Differential Geometry

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Contents

1 C^k manifolds with boundary

The notion of *space* with which one deals in differential geometry is the following.

Definition 1.1. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a topological space.

1. A **chart on** X is a homeomorphism $\phi\colon U \to \tilde{U}$ from an open subset $U \subset X$ to either an open subset $\tilde{U} \subset \mathbf{M}$ with the **model space** $\mathbf{M}$ denoting either $\mathbf{R}^m$ or $[0,\infty) \times \mathbf{R}^{m-1}$.

2. A C^k **atlas on** X is a set $\mathcal{A} = \left\{\phi_\alpha\colon U_\alpha \to \tilde{U}_\alpha : \alpha \in A\right\}$ of charts on X such that:

   (a) The set $\{U_\alpha : \alpha \in A\}$ is an open cover of M:

   $$X = \bigcup_{\alpha \in A} U_\alpha. \tag{1.2}$$

   (b) For every $\alpha, \beta \in A$ the **transition map** $\tau_\beta^\alpha\colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ defined by

   $$\tau_\beta^\alpha \coloneqq \phi_\beta \circ \phi_\alpha^{-1} \tag{1.3}$$

   is C^k (as a map from an open subset of $\mathbf{M}_\alpha$ to $\mathbf{M}_\beta$).

3. A C^k **structure on** X is a C^k atlas $\mathcal{A}$ on X which is **maximal** in the following sense: if $\mathcal{B}$ is a C^k atlas on X with $\mathcal{B} \supset \mathcal{A}$, then $\mathcal{B} = \mathcal{A}$.

4. A C^k **manifold with boundary** is a pair $(X, \mathcal{A})$ consisting of a topological space X which is Hausdorff and paracompact, and a C^k structure $\mathcal{A}$ on X. •

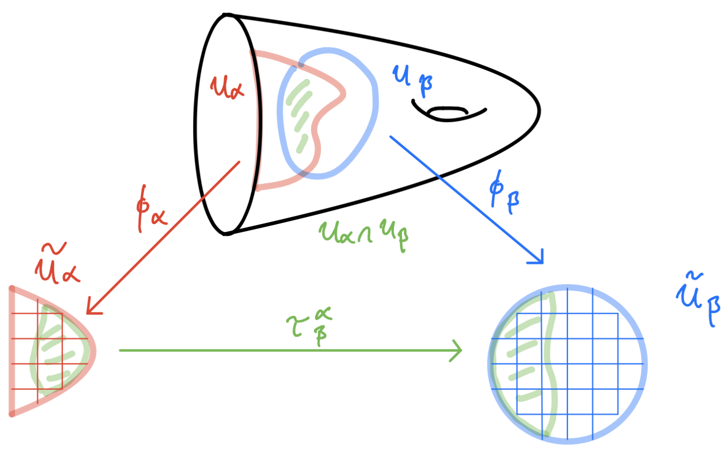

Figure 1.1: An illustration of a transition map.

Notation 1.4. 1. The words **smooth** and **analytic** mean C^∞ and C^ω respectively. If C^k is not specified, then C^∞ is assumed.

2. It is customary to suppress the C^k structure $\mathcal{A}$ and talk about the C^k manifold with boundary X. If it becomes necessary to recover $\mathcal{A}$, then this will be done by referring to it as the C^k **structure of** X. An **admissible chart on** X is an element of $\mathcal{A}$. These are *abuses of notation.* (Like many abuses of notation, this is theoretically problematic, but turns out to be harmless in practice.) ∘

Remark 1.5. For $k = 0$ condition (2b) is vacuous. Therefore, a topological space X admits at most one C^0 structure. If it does and is Hausdorff and paracompact, then it is said to be a **topological manifold with boundary**. The theory developed subsequently requires $k \geqslant 1$. ♣

Remark 1.6. There are topological spaces which admit C^ω atlases but fail to be Hausdorff, paracompact, or both; see Example 1.55, Example 1.59, and Example 1.74. ♣

Example 1.7. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. The empty set $\varnothing$ with $\mathcal{A} = \varnothing$ is a C^k manifold. ♠

Example 1.8. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$ and let $\mathbf{M}$ be a model space. Let $U \subset \mathbf{M}$. The **standard** C^k **structure on** U is the set of those charts ϕ on U for which both ϕ and ϕ^{-1} are C^k. ♠

The maximality condition makes it rather impractical to explicitly write down a C^k structure. Fortunately, to specify a C^k structure, it suffices to exhibit a C^k atlas.

Proposition 1.9. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a topological space. If $\mathcal{A}$ is a C^k atlas, then there is a unique C^k structure $\bar{\mathcal{A}}^k$ on X with $\mathcal{A} \subset \bar{\mathcal{A}}^k$.*

Proof. Denote by $\bar{\mathcal{A}}^k$ the set of charts $\psi\colon V \to \tilde{V}$ on M such that $\mathcal{A} \cup \{\psi\}$ is a C^k atlas. By definition $\bar{\mathcal{A}}^k$ contains every C^k atlas containing $\mathcal{A}$. Therefore, if $\bar{\mathcal{A}}^k$ is a C^k atlas, then it is the unique maximal C^k atlas containing $\mathcal{A}$.

$\bar{\mathcal{A}}^k$ satisfies (2a) in Definition 1.1. To see that it also satisfies (2b) in Definition 1.1, it suffices to prove that if $\psi\colon V \to \tilde{V}$ and $\chi\colon W \to \tilde{W}$ are two charts in $\bar{\mathcal{A}}^k$, then the transition map $\tau \coloneqq \chi \circ \psi^{-1}\colon \psi(V \cap W) \to \chi(V \cap W)$ is C^k. By definition of $\bar{\mathcal{A}}^k$, for every $\alpha \in A$ the transition maps $\phi_\alpha \circ \psi^{-1}\colon \psi(V \cap U_\alpha) \to \phi_\alpha(V \cap U_\alpha)$ and $\chi \circ \phi_\alpha^{-1}\colon \phi_\alpha(V \cap U_\alpha) \to \chi(V \cap U_\alpha)$ are C^k. The restriction of τ to $\psi(V \cap W \cap U_\alpha)$ agrees with the composition $(\chi \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \psi^{-1})$; hence: it is C^k. Since $\mathcal{A}$ satisfies (2a) in Definition 1.1, $\{\psi(V \cap W \cap U_\alpha) : \alpha \in A\}$ is an open cover of $\psi(V \cap W)$. Therefore, τ is C^k. ■

Definition 1.10. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a topological space. Let $\mathcal{A}$ be a C^k atlas on X. The C^k **structure on** X **induced by** $\mathcal{A}$ is the unique maximal C^k atlas $\bar{\mathcal{A}}^k$ containing $\mathcal{A}$. •

Remark 1.11. Let $k, \ell \in \mathbf{N}_0 \cup \{\infty, \omega\}$ with $\ell \leqslant k$. If X is a C^k manifold with boundary, then its C^k structure is contained in a unique C^ℓ structure. Therefore, in accordance with Notation 1.4(2), X might also refer to the corresponding C^ℓ manifold with boundary. ♣

$\sim$

Here are some examples of C^k manifolds with boundary.

Example 1.12. Let $n \in \mathbf{N}_0$. The n**–dimensional sphere** is the subspace

$$S^n \coloneqq \left\{x \in \mathbf{R}^{n+1} : |x| = 1\right\}. \tag{1.13}$$

S^n is Hausdorff and paracompact because it is a subspace of $\mathbf{R}^{n+1}$.

The **hemi-spheres**

$$H_{i,\pm} \coloneqq \{(x_1, \dots, x_{n+1}) \in S^n : \pm x_i > 0\} \tag{1.14}$$

are open, the maps $\phi_{i,\pm} \colon H_{i,\pm} \to B_1(0) \subset \mathbf{R}^n$ defined by

$$\phi_{i,\pm}(x_1, \dots, x_{n+1}) \coloneqq (x_1, \dots, \widehat{x_i}, \dots, x_{n+1}) \tag{1.15}$$

are homeomorphisms, and every $x \in S^n$ is contained in one of the subsets $H_{i,\pm}$.

For $i > j$ the transition map $\tau^{i,\varepsilon}_{j,\delta} \coloneqq \phi_{j,\delta} \circ \phi^{-1}_{i,\varepsilon}$ satisfies

$$\tau^{i,\varepsilon}_{j,\delta}(x_1, \dots, x_n) = \left(x_1, \dots, \widehat{x_j}, \dots, x_{i-1}, \varepsilon\sqrt{1 - |x|^2}, x_i, \dots, x_n\right). \tag{1.16}$$

For $i < j$ there is a similar formula and for $i = j$ the situation is trivial. Therefore,

$$\mathcal{A} \coloneqq \left\{\phi_{i,\varepsilon} : i \in \{0, \dots, n\}, \varepsilon \in \{\pm\}\right\} \tag{1.17}$$

is a C^ω atlas. The smooth structure $\bar{\mathcal{A}}^\infty$ is the **standard smooth structure on** S^n. ♠

Example 1.18. Let $n \in \mathbf{N}_0$. Here is another approach to defining an atlas on S^n. Set

$$U_\pm \coloneqq S^n \backslash (0, \dots, 0, \pm 1). \tag{1.19}$$

The **stereographic projection from** $(0, \dots, 0, \mp 1)$ is the map $\sigma_\pm \colon U_\pm \to \mathbf{R}^n$ defined by

$$\sigma_\pm(x) \coloneqq \frac{(x_1, \dots, x_n)}{1 \mp x_{n+1}}. \tag{1.20}$$

(A moment's thought shows that the straight line through $(0, \dots, 0, \mp 1)$ and $x \in U_\pm$ intersects the hyperplane defined by $x_{n+1} = 0$ in $(\sigma_\pm(x), 0)$.)

These maps are homoeomorphisms. To see this, observe the following. If $y = \sigma_\pm(x)$, then

$$|y|^2 = \frac{1 - x_{n+1}^2}{(1 \mp x_{n+1})^2} = \frac{1 \pm x_{n+1}}{1 \mp x_{n+1}} = \frac{2}{1 \mp x_{n+1}} - 1. \tag{1.21}$$

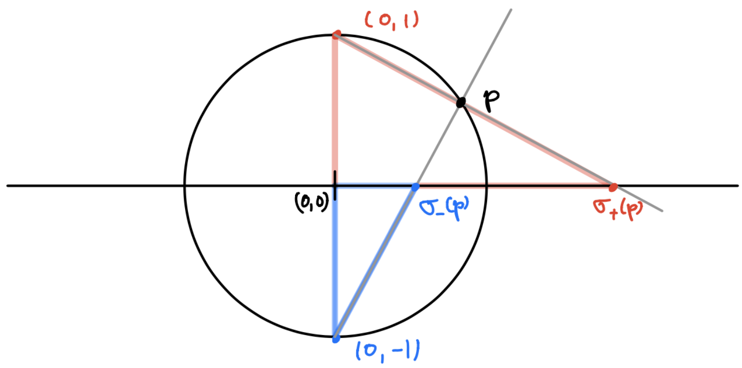

Figure 1.2: The stereographic projections from $(0, \pm 1) \in S^1$.

Therefore,

$$1 \mp x_{n+1} = \frac{2}{|y^2| + 1} \quad \text{and} \quad x_{n+1} = \pm \frac{|y|^2 - 1}{|y^2| + 1}. \tag{1.22}$$

This implies that $\sigma_\pm$ is bijective and its inverse satisfies

$$\sigma_\pm^{-1}(y) = \left(\frac{2y}{|y|^2 + 1}, \pm \frac{|y|^2 - 1}{|y|^2 + 1} \right). \tag{1.23}$$

The transition map $\sigma_\mp \circ \sigma_\pm^{-1} =: \mathbf{R}^n \backslash \{0\} \to \mathbf{R}^n \backslash \{0\}$ is the inversion map

$$\sigma_\mp \circ \sigma_\pm^{-1}(y) = \frac{y}{|y|^2}. \tag{1.24}$$

This can be seen by direct computation or inferred from (1.21); alternatively, it can be deduced from Figure 1.2. Therefore,

$$\mathcal{B} := \{\sigma_+, \sigma_-\} \tag{1.25}$$

is a C^ω atlas on S^n. ♠

Exercise 1.26. Prove (1.24) without computation using only elementary geometry.

Remark 1.27. The atlas $\mathcal{A}$ defined in Example 1.12 and the atlas $\mathcal{B}$ defined in Example 1.18 are not identical. Nevertheless, $\mathcal{A}$ and $\mathcal{B}$ induce the same C^ω structure. ♣

Example 1.28. Denote by $\sim$ the equivalence relation on $[-1, 1] \times \mathbf{R}$ defined by

$$(x, y) \sim (z, w) \quad \text{if and only if} \quad n := w - y \in \mathbf{Z} \text{ and } z = (-1)^n x. \tag{1.29}$$

The **Möbius strip** is the quotient

$$\text{Möb} := ([-1,1] \times \mathbf{R})/\sim . \tag{1.30}$$

Möb is Hausdorff and paracompact.

For $\varepsilon \in \{\pm 1\}$ and $\eta \in \{0, 1/2\}$ set

$$U_{\varepsilon,\eta} := \{[(x,y)] \in \text{Möb} : x \neq \varepsilon \text{ and } y - \eta \in (0,1)\}. \tag{1.31}$$

These are open and cover Möb. For $\varepsilon \in \{\pm 1\}$ and $\eta \in \{0, 1/2\}$ define $\phi_{\varepsilon,\eta} \colon U_{\varepsilon,\eta} \to [0,2) \times (0,1)$ by

$$\phi_{\varepsilon,\eta}([x,y]) := (\varepsilon x + 1, y - \eta). \tag{1.32}$$

These are maps are homeomorphisms. Let $\varepsilon, \delta \in \{\pm 1\}$ and $\eta, \zeta \in \{0, 1/2\}$. If $\varepsilon \neq \delta$ and $\eta = \zeta$, then

$$\phi_{\varepsilon,\eta}(U_{\varepsilon,\eta} \cap U_{\delta,\eta}) = \phi_{\varepsilon,\eta}(U_{\varepsilon,\eta} \cap U_{\delta,\eta}) = (0,2) \times (0,1) \tag{1.33}$$

and the transition maps $\tau^{\varepsilon,\eta}_{\delta,\zeta} := \phi_{\delta,\zeta} \circ \phi^{-1}_{\varepsilon,\eta}$ satisfy

$$\tau^{\varepsilon,\eta}_{\delta,\zeta}(x,y) = (\varepsilon\delta(x-1)+1, y). \tag{1.34}$$

If $\varepsilon \neq \delta$ and $\eta \neq \zeta$, then

$$\phi_{\varepsilon,\eta}(U_{\varepsilon,\eta} \cap U_{\delta,\eta}) = \phi_{\varepsilon,\eta}(U_{\varepsilon,\eta} \cap U_{\delta,\eta}) = (0,2) \times (0,1/2) \cup (0,2) \times (1/2,1) \tag{1.35}$$

and transition maps $\tau^{\varepsilon,\eta}_{\delta,\zeta} := \phi_{\delta,\zeta} \circ \phi^{-1}_{\varepsilon,\eta}$ satisfy

$$\tau^{\varepsilon,\eta}_{\delta,\zeta}(x,y) = \begin{cases} (\varepsilon\delta(x-1)+1, y+1/2) & \text{if } y < 1/2, \\ (\varepsilon\delta(x-1)+1, y-1/2) & \text{if } y > 1/2. \end{cases} \tag{1.36}$$

In the case $\varepsilon = \delta$ and $\eta \neq \zeta$, the domain and codomain of the transition map are $[0,2) \times (0,1/2) \cup [0,2) \times (1/2,1)$ instead but the formula for transition map remains the same. Therefore,

$$\mathcal{A} := \left\{\phi_{\varepsilon,\eta} : \varepsilon \in \{\pm\}, \eta \in \{0, 1/2\}\right\} \tag{1.37}$$

is a C^ω atlas. ♠

Exercise 1.38. Build a (model of a) Möbius band as in Figure 1.3.

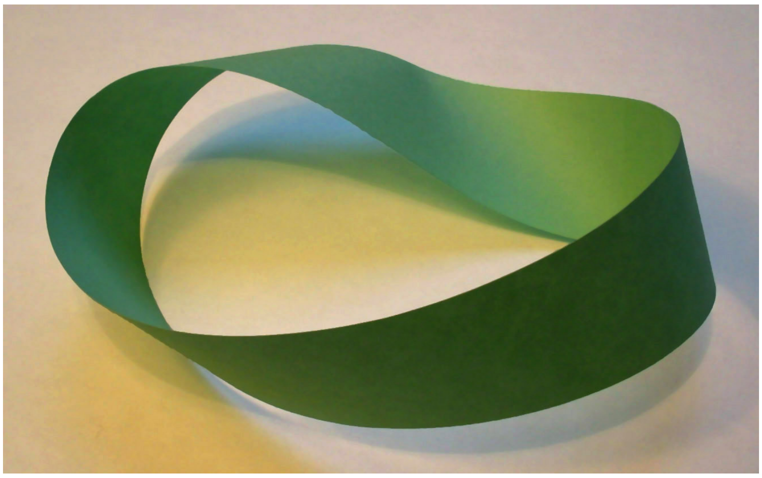

Figure 1.3: A Möbius strip.

Example 1.39. Let $n \in \mathbf{N}_0$ and $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Define the equivalence relation $\sim$ on $\mathbf{K}^{n+1}\backslash\{0\}$ by

$$x \sim y \quad \text{if and only if} \quad x = \lambda y \text{ for some } \lambda \in \mathbf{K}^{\times}. \tag{1.40}$$

The n–**dimensional real, complex, and quaternionic projective space** is the quotient

$$\mathbf{K}P^n := (\mathbf{K}^{n+1}\backslash\{0\})/{\sim} \tag{1.41}$$

for $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$ respectively. ($\mathbf{K}P^n$ is the "moduli space" of 1–dimensional $\mathbf{K}$–linear subspaces of $\mathbf{K}^{n+1}$.) It is customary to write

$$[x_1 : \ldots : x_{n+1}] \tag{1.42}$$

for the equivalence class of $(x_1, \ldots, x_{n+1})$ in $\mathbf{K}P^n$.

$\mathbf{K}P^n$ is Hausdorff and paracompact. (Exercise!)

The subsets

$$U_i := \{[x_1 : \ldots : x_{n+1}] \in \mathbf{K}P^n : x_i \neq 0\} \tag{1.43}$$

are open, the maps $\phi_i \colon U_i \to \mathbf{K}^n$ defined by

$$\phi_i([x_1 : \ldots : x_{n+1}]) := x_i^{-1}(x_1, \ldots, \widehat{x}_i, \ldots, x_{n+1}) \tag{1.44}$$

is a homeomorphism, and every $[x] \in \mathbf{K}P^n$ is contained in one of the subsets U_i.

For $i > j$ the transition function τ_j^i satisfies

$$\tau_j^i(x_1, \ldots, x_n) = x_j^{-1}\big(x_1, \ldots, \widehat{x}_j, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n\big). \tag{1.45}$$

For $i < j$ there is a similar formula and for $i = j$ the situation is trivial. Therefore,

$$\mathcal{A} := \{\phi_i : i = 1, \ldots, n+1\} \tag{1.46}$$

is a C^ω atlas. (Identify $\mathbf{K}^n = \mathbf{R}^{n \dim_{\mathbf{R}} \mathbf{K}}$.) $\bar{\mathcal{A}}$ is the **standard smooth structure on** $\mathbf{K}P^n$. ♠

Question 1.47. Can you define the octonionic projective space $\mathbf{O}P^n$?

Example 1.48. Let $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Let $k, n \in \mathbf{N}$. The **real, complex, and quaternionic Grassmannian of** k**–planes in** $\mathbf{K}^n$ is the quotient

$$\mathrm{Gr}_k(\mathbf{K}^n) := \{T \in \mathrm{Hom}(\mathbf{K}^k, \mathbf{K}^n) : T \text{ is injective}\}/\mathrm{GL}_k(\mathbf{K}) \tag{1.49}$$

for $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, and $\mathbf{K}$ respectively. Here $\mathrm{GL}_k(\mathbf{K})$ acts on $\mathrm{Hom}(\mathbf{K}^k, \mathbf{K}^n)$ by composition on the right. $\mathrm{Gr}_k(\mathbf{K}^n)$ is the "moduli space" of k–dimensional $\mathbf{K}$–linear subspaces of $\mathbf{K}^n$. To see this, observe the following. Let $V \subset \mathbf{K}^n$ be a k–dimensional linear subspace. A choice of basis determines an isomorphism $\mathbf{K}^k \cong V$. The composition $\mathbf{K}^k \cong V \hookrightarrow \mathbf{K}^n$ is a injective linear map $T \in \mathrm{Hom}(\mathbf{K}^k, \mathbf{K}^n)$ with $\operatorname{im} T = V$. A different choice of basis produces an injective linear map $T' \in \mathrm{Hom}(\mathbf{K}^k, \mathbf{K}^n)$ which differs from T by composition on the right with an element of $\mathrm{GL}_k(\mathbf{K}^n)$.

The Grassmannian $\mathrm{Gr}_k(\mathbf{K}^n)$ is Hausdorff and paracompact. (Exercise!) It can be given an analytic structure as follows. Set

$$\mathbf{I} := \{I = (i_1, \ldots, i_k) \in \mathbf{N}^k : 1 \leqslant i_1 < \ldots < i_k \leqslant n\}. \tag{1.50}$$

For $\underline{i} \in \mathbf{I}$ denote by $\tau_{\underline{i}} \in \mathrm{Hom}(\mathbf{K}^n, \mathbf{K}^n)$ the linear map defined by

$$(\tau_{\underline{i}})_{ij} := \begin{cases} 1 & \text{if } i = i_j \\ 0 & \text{otherwise,} \end{cases} \tag{1.51}$$

and denote by $\pi_{\underline{i}} \in \mathrm{Hom}(\mathbf{K}^n, \mathbf{K}^k)$ the composition of $\tau_{\underline{i}}$ with the projection onto $\mathbf{K}^k$. For $\underline{i} \in \mathbf{I}$ set

$$\tilde{U}_{\underline{i}} := \{T \in \mathrm{Hom}(\mathbf{K}^k, \mathbf{K}^n) : \pi_{\underline{i}} \circ T \text{ is invertible}\} \quad \text{and} \quad U_{\underline{i}} := \tilde{U}_i/\mathrm{GL}(\mathbf{K}^k). \tag{1.52}$$

The condition on T is that the rows indicated by $\underline{i}$ form an invertible $k \times k$ matrix. The subsets $U_{\underline{i}}$ are open and every $[T] \in \mathrm{Gr}_k(\mathbf{K}^n)$ is contained in one of the $U_{\underline{i}}$.

If $T \in \tilde{U}_{\underline{i}}$, then

$$(\tau_{\underline{i}} \circ T) \circ (\pi_{\underline{i}} \circ T)^{-1} = \begin{pmatrix} \mathbb{1} \\ B \end{pmatrix} \quad \text{with} \quad B \in \mathrm{Hom}(\mathbf{K}^k, \mathbf{K}^{n-k}). \tag{1.53}$$

This defines a $\mathrm{GL}_k(\mathbf{K})$–equivariant analytic map $\tilde{\phi}_{\underline{i}} \colon \tilde{U}_{\underline{i}} \to \mathrm{End}(\mathbf{K}^k)$. It descends to a homeomorphism $\phi_{\underline{i}} \colon U_{\underline{i}} \to \mathrm{End}(\mathbf{K}^k)$. The transition maps $\phi_{\underline{j}} \circ \phi_{\underline{i}}^{-1}$ are analytic because $\tilde{\phi}_{\underline{j}}$ is analytic and has $\tilde{\phi}_{\underline{i}}$ has a $\mathrm{GL}_k(\mathbf{K})$–equivariant analytic right-inverse. Therefore,

$$\mathcal{A} \coloneqq \{\phi_{\underline{i}} \colon U_{\underline{i}} \to \mathrm{End}(\mathbf{K}^k) : \underline{i} \in \mathbf{I}\} \tag{1.54}$$

is a C^ω atlas. (Identify $\mathrm{Hom}(\mathbf{K}^k, \mathbf{K}^{n-k}) = \mathbf{R}^{k(n-k)\dim_{\mathbf{R}} \mathbf{K}}$.) The smooth structure $\bar{\mathcal{A}}^\infty$ is the **standard smooth structure on** $\mathrm{Gr}_k(\mathbf{K}^n)$. ♠

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Here are some examples of topological spaces X which admit C^ω atlases but fail to be Hausdorff or paracompact.

Example 1.55. Define the equivalence relation $\sim$ on $\mathbf{R} \times \{+1, -1\}$ by

$$(x, i) \sim (y, j) \quad \text{if and only if} \quad x = y \text{ and } (i = j \text{ or } x < 0). \tag{1.56}$$

The **branching line** is the quotient space

$$\Lambda \coloneqq (\mathbf{R} \times \{+1, -1\})/\!\sim . \tag{1.57}$$

Figure 1.4 is an attempt to illustrate Λ and some of its open subsets. The branching line is not Hausdorff because $[0, +1]$ and $[0, -1]$ cannot be separated by disjoint open sets, it is second-countable, and the subsets $\Lambda_\pm \coloneqq (\mathbf{R} \times \{\pm 1\})/\!\sim$ are open, homeomorphic to $\mathbf{R}$, and $\Lambda = \Lambda_+ \cup \Lambda_-$. ♠

(a) An open neighborhood of $[0, +1]$.

(b) An open neighborhood of $[0, -1]$.

Figure 1.4: The branching line.

Remark 1.58. The points $[0, +1], [0, -1] \in \Lambda$, although distinct, are indistinguishable in the following sense. If $f \in C(\Lambda, \mathbf{R})$ is a continuous function on the branched line, then $f([0, +1]) = f([0, -1])$. The failure of Hausdorffness of Λ can be repaired by identifying $[0, +1]$ and $[0, -1]$. The resulting space $\tilde{\Lambda}$ is Hausdorff, second-countable, but $[0, +1] = [0, -1]$ does not have any neighborhood homeomorphic to an open subset of $\mathbf{R}$. ♣

(a) An open neighborhood of $[0, +1]$. (b) An open neighborhood of $[0, -1]$.

Figure 1.5: The line with two origins.

