\colon \Gamma(E) \to \Gamma(F)$ be a first order elliptic differential operator which is asymptotic to*

$$\partial_\ell + A$$

with A denoting a first order self-adjoint *elliptic differential operator on N. The operator $D\colon W^{k+1,2}\Gamma(E) \to W^{k,2}\Gamma(F)$ is Fredholm if and only if A is invertible.*

Sketch proof. The operator D satisfies the estimate

$$\|s\|_{W^{k+1,2}(M)} \lesssim \|Ds\|_{W^{k,2}}(M) + \|s\|_{L^2(M)}.$$

This implies that $\dim\ker D < \infty$ and $\operatorname{im} D$ is closed *if* them embedding $W^{k+1,2}(M) \to L^2(M)$ is compact. For compact manifold the latter is true, but for non-compact manifolds it fails. If one can show that, in fact,

$$\|s\|_{W^{k+1,2}(M)} \lesssim \|Ds\|_{W^{k,2}}(M) + \|s\|_{L^2(K)}$$

for K some compact subset, then the same argument works again.

Such an estimate basically means that D is invertible on the cylindrical end. Since D is asymptotic to $\partial_\ell + A$, this is essentially equivalent to

$$\partial_\ell + A\colon\ W^{k+1,2}(\mathbf{R}\times M) \to W^{k,2}(\mathbf{R}\times M)$$

being invertible.

If A is not invertible, that is A has a non-trivial kernel then for any $x \in \ker A$ we can find a sequence of cut-off functions χ_i such that

$$\|(\partial_\ell + A)\chi_i s\|_{W^{k,2}} = O(1) \quad \text{and} \quad \|\chi_i s\|_{W^{k+1,2}} \to \infty.$$

This is impossible if A is invertible. If A is invertible, however, one can relatively easily write down a formula for the inverse of $\partial_\ell + A$. We can write any compactly supported section s as

$$s(\ell, x) = \sum_{\lambda\in\operatorname{spec} A} f_\lambda(\ell)s_\lambda(x)$$

with s_λ a λ–eigensection for A. Define

$$\left[(\partial_\ell + A)^{-1}s\right](\ell, x) = \sum_{\lambda\in\operatorname{spec} A} g_\lambda(\ell)s_\lambda(x)$$

with

$$g_\lambda(\ell) = \begin{cases} e^{-\lambda\ell}\int_{-\infty}^{\ell} e^{\lambda t} f_\lambda(t)\mathrm{d}t & \text{if } \lambda > 0 \\ -e^{\lambda\ell}\int_{\ell}^{\infty} e^{-\lambda t} f_\lambda(t)\mathrm{d}t & \text{if } \lambda < 0. \end{cases}$$

This formula inverts $\partial + A$. With some work one can show that it induces an inverse of $\partial_\ell + A\colon\ W^{k+1,2}(\mathbf{R}\times M) \to W^{k,2}(\mathbf{R}\times M)$. □

The upshot of the above discussion is that if M is an ACyl spin manifold, then $\not{D}_M$ is Fredholm if and only if $\not{D}_N$ is invertible. Thinking about what was involved in the proof of the Atiyah–Singer index theorem, we see that the formula

$$\lim_{t\to 0} \operatorname{str} k_t(x,x)\mathrm{vol}_M = [\hat{A}(TM)\mathrm{ch}(E)]_n(x)$$

still holds—after all its proof was entirely local. What does not hold, however, is the McKean–Singer formula Theorem 24.1:

$$\operatorname{index} \not{D}^+ \neq \lim_{t\to 0} \int_M \operatorname{str} k_t(x,x)\mathrm{vol}_M!$$

Half of the difference between these two terms is called the **η–invariant**. Here is a general definition

Definition 31.4. Let A be a self-adjoint operator The **η–invariant** of A is the value of

$$\eta_A(0)$$

where η_A is the analytic continuation of

$$s \mapsto \sum_{\lambda\neq 0} \frac{\operatorname{sign}\lambda}{|\lambda|^s}$$

defined for $\operatorname{Re} s \gg 1$.

There is some work involved in showing that $\eta_A(0)$ is well-defined. Roughly speaking, $\eta_A(0)$ is the difference of the number of positive eigenvalues and the number of negative eigenvalues. Of course, this is $\infty - \infty$. In any case, $\eta_A(0)$ is a regularization of this spectral asymmetry.

Theorem 31.5 (Atiyah–Patodi–Singer). *If $\not{D}_{N,E}$ is invertible, then*

$$\operatorname{index} \not{D}^+_{M,E} = \int_M \hat{A}(TM)\mathrm{ch}(E) + \frac{1}{2}\eta_{\not{D}_N}(0).$$

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