Example 1.59. Denote by $\sim$ the equivalence relation on $\mathbf{R} \times \{+1, -1\}$ obtained by replacing $x < 0$ with $x \neq 0$ in (1.56). The **line with two origins** is the quotient space

$$\mathbb{L} := (\mathbf{R} \times \{+1, -1\})/\sim . \tag{1.60}$$

Figure 1.5 is an attempt to illustrate $\mathbb{L}$ and some of its open subsets. The line with two origins is not Hausdorff, it is second-countable, and every $x \in \mathbb{L}$ has a neighborhood homeomorphic an open subset of $\mathbf{R}$. ♠

The following example requires a bit of (set-theoretic) preparation.

Definition 1.61 (von Neumann [vNeu23]). A set S is an **ordinal** if the following hold:

1. If $x \neq y \in S$, then either $x \subset y$ or $y \subset x$; that is: $\subset$ defines a total order on S.
2. If $x \in S$, then $\in S$.
3. The order $\subset$ on S is a **well-order**; that is: every non-empty subset of S contains a least element with respect to the order $\in$. •

Remark 1.62. The empty set $\varnothing$ is an ordinal. If S is an ordinal, then so is

$$S + 1 := S \cup \{S\} = \mathfrak{P}(S). \tag{1.63}$$

Here $\mathfrak{P}(S)$ denotes the **power set of** S; that is: the set of all subsets of S. The process

$$0 := \varnothing, \quad 1 := 0 + 1 = \{\varnothing\}, \quad 2 := (0 + 1) + 1 = \{\varnothing, \{\varnothing\}\}, \quad \ldots \tag{1.64}$$

inductively constructs $\mathbf{N}_0$ together with the order $<$. These are precisely the finite ordinal numbers. A moment's thought shows that $\mathbf{N}_0 := \{0, 1, 2, \ldots\}$ is an ordinal.

In fact, if $\{S_\alpha : \alpha \in A\}$ is a set of ordinal numbers, then

$$\bigcup_{\alpha \in A} S_\alpha \tag{1.65}$$

is an ordinal. ♣

Definition 1.66. Let S be a set. A relation $\preccurlyeq$ on S is a **total order** if for every $x, y, z \in S$:

1. if $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x = y$,

2. if $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$, and

3. $x \preccurlyeq y$ or $y \preccurlyeq x$.

A total order $\preccurlyeq$ is a **well-order** if for every $\varnothing \neq T \subset S$ has a **least element** $\min T$ satisfying $\min T \preccurlyeq x$ for every $x \in T$.

A **ordered set** is a pair $(S, \preccurlyeq)$ consisting of a set S and a total order $\preccurlyeq$ on S. A **well-ordered set** is a pair $(S, \preccurlyeq)$ consisting of a set S and a well-order $\preccurlyeq$ on S. •

Remark 1.67. If $(S, \preccurlyeq)$ is a well-ordered set and $x \in S$, then either x is the greatest element or there is a unique least element greater than x. In the later case, set

$$x + 1 \coloneqq \min\{y \in S : x \prec y\}. \tag{1.68}$$

Here $x \prec y$ if and only if $x \preccurlyeq y$ and $x \neq y$. ♣

Definition 1.69. Let $(S_1, \preccurlyeq_1)$ and $(S_2, \preccurlyeq_2)$ be ordered sets. The **lexicographic order** $\preccurlyeq$ on $S_1 \times S_2$ induces by $\preccurlyeq_1$ and $\preccurlyeq_2$ is the total order defined by

$$(x_1, x_2) \preccurlyeq (y_1, y_2) \quad \text{if and only if} \quad x_1 \preccurlyeq_1 y_1 \text{ and } (x_1 \neq y_1 \text{ or } y_1 \preccurlyeq_2 y_2). \tag{1.70}$$ •

Definition 1.71. If $(S, \preccurlyeq)$ is an ordered set, then the **order topology** $\mathcal{O}_\preccurlyeq$ is the coarsest topology on S with respect to which for every $a, b \in S$ the subset

$$(a, \infty) \coloneqq \{x \in S : a \prec x\} \quad \text{and} \quad (-\infty, b) \coloneqq \{x \in S : x \prec b\} \tag{1.72}$$

are open. •

Example 1.73. The order topology on $(\mathbf{R}, \leqslant)$ agrees with the topology induces by the metric $d(x, y) \coloneqq |x - y|$. ♠

Example 1.74. Let $(\omega_1, \preccurlyeq)$ be an uncountable well-ordered set. It is a fact of set theory that these exist. In fact, the axiom of choice is equivalent to the well-ordering theorem which asserts that every set admits a well-order. If you are familiar with the theory of ordinals, then you can take ω_1 be the first uncountable ordinal.

The **long line** is

$$\mathbf{L} \coloneqq \omega_1 \times (0, 1] \tag{1.75}$$

equipped with the order topology induced by the lexicographic order. Figure 1.6 is an attempt to illustrate $\mathbf{L}$ and one of its open subsets. The long line is Hausdorff, it is not second-countable because it contains a subspace homeomorphic to ω_1, and the subsets

$$I_\alpha \coloneqq \{\alpha\} \times (0, 1] \cup \{\alpha + 1\} \times (0, 1). \tag{1.76}$$

are open (intervals), homeomorphic to $(-1, 1)$, and $\mathbf{L} = \bigcup_{\alpha \in \omega_1} I_\alpha$. ♠

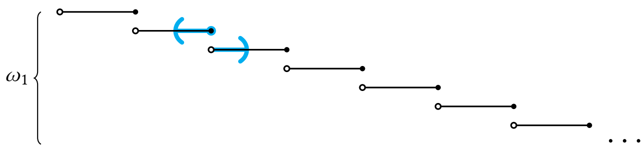

Figure 1.6: The long line.

The spaces Λ, $\mathbb{L}$, $\mathbf{L}$, and their ilk are considered to be too pathological to be manifolds. Part of the theory developed hereafter is still carries over to these spaces, but other important parts break (in interesting ways).

Theorem 1.77 (Brouwer [TODO: add original reference). *; invariance of dimension] Let $m, n \in \mathbf{N}_0$. Let $U \subset [0,\infty) \times \mathbf{R}^{m-1}$ and $V \subset [0,\infty) \times \mathbf{R}^{n-1}$ be non-empty open subsets. If U and V are homeomorphic, then $m = n$.* ■

Exercise 1.78. If you know a bit of algebraic topology, try to prove Theorem 1.77; otherwise, look up a proof or have someone explain the proof to you.

Corollary 1.79. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. For every C^k manifold $(X, \mathcal{A})$ there is a unique locally constant function* $\dim_{\cdot} X \colon X \to \mathbf{N}_0$ *such that* $\dim_x X = m_\alpha$ *whenever $x \in U_\alpha$.* ■

Definition 1.80. In the situation of Corollary 1.79, the **dimension of** X **at** x is $\dim_x X$ is. If $X \neq \varnothing$ and $\dim_{\cdot} X$ is constant, then X is **equi-dimensional** and **of dimension** $\dim X$ with the latter denoting the constant value that $\dim_{\cdot} X$ assumes. •

Theorem 1.81 (Invariance of the boundary). *Let $m \in N_0$. Let $U, V \subset [0,\infty) \times \mathbf{R}^{m-1}$ be open subset. If $\phi \colon U \to V$ is a homeomorphism, then*

$$\phi(U \cap (\{0\} \times \mathbf{R}^{m-1})) = V \cap (\{0\} \times \mathbf{R}^{m-1}). \tag{1.82}$$ ■

Exercise 1.83. If you know a bit of algebraic topology, try to prove Theorem 1.81; otherwise, look up a proof or have someone explain the proof to you.

Exercise 1.84.
1. Construct a homeomorphism $\phi \colon [0,\infty)^2 \to [0,\infty) \times \mathbf{R}$.
2. Prove that ϕ cannot be chosen so that ϕ and ϕ^{-1} are C^1.

Remark 1.85. These above exercise explains why one does not define the notion of a topological manifold with corners, but one could define the notion of a C^k manifold with corners ($k \geqslant 1$); see, e.g., Joyce [Joy16]. ♣

Definition 1.86. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a C^k manifold. Let $x \in X$. If there an $\alpha \in A$ with $\mathbf{M}_\alpha = [0,\infty) \times \mathbf{R}^{m_\alpha - 1}$ and $\phi_\alpha(x) \subset \{0\} \times \mathbf{R}^{m_\alpha - 1}$, then x is **boundary point of** X; otherwise, it is an **interior point of** X. •

Proposition 1.87. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let $(X, \mathcal{A})$ be a C^k manifold with boundary.*

1. *Denote by Set*

$$A_\partial := \{\alpha \in A : \mathbf{M}_\alpha = [0,\infty) \times \mathbf{R}^{m_\alpha - 1} \text{ and } \tilde{U}_\alpha \cap \{0\} \times \mathbf{R}^{m_\alpha - 1} \neq \varnothing\}. \tag{1.88}$$

   For every $\alpha \in A_\partial$ set

$$\tilde{V}_\alpha := \{x \in \mathbf{R}^{m_\alpha - 1} : (0,x) \in \tilde{U}_\alpha\}, \quad V_\alpha := \phi_\alpha^{-1}(\{0\} \times \tilde{V}_\alpha), \tag{1.89}$$

   and define $\psi_\alpha \colon V_\alpha \to \tilde{V}_\alpha$ by $\psi_\alpha := \mathrm{pr}_2 \circ \phi_\alpha$. The set $\mathcal{A}_\partial := \{\psi_\alpha : \alpha \in A_\partial\}$ is a C^k atlas on

$$\partial X := \bigcup_{\alpha \in A} V_\alpha. \tag{1.90}$$

2. *Set $A_\circ := A \backslash A_\partial$. The set $\mathcal{A}_\circ := \{\phi_\alpha : \alpha \in A_\circ\}$ is a C^k atlas on*

$$X^\circ := \bigcup_{\alpha \in A} U_\alpha. \tag{1.91}$$

3. *X is the disjoint union of ∂X and X°.*

Proof. (1) and (2) are obvious. (3) is a consequence of Theorem 1.81. ■

Definition 1.92. In the situation of Proposition 1.87, $(\partial X, \bar{\mathcal{A}}_\partial^k)$ is the **boundary of** $(X, \mathcal{A})$ and $(X^\circ, \bar{\mathcal{A}}_\circ^k)$ is the **interior of** $(X, \mathcal{A})$. A C^k **manifold (without boundary)** is a C^k manifold with boundary with $\partial X = \varnothing$. •

2 C^k maps

Definition 2.1. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be C^k manifolds with boundary. A continuous map $f \colon X \to Y$ is C^k if for every $\alpha \in A$ and $\beta \in B$ the map $\tilde{f}_\beta^\alpha \colon \phi(U_\alpha \cap f^{-1}(V_\beta)) \to \tilde{V}_\beta$ defined by

$$\tilde{f}_\beta^\alpha := \psi_\beta \circ f \circ \phi_\alpha^{-1} \tag{2.2}$$

is C^k (as a map from an open subset of $\mathbf{M}_\alpha$ to $\mathbf{M}_\beta$). The set of C^k maps from M to N is denoted by

$$C^k(X, Y). \tag{2.3}$$

•

Notation 2.4. In the situation of Definition 2.1, for $Y = \mathbf{R}$ one abbreviates

$$C^k(X) \coloneqq C^k(X, \mathbf{R}). \tag{2.5}$$ ○

Remark 2.6. For $k = 0$ the condition on (2.2) in Definition 2.1 is vacuous; that is: a C^0 map is nothing but a continuous map. ♣

The following makes it feasible to verify whether a given map is C^k or not.

Proposition 2.7. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be C^k manifolds with boundary. Let $A' \subset A$ and $B' \subset B$. Let $f\colon M \to N$ be a continuous map. If*

$$\{U_\alpha \cap f^{-1}(V_\alpha) : \alpha \in A', \beta \in B'\} \tag{2.8}$$

is an open cover of X and for every $\alpha \in A'$ and $\beta \in B'$ the map f^α_β defined in (2.2) is C^k, then f is C^k.

Proof. It needs to be proved that if $\phi\colon U \to \tilde{U}$ is a chart in $\mathcal{A}$ and $\psi\colon V \to \tilde{V}$ is a chart in $\mathcal{B}$, then $\psi \circ f \circ \phi^{-1}\colon \phi(U \cap f^{-1}(V)) \to \tilde{V}$ is C^k. For every $\alpha \in A'$ and $\beta \in B'$ the transition maps $\tau_\alpha \coloneqq \phi_\alpha \circ \phi^{-1}\colon \phi(U \cap U_\alpha) \to \phi_\alpha(U \cap U_\alpha)$ and $\sigma^\beta \coloneqq \psi \circ \psi_\beta^{-1}\colon \psi_\beta(V \cap V_\beta) \to \psi(V \cap V_\beta)$, and the map $f^\alpha_\beta\colon \phi(U_\alpha \cap f^{-1}(V_\beta)) \to \tilde{V}_\beta$ are C^k. The restriction of $\psi \circ f \circ \phi^{-1}$ to $\phi(U \cap U_\alpha \cap f^{-1}(V \cap V_\beta))$ agrees with the composition $\sigma^\beta \circ f^\alpha_\beta \circ \tau_\alpha$; hence: it is C^k. By hypothesis, $\{\phi(U \cap U_\alpha \cap f^{-1}(V \cap V_\beta)) : \alpha \in A, \beta \in B\}$ is an open cover of $\phi(U \cap f^{-1}(V))$. Therefore, $\psi \circ f \circ \phi^{-1}$ is C^k. ■

Example 2.9. Let $n \in \mathbf{N}_0$. Let $U \subset \mathbf{R}^{n+1}$ be an open subset containing S^n. Denote by $\iota\colon S^n \hookrightarrow U$ the inclusion. The composition $\iota \circ \phi_{i,\pm}$ of ι with the chart $\phi_{i,\pm}$ defined in (1.15) satisfies

$$\iota \circ \phi_{i,\pm}^{-1}(x_1, \dots, x_n) = \Big(x_1, \dots\dots, x_{i-1}, \pm\sqrt{1 - |x|^2}, x_i, \dots, x_n\Big). \tag{2.10}$$

Therefore, ι is analytic. ♠

Example 2.11. Let $n \in \mathbf{N}_0$ and $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Denote by $\pi\colon \mathbf{K}^{n+1}\backslash\{0\} \twoheadrightarrow \mathbf{K}P^n$ the canonical projection. The composition $\phi_i \circ \pi$ of π with the chart ϕ_i defined in (1.44) satisfies

$$\phi_i \circ \pi(x_1, \dots, x_{n+1}) = x_i^{-1}(x_1, \dots, \widehat{x_i}, \dots, x_{n+1}). \tag{2.12}$$

Therefore, π is analytic. ♠

Exercise 2.13. Let $n \in \mathbf{N}$. Prove that the map $\Pi\colon \mathbf{R}P^n \to \mathrm{End}(\mathbf{R}^{n+1})$ defined by

$$\Pi([x])v \coloneqq \frac{\langle v, x\rangle x}{|x|^2} \tag{2.14}$$

is analytic.

Exercise 2.15. Prove that the **Segre embedding** $\sigma\colon \mathbf{R}P^2 \to \mathbf{R}P^5$ defined by

$$\sigma([x_0 : x_1 : x_2]) \coloneqq [x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_0x_2 : x_1x_2] \tag{2.16}$$

is analytic.

Proposition 2.17. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$, and $(Z, \mathcal{C})$ be C^k manifolds with boundary. If $f\colon X \to Y$ and $g\colon Y \to Z$ are C^k, then $g \circ f$ is C^k.*

Proof. Label the charts in $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ so that $\mathcal{A} = \{\phi_\alpha\colon U_\alpha \to \tilde{U}_\alpha : \alpha \in A\}$, $\mathcal{B} = \{\psi_\beta\colon V_\beta \to \tilde{V}_\beta : \beta \in B\}$, and $\mathcal{C} = \{\chi_\gamma\colon W_\gamma \to \tilde{W}_\gamma : \gamma \in \Gamma\}$. It needs to be proved that for every $\alpha \in A$ and $\gamma \in \Gamma$ the coordinate representation $(g \circ f)^\alpha_\gamma \coloneqq \chi_\gamma \circ (g \circ f) \circ \phi_\alpha^{-1}\colon \phi(U_\alpha \cap (g \circ f)^{-1}(W_\gamma)) \to \tilde{W}_\gamma$ is C^k. By hypothesis, the coordinate representations $f^\alpha_\beta \coloneqq \psi_\beta \circ f \circ \phi_\alpha^{-1}\colon \phi(U_\alpha \cap f^{-1}(V_\beta)) \to \tilde{V}_\beta$ and $g^\beta_\gamma \coloneqq \chi_\gamma \circ g \circ \psi_\beta^{-1}\colon \psi_\beta(V_\beta \cap g^{-1}(W_\gamma)) \to \tilde{W}_\gamma$ are C^k. The restriction of $(g \circ f)^\alpha_\gamma$ to $\phi_\alpha(U_\alpha \cap f^{-1}(V_\beta \cap g^{-1}(W_\gamma)))$ agrees with the composition $g^\beta_\gamma \circ f^\alpha_\beta$; hence: it is C^k. By (2a) in (1), $\{\phi_\alpha(U_\alpha \cap f^{-1}(V_\beta \cap g^{-1}(W_\gamma))) : \beta \in B\}$ is an open cover of $\phi(U_\alpha \cap (g \circ f)^{-1}(W_\gamma))$. Therefore, $(g \circ f)^\alpha_\gamma$ is C^k. ■

Example 2.18. Let $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Set $d \coloneqq \dim_{\mathbf{R}} \mathbf{K}$. Identify $\mathbf{R}^{2d} = \mathbf{K}^d$ and consider $S^{2d-1} \subset \mathbf{K}^d$. The **real, complex, and quaternionic Hopf map** $\eta = \eta_{\mathbf{K}}\colon S^{2d-1} \to \mathbf{K}P^1$ defined by

$$\eta(x, y) \coloneqq [x : y] \tag{2.19}$$

for $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$ respectively. The Hopf map is the composition of the inclusion $\iota\colon S^{2d-1} \hookrightarrow \mathbf{K}^d\backslash\{0\}$ and the projection $\pi\colon \mathbf{K}^d\backslash\{0\} \to \mathbf{K}P^n$. Since ι and π are analytic, so is η. ♠

Example 2.20. Let $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Let $k, n \in \mathbf{N}_0$. Identify $\Lambda^k\mathbf{K}^n = \mathbf{K}^{\binom{n}{k}}$ and set $\mathbf{P}(\Lambda^k\mathbf{K}^n) \coloneqq \mathbf{K}P^{\binom{n}{k}-1}$. Identify $\mathrm{Gr}_k(\mathbf{K}^n)$ with the set of k–dimensional $\mathbf{K}$–linear subspaces of $\mathbf{K}^n$. The **Plücker embedding** $\iota\colon \mathrm{Gr}_k(\mathbf{K}^n) \to \mathbf{P}(\Lambda^k\mathbf{K}^n)$ is defined by

$$\iota(\langle v_1, \dots, v_k\rangle) \coloneqq \mathbf{K}^\times \cdot (v_1 \wedge \ldots \wedge v_k). \tag{2.21}$$

Here $v_1, \dots, v_k$ are linearly-independent vectors in $\mathbf{K}^n$ and $\langle v_1, \dots, v_k\rangle$ denotes their span. The Plücker embedding is analytic. (Exercise!) ♠

Definition 2.22. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$.

1. The **category of C^k manifolds with boundary**, denoted by $\mathbf{Man}_\partial^k$, is the category whose objects are C^k manifolds with boundary and whose morphisms as C^k maps.
2. The **category of C^k manifolds**, denoted by $\mathbf{Man}^k$, is the category whose objects are C^k manifolds with boundary and whose morphisms as C^k maps. •

Remark 2.23. With the above in mind, Remark 1.11 can be rephrased as there being a forgetful functor $U\colon \mathbf{Man}_\partial^k \to \mathbf{Man}_\partial^\ell$. ♣

Proposition 2.24. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$ and let X be a C^k manifold with boundary X. The subset $C^k(X) \subset \mathrm{Map}(X, \mathbf{R})$ is an $\mathbf{R}$–subalgebra; that is: if $f, g \in C^k(X)$ and $\lambda \in \mathbf{R}$, then $fg \in C^k(X)$ and $f + \lambda g \in C^k(X)$.*

Exercise 2.25. Prove Proposition 2.24

Definition 2.26. Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X and Y be C^k manifolds. A map $f\colon X \to Y$ is a C^k **diffeomorphism** if it is C^k, bijective, and f^{-1} is C^k. X and Y are C^k **diffeomorphic** if there is a C^k diffeomorphism $f\colon X \to Y$. •

Example 2.27. The map $\psi\colon \mathbf{R} \to \mathbf{R}$ defined by $\phi(x) := x^3$ is a chart. Therefore, $\mathcal{B} := \overline{\{\phi\}}$ defines a smooth structure on $\mathbf{R}$. Since ϕ^{-1} is not smooth, $\mathcal{B}$ is distinct from the smooth structure on $\mathbf{R}$ defined in Example 1.8. Nevertheless, $(\mathbf{R}, \mathcal{A})$ and $(\mathbf{R}, \mathcal{B})$ are diffeomorphic; indeed, the map ϕ^{-1} defines a diffeomorphism. ♠

Example 2.28. Let $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Denote by $\bar{\cdot}$ the conjugation of $\mathbf{K}$. (This map is the identity for $\mathbf{R}$.) Set $d := \dim_{\mathbf{R}} \mathbf{K}$. Identify $\mathbf{R}^{d+1} = \mathbf{K} \oplus \mathbf{R}$ and consider $S^d \subset \mathbf{K} \oplus \mathbf{R}$. By (1.24), the transition map $\sigma_- \circ \sigma_+\colon \mathbf{K}\backslash\{0\} \to \mathbf{K}\backslash\{0\}$ of the stereographic projections $\sigma_\pm\colon U_\pm \to \mathbf{K}$ defined in (1.20) satisfies

$$\sigma_- \circ \sigma_+(q) = \bar{q}^{-1}. \tag{2.29}$$

Therefore, there is a continuous map $f\colon S^d \to \mathbf{K}P^1$ satisfying

$$f(q, x) = \begin{cases} [\sigma_+(q, x) : 1] & \text{if } x \neq +1 \\ [1 : \overline{\sigma_-(q, x)}] & \text{if } x \neq -1. \end{cases} \tag{2.30}$$

This map is bijective: its inverse satisfies

$$f^{-1}([q : 1]) = \sigma_+^{-1}(q) \quad \text{and} \quad f^{-1}([1 : q]) = \sigma_-^{-1}(\bar{q}). \tag{2.31}$$

Both f and f^{-1} are analytic; therefore, f is an analytic diffeomorphism. ♠

Theorem 2.32 (Classification of 1–manifolds). *Let $k \in \mathbf{N}_0 \cup \{\infty\}$. Every connected 1–dimensional C^k manifold with boundary is C^k diffeomorphic to either S^1, $[0, 1]$, $[0, 1)$, or $\mathbf{R}$.*

Remark 2.33.

1. In dimension two, there also is a classification theorem.
2. In dimension three, Thurston's geometrization conjecture provides a kind of classification. This conjecture has been proved by Perelman [Per02; Per03b; Per03a]; see also Kleiner and Lott [KL08] and Morgan and Tian [MT14].
3. In dimension four and higher, a classification is impossible. Every finitely presented group is the fundamental group of a 4–manifold. However, the isomorphism problem for finitely presented groups is undecidable. ♣

Remark 2.34.

1. Theorem 2.32 implies that if X is topological 1–manifold, then it admits a C^k structure and any two C^k structure on M are C^k diffeomorphic. The same is true in dimension two (again, by classification) and three (by work of Moise [Moi77]).
2. Kervaire [Ker60] produced the first example of a **non-smoothable** topological manifold that is: a topological manifold which does not admit a smooth structure. Kervaire's example is 10–dimensional, but this phenomenon starts to appear in dimension four. Indeed, the E_8–manifold (discovered by Freedman [Fre82, Theorem 1.7]) is a non-smoothable topological 4–manifold. This can be proved using Rokhlin's theorem [Rok52] or using Donaldson theory [Don83].
3. Milnor [Mil56] proved that S^7 admits 27 **exotic smooth structures**; that is: smooth structures not diffeomorphic to the standard one; cf. Example 6.49. This phenomenon too starts appearing in dimension four. Taubes [Tau87] proved that $\mathbf{R}^4$ admits uncountably exotic smooth structures.
4. Whitney [Whi36, Theorem 1] proved that if X admits a C^1 structure $\mathcal{A}$, then there is a smooth structure contained in $\mathcal{A}$ and any two such smooth structures are diffeomorphic. In fact, Grauert [Gra58] and Morrey [Mor58] proved that the same holds with smooth replaced by analytic. ♣

Proof of Theorem 2.32. Let X be a connected 1–dimensional C^k manifold with boundary. Since the following is going to be quite a slog, let us assume that $\partial X = \varnothing$. This saves us some additional case distinctions.

Denote the C^k structure of X by $\overline{\mathscr{A}}$. Choose a countable atlas $\mathscr{A} = \{\phi_\alpha \colon I_\alpha \to (-3,3) : \alpha \in A\} \subset \overline{\mathscr{A}}$ which is minimal in the sense that for every $\beta \in A$ the union $\bigcup_{\alpha \neq \beta} I_\alpha$ is a proper subset of X. This is possible because X is second-countable.

The strategy is to analyze and modify $\mathscr{A}$ until the assertion of the theorem becomes almost self-evident. The following result is the key tool for analyzing $\mathscr{A}$.

Proposition 2.35. *Let $\alpha, \beta \in A$ with $\alpha \neq \beta$. If $J \subset I_\alpha \cap I_\beta$ is a connected component, then $\tau^\alpha_\beta \colon \phi_\alpha(J) \to \phi_\beta(J)$ is either*

1. *increasing and either*
   (a) *$\phi_\alpha(J) = (a, 3)$ and $\phi_\beta(J) = (-3, b)$ or*
   (b) *$\phi_\alpha(J) = (-3, a)$ and $\phi_\beta(J) = (b, 3)$;*

   or
2. *decreasing and either*
   (a) *$\phi_\alpha(J) = (a, 3)$ and $\phi_\beta(J) = (b, 3)$ or*
   (b) *$\phi_\alpha(J) = (-3, a)$ and $\phi_\beta(J) = (-3, b)$.*

Proof. Since $\phi_\alpha(J)$ and $\phi_\beta(J)$ are connected open subset of $(-3,3)$, they are of the form $\phi_\alpha(J) = (a,b)$ and $\phi_\beta(J) = (c,d)$. The transition map $\tau^\alpha_\beta \colon (a,b) \to (c,d)$ is a homeomorphism and, therefore, either increasing or decreasing. Furthermore, since I_α and I_β are not contained in each other, $(a,b) \neq (-3,3)$ and $(c,d) \neq (-3,3)$. After possibly replacing ϕ_α with $-\phi_\alpha$ and ϕ_β with $-\phi_\beta$, we can assume that τ^α_β is increasing and $a \neq -3$.

Since X is Hausdorff, $b = 3$ and $c = -3$. Indeed, assume that $c \neq 3$. Set $x \coloneqq \phi_\alpha^{-1}(a)$ and $y \coloneqq \phi_\beta^{-1}(c)$. By construction, $x \in \partial J \cap I_\alpha$ and $y \in \partial J \cap I_\beta$. Since J is a connected component of $I_\alpha \cap I_\beta$, its boundary does not intersect $I_\alpha \cap I_\beta$. Therefore, $x \neq y$. Since X is Hausdorff, there are neighborhoods U and V of x and y respectively with $U \cap V = \varnothing$. Since ϕ_α and ϕ_β are homeomorphisms, there exists an $0 < \varepsilon < \min\{a, c, b-a, d-c\}$ such that $\phi_\alpha^{-1}(a-\varepsilon, a+\varepsilon) \subset U$ and $\phi_\beta^{-1}(c-\varepsilon, c+\varepsilon) \subset V$. Since τ^α_β is an increasing homeomorphism, there exists a $0 < \delta < \varepsilon$ such that $\phi_\beta(\phi_\alpha^{-1}(a+\delta)) = \tau^\alpha_\beta(a+\delta) \in (c, c+\varepsilon)$—a contradiction to $U \cap V = \varnothing$. Therefore, $c = -3$. This forces $d \neq 3$. A variation of the above argument shows that $b = 3$. ■

Proposition 2.36. *For every pair $\alpha, \beta \in A$ the intersection $I_\alpha \cap I_\beta$ has at most two connected components. If $I_\alpha \cap I_\beta$ has two connected components, then $A = \{\alpha, \beta\}$.*

Proof. Every connected component of $\phi_\alpha(I_\alpha \cap I_\beta) \subset (-3,3)$ is of the form $(-3,a)$ or $(b,3)$. There cannot be three disjoint subsets of this form. Therefore, $I_\alpha \cap I_\beta$ has at most two connected components.

If $I_\alpha \cap I_\beta$ has two connected components, then $\phi_\alpha(I_\alpha \cap I_\beta) = (-3, a) \amalg (b, 3)$ and $\phi_\beta(I_\alpha \cap I_\beta) = (-3, c) \amalg (d, 3)$. Therefore, if $0 < \varepsilon < \min\{a+3, c+3, 3-b, 3-d\}$, then $I_\alpha \cup I_\beta = \phi_\alpha([-3+\varepsilon, 3-\varepsilon]) \cup \phi_\beta([-3+\varepsilon, 3-\varepsilon)]$. Hence, $I_\alpha \cup I_\beta$ is compact. Since X is Hausdorff, $I_\alpha \cup I_\beta$ is a closed subset of X. Since X is connected, $I_\alpha \cup I_\beta = X$. ■

For the sake of a more uniform discusion, we prefer to avoid the situation in Proposition 2.36. If it does occur, then one can split $\phi_\alpha \colon I_\alpha \to (-1, 1)$ as follows. Set $I_\alpha^- := \phi_\alpha^{-1}(-3, b)$ and $I_\alpha^+ := \phi_\alpha^{-1}(a, 3)$ and define $\phi_\alpha^\pm \colon I_\alpha^\pm \to (-1, 1)$ by composing ϕ_α with suitable affine maps. The atlas $\mathscr{A}' := \{\phi_\alpha^+, \phi_\alpha^-, \phi_\beta\}$ avoids the situation in Proposition 2.36 but otherwise has the same properties as $\mathscr{A}$. Therefore, we can (and will) assume that $I_\alpha \cap I_\beta$ has at most one connected component.

Proposition 2.37. *For every $\alpha \in A$ there are at most two $\beta \in A\backslash\{\alpha\}$ with $I_\alpha \cap I_\beta \neq \varnothing$.*

Proof. For every $\beta \in A\backslash\{\alpha\}$ either $\phi_\alpha(I_\alpha \cap I_\beta)$ is empty or it is of the form $(-3, a)$ or $(b, 3)$. There cannot be three disjoint subsets of this form. Therefore, it remains to prove that if $\alpha, \beta, \gamma \in A$ are distinct, then $I_\alpha \cap I_\beta \cap I_\gamma = \varnothing$.

After possibly replacing ϕ_α with $-\phi_\alpha$ and ϕ_β with $-\phi_\beta$, the restriction of τ_β^α to $\phi_\alpha(I_\alpha \cap I_\beta)$ is increasing, $\phi_\alpha(I_\alpha \cap I_\beta) = (a, 3)$, and $\phi_\beta(I_\alpha \cap I_\beta) = (-3, b)$.

The interval $\phi_\alpha(J_{\gamma\alpha})$ is either of the form of the form $(c, 3)$ or $(-3, d)$. In the latter case, since $I_\alpha \cap I_\beta$ and $J_{\gamma\alpha}$ intersect, $d < a$; therefore, $I_\alpha \subset I_\beta \cup I_\gamma$—contradicting to the construction of $\mathscr{A}$. After possibly swapping β and γ, $a < c$.

After possibly replacing ϕ_γ with $-\phi_\gamma$, the restriction of τ_γ^β to $\phi_\beta(I_\beta \cap I_\gamma)$ is increasing. The interval $\phi_\beta(I_\beta \cap I_\gamma)$ is either of the form $(e, 3)$ or $(-3, f)$. The latter case, however, leads to a contradiction with $a < c$. Since $I_\alpha \cap I_\beta$ and $I_\beta \cap I_\gamma$ intersect, $e < d$. Therefore, $I_\beta \subset I_\alpha \cup I_\gamma$—contradicting the construction of $\mathscr{A}$. ■

Denote by Γ the graph with vertices A and with an edge $\{\alpha, \beta\}$ if and only if $I_\alpha \cap I_\beta \neq \varnothing$ and $\alpha \neq \beta$. By Proposition 2.37, Γ is bivalent; moreover, it is non-empty, countable, and connected. A moment's thought shows that such a graph must be isomorphic to one of the following:

1. Let $\ell \in \mathbf{N}$ with $\ell \geqslant 3$. The **cycle C_ℓ of length ℓ** is the graph with vertices $\mathbf{Z}/\ell\mathbf{Z}$ and edges $\{\{n, n+1\} : n \in \mathbf{Z}/\ell\mathbf{Z}\}$.

2. Let $\ell \in \mathbf{N} \cup \{\infty\}$. The **ray R_ℓ of length ℓ** is the graph with vertices $\{n \in \mathbf{N} : n \leqslant \ell\}$ and edges $\{\{n, n+1\} : n \in \mathbf{N}, n \leqslant \ell - 1\}$.

3. The **line** L is the graph with vertices $\mathbf{Z}$ and edges $\{\{n, n+1\} : n \in \mathbf{Z}\}$.

These graphs are canonically oriented. Every vertex n (except ℓ in R_ℓ) has a unique successor $n+1$. Choose an identification Γ with one of the models.

The following result allows us to replace $\mathscr{A}$ with a particularly simple atlas.

Proposition 2.38. *For every $\alpha \in A$ there is a C^k diffeomorphism $\rho_\alpha\colon (-3,3) \to (-3,3)$ such that the atlas*

$$\tilde{\mathscr{A}} := \{\tilde{\phi}_\alpha := \rho_\alpha \circ \phi_\alpha\colon I_\alpha \to (-3,3)\} \tag{2.39}$$

satisfies the following. If $\beta \in A$ and $\gamma \in A$ is its successor, then

$$(2,3) \subset \tilde{\phi}_\beta(I_\beta \cap I_\gamma), \quad \tilde{\phi}_\gamma \circ \tilde{\phi}_\beta^{-1}(2,3) \subset (-3,-2), \tag{2.40}$$

and the transition map $\tilde{\tau}_\gamma^\beta := \tilde{\phi}_\gamma \circ \tilde{\phi}_\beta^{-1}\colon (2,3) \to (-2,-1)$ satisfies

$$\tilde{\tau}_\gamma^\beta(x) = x - 4. \tag{2.41}$$

Proof. For every $-3 < a < b < 3$ there is a C^k diffeomorphism $\rho\colon (-3,3) \to (-3,3)$ such that $\rho(-3,a) = (-3,1)$ and $\rho(b,3) = (2,3)$. After composing with such diffeomorphism and possibly flipping signs, we can assume that the following holds. If $\alpha, \beta, \gamma \in A$, β is the successor of α, and γ is the successor of β, then $\phi_\beta(I_\beta \cap I_\alpha) = (-3,1)$ and $\phi_\beta(I_\beta \cap I_\gamma) = (1,3)$.

Suppose $\beta \in A$ and $\gamma \in A$ is its successor. The transition map $\tau_\gamma^\beta\colon (1,3) \to (-3,1)$ is an increasing C^k diffeomorphism. Choose an increasing C^k diffeomorphism $\rho_\beta\colon (-3,3) \to (-3,3)$ which agrees with the identity map on $(-3,0)$ and satisfies

$$\rho_\beta(x) = \tau_\gamma^\beta(x) + 4 \quad \text{for every} \quad x \in (\tau_\gamma^\beta)^{-1}(-1,2). \tag{2.42}$$

It is easy to construct an increasing homeomorphism $\tilde{\rho}_\beta\colon (-3,3) \to (-3,3)$ which agrees with the identify map on $(-3,1]$ and with $\tau_\gamma^\beta + 4$ on $(1,3)$. This map can be smoothed into the desired ρ_β by modifying it in an arbitrarily small neighborhood of 1. ∎

With the help of Proposition 2.38 it is easy to write down explicit an C^k diffeomorphism from X to either S^1 or $\mathbf{R}$. ∎

3 Tangent spaces and derivatives

Proposition 3.1. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X be a C^k manifold with boundary. Let $x \in X$ and set $m := \dim_x X$. Denote by $\mathscr{A}_x$ the set of charts $\phi\colon U \to \tilde{U}$ on X with $x \in U$. The following hold:*

1. *The relation $\sim$ on $\mathscr{A}_x \times \mathbf{R}^m$ defined by*

$$(\phi, \tilde{v}) \sim (\psi, \tilde{w}) \quad \text{if and only if} \quad \tilde{w} = \mathrm{d}_{\phi(x)}(\psi \circ \phi^{-1})\tilde{v}. \tag{3.2}$$

is an equivalence relation.

2. *The quotient*

$$T_xX := (\mathscr{A}_x \times \mathbf{R}^m)/\sim \tag{3.3}$$

has a unique vector space structure such that the following holds: for every $\phi \in \mathscr{A}_x$ *the map* $\varpi_\phi \colon \mathbf{R}^m \to T_xX$ *defined by*

$$\varpi_\phi(\tilde{v}) := [\phi, \tilde{v}]. \tag{3.4}$$

is an isomorphism; in particular:

$$[\phi, \tilde{v}] + \lambda[\psi, \tilde{w}] = [\chi, \mathrm{d}_{\phi(x)}(\chi \circ \phi^{-1})\tilde{v} + \lambda \mathrm{d}_{\psi(x)}(\chi \circ \psi^{-1})\tilde{w}]. \tag{3.5}$$

Definition 3.6. In the situation of Proposition 3.1, T_xX is called the **tangent space of** X **at** x; its elements are called **tangent vectors to** X **at** x. •

Proof of Proposition 3.1. The relation $\sim$ is reflexive because $\mathrm{d}_{\phi(x)}\mathrm{id}_{\tilde{U}} = \mathbf{1}$. It is symmetric because

$$\left(\mathrm{d}_{\phi(x)}(\psi \circ \phi^{-1})\right)^{-1} = \mathrm{d}_{\psi(x)}(\phi \circ \psi^{-1}). \tag{3.7}$$

It is transitive because

$$\mathrm{d}_{\phi(x)}(\chi \circ \phi^{-1}) = \mathrm{d}_{\psi(x)}(\chi \circ \psi^{-1}) \circ \mathrm{d}_{\phi(x)}(\psi \circ \phi^{-1}). \tag{3.8}$$

This proves (1).

For every $\phi \in \mathscr{A}_x$ the map $\varpi_\phi \colon \mathbf{R}^n \to T_xX$ defined by (3.4) is a bijection. Its inverse satisfies

$$\varpi_\phi^{-1}[\psi, \tilde{v}] = \mathrm{d}_{\psi(x)}(\phi \circ \psi^{-1})\tilde{v}. \tag{3.9}$$

(This is well-defined because of (3.8).)

Define $+\colon T_xX \times T_xX \to \mathbf{R}$ and $\cdot\colon \mathbf{R} \times T_xX \to T_xX$ by

$$v + w := \varpi_\phi(\varpi_\phi^{-1}(v) + \varpi_\phi^{-1}(w)) \quad \text{and} \quad \lambda \cdot v := \varpi_\phi(\lambda \cdot \varpi_\phi^{-1}(v)). \tag{3.10}$$

This equips T_xX with the structure of a vector space. This structure does not depend on the choice of ϕ because $\varpi_\psi^{-1} \circ \varpi_\phi = \mathrm{d}_{\phi(x)}(\psi \circ \phi^{-1})$ is an isomorphism. It satisfies (3.5) by construction. This proves (2). ■

Notation 3.11. For an open subset $U \subset \mathbf{R}^n$ with its standard C^k structure there is a preferred chart: $\phi = \mathrm{id}_U$. This induces a preferred isomorphism $\varpi_\phi \colon \mathbf{R}^n \cong T_xU$. It is customary to identify

$$\mathbf{R}^n = T_xU \tag{3.12}$$

via $\varpi := \varpi_{\mathrm{id}_U}$ and set

$$\frac{\partial}{\partial x^i} = \partial_{x^i} := \varpi(e_i). \tag{3.13}$$

If the coordinates of $\mathbf{R}^n$ have been labeled in some other way (as it is sometimes convenient), then this notation is adjusted correspondingly. For example, it is customary to write $\frac{\partial}{\partial t}$ or ∂_t for the image of $1 \in \mathbf{R}$ under ϖ. ○

Proposition 3.14. *Let X and Y be C^1 manifolds. Let $f\colon X \to Y$ be a C^1 map. Let $x \in X$. Let $\mathscr{A}_x$ be as in Proposition 3.1 and define $\mathscr{B}_{f(x)}$ analogously. There exists a unique linear map $T_x f\colon T_x X \to T_{f(x)} Y$ such that for every $\phi \in \mathscr{A}_x$ and $\psi \in \mathscr{B}_{f(x)}$*

$$T_x f([\phi, \tilde{v}]) = [\psi, \mathrm{d}_{\phi(x)}(\psi \circ f \circ \phi^{-1})\tilde{v}]. \tag{3.15}$$

Proof. Choose $\phi \in \mathscr{A}_x$ and $\psi \in \mathscr{B}_{f(x)}$. By Proposition 3.1, the map $T_x f\colon T_x X \to T_{f(x)} Y$ defined by

$$T_x f := \varpi_\psi \circ \mathrm{d}_{\phi(x)}(\psi \circ f \circ \phi^{-1}) \circ \varpi_\phi^{-1} \tag{3.16}$$

is linear. By the chain rule, it does not depend on the choice of ϕ and ψ. ■

Definition 3.17. In the situation of Proposition 3.14, the map

$$T_x f\colon T_x X \to T_{f(x)} Y \tag{3.18}$$

is called the **derivative of f at x**. •

Notation 3.19. In the situation of Proposition 3.14 with $Y = \mathbf{R}$ it is customary to write

$$\mathrm{d}_x f \in \mathrm{Hom}(T_x X, \mathbf{R}) \tag{3.20}$$

instead of $T_x f$ and define the derivative of f in the direction of $v \in T_x X$ by

$$vf = v(f) := \mathrm{d}_x f(v). \tag{3.21}$$

This notation meshes well with (23.8). ○

Exercise 3.22. Let $n \in \mathbf{N}$. Denote by $\iota\colon S^n \to \mathbf{R}^{n+1}$ the inclusion. Prove that with respect to the identification $T_x\mathbf{R}^{n+1} = \mathbf{R}^{n+1}$ for every $x \in S^n$

$$\operatorname{im} T_x \iota = \left\{ v \in \mathbf{R}^{n+1} : \langle x, v \rangle = 0 \right\}. \tag{3.23}$$

Proposition 3.24. *Let X, Y, and Z be C^1 manifolds. Let $x \in X$. If $f\colon X \to Y$ and $g\colon Y \to Z$ are C^1 maps, then*

$$T_{f(x)} g \circ T_x f = T_x(g \circ f) \tag{3.25}$$

Proof. This is a consequence of the chain rule. ∎

Corollary 3.26. *Let X and Y be C^1 manifolds. If $f\colon X \to Y$ is a C^1 diffeomorphism, then for every $x \in X$ the derivative $T_x f$ of f is invertible.* ∎

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Proposition 3.27. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X be a C^k manifold with boundary. If $f, g \in C^k(X)$, $v, w \in T_x X$, and $\lambda \in \mathbf{R}$, then*

$$(v + \lambda w)f = v(f) + \lambda w(f), \tag{3.28}$$

$$v(f + \lambda g), = v(f) + \lambda v(g), \quad \textit{and} \tag{3.29}$$

$$v(fg) = v(f)g(x) + f(x)v(g). \tag{3.30}$$

Exercise 3.31. Prove Proposition 3.27.

Exercise 3.32. Find a map $S\colon C^\infty(X) \to \mathbf{R}$ which fails to be linear but satisfies

$$S(fg) = S(f)g(x) + f(x)S(g). \tag{3.33}$$

Proposition 3.27 opens up a different—purely algebraic—perspective on tangent vectors *for* $k = \infty$.

Definition 3.34. Let A be an $\mathbf{R}$–algebra and let X be an A–bimodule. A **derivation of** A **into** X is a linear map $\delta\colon A \to X$ such that

$$\delta(ab) = \delta(a)b + a\delta(b). \tag{3.35}$$

The vector space of derivations of A with values in X is denoted by $\mathrm{Der}(A, X)$. •

Example 3.36. Let X be a smooth manifold with boundary. $C^\infty(X)$ is a commutative $\mathbf{R}$–algebra. For every $x \in X$ the real numbers $\mathbf{R}$ becomes a $C^\infty(X)$–module with

$$f \cdot \lambda := f(x)\lambda. \tag{3.37}$$

Denote this $C^\infty(X)$–module by $\underline{\mathbf{R}}_x$. By Proposition 3.27, every $v \in T_x X$ defines an element of $\mathrm{Der}(C^\infty(X), \underline{\mathbf{R}}_x)$. ♠

Proposition 3.38. *Let $k \in \mathbf{N} \cup \{\infty\}$. Let X be a smooth manifold with boundary. The map $\Upsilon = \Upsilon_x\colon T_x X \to \mathrm{Der}(C^\infty(X), \underline{\mathbf{R}}_x)$ defined by*

$$\Upsilon(v)(f) := v(f) \tag{3.39}$$

is an isomorphism.

Proof. Let us begin by proving this for $X = \mathbf{R}^n$ and $x = 0$. In this case

$$\Upsilon(v)(f) = \sum_{i=1}^{n} v^i \cdot \frac{\partial f}{\partial x^i}(0). \tag{3.40}$$

In particular, $\Upsilon(v)(x^i) = v^i$. Therefore, Υ is injective. To prove that Υ is surjective, let $\delta \in \mathrm{Der}(C^\infty(\mathbf{R}^n), \underline{\mathbf{R}}_0)$. Since $\delta(1) = \delta(1^2) = 2\delta(1)$, δ vanishes on constant functions. Every $f \in C^\infty(\mathbf{R}^n)$ satisfies

$$f(x) = f(0) + \sum_{i=1}^{n} r_i(x) \cdot x^i \quad \text{with} \quad r_i(x) := \int_0^1 \frac{\partial f}{\partial x^i}(tx)\, \mathrm{d}t. \tag{3.41}$$

Evidently,

$$r_i(0) = \frac{\partial f}{\partial x^i}(0). \tag{3.42}$$

Therefore and since δ is a derivation,

$$\delta(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(0) \cdot \delta(x^i). \tag{3.43}$$

Hence, for $v \in \mathbf{R}^m$ with $v_i := \delta(x^i)$

$$\Upsilon(v) = \delta. \tag{3.44}$$

To transfer the result from $\mathbf{R}^n$ to X, observe the following. If $\delta \in \mathrm{Der}(C^\infty(X), \underline{\mathbf{R}}_x)$ and $f \in C^\infty(X)$ vanishes in a neighborhood of x, then $\delta(f) = 0$. Indeed, if $\chi \in C^\infty(X)$ is supported away from x and equal to one on the support of f, then

$$\delta(f) = \delta(\chi f) = \delta(\chi) f(0) + \chi(0)\delta(f) = 0. \tag{3.45}$$

Let $\phi\colon U \to \tilde{U}$ be a chart with $x \in U$ and $\phi(x) = 0$. Let $\tilde{V}$ be an open neighborhood of 0 with $\tilde{V} \subset \tilde{U}$. Choose $\chi \in C^\infty(X)$ supported in U and equal to one on V. Set $\tilde{\chi} := \chi \circ \phi^{-1} \in C^\infty(\mathbf{R}^n)$. Define $\sigma\colon C^\infty(\mathbf{R}^n) \to C^\infty(X)$ and $\tau\colon C^\infty(X) \to C^\infty(\mathbf{R}^n)$ by

$$\sigma(f) = \chi \cdot (f \circ \phi) \quad \text{and} \quad \tau(f) = \tilde{\chi} \cdot (f \circ \phi^{-1}). \tag{3.46}$$

The map $\tau^*\colon \mathrm{Der}(C^\infty(\mathbf{R}^n), \underline{\mathbf{R}}_0) \to \mathrm{Der}(C^\infty(X), \underline{\mathbf{R}}_x)$ induced by τ fits into the commutative diagram

$$\begin{tikzcd} \mathbf{R}^n \arrow[r, "\Upsilon"] \arrow[d, "\varpi_\phi"] & \mathrm{Der}(C^\infty(\mathbf{R}^n), \underline{\mathbf{R}}_0) \arrow[d, "\tau^*"] \\ T_xX \arrow[r, "\Upsilon"] & \mathrm{Der}(C^\infty(X), \underline{\mathbf{R}}_x). \end{tikzcd} \tag{3.47}$$

Therefore, it remains to prove that τ^* is an isomorphism; indeed, σ^* is its inverse. To see this, observe that

$$\sigma \circ \tau(f) = \chi^2 f \quad \text{and} \quad \tau \circ \sigma(f) = \tilde{\chi}^2 f. \tag{3.48}$$

Since $\chi^2 f$ and f agree on V, for every $\delta \in \mathrm{Der}(C^\infty(X), \underline{\mathbf{R}}_x)$

$$(\tau^* \sigma^* \delta)(f) = \delta(\sigma \circ \tau(f)) = \delta(f). \tag{3.49}$$

Therefore, $\tau^*\sigma^*\delta = \delta$. Similarly, $\sigma^*\tau^*\delta = \delta$. ■

Proposition 3.50. *Let X be a smooth manifold and $x \in X$. Denote by*

$$\mathfrak{m}_x := \ker(\mathrm{ev}_x \colon C^\infty(X) \to \underline{\mathbf{R}}_x) \tag{3.51}$$

the ideal of smooth functions on X vanishing at x. The inclusion $\mathfrak{m}_x \hookrightarrow C^\infty(X)$ induces an isomorphism

$$\mathrm{Der}(C^\infty(X), \underline{\mathbf{R}}_x) \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^*. \tag{3.52}$$

Proof. The composition

$$\mathrm{Der}(C^\infty(X), \underline{\mathbf{R}}_x) \hookrightarrow C^\infty(X)^* \to \mathfrak{m}_x^* \tag{3.53}$$

is injective since every $\delta \in \mathrm{Der}(C^\infty(X), \underline{\mathbf{R}}_x)$ vanishes on constants. Since every $\delta \in \mathrm{Der}(C^\infty(X), \underline{\mathbf{R}}_x)$ vanishes on $\mathfrak{m}_x^2$, the above composition descends to an injective map $\mathrm{Der}(C^\infty(X), \underline{\mathbf{R}}_x) \hookrightarrow (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. To see that this map is surjective, lift $\lambda \in (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ to $\delta \in C^\infty(X)^*$ defined by

$$\delta(f) := \lambda(f - f(x) + \mathfrak{m}_x^2). \tag{3.54}$$

Since

$$fg - f(x)g(x) = (f - f(x))g(x) + f(x)(g - g(x)) + (f - f(x))(g - g(x)), \tag{3.55}$$

the linear map δ satisfies

$$\delta(fg) = \delta(f)g(x) + f(x)\delta(g). \tag{3.56}$$

Therefore, $\delta \in \mathrm{Der}(C^\infty(X), \underline{\mathbf{R}}_x)$. ■

The preceding discussion fails for $k \in \mathbf{N}$ instead of ∞ (provided $\dim_x X \geqslant 1$). The map $\Upsilon \colon T_xX \to \mathrm{Der}(C^k(X), \underline{\mathbf{R}}_x)$ defined by (3.39) is injective, but not surjective. The proof of Proposition 3.38 fails because the integral in (3.41) is C^{k-1} but might not be C^k. Indeed, Υ has no chance of being surjective.

Theorem 3.57 (Newns and Walker [NW56]). *In the above situation,* $\mathrm{Der}(C^k(X), \underline{\mathbf{R}}_x)$ *is infinite-dimensional.*

Proof. The following elementary argument is due to Taylor [Tay73].

The proof of Proposition 3.50 shows that $\mathrm{Der}(C^k(X), \underline{\mathbf{R}}_x) \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ for $\mathfrak{m}_x \coloneqq \ker \mathrm{ev}_x \subset C^k(X)$. It suffices to prove that $\mathfrak{m}_0/\mathfrak{m}_0^2$ is infinite-dimensional for $X = \mathbf{R}$.

For every $f \in \mathfrak{m}_0$ its **vanishing order** $\mathrm{ord}(f) \in [0, \infty]$ is defined by

$$\mathrm{ord}(f) \coloneqq \sup\left\{ s \in [0, \infty) : \lim_{x \to 0} |x|^{-s} f(x) = 0 \right\}. \tag{3.58}$$

The proof relies on the following observation which itself follows from Taylor expansion with remainder.

Lemma 3.59. *If* $f \in \mathfrak{m}_0^2$, *then* $\mathrm{ord}(f) > k + 1$ *or* $\mathrm{ord}(f) \in \mathbf{N}$. ■

The uncountable set

$$\left\{ |x|^s + \mathfrak{m}_0^2 : s \in (k, k+1) \right\} \subset \mathfrak{m}_0/\mathfrak{m}_0^2 \tag{3.60}$$

is linearly independent. Indeed, otherwise there would be a linear combination

$$\sum_{i=1}^n a_i |x|^{s_i} = f \in \mathfrak{m}_0^2 \tag{3.61}$$

with $a_1, \ldots, a_n$ non-zero and $s_1 < \ldots < s_n$. Therefore, $\mathrm{ord}(f) = s_i \in (k, k+1)$; contradicting the lemma. ■

Remark 3.62. The situation can be improved by restricting to continuous derivations of $C^k(X)$ into $\underline{\mathbf{R}}_x$; however: this subspace of $\mathrm{Der}(C^k(X), \underline{\mathbf{R}}_x)$ cannot be characterized purely algebraically. ♣

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Proposition 3.63. *Let* $k \in \mathbf{N} \cup \{\infty\}$. *Let* X *be a* C^k *manifold with boundary. Let* $x \in X$. *Set*

$$\Gamma_x \coloneqq \{\gamma \in C^k(\mathbf{R}, X) : \gamma(0) = x\}. \tag{3.64}$$

and define $\kappa = \kappa_x \colon \Gamma_x \to T_x X$ *by*

$$\kappa(\gamma) \coloneqq T_0\gamma(\partial_t). \tag{3.65}$$

The following hold:

1. *The map* κ *is surjective.*

2. *If $\gamma, \delta \in \Gamma_x$, then $\kappa(\gamma) = \kappa(\delta)$ if and only if for every $f \in C^k(X)$*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f \circ \gamma(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f \circ \delta(t). \tag{3.66}$$

Proof. Let $v = [\phi, \tilde{v}]$. Choose $\tilde{\gamma} \in C^k(\mathbf{R}, \tilde{U})$ such that $\partial_t \tilde{\gamma}(0) = \tilde{v}$. By construction, $\gamma := \phi^{-1} \circ \tilde{\gamma} \in C^k(\mathbf{R}, X)$ satisfies $\kappa(\gamma) = v$. This proves (1).

Let $\gamma, \delta \in \Gamma_x$. Choose $\phi \in \mathscr{A}_x$ with $\phi(x) = 0$. Set $\tilde{\gamma} := \phi \circ \gamma$ and $\tilde{\delta} := \phi \circ \delta$. By definition, $\kappa(\gamma) = \kappa(\delta)$ if and only if

$$\partial_t \tilde{\gamma}(0) = \sum_{i=1}^{n} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} x^i \circ \gamma(t) \cdot \partial_{x^i} = \sum_{i=1}^{n} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} x^i \circ \delta(t) \cdot \partial_{x^i} = \partial_t \tilde{\delta}(0). \tag{3.67}$$

This implies (2). ■

Define an equivalence relation $\sim$ on Γ_x be declaring that $\gamma \sim \delta$ if and only if (3.66) holds for every $f \in C^k(X)$. The map κ induces a bijection $\Gamma_x/\!\sim \,\longrightarrow T_xX$. Therefore, tangent vector to X at x can be understood *kinematically* as equivalence classes of paths γ passing through x at $t = 0$. The cone structure on T_xX can easily be seen to arise from the rescaling action on Γ_x. However, the vector space structure on T_xX is not obvious from this description.

4 Product manifolds

Proposition 4.1. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let $I \in \mathbf{N}$. Let $X_1, \ldots, X_I$ be C^k manifolds with boundary with at most one X_i having non-empty boundary. For $i \in \{1, \ldots, I\}$ denote by*

$$\mathrm{pr}_i \colon X_1 \times \cdots \times X_I \twoheadrightarrow X_i \tag{4.2}$$

the canonical projection maps. There exists a unique C^k structure $\mathscr{A}_\times$ on $X_1 \times \cdots \times X_I$ satisfying the following universal property:

1. *For every $i \in \{1, \ldots, I\}$ the map pr_i is C^k.*

2. *If Y is a C^k manifold with boundary and $f \colon Y \to X_1 \times \cdots \times X_I$ is a continuous map, then f is C^k with respect to $\mathscr{A}_\times$ if and only if for every $i \in \{1, \ldots, I\}$ the map $\mathrm{pr}_i \circ f \colon Y \to X_i$ is C^k.*

Definition 4.3. In the situation of Proposition 4.1, the C^k manifold with boundary $X_1 \times \cdots \times X_I$ is called the **product of** $X_1, \ldots, X_I$. •

Remark 4.4. The situation of Proposition 4.1 is illustrated by the diagram

$$
\begin{tikzcd}
Y \arrow[d, "f"'] \arrow[dr, dashed] & \\
X_1 \times \cdots \times X_I \arrow[r, "\mathrm{pr}_i"'] & X_i.
\end{tikzcd} \tag{4.5}
$$

The map $\eta\colon C^k(Y, X_1 \times \cdots \times X_I) \cong C^k(Y, X_1) \times \cdots \times C^k(Y, X_I)$ defined by

$$\eta(f) := (\mathrm{pr}_1 f, \ldots, \mathrm{pr}_I f) \tag{4.6}$$

is a bijection. ♣

Proof of Proposition 4.1. Denote by $\mathscr{A}_i = \{\phi^i_\alpha\colon U^i_\alpha \to \tilde{U}^i_\alpha : \alpha \in A_i\}$ the C^k structure on X_i. Set $A_\times := A_1 \times \cdots \times A_I$ and for every $\alpha = (\alpha_1, \ldots, \alpha_I) \in A_\times$ set

$$\phi_\alpha := \phi^1_{\alpha_1} \times \cdots \times \phi^I_{\alpha_I}\colon U_\alpha := U^1_{\alpha_1} \times \cdots \times U^I_{\alpha_I} \to \tilde{U}_\alpha := \tilde{U}^1_{\alpha_1} \times \cdots \times \tilde{U}^I_{\alpha_I}. \tag{4.7}$$

Without loss of generality, $\mathbf{M}_{\alpha_i} = \mathbf{R}^{m_{\alpha_i}}$ for every $\alpha_i \in A_i$ with $i \geqslant 2$. Therefore, $\tilde{U}_\alpha \subset \mathbf{M}_\alpha$ is an open subset with $\mathbf{M}_\alpha$ denoting either $\mathbf{R}^{m_\alpha}$ or $[0, \infty) \times \mathbf{R}^{m_\alpha - 1}$ with $m_\alpha := \sum_{i=1}^{i} m^i_{\alpha_i}$. Set

$$\mathscr{A}^\circ_\times := \{\phi_\alpha : \alpha \in A_\times\}. \tag{4.8}$$

Since

$$\phi_\alpha \circ \phi_\beta^{-1} = \phi^1_{\alpha_1} \circ (\phi^1_{\beta_1})^{-1} \times \cdots \times \phi^I_{\alpha_I} \circ (\phi^I_{\beta_I})^{-1}. \tag{4.9}$$

$\mathscr{A}^\circ_\times$ is a C^k atlas. Set $\mathscr{A}_\times := \overline{\mathscr{A}^\circ_\times}$. Since

$$\phi^i_{\alpha_i} \circ \mathrm{pr}_i \circ \phi_\alpha^{-1}(x_1, \ldots, x_I) = \mathrm{pr}_i(x_1, \ldots, x_I) = x_i \tag{4.10}$$

the maps pr_i are C^k with respect to the C^k structure $\mathscr{A}_\times$.

Let Y be a C^k manifold with boundary with C^k structure $\mathscr{B} = \{\psi_\beta\colon V_\beta \to \tilde{V}_\beta : \beta \in B\}$. Let $f\colon Y \to X_1 \times \cdots \times X_I$ be a continuous map. If f is C^k with respect to $\mathscr{A}_\times$, then so is $\mathrm{pr}_i \circ f$ for every $i \in \{1, \ldots, I\}$. Conversely, if $\mathrm{pr}_i \circ f$ is C^k for every $i \in \{1, \ldots, I\}$, then f is C^k with respect to $\mathscr{A}_\times$ because for every $\alpha \in A_\times$ and $\beta \in B$

$$\phi_\alpha \circ f \circ \psi_\beta^{-1} = (\phi_{\alpha_1} \circ \mathrm{pr}_1 \circ f \circ \psi_\beta^{-1}, \ldots, \phi_{\alpha_I} \circ \mathrm{pr}_I \circ f \circ \psi_\beta^{-1}). \tag{4.11}$$

Therefore, $\mathscr{A}_\times$ satisfies the universal property.

Let $\mathscr{B} = \{\psi_\beta\colon V_\beta \to \tilde{V}_\beta : \beta \in B\}$ be a C^k structure on $X_1 \times \cdots \times X_I$ satisfying the universal property. To prove that $\mathscr{B} = \mathscr{A}_\times$ it suffices to show that identity map

$$\delta := \mathrm{id}_{X_1 \times \cdots \times X_I}\colon (X_1 \times \cdots \times X_I, \mathscr{A}_\times) \to (X_1 \times \cdots \times X_I, \mathscr{B}) \tag{4.12}$$

is a C^k diffeomorphism. By (2) for $\mathscr{B}$, δ map is C^k if and only if

$$\mathrm{pr}_i \circ \delta = \mathrm{pr}_i \colon \ (X_1 \times \cdots \times X_I, \mathscr{A}_\times) \longrightarrow (X_i, \mathscr{A}_i) \tag{4.13}$$

is C^k. By (1) for $\mathscr{A}_\times$, this is the case. The same reasoning with the roles of $\mathscr{A}_\times$ and $\mathscr{B}$ reversed shows that δ^{-1} is C^k as well. ■

Remark 4.14. Proposition 4.1 fails if more than one X_i has non-empty boundary. This can be rectified by working with C^k manifolds with corners. ♣

Corollary 4.15. *Assume the situation of Proposition 4.1. The following hold:*

1. *If Y is a C^k manifold and $f_i \colon Y \longrightarrow X_i$ are C^k, then the map $(f_1, \ldots, f_I) \colon Y \longrightarrow X_1 \times \cdots \times X_I$ is C^k.*
2. *If Y_i $(i = 1, \ldots, I)$ are C^k manifolds and $f_i \colon Y_i \longrightarrow X_i$ $(i = 1, \ldots, I)$ are C^k, then the map $f_1 \times \cdots \times f_I \colon Y_1 \times \cdots \times Y_I \longrightarrow X_1 \times \cdots \times X_I$ is C^k.*

■

Corollary 4.16. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a C^k manifold. The diagonal map $\Delta \colon X \longrightarrow X \times X$ is C^k.* ■

Proof of Proposition 2.24. Let $f, g \in C^k(X)$ and $\lambda \in \mathbf{R}$ Both fg and $f + \lambda g$ are compositions of the form

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R} \tag{4.17}$$

with the final map denoting either the map $(x, y) \mapsto xy$ or the map $(x, y) \mapsto x + \lambda y$. ■

Proposition 4.18. *In the situation of Proposition 4.1 for every $x = (x_1, \ldots, x_I) \in X_1 \times \cdots \times X_I$ the following hold:*

1. *The map $\varsigma \colon T_x(X_1 \times \cdots \times X_I) \longrightarrow T_{x_1}X_1 \oplus \cdots \oplus T_{x_I}X_I$ defined by*
$$\varsigma(v) \coloneqq \big(T_x\mathrm{pr}_1(v), \ldots, T_x\mathrm{pr}_I(v)\big) \tag{4.19}$$
is an isomorphism.
2. *If $f_i \colon Y_i \longrightarrow X_i$ $(i = 1, \ldots, I)$ are C^k, then*
$$\varsigma \circ T_x(f_1 \times \cdots \times f_I) = (T_{x_1}f_1 \oplus \cdots \oplus T_{x_I}f_I) \circ \varsigma. \tag{4.20}$$

Notation 4.21. In the situation of Proposition 4.18, it is customary to identify

$$\begin{aligned} T_x(X_1 \times \cdots \times X_I) &= T_{x_1}X_1 \oplus \cdots \oplus T_{x_I}X_I \quad \text{and} \\ T_x(f_1 \times \cdots \times f_I) &= T_{x_1}f_1 \oplus \cdots \oplus T_{x_I}f_I. \end{aligned}$$ ○

Proof of Proposition 4.18. Define $\iota_i\colon X_i \to X_1 \times \cdots \times X_I$ by

$$\iota_i(\cdot) \coloneqq (x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_I). \tag{4.22}$$

Since $\mathrm{pr}_i \circ \iota_i = \mathrm{id}_{X_i}$, the map $T_{x_1}X_1 \oplus \cdots \oplus T_{x_I}X_I \to T_x(X_1 \times \cdots \times X_I)$ given by

$$(v_1, \dots, v_I) \mapsto \sum_{i=1}^{I} \mathrm{d}_x \iota_i(v_i) \tag{4.23}$$

is an inverse of ς. This proves (1).

Proposition 3.24 implies (2). ■

Proof of Proposition 3.27. (3.28) is trivial.

Denote by $a_\lambda\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ the map defined by $a_\lambda(x, y) \coloneqq x + \lambda y$. Denote by $m\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ the map defined by $m(x, y) \coloneqq xy$. Trivially and by the Leibniz rule,

$$T_{(x,y)}a_\lambda(\hat{x}, \hat{y}) = \hat{x} + \lambda\hat{y} \quad \text{and} \quad T_{(x,y)}m(\hat{x}, \hat{y}) = x\hat{y} + y\hat{x}.$$

Therefore, by Proposition 3.24,

$$\begin{aligned} v(f + \lambda g) &= T_x(a_\lambda \circ (f, g) \circ \Delta)(v) \\ &= T_{(f(x),g(x))}a_\lambda \circ (T_x f \oplus T_x g) \circ T_x\Delta(v) \\ &= T_x f(v) + \lambda T_x g(v) \\ &= v(f) + \lambda v(g). \end{aligned} \tag{4.24}$$

This proves (3.29) Similarly,

$$\begin{aligned} v(fg) &= T_x(m \circ (f, g) \circ \Delta)(v) \\ &= T_{(f(x),g(x))}m \circ (T_x f \oplus T_x g) \circ T_x\Delta(v) \\ &= T_x f(v)g(x) + f(x)T_x g(v) \\ &= v(f)g(x) + f(x)v(g). \end{aligned} \tag{4.25}$$

This proves (3.30) ■

5 The inverse function theorem

Theorem 5.1 (Inverse Function Theorem). *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X and Y be C^k manifolds without boundary and let $f\colon X \to Y$ be C^k. Let $x \in X$. If $T_x f\colon T_x M \to T_{f(x)}N$ is invertible, then there is an open neighborhood U of $x \in X$ such that $V \coloneqq f(U)$ is open and $f|_U\colon U \to V$ is a C^k diffeomorphism.*

Remark 5.2. This result is false if X permitted to have a boundary and $x \in \partial X$. To see this, consider the inclusion $\iota\colon [0,\infty) \times \mathbf{R}^{m-1} \hookrightarrow \mathbf{R}^m$ and $x = 0$. ♣

This follows from the following result.

Lemma 5.3 (Quantitative Inverse Function Theorem). *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$ and $n \in \mathbf{N}$. Let $r, \varepsilon > 0$ with $\varepsilon < 1$. Let $\Lambda \in \mathrm{GL}(\mathbf{R}^n)$. Let $v\colon \bar{B}_r(0) \to \mathbf{R}^n$ be C^k with $v(0) = 0$ and*

$$|v(x) - v(y)| \leqslant \varepsilon |x - y|. \tag{5.4}$$

Set

$$f \coloneqq \Lambda \circ (\mathbf{1} + v)\colon B_r(0) \to \mathbf{R}^n \quad \text{and} \quad V \coloneqq f(B_r(0)). \tag{5.5}$$

The set V is open and $f\colon B_r(0) \to V$ is a C^k diffeomorphism; moreover: $\Lambda B_{(1-\varepsilon)r}(0) \subset V \subset \Lambda B_{(1+\varepsilon)r}(0)$,

Proof of Theorem 5.1. It suffices to prove the result for $X = Y = \mathbf{R}^n$, $x = 0$, and $f(x) = 0$. By Taylor expansion,

$$f = \mathrm{d}_0 f + R \tag{5.6}$$

with the remainder term satisfying $R(0) = 0$ and

$$\lim_{r \to 0} \sup_{x \neq y \in B_r(0)} \frac{|R(x) - R(y)|}{|x - y|} = 0. \tag{5.7}$$

Therefore, Lemma 5.3 with $\Lambda = \mathrm{d}_0 f$ applies provided $r \ll 1$. ■

Remark 5.8. If (5.7) can be made effective (e.g.: if f is C^2), then the size of U and V are controlled by Lemma 5.3. ♣

Proof of Lemma 5.3. It suffices to consider the case $\Lambda = \mathbf{1}$.

Step 1. *The map f is injective and f^{-1} is Lipschitz continuous.*

For every $x, y \in B_r(0)$

$$|x - y| \leqslant |f(x) - f(y)| + |v(x) - v(y)| \leqslant |f(x) - f(y)| + \varepsilon |x - y|; \tag{5.9}$$

hence: $|x - y| \leqslant (1 - \varepsilon)^{-1} |f(x) - f(y)|$.

Step 2. *For every $x_0 \in B_r(0)$*

$$B_{(1-\varepsilon)s}(f(x_0)) \subset f(B_s(x_0)) \quad \text{with} \quad s \coloneqq r - |x_0|. \tag{5.10}$$

In particular, V is open.

For $z \in \bar{B}_{(1-\varepsilon)s}(f(x_0))$ define $\Phi_z\colon \bar{B}_r(x) \to \mathbf{R}^n$ by

$$\Phi_z(x) := z - v(x). \tag{5.11}$$

This map has the following properties:

1. $\Phi_z(x) = x$ if and only if $f(x) = z$.
2. Φ_z is a contraction of $\bar{B}_s(x_0)$; indeed: for every $x, y \in \bar{B}_s(x_0)$

   $$|\Phi_z(x) - \Phi_z(y)| \leqslant \varepsilon|x - y|; \tag{5.12}$$

   moreover, from

   $$\Phi_z(x)-x_0 = z-v(x)-x_0 = z-(x_0+v(x_0))+v(x_0)-v(x) = z-f(x_0)+v(x_0)-v(x) \tag{5.13}$$

   it follows that

   $$|\Phi_z(x) - x_0| \leqslant (1-\varepsilon)s + \varepsilon|x - x_0| \leqslant s. \tag{5.14}$$

Therefore, by Banach's fixed-point theorem, for every $z \in \bar{B}_s(0)$ there is a unique $x \in \bar{B}_r(0)$ satisfying $f(x) = z$.

Step 3. *The map $f^{-1}\colon V \to B_r(0)$ is C^k.*

For every $x \in B_r(x)$, $\mathrm{d}_x f = \mathbf{1} + \mathrm{d}_x v$ is invertible since $|\mathrm{d}_x v| \leqslant \varepsilon$. Let $z, w \in V$. Set $x := f^{-1}(z), y := f^{-1}(w)$. Since

$$\begin{aligned} \frac{\left|f^{-1}(z) - f^{-1}(w) - (\mathrm{d}_x f)^{-1}(z-w)\right|}{|z-w|} &= \frac{|x - y - (\mathrm{d}_x f)^{-1}(f(x) - f(y))|}{|z-w|} \\ &\lesssim \frac{|\mathrm{d}_x f(x-y) - (f(x) - f(y))|}{|x-y|}, \end{aligned} \tag{5.15}$$

the map f^{-1} is differentiable; moreover:

$$\mathrm{d}_z f^{-1} = (\mathrm{d}_{f^{-1}(z)} f)^{-1}. \tag{5.16}$$

The right-hand side is continuous; hence, f^{-1} is C^1. In fact, if $k \geqslant 2$, then the right-hand side is C^1; hence: f^{-1} is C^2, and so on.

If $k = \omega$, then it can be seen that f^{-1} is C^ω by using the following: a smooth function $f\colon U \subset \mathbf{R}^m \to \mathbf{R}^n$ is analytic if and only if for every compact $K \subset C$ there is a constant $c = c(K) > 0$ such that for every $\alpha \in \mathbf{N}_0^m$

$$\sup_{x \in K} |\partial^\alpha f(x)| \leqslant c^{|\alpha|+1} \alpha!. \tag{5.17}$$

■

Exercise 5.18. Prove Lemma 5.3 for $k = \omega$.

∼

Definition 5.19. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X, Y be C^k manifolds with boundary. A C^k map $f\colon X \to Y$ is a **local diffeomorphism** if every $x \in X$ has an open neighborhood U such that $f|_U\colon U \to f(U)$ is a C^k diffeomorphism. •

Corollary 5.20. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X, Y be C^k manifolds without boundary. A C^k map $f\colon X \to Y$ is a local diffeomorphism if and only if for every $x \in X$ the map $T_x f$ is invertible.* ■

Definition 5.21. Let X and Y be topological space. A map $p\colon X \to Y$ is a **covering map** if for every $y \in Y$ there is an open neighborhood U of X and a discrete space D and a homeomorphism

$$\phi\colon p^{-1}(U) \to U \times D \tag{5.22}$$

satisfying

$$\mathrm{pr}_1 \circ \phi = p. \tag{5.23}$$

•

Proposition 5.24. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X, Y be C^k manifolds with boundary. Suppose that Y is connected. Let $f\colon X \to Y$ be a local diffeomorphism. If f is proper, then f is a covering map.*

The proof requires the following preparation.

Proposition 5.25. *Let X and Y be topological spaces. Suppose that Y is locally compact and Hausdorff. If $f\colon X \to Y$ is a proper continuous map, then it is closed.*

Proof of Proposition 5.25. Let $A \subset X$ be a closed set. Let $x \in \overline{f(A)}$. Since Y is locally compact, x has a compact neighborhood K. Since f is proper, $f^{-1}(K)$ is a compact. Therefore, $A \cap f^{-1}(K)$ is compact. Since f is continuous, $f(A \cap f^{-1}(K)) = f(A) \cap K$ is compact; indeed: since Y is Hausdorff, it is closed. Therefore, $x \in \overline{f(A) \cap K} = f(A) \cap K \subset f(A)$. Since $x \in \overline{f(A)}$ is arbitrary, $f(A)$ is closed. ■

Proof of Proposition 5.24. Let $y \in Y$. Set

$$D \coloneqq f^{-1}(y). \tag{5.26}$$

Since f is a local diffeomorphism, every $x \in D$ has a neighborhood U_x such that

$$f|_{U_x}\colon U_x \to \tilde{U}_x \coloneqq f(U_x) \tag{5.27}$$

is a diffeomorphism. In particular, D is discrete. Since f is proper, D is compact; hence: finite.

After possibly shrinking the U_x, it can be assumed that for every $x \neq y \in D$

$$U_x \cap U_y = \varnothing. \tag{5.28}$$

The set

$$A \coloneqq X \backslash \bigcup_{x \in D} U_x \tag{5.29}$$

is closed and disjoint from D. Therefore, by Proposition 5.25,

$$\tilde{V} \coloneqq \bigcap_{x \in X} \tilde{U}_x \backslash f(A) \tag{5.30}$$

is an open neighborhood of $y \in Y$. For every $x \in D$ set

$$V_x \coloneqq f^{-1}(\tilde{V}) \cap U_x. \tag{5.31}$$

By construction, for every $x \in X$

$$f|_{V_x} \colon V_x \to \tilde{V} \tag{5.32}$$

is a diffeomorphism and

$$f^{-1}(V) = \bigcup_{x \in X} V_x. \tag{5.33}$$

The map $\phi \colon f^{-1}(V) \to V \times D$ defined by

$$\phi|_{V_x}(z) \coloneqq (f(z), x) \tag{5.34}$$

is the required homoemorphism. ■

Example 5.35. Let $n \in \mathbf{N}$. The map $\pi \colon S^n \to \mathbf{R}P^n$ defined by

$$\pi(x) \coloneqq [x] \tag{5.36}$$

is a covering map. ♠

Example 5.37 (Milnor [Mil97, p.8]). The **fundamental theorem of algebra** asserts that every non-constant polynomial $p \in \mathbf{C}[z]$ has a zero. This can be proved as follows. Let

$$p(z) = \sum_{k=0}^{d} a_k z^k \in \mathbf{C}[z]$$

be a polynomial of degree $d \geqslant 1$. Define $f\colon \mathbf{C}P^1 \to \mathbf{C}P^1$ by

$$f([z:w]) := \left[\sum_{k=0}^{d} a_k z^k w^{d-k} : w^d\right].$$

A moment's thought shows that, $[1:0]$ is a critical point of f, and $[z:1]$ is a critical point of f if and only if $p'(z) = \sum_{k=0}^{d-1} k a_{k-1} z^k = 0$. Therefore,

$$\Delta := f(\operatorname{crit} f) \subset \mathbf{C}P^1. \tag{5.38}$$

the set of critical values of f, is finite. Trivially, $\operatorname{im} f \supset \Delta$.

By Proposition 5.24, the map $f\colon f^{-1}(\mathbf{C}P^1\backslash\Delta) \to \mathbf{C}P^1\backslash\Delta$ is a covering map. Since $\mathbf{C}P^1\backslash\Delta$ is connected,

$$N := \# f^{-1}(y) \in \mathbf{N}_0 \tag{5.39}$$

is independent of $y \in \mathbf{C}P^1\backslash\Delta$. Evidently, $N \neq 0$. Therefore, $\operatorname{im} f \supset \mathbf{C}P^1\backslash\Delta$.

This proves that f is surjective; in particular: $[0:1] \in \operatorname{im} f$; hence: f has a zero. ♠

Theorem 5.40 (Normal forms of smooth maps). *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X and Y be C^k manifolds without boundary. Let $f\colon X \to Y$ be C^k. Let $x \in X$ and set $r := \operatorname{rk} T_x f$, $m := \dim_x X$, and $n := \dim_{F(x)} Y$. There are an admissible chart $\phi\colon U \to \tilde{U}$ of X with $x \in U$ and $\phi(x) = 0$ and and admissible chart $\psi\colon V \to \tilde{V}$ of Y with $f(U) \subset V$ and $\psi(f(x)) = 0$ such that $\tilde{f} := \psi \circ f \circ \phi^{-1}\colon \tilde{U} \to \tilde{V}$ is of the form*

$$\tilde{f}(y,z) = (y, g(y,z)) \tag{5.41}$$

with $g\colon \tilde{U} \to \mathbf{R}^{n-r}$ satisfying

$$g(\phi(x)) = 0 \quad \text{and} \quad \mathrm{d}_{\phi(x)} g = 0; \tag{5.42}$$

moreover:

1. *if $T_x f$ is injective, then $\tilde{f}(y) = (y, 0)$;*
2. *if $T_x f$ is surjective, then $\tilde{f}(y,z) = y$; and*
3. *if $y \mapsto \operatorname{rk} T_y f$ is constant in neighborhood of x, then $\tilde{f}(y,z) = (y,0)$.*

Proof. Without loss of generality, $X = \mathbf{R}^m$, $Y = \mathbf{R}^m$, $x = f(x) = 0$ and

$$\mathrm{d}_0 f = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.43}$$

Suppose that $\mathrm{d}_0 f$ is injective. The C^k map $F\colon\ \mathbf{R}^n = \mathbf{R}^m \oplus \mathbf{R}^{n-m} \to \mathbf{R}^n$ defined by

$$F(y,z) \coloneqq f(y) + (0,z) \tag{5.44}$$

satisfies $\mathrm{d}_0 F = \mathbf{1}$. By Theorem 5.1, there is an open neighborhood V of $0 \in \mathbf{R}^n$ such that $F|_V\colon\ V \to \tilde{V} \coloneqq F(V)$ is a C^k diffeomorphism. Set $\psi \coloneqq (F|_V)^{-1}$ and $U \coloneqq f^{-1}(V)$. for every $y \in U$

$$\psi \circ f(y) = (y,0). \tag{5.45}$$

This proves (1).

Suppose that $\mathrm{d}_0 f$ is surjective. The C^k map $F\colon\ \mathbf{R}^m = \mathbf{R}^n \oplus \mathbf{R}^{n-m} \to \mathbf{R}^m = \mathbf{R}^n \oplus \mathbf{R}^{n-m}$ defined by

$$F(y,z) \coloneqq (f(y,z), z) \tag{5.46}$$

satisfies $\mathrm{d}_0 F = \mathbf{1}$. By Theorem 5.1, there is an open neighborhood U of $0 \in \mathbf{R}^n$ such that $F|_U\colon\ U \to \tilde{U} \coloneqq F(U)$ is a C^k diffeomorphism. Set $\phi \coloneqq F|_U$. For every $(y,z) \in \tilde{U}$

$$f \circ \phi^{-1}(y,z) = y. \tag{5.47}$$

This proves (2).

Decompose $f = (f_1, f_2)$ with $f_1\colon\ \mathbf{R}^m \to \mathbf{R}^r$ and $f_2\colon\ \mathbf{R}^m \to \mathbf{R}^{n-r}$. By construction, $\mathrm{d}_0 f_1$ is surjective. Define ϕ as above with f_1 instead of f. For every $(y,z) \in \tilde{U}$

$$f \circ \phi^{-1}(y,z) = (f_1 \circ \phi^{-1}(y,z), f_2 \circ \phi^{-1}(y,z)) = (y, \tilde{g}(y,z)) \quad \text{with} \quad \tilde{g} \coloneqq f_2 \circ \phi^{-1}. \tag{5.48}$$

From

$$\mathrm{d}_{(y,z)}(f \circ \phi^{-1}) = \begin{pmatrix} \mathbf{1} & 0 \\ \frac{\partial \tilde{g}}{\partial y}(y,z) & \frac{\partial \tilde{g}}{\partial z}(y,z) \end{pmatrix} \tag{5.49}$$

it follows that

$$\operatorname{rk} \mathrm{d}_{(y,z)} f = r + \operatorname{rk} \frac{\partial \tilde{g}}{\partial z}(y,z). \tag{5.50}$$

Therefore and since $\operatorname{rk} \mathrm{d}_0 f = r$,

$$\frac{\partial \tilde{g}}{\partial z}(0,0). \tag{5.51}$$

Set $V := \{(y,z) := (y,0) \in \tilde{U}\}$. The map $\psi\colon \tilde{U} \to \mathbf{R}^n$ defined by

$$\psi(y,z) := (y, z - \tilde{g}(y,0)) \tag{5.52}$$

satisfies $\mathrm{d}_0\psi = \mathbf{1}$. Therefore, after possibly shrinking V, the map $\psi\colon V \to \tilde{V} := \psi(V)$ is a C^k diffeomorphism. Define $g\colon \tilde{U} \to \mathbf{R}^{n-r}$ by

$$(y, g(y,z)) := \psi \circ f \circ \phi^{-1}(y,z). \tag{5.53}$$

By construction,

$$\mathrm{d}_0 g = \begin{pmatrix} -\frac{\partial\tilde{g}}{\partial y}(0,0) & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ \frac{\partial\tilde{g}}{\partial y}(0,0) & 0 \end{pmatrix} = 0. \tag{5.54}$$

It remains to prove (3). If $(y,z) \mapsto \operatorname{rk} \mathrm{d}_{(y,z)} f$ is constant, then $\frac{\tilde{g}}{\partial z}(y,\cdot)$ vanishes; hence, after possibly shrinking $\tilde{U}$, $\tilde{g}$ is independent of z. Therefore, by construction of ψ,

$$g(y,z) = \tilde{g}(y,z) - \tilde{g}(y,0) = 0. \tag{5.55}$$

■

Remark 5.56. There is a variant of this if X and Y have boundary, but the ϕ and ψ are not quite charts. ♣

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Definition 5.57. Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X be a C^k manifold with boundary. Let G be a group.

1. A G–**action on** X is a homomorphism $\rho\colon G \to \mathrm{Diff}(X)$.
2. A point $x \in X$ is a **fixed-point of** ρ if for every $g \in G$

   $$\rho_g(x) = x. \tag{5.58}$$

   The set of fixed-points is denoted by X^G. •

Proposition 5.59 (Linearization of finite group actions). *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X be a C^k manifold without boundary. Let G be a finite group. Let $\rho\colon G \to \mathrm{Diff}(X)$ be a G–action. For every fixed-point $x \in X^G$ there are open neighborhoods U of $x \in X$ and $\tilde{U}$ of $0 \in T_xX$, and a C^k diffeomorphism $\phi\colon U \to \tilde{U} \subset T_xX$ such that $\phi(x) = 0$ and for every $g \in G$*

$$\rho_g(U) = U, \quad T_x\rho_g(\tilde{U}) = \tilde{U}, \quad \textit{and} \quad \phi \circ \rho_g \circ \phi^{-1} = T_x\rho_g. \tag{5.60}$$

Proof. Let $\psi\colon V \to \tilde{V} \subset T_xX$ be a C^k diffeomorphism with

$$\psi(x) = 0 \quad \text{and} \quad T_x\psi = \mathbf{1}. \tag{5.61}$$

Set

$$U := \bigcap_{g\in G} \rho_g(V). \tag{5.62}$$

By construction, $x \in U \subset V$ and for every $g \in G$

$$\rho_g(U) = U. \tag{5.63}$$

Define $\phi\colon U \to T_xX$ by

$$\phi = \frac{1}{|G|}\sum_{g\in G} T_x\rho_g \circ \psi \circ \rho_g^{-1}. \tag{5.64}$$

Since

$$T_x\phi = \frac{1}{|G|}\sum_{g\in G} T_x\rho_g \circ \mathbf{1} \circ T_x\rho_g^{-1} = \mathbf{1}, \tag{5.65}$$

after possibly (G–invariantly) shrinking U, $\phi\colon U \to \tilde{U} := \phi(U)$ is a C^k diffeomorphism. To see that ϕ is G–equivariant, observe that for every $h \in G$

$$\begin{aligned}
\phi \circ \rho_h &= \frac{1}{|G|}\sum_{g\in G} T_x\rho_g \circ \psi \circ \rho_{h^{-1}g}^{-1} \\
&= \frac{1}{|G|}\sum_{h^{-1}g\in G} T_x\rho_h \circ T_x\rho_{h^{-1}g} \circ \circ\psi\rho_{h^{-1}g}^{-1} \\
&= T_x\rho_h \circ \phi.
\end{aligned} \tag{5.66}$$

This finishes the proof. ∎

Remark 5.67. A version of this holds for C^k manifolds with boundary and $x \in \partial X$. This requires replacing T_xX with the inward pointing half-space I_xX (which I have not defined, yet). ♣

6 Submanifolds

Definition 6.1. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a C^k manifold with boundary. A subset $Y \subset X$ is a C^k **submanifold** if for every $x \in Y$ there is an admissible chart $\phi\colon U \to \tilde{U} \subset \mathbf{M}$ of X with $x \in U$ and $n \in \{0, \ldots, m := \dim_x X\}$ such that

$$\phi(Y \cap U) = \mathbf{S} \cap \tilde{U} \tag{6.2}$$

with $\mathbf{S} \subset \mathbf{M}$ denoting either $\mathbf{R}^n \times \{0\}$ or $[0,\infty) \times \mathbf{R}^{n-1} \times \{0\}$. •

Proposition 6.3. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a C^k manifold without boundary. Let $Y \subset X$ be a C^k submanifold. There is a unique C^k structure $\mathscr{A}_\subset$ on Y satisfying the following universal property:*

1. *The inclusion map $\iota\colon\ Y \hookrightarrow X$ is C^k.*

2. *If Z is a C^k manifold and $f\colon\ Z \to Y$ is a continuous map, then f is C^k with respect to $\mathscr{A}_\subset$ if and only if $\iota \circ f\colon\ Z \to X$ is C^k.*

Exercise 6.4. Prove this.

Definition 6.5. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a C^k manifold with boundary. Let $Y \subset X$ be a C^k submanifold.

1. Y is **neat** if $\partial Y = Y \cap \partial X$.

2. The **codimension of** Y is the map $\operatorname{codim}_\cdot\colon\ Y \to \mathbf{N}_0$ defined by

$$\operatorname{codim}_x Y := \dim_x X - \dim_x Y. \tag{6.6}$$ •

Example 6.7. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a C^k manifold with boundary. If $U \subset X$ is open, then it is a submanifold of codimension zero ♠

Example 6.8. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a C^k manifold with boundary. $\partial X \subset X$ is a submanifold of codimension one. ♠

∼

Definition 6.9. Let $k \in \mathbf{N} \cup \{\infty\}$. Let X and Y be C^k manifolds. Let $f\colon\ X \to Y$ be a C^k map.

1. An $x \in X$ is a **regular point of** f if $T_x f$ is surjective; otherwise, it is a **critical point of** f. The set of critical points of f is denoted by

$$\operatorname{crit} f := \{x \in X : T_x f \text{ is not surjective}\}. \tag{6.10}$$

2. A $y \in Y$ is a **critical value of** f if is contained in $f(\operatorname{crit} f)$; otherwise, it is a **regular value of** f. •

Remark 6.11. In the situation of Definition 6.9, every $y \notin \operatorname{im} f$ is a regular value of f. In fact, if $\dim X < \dim Y$, then $y \in Y$ is a regular value of f if and only if $y \notin \operatorname{im} f$. ♣

Theorem 6.12 (Regular Value Theorem). *Let $k \in \mathbf{N} \cup \{\infty\}$. Let X and Y be C^k manifolds with boundary. Let $f\colon\ X \to Y$ be C^k. If $y \in Y$ is a regular value of f and $\partial f := f|_{\partial X}$, then*

1. *$Z := f^{-1}(y)$ is a C^k neat submanifold of X.*
2. *Denote by $\iota\colon Z \to X$ the inclusion. The sequence*

$$0 \to T_xZ \xrightarrow{T_x\iota} T_xX \xrightarrow{T_xf} T_{f(x)}Y \to 0 \tag{6.13}$$

is exact; that is: $T_x\iota$ defines an isomorphism

$$T_xZ \cong \ker T_xf. \tag{6.14}$$

Proof of Theorem 6.12. Let $x \in f^{-1}(y)$. If $x \notin \partial X$, then, by Theorem 5.40 (2), there is an admissible chart $\phi\colon U \to \tilde{U}$ of X with $x \in U$ and $\phi(x) = 0$ and and admissible chart $\psi\colon V \to \tilde{V}$ of Y with $f(U) \subset V$ and $\psi(y) = 0$ such that $\tilde{f} := \psi \circ f \circ \phi^{-1}$ satisfies

$$\tilde{f}(x_1, \ldots, x_m) = (x_1, \ldots, x_n) \tag{6.15}$$

with $m := \dim_x X$ and $n := \dim_y Y$. Therefore,

$$\phi(f^{-1}(y) \cap U) = (\mathbf{R}^n \times \{0\}) \cap \tilde{U}. \tag{6.16}$$

This proves the assertion if $\partial X = \varnothing$.

If $x \in \partial X$, then there is a neighborhood U of $x \in X$ which is embedded into a C^k manifold W without boundary and a C^k map $b \in C^k(W)$ such that 0 is regular value of b and $U = b^{-1}([0,\infty))$. Furthermore, there is a C^k map $g\colon W \to Y$ with $g|_U = f$ and such that y is a regular value of g. By the above, $g^{-1}(y)$ is a submanifold of W. 0 is a regular value of $b|_{f^{-1}(y)}$. Indeed, for every $z \in f^{-1}(y) \cap b^{-1}(0)$

$$T_zf(\ker T_zb) = T_{f(z)}Y; \tag{6.17}$$

in particular: $\ker T_xf \not\subset \ker T_xb$. The result thus follows from the following proposition. ∎

Proposition 6.18. *Let $k \in \mathbf{N} \cup \{\infty\}$. Let X be C^k manifold without boundary. Let $f \in C^k(X)$. If $a \in \mathbf{R}$ is a regular value of f, then $Y := f^{-1}([a,\infty))$ is a submanifold with boundary $f^{-1}(a)$.*

Proof. If $x \in f^{-1}(a,\infty)$, then there is an open neighborhood U of $x \in X$ such that $U \subset f^{-1}(a,\infty)$. Therefore, it remains to consider $x \in f^{-1}(a)$ By Theorem 5.40 (2), there is an admissible chart $\phi\colon U \to \tilde{U}$ of X with $x \in U$ and $\phi(x) = 0$ and and admissible chart $\psi\colon V \to \tilde{V}$ of $\mathbf{R}$ with $f(U) \subset V$ and $\psi(a) = 0$ such that $\tilde{f} := \psi \circ f \circ \phi^{-1}$ satisfies

$$\tilde{f}(x_1, \ldots, x_m) = x. \tag{6.19}$$

Without loss of generality, ψ is monotone increasing. Therefore, $f(y) \geqslant a$ if and only if $y_1 \geqslant 0$ for $(y_1, \ldots, y_m) := \phi(y)$. Consequently,

$$\phi(Y \cap U) = ([0,\infty) \times \mathbf{R}^{m-1}) \cap \tilde{U}. \tag{6.20}$$

∎

Remark 6.21. If X itself has a boundary, then $f^{-1}([a,\infty]$ might have corners. ♣

Theorem 6.12 is incredibly useful for constructing C^k manifolds.

Example 6.22. Let $n \in \mathbf{N}_0$. Define $f\colon \mathbf{R}^{n+1} \to \mathbf{R}$ by

$$f(x_1,\dots,x_{n+1}) \coloneqq |x|^2 = \sum_{i=1}^{n+1} x_i^2. \tag{6.23}$$

Since

$$\mathrm{d}_x f(\hat{x}) = \langle x, \hat{x}\rangle = 2\sum_{i=1}^{n+1} x_i \hat{x}_i, \tag{6.24}$$

every $y \in \mathbf{R}\backslash\{0\}$ is a regular value of f. The level set $f^{-1}(1)$ is S^n; indeed, for every $y > 0$, $f^{-1}(y)$ is diffeomorphic to S^n. For $y < 0$, $f^{-1}(y) = \varnothing$. ♠

Example 6.25. Let $n \in \mathbf{N}$. The **orthogonal group in dimension** n is the subgroup of $\mathrm{GL}(\mathbf{R}^n)$ defined by

$$\mathrm{O}(n) \coloneqq \{A \in \mathrm{End}(\mathbf{R}^n) : A^t A = \mathbf{1}\}. \tag{6.26}$$

$\mathrm{O}(n)$ is a submanifold of $\mathrm{End}(\mathbf{R}^n)$ of dimension

$$\dim \mathrm{O}(n) = \frac{n(n-1)}{2}. \tag{6.27}$$

To see this, define $f\colon \mathrm{End}(\mathbf{R}^n) \to \mathrm{Sym}(\mathbf{R}^n)$ by

$$f(A) \coloneqq AA^t \tag{6.28}$$

and observe that $\mathbf{1}$ is a regular value of f. Indeed,

$$\mathrm{d}_A f(\hat{A}) = \hat{A}^t A + A^t \hat{A}; \tag{6.29}$$

therefore, for every $B \in \mathrm{Sym}(\mathbf{R}^n)$

$$\mathrm{d}_A f(\tfrac{1}{2}AB) = B. \tag{6.30}$$

♠

Example 6.31. Let $n \in \mathbf{N}$. The **special orthogonal group in dimension** n is the subgroup of $\mathrm{GL}(\mathbf{R}^n)$ defined by

$$\mathrm{SO}(n) \coloneqq \{A \in \mathrm{End}(\mathbf{R}^n) : A^t A = \mathbf{1}, \det A = 1\}. \tag{6.32}$$

For every $A \in \mathrm{O}(n)$

$$1 = \det(A^t A) = \det(A)^2; \tag{6.33}$$

hence: $\det(A) \in \{1,-1\}$. Therefore, $\mathrm{SO}(n) \subset \mathrm{O}(n)$ is open and closed. Indeed, $\mathrm{SO}(n)$ is the connected component of $\mathrm{O}(n)$ which contains $\mathbf{1}$. ♠

Example 6.34. Let $n \in \mathbf{N}$. The **unitary group in dimension** n is the subgroup of $\mathrm{GL}(\mathbf{C}^n)$ defined by

$$\mathrm{U}(n) := \{A \in \mathrm{End}_{\mathbf{C}}(\mathbf{C}^n) : A^*A = \mathbf{1}\}. \tag{6.35}$$

$\mathrm{U}(n)$ is a submanifold of $\mathrm{End}_{\mathbf{C}}(\mathbf{C}^n)$ of dimension

$$\dim \mathrm{U}(n) = n^2. \tag{6.36}$$

Indeed, $\mathbf{1}$ is a regular value of the map $f\colon \mathrm{End}_{\mathbf{C}}(\mathbf{C}^n) \to \mathrm{Herm}(\mathbf{C}^n)$ by

$$f(A) := A^*A \tag{6.37}$$

and

$$\dim \mathrm{End}_{\mathbf{C}}(\mathbf{C}^n) - \dim \mathrm{Herm}(\mathbf{C}^n) = 2n^2 - \left(2\frac{n(n-1)}{2} + n\right) = n^2. \tag{6.38}$$

♠

Example 6.39. Let $n \in \mathbf{N}$. The **special unitary group in dimension** n is the subgroup of $\mathrm{GL}(\mathbf{C}^n)$ defined by

$$\mathrm{SU}(n) := \{A \in \mathrm{End}_{\mathbf{C}}(\mathbf{C}^n) : A^*A = \mathbf{1}, \det A = 1\}. \tag{6.40}$$

$\mathrm{U}(n)$ is a submanifold of $\mathrm{SU}(n)$ of dimension

$$\dim \mathrm{SU}(n) = n^2 - 1. \tag{6.41}$$

For every $A \in \mathrm{U}(n)$, $\det A \in S^1 = \{z \in \mathbf{C} : |z| = 1\}$. 1 is a regular value of

$$\det\colon \mathrm{U}(n) \to S^1 \tag{6.42}$$

Indeed, for every $A \in \mathrm{SU}(n)$

$$T_A\mathrm{U}(n) = A \cdot \{\hat{A} \in \mathrm{End}_{\mathbf{C}}(\mathbf{C}^n) : A^* + A = 0\}, \quad T_1S^1 = i\mathbf{R}, \tag{6.43}$$

and

$$\mathrm{d}_A \det(A\hat{A}) = \mathrm{tr}(\hat{A}). \tag{6.44}$$

In particular, $\mathrm{d}_A \det(iA) = ni$. ♠

∼

Exercise 6.45 (Constant rank theorem). Let $k \in \mathbf{N} \cup \{\infty\}$. Let X and Y be C^k manifolds with boundary. Let $f\colon X \to Y$ be C^k. Suppose that the map $x \mapsto \mathrm{rk}\, T_xf$ is locally constant. Prove that for every $y \in Y$ the level set $f^{-1}(y)$ is a C^k submanifold without boundary.

$\sim$

Example 6.46. Let $p_1, \dots, p_r$ be r homogeneous polynomials in $n+1$ complex variables $z_0, \dots, z_n$. The zero locus

$$Z := Z(p_1, \dots, p_r) := \{[z_0 : \dots : z_n] \in \mathbf{C}P^n : p_i(z_0, \dots, z_n) = 0\}$$

is a (complex) submanifold of $\mathbf{C}P^n$ if for all $(z_0, \dots, z_n) \in \mathbf{C}^{n+1}$ with $[z_0 : \dots : z_n] \in Z$, the map $(\mathrm{d}p_1, \dots, \mathrm{d}p_r) \colon \mathbf{C}^{n+1} \to \mathbf{C}^r$ is surjective. (Exercise!) Such a Z is called a **complete intersection**. ♠

Example 6.47. The quadric

$$Q = \{[z_0 : z_1 : z_2 : z_3] \in \mathbf{C}P^3 : z_0 z_3 - z_1 z_2 = 0\}$$

is a complex submanifold of $\mathbf{C}P^3$. If fact, it can be shown that Q is diffeomorphic (in fact: biholomorphic) to $\mathbf{C}P^1 \times \mathbf{C}P^1$. ♠

Example 6.48. The Fermat quartic

$$Q = \{[z_0 : z_1 : z_2 : z_3] \in \mathbf{C}P^3 : z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$$

is a complex submanifold of $\mathbf{C}P^3$. This is a $K3$ surface, part of a particularly interesting class of complex manifolds. ♠

Example 6.49. Let $a_1, \dots, a_n \in \{2, 3, \dots\}$ The **Brieskorn manifold**

$$\Sigma(a_1, \dots, a_n) := \left\{z = (z_1, \dots, z_n) \in \mathbf{C}^n : z_1^{a_1} + \dots + z_n^{a^n} = 0 \text{ and } |z| = 1\right\} \tag{6.50}$$

is a submanifold of dimension $2n-3$. To see this, define $f \colon \mathbf{C}^n \to \mathbf{C} \oplus \mathbf{R}$ by

$$f(z) := \left(z_1^{a_1} + \dots + z_n^{a_n}, |z|^2\right) \tag{6.51}$$

and observe that $(0, 1)$ is a value of f. Indeed, if $z \in \Sigma(a_1, \dots, a_n)$, then at least two of its components are non-zero. From this easy to see that $\mathrm{d}_z f$ is surjective.

The hypersurface $\{(z_1, \dots, z_n) \in \mathbf{C}^n : z_1^{a_1} + \dots + z_n^{a^n} = 0\}$ has a singularity at 0. However, it might still be a topological manifold. This is the case if and only if $\Sigma(a_1, \dots, a_n)$ is homeomorphic to S^{2n-3}. Building on work of Phạm [Phạ65], Brieskorn [Bri66, Satz 1] proved a criterion for $\Sigma(a_1, \dots, a_n)$ to be homeomorphic to S^{2n-3}. In this case Brieskorn [Bri66, Satz 2 and 3] also determined the diffeomorphism type. The Brieskorn manifolds $\Sigma(2, 2, 2, 3, 6k-1)$ with $k = 1, \dots, 28$ are homeomorphic to S^7 but not diffeomorphic to each other [Bri66]. These are all diffeomorphism types of S^7; cf. Milnor [Mil56]. ♠

$\sim$

Proposition 6.52. *In the situation of Proposition 5.59, the fixed-point locus X^G is a C^k submanifold.*

Proof. By Proposition 5.59, it suffices to prove this for $X = \mathbf{R}^m$ and $\rho\colon G \to \mathrm{GL}(\mathbf{R}^m)$. In this case,

$$X^G = \bigcap_{g\in G} \ker \rho_g \tag{6.53}$$

is a linear subspace. ■

~

Definition 6.54. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X, Y be C^k manifolds with boundary.

1. A C^k map $\iota\colon X \to Y$ is an **immersion** if for every $x \in X$ the map $T_x\iota\colon T_xX \to T_{\iota(x)}Y$ is injective.
2. It is an **embedding** if it is an injective immersion and the map $X \to \iota(X)$ is a homeomorphism. •

Notation 6.55. If $\iota\colon X \to Y$ is an immersion, then this can be indicated by denoting it as $\iota\colon X \looparrowright Y$. If it is an embedding, then this can be indicated by denoting it as $\iota\colon X \hookrightarrow Y$. ○

Proposition 6.56. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a C^k manifold with boundary.*

1. *If $Y \subset X$ is a C^k submanifold, then the inclusion $\iota\colon Y \hookrightarrow X$ is an C^k embedding.*
2. *If Y is a C^k manifold without boundary and $\iota\colon Y \hookrightarrow X$ is an C^k embedding, then $\iota(Y)$ is a C^k submanifold.*

Proof. This is a consequence of Theorem 5.40(1). ■

Remark 6.57. Every connected C^k manifold X admits an embedding into $\mathbf{R}^m$ for some $m = m(X) \in \mathbf{N}_0$; see Theorem 10.1. ♣

Example 6.58. The map $\iota\colon \mathbf{R}/2\pi\mathbf{Z} \to \mathbf{R}^2$ defined by

$$\iota(t) \coloneqq (\sin t, \sin 2t) \tag{6.59}$$

is an immersion, but not injective; hence: not an embedding. The image of ι is precisely

$$f^{-1}(0) \quad \text{with} \quad f(x,y) \coloneqq 4x^2(x^2-1) + y^2. \tag{6.60}$$

This is illustrated in Figure 6.1. 0 is not a regular value of f; indeed, $\mathrm{d}_0 f = 0$. ♠

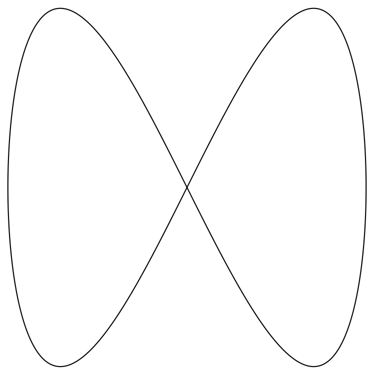

Figure 6.1: The lemniscate $4x^2(x^2-1)+y^2=0$.

Example 6.61. The restriction of the map ι defined by (6.59) to $(-\pi,\pi)$ is an injective immersion, but not an embedding; indeed, there is no open subset $U \subset \mathbf{R}^2$ such that $\operatorname{im}\iota \cap U = \iota(-\pi/2, \pi/2)$. ♠

Example 6.62. For $\mu \in \mathbf{R}$ define $\iota_\mu(t)\colon \mathbf{R} \to T^2$ by

$$\iota_\mu(t) := [(t, \mu t)]. \tag{6.63}$$

This map is an immersion.

1. If $\mu = a/b \in \mathbf{Q}$ with a and b coprime, then $\iota_\mu(t)$ descends to an embedding

   $$\mathbf{R}/b\mathbf{Z} \hookrightarrow T^2. \tag{6.64}$$

2. If $\mu \notin \mathbf{Q}$, then ι_μ is injective, but not an embedding; $\operatorname{im}\iota_\mu$ is dense in T^2. ♠

Proposition 5.25 implies the following.

Corollary 6.65. *Let X and Y be C^k manifold. If $\iota\colon X \looparrowright Y$ is a proper injective immersion, then it is a proper embedding.* ■

Example 6.66. The map $f\colon S^2 \to \mathbf{R}^4$ defined by

$$f(x,y,z) := (xy, yz, zx, x^2 - y^2) \tag{6.67}$$

is an immersion. The is unique map $g\colon \mathbf{R}P^2 \to \mathbf{R}^4$ such that

$$f = g \circ \pi \tag{6.68}$$

with $\pi\colon S^2 \to \mathbf{R}P^2$ denoting the projection map. The map g is an injective immersion; hence: an embedding. ♠

Example 6.69. The map $\iota\colon (0,\infty) \to \mathbf{R}^2$ defined by

$$\iota(r) := r(\cos(1/r), \sin(1/r))$$

is an embedding because $\iota^{-1}(x,y) = \sqrt{x^2+y^2}$ is continuous, but it is not proper; indeed: $\iota^{-1}(\bar{B}_1(0)) = (0,1]$ is not compact. The image of ι is the decelerating spiral depicted in Figure 6.2 ♠

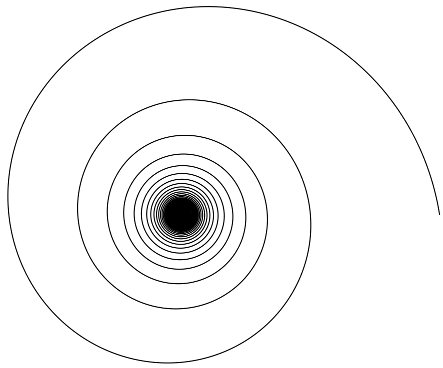

Figure 6.2: Decelerating spiral.

Definition 6.70. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X, Y be C^k manifolds with boundary. Let $\iota\colon Y \looparrowright X$ be an immersion. The **normal space** of ι at $x \in Y$ is

$$N_x\iota := T_{\iota(x)}X / T_x\iota(T_xY).$$

The **codimension of** ι is the map $\operatorname{codim}\iota\colon Y \to \mathbf{N}_0$ defined by

$$\operatorname{codim}_x \iota = \dim N_x\iota. \tag{6.71}$$ •

Definition 6.72. Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X be a C^k manifold with boundary. A C^k map $r\colon X \to X$ is a C^k **retraction** if

$$r \circ r = r. \tag{6.73}$$ •

Proposition 6.74. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X be a C^k manifold without boundary. If $r\colon X \to X$ is a C^k retraction, then $\operatorname{im} r \subset X$ is a submanifold.*

Proof. The functions $\rho\colon X \to \mathbf{N}_0$ and $\sigma\colon X \to \mathbf{N}_0$ defined by

$$\rho(x) \coloneqq \operatorname{rk} T_x r \quad \text{and} \quad \sigma(x) \coloneqq \operatorname{rk}(\mathbf{1} - T_x r) \tag{6.75}$$

are lower semi-continuous. If $x \in \operatorname{im} r$, then

$$r(x) = x \quad \text{and} \quad T_x r \circ T_x r = T_x r; \tag{6.76}$$

therefore, $\operatorname{im} T_x r = \ker(\mathbf{1} - T_x r)$ and $\rho(x) + \sigma(x) = \dim_x X$. Consequently, ρ is locally constant on $\operatorname{im} r$.

Choose a neighborhood U of $x \in X$ such that $\rho|_U \geqslant \rho(x)$ and equality holds on $\operatorname{im} r \cap U$. Since

$$\operatorname{rk} T_y r = \operatorname{rk} T_y(r \circ r) = \operatorname{rk}(T_{r(y)} r \circ T_y r) \leqslant \operatorname{rk} T_{r(y)} r = \rho(x),$$

$\rho|_U$ is constant.

The assertion thus follows from Theorem 5.40 (3). ■

∼

Proposition 6.77. *Let X, Y be finite-dimensional real vector spaces. For $r \in \mathbf{N}_0$ the subset*

$$\mathscr{H}_r \coloneqq \{L \in \operatorname{Hom}(X, Y) : \operatorname{rk} L = r\} \tag{6.78}$$

is a submanifold of codimension

$$\operatorname{codim} \mathscr{H}_r = (\dim X - r)(\dim Y - r). \tag{6.79}$$

Proof. Let $L \in \mathscr{H}_r$. Choose direct sum decompositions

$$X = X_1 \oplus X_2 \quad \text{and} \quad Y = Y_1 \oplus Y_2 \quad \text{with} \quad X_2 \coloneqq \ker L \quad \text{and} \quad Y_1 \coloneqq \operatorname{im} L. \tag{6.80}$$

Every $\Lambda \in \operatorname{Hom}(X, Y)$ decomposes accordingly as

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}. \tag{6.81}$$

By construction, L_{11} is invertible. Choose an open neighborhood U of L in $\operatorname{Hom}(X, Y)$ such that Λ_{11} is invertible for every $\Lambda \in U$.

The the map $f\colon U \to \operatorname{Hom}(X_2, Y_2)$ defined by

$$f(\Lambda) \coloneqq \Lambda_{22} - \Lambda_{21}\Lambda_{11}^{-1}\Lambda_{12} \tag{6.82}$$

has the property that for every $\Lambda \in U$

$$\begin{pmatrix} \Lambda_{11}^{-1} & 0 \\ -\Lambda_{21}\Lambda_{11}^{-1} & \mathbf{1} \end{pmatrix} \circ \Lambda \circ \begin{pmatrix} \mathbf{1} & -\Lambda_{11}^{-1}\Lambda_{12} \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & f(\Lambda) \end{pmatrix}. \tag{6.83}$$

Therefore,

$$\mathscr{H}_r \cap U = f^{-1}(0). \tag{6.84}$$

Since

$$\mathrm{d}_L f(\hat{\Lambda}) = \hat{\Lambda}_{22}, \tag{6.85}$$

after possibly shrinking U, 0 is a regular value of f. This proves the assertion since

$$\dim \mathrm{Hom}(X_2, Y_2) = (\dim X - r)(\dim Y - r). \tag{6.86}$$

■

Example 6.87. Let V be a finite dimensional vector space. The **Stiefel manifold** $\mathrm{St}_k^*(V)$ defined by

$$\mathrm{St}_k^*(V) := \big\{(v_1, \dots, v_k) \in V^k : v_1, \dots, v_k \text{ are linearly independent}\big\} \tag{6.88}$$

is a submanifold of V^k. ♠

Example 6.89. Let V be a finite dimensional Euclidean vector space. The **(orthonormal) Stiefel manifold** $\mathrm{St}_k(V)$ defined by

$$\mathrm{St}_k(V) := \big\{(v_1, \dots, v_k) \in V^k : \langle v_i, v_j \rangle = \delta_{ij}\big\}. \tag{6.90}$$

$\mathrm{St}_k(V)$ is a submanifold of $V^k = \mathrm{Hom}(\mathbf{R}^k, V)$; indeed: $\mathbf{1}$ is a regular value of the map $f\colon \mathrm{Hom}(\mathbf{R}^k, V) \to \mathrm{Sym}(\mathbf{R}^k)$ defined by

$$f(A) := A^* A. \tag{6.91}$$

♠

Example 6.92. The Gram–Schmidt process defines an analytic map $\Psi\colon \mathrm{St}_k^*(V) \to \mathrm{GL}(\mathbf{R}^k)$ such that

$$A\Psi(A)^* \in \mathrm{St}_k(V) \tag{6.93}$$

and $\Psi(A)$ is upper triangular. ♠

Example 6.94. Let V be a finite dimensional Euclidean vector space. The **Grassmannian of k–planes in V** is defined by

$$\mathrm{Gr}_k(V) := \big\{\pi \in \mathrm{Sym}(V) : \pi^2 = \pi \text{ and } \operatorname{tr} \pi = k\big\}. \tag{6.95}$$

$\mathrm{Gr}_k(V)$ is bijective to the set of k–dimensional linear subspaces in V; indeed: π is the orthogonal projection onto $\operatorname{im} \pi$. $\mathrm{Gr}_k(V)$ is a submanifold of $\mathrm{Sym}(V)$. ♠

Example 6.96. Let $n \in \mathbf{N}$. Denote by $\Delta \subset \mathbf{R}^{n\times n}$ the subspace of diagonal matrices. Define $f\colon \mathrm{O}(n) \times \mathbf{R}^n \to \mathrm{Sym}(\mathbf{R}^n)$ by

$$f(\Phi, \Lambda) := \Phi\Lambda\Phi^t. \tag{6.97}$$

The map f is surjective. $A \in \mathrm{Sym}(\mathbf{R}^n)$ is a regular value of f if and only if A has n distinct eigenvalues. Indeed, $\mathrm{d}_{\Phi,\Lambda}f$ is surjective if and only if it is injective. By a direct computation,

$$T_\Phi \mathrm{O}(n) = \{\Phi\phi \in \mathbf{R}^{n\times n} : \phi^* + \phi = 0\}. \tag{6.98}$$

and

$$\mathrm{d}_{\Phi,\Lambda} f(\Phi\phi, \lambda) = \Phi([\phi, \Lambda] + \lambda)\Phi^*. \tag{6.99}$$

The terms $[\phi, \Lambda]$ and λ are perpendicular. Obviously, the map $\lambda \mapsto \lambda$ is injective.e A moment's thought shows that the map $\phi \mapsto [\phi, \Lambda]$ is injective if and only if Λ has n distinct eigenvalues. ♠

7 Sard's Theorem

Definition 7.1. Let X be a metric space. Let $s \in [0, \infty)$. Set

$$c_H^s := \frac{\pi^{s/2}}{\Gamma(s/2 + 1)} \tag{7.2}$$

Let $A \subset X$. The s**–dimensional Hausdorff measure of** A is defined by

$$\mathscr{H}^s(A) := \sup_{\delta > 0} \mathscr{H}^s_\delta(A) \tag{7.3}$$

with

$$\mathscr{H}^s_\delta(A) := c_H^s \inf\left\{ \sum_{i=1}^\infty r_i^s : A \subset \bigcup_{i=1}^\infty B_{r_i}(x_i) \text{ and } r_i \leqslant \delta \right\} \in [0, \infty]. \tag{7.4}$$ •

If $\mathscr{H}^s(A) = 0$, then A is said to have $\mathscr{H}^s$**–measure zero**.

Proposition 7.5. *Let $s \in [0, \infty)$. Let X be a metric space. For every countable set $\{A_n \subset X : n \in \mathbf{N}\}$ of subsets of X*

$$\mathscr{H}^s\left(\bigcup_{n=1}^\infty A_n\right) \leqslant \sum_{n=1}^\infty \mathscr{H}^s(A_n). \tag{7.6}$$

Proposition 7.7. *Let $s \in [0,\infty)$. Let X, Y be metric spaces. Let $f\colon X \to Y$ be Lipschitz. If $A \subset X$ has $\mathscr{H}^s$–measure zero, then $f(A) \subset Y$ has $\mathscr{H}^s$–measure zero.*

Corollary 7.8. *Let $m, n \in \mathbf{N}_0$ and $s \in [0,\infty)$. Let $U \subset \mathbf{R}^m$ be an open subset. Let $f\colon U \to \mathbf{R}^n$ be C^1. If $A \subset X$ has $\mathscr{H}^s$–measure zero, then $f(A) \subset Y$ has $\mathscr{H}^s$–measure zero.* ■

Definition 7.9. Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X be a C^k manifold with boundary.

1. A subset $A \subset X$ is of **measure zero** for every $x \in A$ there is a chart $\phi\colon U \to \tilde{U} \subset \mathbf{R}^m$ on X with $x \in U$ and $m \coloneqq \dim_x X$ such that $\phi(A \cap U)$ has $\mathscr{H}^m$–measure zero.
2. Let (♣) be a property of points $x \in X$. (♣) holds for **almost every** $x \in X$ if $\{x \in X : (♣) \text{ does not hold for } x\}$ has measure zero. •

Proposition 7.10. *Let $n \in \mathbf{N}$ and $r \in \{1, \dots, n-1\}$. Let $A \subset \mathbf{R}^n = \mathbf{R}^r \times \mathbf{R}^{n-r}$. If for every $y \in \mathbf{R}^n$ the slice*

$$A_y \coloneqq \{z \in \mathbf{R}^{n-r} : (z,y) \in A\} \tag{7.11}$$

has $\mathscr{H}^{n-r}$–measure zero, then A has $\mathscr{H}^n$–measure zero. ■

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Theorem 7.12 (Sard [Sar42]). *Let $k \in \mathbf{N} \cup \{\infty\}$. Let X and Y be connected C^k manifolds with boundary and let $f\colon X \to Y$ be C^k. If $k \geqslant \dim X - \dim Y + 1$, then the set of critical values of f in Y has measure zero.*

Example 7.13. Define $f\colon \mathbf{C}^2 \to \mathbf{C}$

$$f(x,y) \coloneqq xy. \tag{7.14}$$

The set of critical values of f is $\{0\}$. ♠

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Definition 7.15. Let X be a topological space. A subset $A \subset X$ is **meager** if it is a countable union of nowhere dense subsets. •

Proposition 7.16. *In the situation of Theorem 7.12, the set of critical values of f is meager.*

Definition 7.17. Let X be a topological space. A **exhaustion by compact sets of** X is a sequence of compact subsets $(K_n)_{n\in\mathbf{N}}$ such that

$$K_n \subset K_{n+1}^\circ \quad \text{for every} \quad n \in \mathbf{N} \quad \text{and} \quad X = \bigcup_{n\in\mathbf{N}} K_n. \tag{7.18}$$ •

Proposition 7.19. *If X is a topological space which is locally compact, paracompact, and path-connected, then it admits an exhaustion by compact subsets.*

Proof. Since X is locally compact, it has an open cover $\mathscr{U}$ such that every $U \in \mathscr{U}$ is contained in a compact subset; indeed: $\overline{U}$ is compact because X is Hausdorff. Since X is paracompact, after passing to a refinement, $\mathscr{U}$ can be assumed to be locally finite.

Fix $U_1 \in \mathscr{U}$. For every $U \in \mathscr{U}$ there are $U_1, U_2, \dots, U_n = U \in \mathscr{U}$ such that

$$U_i \cap U_{i-1} \neq \varnothing \quad \text{for every} \quad i \in \{2, \dots, n\}. \tag{7.20}$$

To see this, observe the following. Since X is path-connected, there is a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) \in U_1$ and $\gamma(1) \in U$. The subset $\gamma([0,1])$ is compact and, therefore, covered by U_1, U, and finitely many further elements of $\mathscr{U}$. The desired $U_1, U_2, \dots, U_n$ can be extracted out of these.

Define $\ell\colon \mathscr{U} \to \mathbf{N}$ by declaring $\ell(U)$ to be the minimal n for which there are $U_1, U_2, \dots, U_n = U$ as above. For every $n \in \mathbf{N}$ the subset $\ell^{-1}(n)$ is finite and, therefore,

$$K_n \coloneqq \bigcup_{U \in \ell^{-1}(n)} \overline{U} \tag{7.21}$$

is compact. This is proved by induction. For $n = 1$ it is trivial. Suppose $\ell^{-1}(n)$ is finite. Since K_n is compact and $\mathscr{U}$ is locally finite, only finitely many $U \in \mathscr{U}$ intersect K_n. Therefore, $\ell^{-1}(n+1)$ is finite.

It follows from the above that

$$K_n \subset \bigcup_{U \in \ell^{-1}(n+1)} U \subset K_{n+1}^\circ \quad \text{for every} \quad n \in \mathbf{N} \quad \text{and} \quad X = \bigcup_{n \in \mathbf{N}} K_n. \tag{7.22}$$

■

Proof of Proposition 7.16 assuming Theorem 7.12. Denote by $\mathrm{crit}(f) \subset X$ the set of critical points of f. Since X is connected, by Proposition 7.19, it admits a exhaustion by compact subsets $(K_n)_{n \in \mathbf{N}}$. For every $n \in \mathbf{N}$ the subset $f(K_n \cap \mathrm{crit}(f))$ is compact. Since Y is Hausdorff, $f(K_n \cap \mathrm{crit}(f))$ is closed. Its interior must be empty because a non-empty open subset of Y cannot have measure zero. Therefore, $f(\mathrm{crit}(f))$ is meager. ■

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Proof of Theorem 7.12 for $\dim X < \dim Y$. This is an immediate consequence of Corollary 7.8 because $\{0\} \times \mathbf{R}^m \subset \mathbf{R}^n$ has $\mathscr{H}^n$–measure zero provided $m < n$. ■

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The proof of Theorem 7.12 for $\dim X \geqslant \dim Y$ reduces to the following result.

Lemma 7.23. *Let $m, n \in \mathbf{N}_0$ with $m \geqslant n$ and $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let $U \subset \mathbf{R}^m$ be an open subset. Let $f\colon U \to \mathbf{R}^n$ be C^k. Set*

$$Z_1^f \coloneqq \{x \in U : \mathrm{d}_x f = 0\} \tag{7.24}$$

If $k \geqslant \frac{m}{n}$, then $f(Z_1^f)$ has $\mathcal{H}^n$–measure zero.

Proof of Theorem 7.12 for $\dim X \geqslant \dim Y$ assuming Lemma 7.23. Denote by $\mathrm{crit}(f)$ the set of critical points of f. Set $m \coloneqq \dim X$ and $n \coloneqq \dim Y$. For $r \in \{0, \ldots, n-1\}$ set

$$\mathrm{crit}_r(f) \coloneqq \{x \in X : \mathrm{rk}\, T_x f = r\}.$$

The set $\mathrm{crit}(f)$ decomposes as $\mathrm{crit}_0(f) \cup \cdots \cup \mathrm{crit}_{n-1}(f)$. Therefore, it suffices to show that the subsets $f(\mathrm{crit}_r(f))$ have measure zero. By Lemma 7.23, $f(\mathrm{crit}_0(f)) = f(Z_1^f)$ has measure zero.

Let $r \in \{1, \ldots, n-1\}$ and let $x \in \mathrm{crit}_r(f)$. By Theorem 5.40, there are a chart $\phi\colon U \to \tilde{U}$ on X with $x \in U$ and a chart $\psi\colon V \to \tilde{V}$ with $f(x) \in V$ on Y such that map $\tilde{f} \coloneqq \psi \circ f \circ \phi^{-1}\colon \tilde{U} \to \tilde{V}$ is of the form

$$\tilde{f}(x_1, \ldots, x_m) = (x_1, \ldots, x_r, g(x_1, \ldots, x_m)) \tag{7.25}$$

for a C^k map $g\colon \tilde{U} \to \tilde{\mathbf{R}}^{n-r}$. By construction,

$$f(\mathrm{crit}_r(f) \cap U) = \psi^{-1}(\tilde{f}(\mathrm{crit}_r(\tilde{f}))). \tag{7.26}$$

Since Y is second-countable, $f(\mathrm{crit}_r(f))$ can be covered by countably many sets of the above form. Therefore, it remains to prove that $\tilde{f}(\mathrm{crit}_r(\tilde{f}))$ has measure zero.

For every $y \in \mathbf{R}^r$ define $g_y\colon \tilde{U}_y \to \mathbf{R}^{n-r}$ by

$$g_y(z) \coloneqq g(y, z). \tag{7.27}$$

This map is C^k with

$$k \geqslant m - n + 1 \geqslant \frac{m-r}{n-r}. \tag{7.28}$$

By Lemma 7.23, $g_y(Z_1^{g_y})$ has $\mathcal{H}^{n-r}$–measure zero for every $y \in \mathbf{R}^r$. It is immediate from (7.25) that $(y, z) \in \mathrm{crit}_r(\tilde{f})$ if and only if $\mathrm{d}_z g_y = 0$; hence: for every $y \in \mathbf{R}^r$

$$(\tilde{f}(\mathrm{crit}_r(\tilde{f})))_y = g_y(Z_1^{g_y}). \tag{7.29}$$

Therefore, by Proposition 7.10, $\tilde{f}(\mathrm{crit}_r(\tilde{f}))$ has $\mathcal{H}^n$–measure zero. ■

Proposition 7.30. *Let $m, n \in \mathbf{N}_0$ and $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let $U \subset \mathbf{R}^m$ be an open subset. Let $f \colon U \to \mathbf{R}^n$ be C^k. Set*

$$Z_k^f := \left\{x \in U : \partial^\alpha f(x) = 0 \text{ for every } \alpha \in \mathbf{N}_0^m \text{ with } |\alpha| \leqslant k\right\} \tag{7.31}$$

If $k + 1 > \frac{m}{n}$, then $f(Z_k^f)$ has $\mathcal{H}^n$–measure zero.

Proof. Z_k^f can be covered by countably compact cubes contained in U and of the form $Q = [0, \ell]^m + x_0$ with $x_0 \in \mathbf{R}^m$. Therefore, it suffices to prove that $f(Z_k^f \cap Q)$ has $\mathcal{H}^n$–measure zero.

By Taylor's Theorem and since Q is compact, there is a constant $c_f > 0$ such that for every $x \in Z_k^f \cap Q$ and $y \in Q$

$$|f(x) - f(y)| \leqslant c_f |x - y|^{k+1}. \tag{7.32}$$

For every $N \in \mathbf{N}_0$ the cube Q can be subdivided into N^m into compact cubes Q^* of side-length ℓ/N. If Q^* intersects Z_k^f, then $f(Q^*)$ is contained in a ball of radius $c_f(\ell/N)^{k+1}$. Therefore,

$$\mathcal{H}^n(f(Z_k^f \cap Q)) \leqslant c_H^s N^m c_f^n (\ell/N)^{n(k+1)} = c_H^s c_f^n \ell^{n(k+1)} N^{m-n(k+1)}. \tag{7.33}$$

The exponent of N is negative and N can be chosen arbitrarily large. Therefore, $f(Z_k^f \cap Q)$ has $\mathcal{H}^n$–measure zero. ∎

Proof of Lemma 7.23. This is proved by induction on m. For $m = n$ it follows from Proposition 7.30.

For $s \in \{1, \ldots, k\}$ set

$$Z_s^f := \{x \in U : \partial^\alpha f(x) = 0 \text{ for every } \alpha \in \mathbf{N}_0^m \text{ with } |\alpha| \leqslant s\}. \tag{7.34}$$

By Proposition 7.30, $f(Z_k^f)$ has $\mathcal{H}^n$–measure zero. Therefore, it suffices to show that $f(Z_s^f \backslash Z_{s+1}^f)$ has $\mathcal{H}^n$–measure zero for every $s \in \{1, \ldots, k-1\}$.

Let $x \in Z_s^f \backslash Z_{s+1}^f$. After possibly relabeling the coordinates, there is an $\alpha \in \mathbf{N}_0^m$ with $|\alpha| = s$ such that $\partial^\alpha f_1$ vanishes at x but $\partial_1 \partial^\alpha f_1(x) \neq 0$. Define $\phi \colon U \to \mathbf{R}^n$ by

$$\phi(x_1, \ldots, x_m) := (\partial^\alpha f_1(x_1, \ldots, x_m), x_2, \ldots, x_m). \tag{7.35}$$

There is an open neighborhood V of $x \in U$ such that $\phi \colon V \to \tilde{V} := \phi(V)$ is a C^{r-s} diffeomorphism. By construction,

$$\phi(Z_s^f \cap V) \subset \{0\} \times \mathbf{R}^{m-1}. \tag{7.36}$$

There *obviously* is a C^{r-s} map $g\colon \tilde V \to \mathbf{R}^n$ such that for every $y \in \phi(Z_s^f \cap V)$

$$g(y) = f \circ \phi^{-1}(y) \quad \text{and} \quad \mathrm{d}_y g = 0. \tag{7.37}$$

Set $\tilde V_0 := \{z \in \mathbf{R}^{m-1} : (0,z) \in \tilde V\}$ and, given g define $g_0\colon \tilde V_0 \to \mathbf{R}^n$ by $g_0(z) := g(0,z)$. By construction of g,

$$f(Z_s^f \cap V) \subset g_0(Z_1^{g_0}). \tag{7.38}$$

If g could be chosen C^r (in a non-obvious fashion), then it would follow by induction that the latter set has $\mathscr{H}^n$–measure zero. Theorem 7.39 precisely says that this can be achieved. Without this result, the proof still goes through for $k = \infty$ (or, in fact, a stricter lower bound for k than $\frac{m}{n}$). ■

Theorem 7.39 (Kneser [Kne51, Hilfssatz 1] and Glaeser [Gla58]; Rough Composition Theorem). *Let $U \subset \mathbf{R}^m$ and $V \subset \mathbf{R}^n$ be open subsets. Let $A \subset U$ be closed and let $A^* \subset V$. Let $f\colon V \to \mathbf{R}^m$ be C^k and satisfying $\partial^\alpha f = 0$ on A. Let $\phi\colon V \to U$ be C^{r-s} with $\phi(A^*) \subset A$. There is a C^r map $g\colon V \to \mathbf{R}^m$ satisfying:*

1. *$g(x) = f \circ \phi(x)$ for $x \in A^*$, and*
2. *$\partial^\alpha g(x) = 0$ for $|\alpha| \leqslant s$ for $x \in A^*$.*

Proof. See the references given or [AR67, Section 14]. ■

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TBD: In the lecture we have discussed Hirsch's/Milnor's proof of the Brouwer fixed-point theorem [Mil97, p. 8]. I might add this later.

8 Transversality

Definition 8.1. Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X, Y be C^k manifolds with boundary. Let $f\colon X \to Y$ be a C^k and let $Z \subset Y$ be a C^k submanifold. The map f is **transverse to** Z if for every $x \in X$

$$\operatorname{im} T_x f + T_{f(x)} Z = T_{f(x)} Y. \tag{8.2}$$ •

Notation 8.3. In the situation of Definition 8.1, if f is transverse to Z then this is denoted by

$$f \pitchfork Z. \tag{8.4}$$ ∘

Remark 8.5. The condition (8.2) is equivalent to the composition

$$T_x X \xrightarrow{T_x f} T_x Y \twoheadrightarrow N_{f(x)} Z \tag{8.6}$$

being surjective. ♣

Example 8.7. The map $f\colon \mathbf{R} \to \mathbf{R}^2$ defined by

$$f(t) := (t, \cos(t)) \tag{8.8}$$

is transverse to $\mathbf{R} \times \{s\}$ if and only if $s \notin \{1, -1\}$. ♠

TBD: pictures

Definition 8.9. Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X, Y be a C^k manifolds without boundary. A C^k map $f\colon X \to Y$ is a **submersion** if for every $x \in X$ the map $T_x f\colon T_x X \to T_{f(x)} Y$ is surjective. ●

Example 8.10. The Hopf map $\eta\colon S^3 \to S^2$ is a submersion. ♠

Proposition 8.11. *A submersion $f\colon X \to Y$ is transverse to every submanifold $Z \subset Y$.*

Theorem 8.12. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X, Y be C^k manifolds with boundary. Let $f\colon X \to Y$ be C^k and let $Z \subset Y$ be a C^k submanifold* without boundary. *If f and $\partial f := f|_{\partial X}$ are transverse to Z, then*

$$f^{-1}(Z) \subset X \tag{8.13}$$

is a neat C^k submanifold with

$$\operatorname{codim}_x f^{-1}(Z) = \operatorname{codim}_{f(x)} Z. \tag{8.14}$$

Proof. Let $x \in f^{-1}(Z)$ and $y := f(x)$. Set $m := \dim_y Y$ and $n := \dim_y Z$. Since $Z \subset Y$ is a submanifold, there is an admissible chart $\phi\colon U \to \tilde{U}$ with $y \in U$ and $\tilde{U} \subset \mathbf{R}^m$ open such that

$$\phi(Z \cap U) = (\mathbf{R}^n \times \{0\}) \cap \tilde{U}. \tag{8.15}$$

Denote by $\mathrm{pr}_2\colon \mathbf{R}^m = \mathbf{R}^n \times \mathbf{R}^{m-n} \to \mathbf{R}^{m-n}$ the canonical projection. Set $V := f^{-1}(U)$. By construction,

$$f^{-1}(Z) \cap V = g^{-1}(0) \quad \text{with} \quad g := \mathrm{pr}_2 \circ \phi \circ f|_V\colon V \to \mathbf{R}^{m-n}. \tag{8.16}$$

Therefore and by Theorem 6.12, it suffices to prove that 0 is a regular value of g and $\partial g := g|_{V \cap \partial X}$.

For every $x \in g^{-1}(0)$

$$T_x g = T_{f(x)}(\mathrm{pr}_2 \circ \phi) \circ T_x f \quad \text{and} \quad T_{f(x)} Z = \ker T_{f(x)}(\mathrm{pr}_2 \circ \phi). \tag{8.17}$$

Since f is transverse to Z,

$$\operatorname{im} T_x g = T_{f(x)}(\mathrm{pr}_2 \circ \phi)(\operatorname{im} T_x f + T_{f(x)} Z) = T_{f(x)}(\mathrm{pr}_2 \circ \phi)(T_{f(x)} Y). \tag{8.18}$$

Since $T_{f(x)}(\mathrm{pr}_2 \circ \phi)$ is surjective, $T_x g$ is surjective. Therefore, 0 is a regular value of g. Similarly, 0 is a regular value of ∂g. ■

Example 8.19. If $\gamma \subset S^2$ is a submanifold, then $\eta^{-1}(\gamma) \subset S^3$ is a submanifold. These are all diffeomorphic to T^2. ♠

Remark 8.20. If $\dim X < \operatorname{codim} Z$, then f is transverse to Z if and only if $\operatorname{im} f \cap Z = \varnothing$. ♣

Theorem 8.21. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let $\mathscr{P}, X, Y$ be C^k manifolds without boundary. Let $Z \subset Y$ be a C^k submanifold. Let $F \colon \mathscr{P} \times X \to Y$ be a C^k map. If is transverse to Z, then for almost every $p \in \mathscr{P}$ the map $f_p \coloneqq F(p, \cdot) \colon X \to Y$ is transverse to Z.*

Lemma 8.22. *If*

$$\begin{tikzcd} & A' \arrow[d, "i'"'] \arrow[dr, "f'"] & \\ A \arrow[r, "i"] \arrow[dr, "f"'] & B = B' \arrow[r, two heads, "j"'] \arrow[d, "j'"] & C \\ & C' & \end{tikzcd} \tag{8.23}$$

is a commutative diagram with j and f surjective, $\operatorname{im} i \subset \ker j$ *and* $\ker j' \subset \operatorname{im} i'$, *then f' is surjective.*

Proof. Let $c \in C$. Since j is surjective, there is a $b \in B$ with $j(b) = c$. Since f' is surjective, there is an $a \in A$ with $g(a) = j'(b)$. Since $\operatorname{im} i \subset \ker j$, $b' \coloneqq b - i(a)$ satisfies $j(b') = c$ and $j'(b) = 0$. Since $\ker j' \subset \operatorname{im} i'$, there is an $a' \in A$ with $i'(a') = b'$. Therefore, $f(a') = j(i'(a')) = j(b') = c$. ■

Exercise 8.24. Prove Lemma 8.22 using the Snake Lemma.

Proof of Theorem 8.21. By Theorem 8.12, $F^{-1}(Z)$ is a submanifold. Denote by

$$\pi \colon F^{-1}(Z) \to \mathscr{P} \quad \text{and} \quad \iota \colon F^{-1}(Z) \to \mathscr{P} \times X \tag{8.25}$$

the restriction of the canonical projection onto and the inclusion map respectively. Lemma 8.22 applied to

$$
\begin{tikzcd}
& T_xX \arrow[d, hook] \arrow[dr, "T_x f_p"] & \\
T_{(p,x)}F^{-1}(Z) \arrow[r, hook, "T_{(p,x)}\iota"] \arrow[dr, "T_{(p,x)}\pi"'] & T_p\mathscr{P} \oplus T_xX \arrow[r, two heads, "T_{(p,x)}F"'] \arrow[d, two heads] & T_yY/T_yZ \\
& T_p\mathscr{P} &
\end{tikzcd}
\tag{8.26}
$$

with $y := F(p,x)$ implies that p is a regular value of π if and only of f_p intersects Z transversely. By Theorem 7.12, almost every $p \in \mathscr{P}$ is a regular value of π. ■

Here is a typical application of Theorem 8.21.

Proposition 8.27. *Let $k \in (2 + \mathbf{N}_0) \cup \{\infty, \omega\}$. Let $m, n \in \mathbf{N}_0$. Let $U \subset \mathbf{R}^m$ be an open subset. Let $f\colon U \to \mathbf{R}^n$ be C^k.*

1. *If $2m \leqslant n + 1$, then for almost every $A \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n)$ the map $f + A\colon U \to \mathbf{R}^n$ is an injective immersion.*
2. *If $2m \leqslant n$, then for almost every $A \in \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n)$ the map $f + A\colon U \to \mathbf{R}^n$ is an immersion.*

Proof. The map $f + A$ is an immersion if and only 0 is not contained in the image of the map $G_A\colon U \times S^{m-1} \to \mathbf{R}^n$ defined by

$$G_A(x, v) := \mathrm{d}_x f(v) + Av. \tag{8.28}$$

If $2m - 1 < n$, then the latter is equivalent to 0 being a regular value of G_A. The map $G\colon \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n) \times U \times S^{m-1} \to \mathbf{R}^m$ defined by

$$G(A; x, v) := G_A(x, v) \tag{8.29}$$

is a submersion. Therefore, the second assertion follows from Theorem 8.21.

The map $f + A$ is an injective if and only 0 is not contained in the image of the map $H_A\colon (U \times U)\backslash\Delta \to \mathbf{R}^m$ defined by

$$H_A(x, y) := f(x) + Ax - (f(y) - Ay). \tag{8.30}$$

If $2m < n$, then the latter is equivalent to 0 being a regular value of H_A. The map $H\colon \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n) \times (U \times U)\backslash\Delta \to \mathbf{R}^m$ defined by

$$H(A; x, v) := H_A(x, v) \tag{8.31}$$

is a submersion. Therefore, the first assertion follows from Theorem 8.21. ■

Locally, this shows that any map can be wiggled into an (injective) immersion provided the dimension allow for it. This is also true globally, but requires an additional tool (partitions of unity) that we do not have yet.

Theorem 8.32. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X, Y be C^k manifolds without boundary. Let $f\colon X \to Y$ be a C^k map. If X is compact, then there is an $n \in \mathbf{N}$ and a C^k map $F\colon B_1^n(0) \times X \to Y$ which is a submersion and satisfies $F(0,\cdot) = f$. In particular, F is transverse to every submanifold $Z \subset Y$.*

Proof sketch. The basic idea is that if $Y = \mathbf{R}^n$, then then map $F(y,x) := f(x) + y$ has the desired property. Using cut-off functions one can develop a full proof based on this idea, but it is best to postpone the full proof for a few weeks until we have some more technology. ■

Theorem 8.21 and Theorem 8.32 combined imply that $f\colon X \to Y$ can be made transverse to Z by slightly wiggling it.

Theorem 8.33. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X, Y be C^k manifolds without boundary. Suppose that X is compact. Let $Z \subset Y$ be a submanifold. Let $F\colon B_1(0) \times X \to Y$ be a C^k map. If $f = F(0,\cdot)$ is transverse to Z, then there is an $\varepsilon > 0$ such that for every $t \in B_\varepsilon(0)$ the map $F(t,\cdot)$ is transverse to Z.*

Proof. Let $x \in X$ and set $y := F(0,x)$. Choose an admissible chart $\psi\colon V \to B_1(0)$ of Y with $\psi(y) = 0$ such that $\psi(Z \cap V) = (\mathbf{R}^m \times \{0\}) \cap \tilde{V}$. Choose an admissible chart $\psi\colon U \to B_1(0)$ of X with $\phi(x) = 0$ and $r \in (0,1]$ such $F(B_r(0) \times U) \subset V$. Decompose

$$\tilde{F} = (\tilde{F}_1, \tilde{F}_2). \tag{8.34}$$

Since $F(0,\cdot)$ is transverse to Z, for every $x \in U$ the map $\mathrm{d}_{0,x}\tilde{F}_2$ is surjective. By continuity, there are $\varepsilon, \rho > 0$ such that $\mathrm{d}_{t,x}\tilde{F}_2$ is surjective for every $t \in B_\varepsilon(0)$ and $x \in B_\rho(0)$. By compactness this finishes the proof. ■

Theorem 8.33 says that transversality to Z cannot be spoiled by slightly wiggling f.

Remark 8.35. It might irk you that X needs to be assumed to be compact. (I certainly find it terribly annoying.) This hypothesis cannot be removed, but there is a better way to do this using the Whitney's strong C^k topology; see Golubitsky and Guillemin [GG80]. ♣

Definition 8.36. Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X be a C^k manifolds with boundary. Let $Y, Z \subset X$ be a C^k submanifolds without boundary. Y and Z **intersect transversely** if for every $x \in Y \cap Z$

$$T_x Y + T_x Z = T_x X. \tag{8.37}$$ •

Notation 8.38. In the situation of Definition 8.36 if Y and Z intersect transversely, then this is denoted by

$$Y \pitchfork Z \tag{8.39}$$ ◦

Proposition 8.40. *Assume the situation of Definition 8.36. Suppose that $\partial X = \partial Y = \partial Z = \varnothing$. If Y and Z intersect transversely, then $Y \cap Z \subset X$ is a submanifold without boundary and*

$$\operatorname{codim}(Y \cap Z) = \operatorname{codim} Y + \operatorname{codim} Z. \tag{8.41}$$

Proof. The inclusion $\iota \colon Y \to X$ is tranverse to Z. Therefore, $Y \cap Z = \iota^{-1}(Z)$ is a submanifold without boundary of Y; hence, a submanifold without boundary of X. ■

Example 8.42. For every $t \in \mathbf{R}$ the cylinder

$$C_t \coloneqq \{(x_1, x_2, x_3) \in \mathbf{R}^3 : (x_1 - t)^2 + x_2^2 = 1/2\} \tag{8.43}$$

is a submanifold of $\mathbf{R}^3$. C_t and S^2 intersect transversely provided $t \notin \{-1/2, 1/2\}$; moreover:

1. If $t \in (-1/2, 1/2)$, then $C_t \cap S^2$ is diffeomorphic to $S^1 \amalg S^1$.
2. If $t \in (-\infty, -1/2) \cup (1/2, \infty)$, then $C_t \cap S^2$ is diffeomorphic to S^1.
3. If $t \in \{-1/2, 1/2\}$, then $C_t \cap S^2$ is not a submanifold; but there is an immersion $\iota \colon \mathbf{R} \to \mathbf{R}^3$ with $\operatorname{im} \iota = C_t \cap S^2$. ♠

Exercise 8.44. Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X, Y be C^k manifolds. Let $f \colon X \to Y$ be C^k. The **graph of** f is the subset

$$\operatorname{graph} f \coloneqq \{(x, f(x)) : x \in X\}. \tag{8.45}$$

Let $Z \subset Y$ be a submanifold. Prove that f is transverse to Z if and only if $\operatorname{graph} f$ is transverse to $X \times Z$.

∼

Definition 8.46. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X, Y, Z be C^k manifolds with boundary. Let $f\colon X \to Z$ and $g\colon Y \to Z$ be C^k. The **fibred product of X and Y over Z with respect to f and g** is the subset

$$X \times_Z Y := \{(x,y) \in X \times Y : f(x) = g(y)\}. \tag{8.47}$$

•

Definition 8.48. Assume the situation of Definition 8.46. The maps f and g are **transverse** if for every $(x,y) \in X \times_Z Y$

$$\operatorname{im} T_x f + \operatorname{im} T_y g = T_z Z \quad \text{with} \quad z := f(x) = g(y). \tag{8.49}$$

•

Notation 8.50. In the situation of Definition 8.48, if f and g are transverse, then this is denoted by

$$f \pitchfork g \tag{8.51}$$

∘

Proposition 8.52. *Assume the situation of Definition 8.46. Suppose that $\partial X = \partial Y = \partial Z = \varnothing$. If f and g are transverse, then $X \times_Z Y$ is a submanifold.*

Proof. Denote the diagonal in $Z \times Z$ by Δ. Since

$$X \times_Z Y = (f \times g)^{-1}(\Delta), \tag{8.53}$$

it suffices to prove that $f \times g$ is transverse to Δ; that is: for every $(x,y) \in X \times_Z Y$

$$\operatorname{im} T_x f \oplus \operatorname{im} T_x g + T_z\Delta = T_z Z \oplus T_z Z \quad \text{with} \quad z := f(x) = g(y). \tag{8.54}$$

Since f and g are transverse, every $w_i \in T_x Z$ can be written as

$$w_i = T_x f(u_i) + T_y g(v_i) \tag{8.55}$$

$(i = 1, 2)$. Therefore,

$$\begin{aligned} w_1 &= T_x f(u_1 - u_2) + T_x f(u_2) + T_y g(v_1) \quad \text{and} \\ w_2 &= T_y g(v_2 - v_1) + T_x f(u_2) + T_y g(v_1). \end{aligned} \tag{8.56}$$

This proves that $f \times g$ is transverse to Δ. ■

Remark 8.57. In the above situation, $X \times_Z Y$ satisfies a universal property analogous to the one of $X \times Y$ and illustrated in the following diagram:

$$\begin{tikzcd} W \arrow[dr, "h"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\ & X \times_Z Y \arrow[r, "\mathrm{pr}_Y"] \arrow[d, "\mathrm{pr}_X"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z. \end{tikzcd} \tag{8.58}$$

♣

~

Here is a short digression:

Definition 8.59. Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X, Y be a C^k manifolds without boundary. A property (♣) of a C^k maps $f\colon X \to Y$ is **stable** if for every C^k map $F\colon B_1(0) \times X \to Y$ with $F(0, \cdot)$ satisfying (♣) there is an $\varepsilon > 0$ such that for every $t \in (-\varepsilon, \varepsilon)$ the map $F(\varepsilon, \cdot)$ satisfies (♣) as well. •

Informally speaking: (♣) is stable if it cannot be spoiled by slightly wiggling. (Again it would be better to introduce the strong C^k topology and prove that these properties are open in that topology.)

Theorem 8.60. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X, Y be C^k manifolds with boundary. Let $Z \subset Y$ be a C^k submanifold. If X is compact, then the following properties for C^k maps $f\colon X \to Y$ are stable:*

1. $\mathrm{rk}\, T_x f \geqslant r$ *for every* $x \in X$,
2. *being an immersion,*
3. *being a submersion, and*
4. *being a (local) diffeomorphism.*

Proof. It suffices to prove (1). To prove (1) it suffices to prove it for $X = \bar{B}_1(0)$ and $Y = V \subset \mathbf{R}^n$ open. Let $F\colon B_1(0) \times \bar{B}_1(0) \to V$ be C^k. The map $\frac{\partial F}{\partial x}\colon B_1(0) \times \bar{B}_1(0) \to \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^n)$ is continuous. The set of matrices $A \in \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^n)$ with rank at least r is open. This implies (1).

(1) implies that that being an immersion/submersion/local diffeomorphism are stable. ■

Remark 8.61. Being an injective immersion is also stable if X is compact. So is being a surjective submersion if X is compact and $\partial X = \emptyset$. ♣

9 Partitions of Unity

Definition 9.1. Let $k \in \mathbf{N}_0 \cup \{\infty\}$. Let X be a C^k manifold with boundary. Let $\mathscr{V} = \{V_i : i \in I\}$ be an open cover of X. A **partition of unity subordinate to** $\mathscr{V}$ is a set of non-negative C^k functions $\{\rho_i : i \in I\}$ such that:

1. For every $i \in I$, $\mathrm{supp}\, \rho_i \subset V_i$.
2. For every $x \in X$ there is an open neighborhood U with $U \cap \mathrm{supp}\, \rho_i \neq \emptyset$ for at most finitely many $i \in I$.

3. For every $x \in X$

$$\sum_{i \in I} \rho_i(x) = 1. \tag{9.2}$$ •

Theorem 9.3. *Let $k \in \mathbf{N}_0 \cup \{\infty\}$. Let X be a C^k manifold with boundary. For every open cover $\mathscr{V} = \{V_i : i \in I\}$ there is a C^k partition of unity subordinate to $\mathscr{V}$.*

Remark 9.4. Theorem 9.3 is false for $k = \omega$. ♣

The proof of Theorem 9.3 relies on the following.

Lemma 9.5. *Let $k \in \mathbf{N}_0 \cup \{\infty\}$. Let X be a C^k manifold with boundary. For every open cover $\mathscr{V}$ of X there is an atlas $\mathscr{A}_{\mathscr{V}} = \{\phi_\alpha \colon U_\alpha \to \mathbf{M}_\alpha \cap B_2(0) : \alpha \in A_{\mathscr{V}}\} \subset \mathscr{A}$ such that $\{U_\alpha : \alpha \in A_{\mathscr{V}}\}$ is a locally finite refinement of $\mathscr{V}$ and $\{\phi_\alpha^{-1}(\mathbf{M}_\alpha \cap B_1(0)) : \alpha \in A_{\mathscr{V}}\}$ is an open cover of X.*

Proof of Theorem 9.3 assuming Lemma 9.5. Let $\mathscr{A}_{\mathscr{V}}$ be as in Lemma 9.5. Choose a smooth function $\beta \colon [0, 2] \to [0, 1]$ with

$$\beta|_{[0,1]} = 1 \quad \text{and} \quad \operatorname{supp} \beta \subset [0, 2). \tag{9.6}$$

Choose a map $j \colon A_{\mathscr{V}} \to I$ such that for every $\alpha \in A_{\mathscr{V}}$

$$U_\alpha \subset V_{j(\alpha)}. \tag{9.7}$$

For every $i \in I$ define $\tilde{\rho}_i \in C^k(X)$ by

$$\tilde{\rho}_i(x) \coloneqq \sum_{\alpha \in j^{-1}(i)} \beta \circ |\phi_\alpha|(x) \tag{9.8}$$

for $x \in V_i$ and $\tilde{\rho}_i(x) \coloneqq 0$ otherwise; furthermore set

$$\rho_i \coloneqq \frac{\tilde{\rho}_i}{\sum_{i \in I} \tilde{\rho}_i}. \tag{9.9}$$

By construction, $\{\rho_i : i \in I\}$ is the desired partition of unity. ■

The proof of Lemma 9.5 is straight-forward if X is compact. The general case requires Proposition 7.19.

Proof of Lemma 9.5. Without loss of generality, X is connected. Since X is locally path-connected, it is path-connected. Moreover, X is locally compact. Therefore, Proposition 7.19 applies.

Let $(K_n)_{n\in\mathbf{N}}$ be an exhaustion by compact sets of X . Set $K_0 = K_{-1} \coloneqq \varnothing$. Observe that

$$X = \bigcup_{n\in\mathbf{N}} K_n = \bigcup_{n\in\mathbf{N}} K_n \backslash K_{n-1}^\circ, \tag{9.10}$$

For every $n \in \mathbf{N}$ denote by $A_n \subset A$ the subset of those $\alpha \in A$ with $\tilde{U}_\alpha = B_2(0)$ and such that there is a $U \in \mathscr{U}$ with

$$U_\alpha \subset U \cap (K_{n+1} \backslash K_{n-2}^\circ). \tag{9.11}$$

A moment's thought shows that

$$K_n \backslash K_{n-1}^\circ \subset \bigcup_{\alpha\in A_n} \phi_\alpha^{-1}(B_1(0)). \tag{9.12}$$

Since $K_n \backslash K_{n-1}^\circ$ is compact, the same holds for a finite subset $A_n' \subset A_n$. Set

$$A_{\mathscr{V}} \coloneqq \bigcup_{n\in\mathbf{N}} A_n' \quad \text{and} \quad \mathscr{A}_{\mathscr{V}} \coloneqq \{\phi_\alpha : \alpha \in A_{\mathscr{V}}\}. \tag{9.13}$$

Every $x \in X$ is contained in an open subset of the form $K_{n+1}^\circ \backslash K_{n-2}$ for some $n \in \mathbf{N}$. There are only finitely many $\alpha \in A_{\mathscr{V}}$ with $U_\alpha \cap (K_{n+1}^\circ \backslash K_{n-2}) \neq \varnothing$. Therefore, $\mathscr{A}_{\mathscr{V}} \subset \mathscr{A}$ is the desired atlas. ■

Corollary 9.14. *Let $k \in \mathbf{N}_0 \cup \{\infty\}$. Let X be a C^k manifold. If $U, V \subset X$ be open subsets with $\overline{U} \subset V$, then there is a C^k function $\chi\colon X \to [0,1]$ which is equal to one on U and supported in V.* ■

Lemma 9.15. *Let $k \in \mathbf{N}_0 \cup \{\infty\}$. If X is a σ−compact C^k manifold, then there is a proper C^k function $r\colon X \to [0,\infty)$.*

Proof. Let $(K_n)_{n\in\mathbf{N}}$ be an exhaustion by compact subsets of X. Choose a partition of unity $\{\rho_n : n \in \mathbf{N}\}$ subordinate to $\{K_n^\circ : n \in \mathbf{N}\}$. Define $r\colon X \to [0,\infty)$ by

$$r \coloneqq \sum_{n\in\mathbf{N}} n \cdot \rho_n. \tag{9.16}$$

To see that r is proper, observe the following. If $x \notin K_m^\circ$, then

$$r(x) \geqslant \sum_{n=m}^{\infty} n \cdot \rho_n(x) \geqslant m \cdot \sum_{n=m}^{\infty} \rho_n(x) = m. \tag{9.17}$$

Therefore,

$$r^{-1}([a,b]) \subset K_{\lceil b \rceil}; \tag{9.18}$$

hence: $r^{-1}([a,b])$ is a closed subset of a compact subset and thus itself compact. ■

The following demonstrates the flexibility of smooth functions.

Proposition 9.19. *Let $k \in \mathbf{N}_0 \cup \{\infty\}$. Let X be a C^k manifold. For every closed subset $A \subset X$ there is a C^k function $f\colon X \to \mathbf{R}$ with $f^{-1}(0) = A$.*

Proof. Let $U \subset \mathbf{R}^m$ be an open subset and let $A \subset U$ be closed. There is a sequence $(B_{r_n}(x_n))_{n\in\mathbf{N}}$ such that

$$U\backslash A = \bigcup_{n\in\mathbf{N}} B_{r_n}(x_n). \tag{9.20}$$

For every $n \in \mathbf{N}$ choose a smooth function $f_n\colon U \to [0,\infty)$ which vanishes outside of $B_{r_x}(x)$ and is positive on $B_{r_x}(x)$. Set

$$c_n \coloneqq \|f_n\|_{C^k} = \sup_{x\in U}\sup_{|\alpha|\leqslant n} |\partial^\alpha f_n(x)|. \tag{9.21}$$

Define $f\colon U \to [0,\infty)$ by

$$f(x) \coloneqq \sum_{n\in\mathbf{N}} \frac{1}{2^n c_n} f_n(x). \tag{9.22}$$

A moment's thought shows that f is smooth. By construction, $f^{-1}(0) = A$.

Choose an open cover $\mathscr{U} = \{U_\alpha : \alpha \in A\}$ of X consisting of the domains of admissible charts. By the above, for every $\alpha \in A$ there is a C^k function $f_\alpha\colon U_\alpha \to [0,\infty)$ with $f_\alpha^{-1}(0) = U_\alpha \cap C$. Choose a partition of unity $\{\rho_\alpha : \alpha \in A\}$ subordinate to $\mathscr{U}$. The function $f\colon X \to [0,\infty)$ defined by

$$f \coloneqq \sum_{\alpha\in A} \rho_\alpha \cdot f_\alpha \tag{9.23}$$

is C^k and satisfies $f^{-1}(0) = A$. ∎

Proposition 9.24. *Let $k \in \mathbf{N}_0 \cup \{\infty\}$. Let X be a C^k manifold with boundary. Let $Z \subset X$ be a submanifold which is closed as a subset. For every $f \in C^k(Z)$ there is a $F \in C^k(X)$ satisfying*

$$F|_Z = f. \tag{9.25}$$

Proof. Choose a cover $\mathscr{U} = \{U_0\} \cup \{U_i : i \in I\}$ of X with $U_0 \coloneqq X\backslash Z$ and such that for every $i \in$ there is an admissible chart $\phi_i\colon U_i \to \tilde{U}_i$ of X as in Definition 6.1. For every $i \in I$ the C^k function $F_i\colon U_i \to \mathbf{R}$ defined by

$$F_i \coloneqq f \circ \phi \circ \mathrm{pr}_1 \circ \phi^{-1} \tag{9.26}$$

with $\mathrm{pr}_1\colon\ \mathbf{R}^m = \mathbf{R}^n \times \mathbf{R}^{m-n} \to \mathbf{R}^n$ denoting the canonical projection satisfies

$$F_i|_Z = f. \tag{9.27}$$

Choose a C^k partition of unity $\{\rho_i : i \in \{0\} \cup I\}$ subordinate to $\mathscr{U}$. Since $U_0 = X\backslash Z$,

$$\sum_{i\in I} \rho_i|_Z = 1. \tag{9.28}$$

Therefore, the C^k function $F\colon\ X \to \mathbf{R}$ defined by

$$F \coloneqq \sum_{i\in I} \rho_i \cdot F_i \tag{9.29}$$

satisfies

$$F|_Z \coloneqq \sum_{i\in I} \rho_i \cdot F_i|_Z = \sum_{i\in I} \rho_i \cdot f = f. \tag{9.30}$$ ■

10 The Whitney embedding theorem

Theorem 10.1 (Whitney [Whi36, Theorems 1 and 3]; weak Whitney Embedding/Immersion Theorem). *Let $k \in \mathbf{N} \cup \{\infty\}$ and $m \in \mathbf{N}$. For every σ–compact C^k manifold X of dimension m there are a proper C^k embedding $\iota\colon\ X \hookrightarrow \mathbf{R}^{2m+1}$ and a proper C^k immersion $\jmath\colon\ X \looparrowright \mathbf{R}^{2m}$.*

Remark 10.2. The following is due to Arnold [Arn98]:

> *An "abstract" smooth manifold is a smooth submanifold of a Euclidean space considered up to a diffeomorphism. There are no "more abstract" finite-dimensional smooth manifolds in the world (Whitney's theorem). Why do we keep on tormenting students with the abstract definition?*

There a numerous responses to Arnold's question; some of which are discussed at `https://math.stackexchange.com/questions/26551/why-abstract-manifolds`. ♣

Remark 10.3. Whitney's embedding theorem says that there are proper $f_1, \ldots, f_{2m+1} \in C^k(X)$ which separate the points of X as well as its tangent vectors; more precisely:

1. For every $x \neq y \in X$ there is an $i \in \{1, \ldots, 2m+1\}$ such that

$$f_i(x) \neq f_i(y). \tag{10.4}$$

2. For every $0 \neq v \in T_xX$ there is an $i \in \{1, \ldots, 2m+1\}$ such that

$$\mathrm{d}_x f_i(v) \neq 0. \tag{10.5}$$ ♣

Proposition 10.6. *Let $k \in \mathbf{N} \cup \{\infty\}$. Let X be a C^k manifold. If X admits a finite atlas, then there is a $M \in \mathbf{N}$ and an injective immersion $\iota\colon X \hookrightarrow \mathbf{R}^M$.*

Proof. Denote by $\{\phi_\alpha\colon U_\alpha \to \tilde{U}_\alpha : \alpha \in A\}$ a finite atlas of X. Choose a C^k partition of unity $\{\rho_\alpha : \alpha \in A\}$ subordinate to $\{U_\alpha : \alpha \in A\}$. Define $\iota\colon X \to \prod_{\alpha\in A} \mathbf{R}^{m_\alpha} \times \mathbf{R}$ by

$$\iota(x) := \big(\rho_\alpha \cdot \phi_\alpha(x), \rho_\alpha(x)\big)_{\alpha\in A}. \tag{10.7}$$

The map ι is injective. Indeed, if $x, y \in X$ satisfy $\iota(x) = \iota(y)$, then exists an $\alpha \in A$ such that $\rho_\alpha(x) = \rho_\alpha(y) \neq 0$. Therefore, $x, y \in U_\alpha$. Since $\phi_\alpha(x) = \phi_\alpha(y)$, $x = y$.

The map ι is an immersion. Indeed,

$$T_x\iota = (\mathrm{d}_x\rho_\alpha \cdot \phi_\alpha(x) + \rho_\alpha(x) \cdot T_x\phi_\alpha(v), \mathrm{d}_x\rho_\alpha)_{\alpha\in A} \tag{10.8}$$

is injective. ■

Proposition 10.9. *Let $k \in \mathbf{N} \cup \{\infty\}$ and $m, M \in \mathbf{N}$. Define $\Pi_{\cdot}\colon\ \mathbf{R}P^{M-1} \to \mathrm{End}(\mathbf{R}^M)$ by*

$$\Pi_{[v]}(x) \coloneqq x - \frac{\langle v, x\rangle v}{|v|^2}. \tag{10.10}$$

Let X be a σ–compact C^k manifold of dimension m. Let $\iota\colon\ X \hookrightarrow \mathbf{R}^M$ be an injective immersion. The following hold:

1. *If $M > 2m + 1$, then for almost every $[v] \in \mathbf{R}P^{M-1}$ the map*

   $$\Pi_{[v]} \circ \iota\colon\ X \to \langle v\rangle^{\perp} \tag{10.11}$$

   is injective.

2. *If $M > 2m$, then for almost every $[v] \in \mathbf{R}P^{M-1}$ the map*

   $$\Pi_{[v]} \circ \iota\colon\ X \to \langle v\rangle^{\perp} \tag{10.12}$$

   is an immersion.

Proof. Since X can be covered by countably many charts, it suffices to prove this result for X an open subset $U \subset \mathbf{R}^m$.

If the composition $\Pi_{[v]} \circ \iota$ fails to be injective, then there are $x \neq y \in U$ with $\iota(x) - \iota(y) \in \langle v\rangle$. The latter is equivalent to $[v]$ being contained in the image of the map $L\colon\ (U \times U)\backslash\Delta \to \mathbf{R}P^M$ defined by

$$L(x, y) \coloneqq \mathbf{R} \cdot (\iota(x) - \iota(y)). \tag{10.13}$$

By Theorem 7.12, $\operatorname{im} L$ has measure zero provided

$$\dim(U \times U)\backslash\Delta = 2m < M - 1 = \dim \mathbf{R}P^{M-1}, . \tag{10.14}$$

This proves (1).

If $\Pi_{[v]} \circ \iota$ fails to be an immersion, then there are an $x \in U$ and a $[w] \in \mathbf{R}P^{m-1}$ such that

$$[\mathrm{d}_x\iota(w)] = [v]. \tag{10.15}$$

The latter is equivalent to $[v]$ being contained in the image of the map $\Lambda\colon\ U \times \mathbf{R}P^{m-1} \to \mathbf{R}P^{M-1}$ defined by

$$\Lambda(x, [w]) \coloneqq [\mathrm{d}_x\iota(w)] \tag{10.16}$$

By Theorem 7.12, $\operatorname{im}\Lambda$ has measure zero provided

$$\dim U \times \mathbf{R}P^{m-1} = 2m - 1 < M - 1 = \dim \mathbf{R}P^{M-1}. \tag{10.17}$$

This proves (2). ■

Proposition 10.18. *Let $k \in \mathbf{N} \subset \{\infty\}$ and $m \in \mathbf{N}$. Let X be a σ−compact C^k manifold of dimension m.*

1. *If $\iota\colon X \looparrowright \mathbf{R}^{2m+1}$ is an injective immersion, then there is a proper embedding $\jmath\colon X \hookrightarrow \mathbf{R}^{2m+1}$.*
2. *If $\iota\colon X \looparrowright \mathbf{R}^{2m}$ is an immersion, then there is a proper immersion $\jmath\colon X \hookrightarrow \mathbf{R}^{2m}$.*

Proof. Let $\iota\colon X \looparrowright \mathbf{R}^{2m+1}$ be an injective immersion. After composing ι with a smooth diffemorphism $\mathbf{R}^{2m+1} \to B_1(0)$, it can be assumed that $\operatorname{im}\iota \subset B_1(0)$. By Lemma 9.15, there is a proper C^k function $r\colon X \to [0,\infty)$. The C^k map $\hat\iota\colon X \to B_1(0) \times \mathbf{R} \subset \mathbf{R}^{2m+2}$ defined by

$$\hat\iota(x) \coloneqq (\iota(x), r(x)) \tag{10.19}$$

is an injective immersion. By Proposition 10.9, there is a $[v] \in \mathbf{R}^{2m+1}$ not equal to $[(0,\dots,0,1)]$ such that

$$\jmath \coloneqq \Pi_{[v]} \circ \hat\iota \tag{10.20}$$

is an injective immersion.

By Corollary 6.65, it remains to prove that $\jmath$ is proper. Choose a representative $v = (v_1,\dots,v_{2m+2}) \in S^{2m+1}$ of $[v] \in \mathbf{R}P^{2m+1}$. By a direct computation, the $(2m+2)$−nd component of $\jmath$ satisfies

$$\jmath_{2m+2}(x) = (1 - v_{2m+2}^2)\cdot r(x) - \langle \iota(x), (v_1,\dots,v_{2m+1})\rangle \cdot v_{2m+2}; \tag{10.21}$$

hence:

$$|r(x)| \leqslant c(|\jmath_{2m+2}(x)| + 1) \quad \text{with} \quad c \coloneqq \frac{1}{1 - v_{2m+2}^2}. \tag{10.22}$$

This implies that $\jmath$ is proper. Indeed, if $K \subset \langle v\rangle^\perp$ is compact, then there is an $R > 0$ such that $K \subset \bar B_R(0)$. Therefore,

$$\jmath^{-1}(K) \subset r^{-1}(\bar B_{cR+1}(0)). \tag{10.23}$$

Since r is proper, $r^{-1}(\bar B_{cR+1}(0))$ is compact. Hence, the closed subset $\jmath^{-1}(K)$ is compact. This proves (1).

The proof of (2) is similar. ■

Proof of Theorem 10.1. By Lemma 9.15, there is a proper C^k function $r\colon X \to [0,\infty)$. For every $n \in \mathbf{N}$ choose an open subset U_n satisfying

$$r^{-1}[n-1,n] \subset U_n \subset r^{-1}(n - \tfrac{3}{2}, n + \tfrac{1}{2}) \tag{10.24}$$

and covered by finitely many charts. By construction,

$$U_{n_1} \cap U_{n_2} = \varnothing \quad \text{unless} \quad |n_1 - n_2| \leqslant 1. \tag{10.25}$$

By Proposition 10.6 and Proposition 10.9, for every $n \in \mathbf{N}$ there is an injective immersion $\iota_n \colon U_n \hookrightarrow \mathbf{R}^{2m+1}$. For every $n \in \mathbf{N}$ choose a C^k function $\chi_n \colon X \to [0,1]$ which is equal to one on $r^{-1}[n-1, nP]$ and supported in U_n. Define $\iota \colon X \to \mathbf{R}^{2m+1} \times \mathbf{R}^{2m+1}$ by

$$\iota(x) := \left(\sum_{n \in 2\mathbf{N}-1} \chi_n(x) \cdot \iota_n(x), \sum_{n \in 2\mathbf{N}} \chi_n(x) \cdot \iota_n(x) \right).$$

Each of the sums has precisely one non-zero term because of (10.25).

The map ι is injective. Indeed, if $x, y \in X$ with $\iota(x) = \iota(y)$, then there exists an $n \in \ N$ such that $x, y \in r^{-1}[n-1, n] \subset U_n$. Since ι_n is injective, $x = y$. The map ι is an immersion. Indeed, if $x \in r^{-1}[n-1, n]$, then one of the components of $T_x\iota$ is $T_x\iota_n$, which is injective by construction.

The proof is finished by appealing to Proposition 10.9 and Proposition 10.18. ■

Remark 10.26. *What is the smallest $e(m) \in \mathbf{N}$ such that every σ–compact manifold embeds into $\mathbf{R}^{e(m)}$?* This is an open question, but a lot is known about it.

1. Theorem 10.1 asserts that $e(m) \leqslant 2m + 1$. Whitney [Whi44, Theorem 3] proved that, in fact, $e(m) \leqslant 2m$. The idea of the proof is as follows. The injective immersion $\jmath \colon X \looparrowright \mathbf{R}^{2m}$ can be perturbed slightly to have transverse self-intersections. The number of self-intersections can be assumed to be even because its is possible to introduce an additional transverse self-intersection point. The self-intersections can then be paired up and canceled. This is called the **Whitney trick**. It is easy to visualize for $m = 1$ and explained in Wikipedia. Wall [Wal16, Section 6.3] discusses the Whitney trick in detail.

2. $\mathbf{R}P^{2^n}$ cannot be embedded into $\mathbf{R}^{2^{n+1}-1}$. Therefore, Whitney's bound is sharp for $m = 2^n$. If m is not of this form, then $e(m) \leqslant 2m - 1$.

3. The answer to the question with embedding replaced by immersion is know. For $m \in \mathbf{N}$ and denote by $\nu(m)$ the number of ones appearing in the binary expansion of m; that is: if $m = 2^{b_1} + \cdots + 2^{b_n}$, then $\nu(m) = n$. It is known that $\mathbf{R}P^{2^{b_1}} \times \cdots \times \mathbf{R}P^{2^{b_n}}$ cannot be immersed into $\mathbf{R}^{2m-\nu(n)-1}$. (This requires the theory of characteristic classes; cf. Milnor and Stasheff [MS74].) However, the following holds.

   Theorem 10.27 (Cohen [Coh85]; The Immersion Conjecture). *Let $m \in \mathbf{N}$. If X is a compact manifold of dimension m, then there is an immersion $\jmath \colon X \looparrowright \mathbf{R}^{2n-\nu(n)}$.*

   Therefore, the analogue of $e(m)$ for immersions is $2m - \nu(m)$. ♣

11 Tubular neighborhoods

Proposition 11.1. *For every submanifold $X \subset \mathbf{R}^N$ without boundary*

$$NX := \{(x, v) \in X \times \mathbf{R}^N : v \perp T_x X\} \tag{11.2}$$

is a submanifold of $\mathbf{R}^N \times \mathbf{R}^N$.

Proof. Let $\phi\colon\ U \to \tilde{U}$ be and admissible chart of $\mathbf{R}^N$ such that

$$\phi(Z \cap U) = (\mathbf{R}^m \times \{0\}) \cap \tilde{U}. \tag{11.3}$$

By construction, for every $x \in U$

$$T_x X = (\mathrm{d}_x \phi)^{-1}(\mathbf{R}^m \times \{0\}). \tag{11.4}$$

Therefore, $v \in T_x X^{\perp}$ of and only if for every $w \in \mathbf{R}^m \times \{0\} \subset \mathbf{R}^N$.

$$0 = \langle v, (\mathrm{d}_x \phi)^{-1} e_i \rangle = \langle e_i, \big((\mathrm{d}_x \phi)^{-1}\big)^* v \rangle \tag{11.5}$$

Set $V := U \times \mathbf{R}^N$, $\tilde{V} := \tilde{U} \times \mathbf{R}^N$, and define $\Phi\colon\ V \to \mathbf{R}^N \times \mathbf{R}^N$ by

$$\Phi(x, v) := \big(\phi(x), \big((\mathrm{d}_x \phi)^{-1}\big)^* v\big). \tag{11.6}$$

Evidently, Φ is a diffeomorphism. Moreover, by the preceding discussion,

$$\Phi(NX \cap V) = \big((\mathbf{R}^m \times \{0\}) \times (\{0\} \times \mathbf{R}^{N-m})\big) \cap \tilde{V}. \tag{11.7}$$

■

Definition 11.8. Let $X \subset \mathbf{R}^N$ be submanifold without boundary. Define $\exp\colon\ NX \to \mathbf{R}^N$ by

$$\exp(x, v) := x + v. \tag{11.9}$$

An open subset $U \subset \mathbf{R}^N$ is a **tubular neighborhood of** X if there is a continuous function $\varepsilon\colon\ X \to (0, \infty)$ such that the restriction of exp to

$$V := \{(x, v) \in NX : |v| < \varepsilon(x)\}. \tag{11.10}$$

is a diffeomorphism from V to U. •

Proposition 11.11. *Every submanifold $X \subset \mathbf{R}^N$ without boundary admits a tubular neighborhood.*

Proof. For every $x \in X$,

$$T_{x,0} \exp = \mathbf{1}. \tag{11.12}$$

Therefore, there is an open subset V of NX with $(X, 0) \subset V$ such that, for every $(x, v) \in V$, $T_{x,v} \exp$ is invertible. For every $x \in X$ choose $\delta = \delta(x) > 0$ such that the restriction of exp to $NX \cap (B_\delta(x) \times B_\delta(0))$ is a diffeomorphism onto its image. The function $\delta\colon X \to (0, \infty)$ can be chosen continuously.

Set

$$V := \left\{(x, v) \in NX : |v| < \tfrac{1}{2}\delta(x)\right\}. \tag{11.13}$$

The map $\exp\colon V \to U := \exp(V)$ is a local diffeomorphism. It remains to prove that it is injective. Suppose that $(x, v), (z, w) \in V$ satisfy

$$\exp(x, v) = \exp(z, w). \tag{11.14}$$

Without loss of generality, $\delta(z) \leqslant \delta(x)$ By the triangle inequality,

$$|x - z| = |v - w| < \delta(x).$$

By construction of δ, $(x, v) = (z, w)$. ■

Lemma 11.15. *Let $X \subset \mathbf{R}^m$ be a submanifold without boundary. There is a C^k submersion $r\colon U \to X$ with $r|_X = \mathrm{id}_X$.*

Proof. Let U be a tubular neighborhood of X and define $r\colon U \to X$ by

$$r := \mathrm{pr}_1 \circ \exp^{-1}. \tag{11.16}$$ ■

Proposition 11.17. *Let X and Y be smooth manifolds. Suppose that X is compact. Let $f\colon X \to Y$ be a smooth map. If $\dim X \leqslant 2\dim Y + 1$, then there exists an $N \in \mathbf{N}_0$, an open subset $\mathscr{P} \subset \mathbf{R}^N$, and a smooth map $F\colon \mathscr{P} \times X \to Y$ such that for almost every $p \in B_1^N(0)$ the map $f_p := F(p, \cdot)\colon X \to Y$ is an injective immersion.*

Proof. By Theorem 10.1, without loss of generality, X and Y are submanifolds of $\mathbf{R}^n$. Choose a neighborhood U of $Y \subset \mathbf{R}^n$ and a submersion $r\colon U \to Y$ with $r|_Y = \mathrm{id}_Y$.

define $\tilde{G}\colon \mathbf{R}^n \times \mathrm{End}(\mathbf{R}^n) \times X \to \mathbf{R}^n$ by

$$G(v, A; x) := f(x) + v + Ax. \tag{11.18}$$

Here $Q \in \mathrm{Hom}(S^2\mathbf{R}^n, \mathbf{R}^n)$ is considered as a quadratic map $\mathbf{R}^n \to \mathbf{R}^n$. Choose a non-empty open subset $\mathscr{P} \subset \mathbf{R}^n \times \mathrm{End}(\mathbf{R}^n)$ such that

$$\tilde{G}(\mathscr{P} \times X) \subset U. \tag{11.19}$$

Set

$$G := \tilde{G}|_{\mathscr{P}\times X} \quad \text{and} \quad F := r \circ G. \tag{11.20}$$

The map $\Psi\colon \mathscr{P} \times ((X \times X)\backslash\Delta_X) \to \mathbf{R}^n \times \mathbf{R}^n$ defined by

$$\Psi(v, A; x, y) \to (G(v, A; x), G(v, A; y)) \tag{11.21}$$

is a submersion. To see this, observe that

$$(\hat{v}, \hat{A}) \mapsto \mathrm{d}_{v,A;x,v}\Psi(\hat{v}, \hat{A}; 0, 0) = (\hat{v} + \hat{A}x, \hat{v} + \hat{A}y) \tag{11.22}$$

is surjective provided $x \neq y$. Therefore, the map $\Phi\colon \mathscr{P} \times ((X \times X)\backslash\Delta_X) \to Y \times Y$ defined by

$$\Phi(v, A; x, y) \to (F(v, A; x), F(v, A; y)) \tag{11.23}$$

is surjective. Consequently, for almost every $p \in \mathscr{P}$ the map $\Phi_p := \Phi(p, \cdot)$ is transverse to Δ_Y. Since $2\dim X + \dim \Delta_Y < 2\dim X$, this is equivalent to $f_p := F(p, \cdot)$ being injective.

It remains to prove that for almost every $p \in \mathscr{P}$ the map f_p is an immersion. To this end, it suffices to assume that X is an open subset $U \subset \mathbf{R}^d$ with $d := \dim X$. The map $K\colon \mathscr{P} \times X \times S^{d-1} \to \mathbf{R}^n \times \mathbf{R}^n$ defined by

$$K(p; x, \xi) := (g_p(x), \mathrm{d}_x g_p(\xi)) \tag{11.24}$$

with $g_p := G(p, \cdot)$ is a submersion. To see this, observe that

$$(\hat{v}, \hat{A}, 0) \mapsto \mathrm{d}_{(p;x,\xi)}K(\hat{v}, \hat{A}; 0, 0) = (\hat{v} + \hat{A}x, \hat{A}\xi) \tag{11.25}$$

is surjective. Observe that, $TY := \{(x, \xi) : X \times \mathbf{R}^n : v \in T_x y\}$ is a submanifold of dimension $2\dim Y$. Therefore, $\Lambda\colon \mathscr{P} \times X \times S^{d-1} \to TY$ defined by

$$\Lambda(p; x, \xi) := (f_p(x), \mathrm{d}_x f_p(\xi)) \tag{11.26}$$

is transverse to the zero section $Y \times \{0\} \subset TY$. Consequently, for almost every $p \in \mathscr{P}$ the map $\Lambda_p := \Lambda(p, \cdot)$ is transverse to $Y \times \{0\}$. Since $2\dim X + \dim \Delta_Y \leqslant 2\dim Y$, this is equivalent to $f_p := F(p, \cdot)$ being an immersion. ■

12 Cobordism

Definition 12.1. Let X be a smooth manifold with boundary. A **collar for** X is an embedding $\kappa\colon [0, \infty) \times \partial X \hookrightarrow X$ satisfying

$$\kappa(0, \cdot) = \mathrm{id}_{\partial X}. \tag{12.2}$$

•

Theorem 12.3 (Collaring Theorem). *Every smooth manifold with boundary admits a collar.*

Proof. Omitted since it is best to prove this using the theory of flows developed in Section 17. ∎

Remark 12.4. A variant of Theorem 12.3 for topological manifolds holds. This result is due to Brown [Bro62, §IV Theorem 2]; see also Connelly [Con71]. ♣

Proposition 12.5. *Let X_1 and X_2 be smooth manifolds with boundary. Let $\phi\colon \partial X_1 \to \partial X_2$ be a diffeomorphism.*

1. *The topological space*

$$X \coloneqq X_1 \cup_\phi X_2 \tag{12.6}$$

is Hausdorff and paracompact.

2. *Let κ_i be a collar of X_i. Define $\kappa\colon \mathbf{R} \times \partial X_1 \to X$ by*

$$\kappa(t,x) \coloneqq \begin{cases} \kappa_1(-t,x) & \text{if } t \leqslant 0, \\ \kappa_2(t,\phi(x)) & \text{if } t \geqslant 0. \end{cases} \tag{12.7}$$

Denote by $\iota_i\colon X_i^\circ \to X$ the canonical inclusions. There is a unique smooth structure on X with respect to which ι_i and κ are embeddings.

3. *Up to diffeomorphism, the smooth structure in (2) depends only on (the isotopy class of) ϕ.*

Proof. (1) is straight-forward.

Denote by $\mathscr{A}_i^\circ = \{\phi_\alpha : \alpha \in A_i^\circ\}$ the smooth structure of X_i° and by $\mathscr{A}_\partial = \{\phi_\alpha : \alpha \in A_\partial\}$ the smooth structure of $\mathbf{R} \times \partial X_1$. For $\alpha \in A_i^\circ$ and $\beta \in A_\partial$ set

$$\psi_\alpha \coloneqq \phi_\alpha \circ \iota_i^{-1} \quad \text{and} \quad \psi_\beta \coloneqq \phi_\beta \circ \kappa^{-1}, \tag{12.8}$$

and observe that

$$\psi_\alpha \circ \psi_\beta^{-1} = \begin{cases} \phi_\alpha \circ \iota_1^{-1} \circ \kappa_1 \circ (-\mathrm{id}_{(-1,0)} \times \mathrm{id}_{\partial X_1}) \circ \phi_\beta & \text{if } \alpha \in A_1^\circ, \\ \phi_\alpha \circ \iota_2^{-1} \circ \kappa_2 \circ (\mathrm{id}_{(0,1)} \times \phi) \circ \phi_\beta & \text{if } \alpha \in A_2^\circ \end{cases} \tag{12.9}$$

is a diffeomorphism. Therefore,

$$\mathscr{A} \coloneqq \{\psi_\alpha : \alpha \in A \coloneqq A_1^\circ \cup A_2^\circ \cup A_\partial\} \tag{12.10}$$

is a smooth atlas on X. By construction, ι_i and κ are embeddings with respect to $\bar{\mathscr{A}}$. Evidently, every smooth structure with respect to which ι_i and κ are embeddings must contain $\mathscr{A}$. This proves (2).

The proof of (3) is omitted. ∎

Figure 12.1: Gluing along the boundary.

Definition 12.11. In the situation of Proposition 12.5, X is said to be obtained by **gluing X_1 and X_2 along the boundary via ϕ**. (This is an abuse of notation.) •

~

Let X_i be smooth manifolds of dimension m. Set

$$D = D^m := \bar{B}_1^m(0). \tag{12.12}$$

Let $e_i\colon D \hookrightarrow X_i$ be embeddings. $X_i \backslash e_i(D^\circ)$ is a manifold with boundary diffeomorphic to S^{m-1}. Gluing these along the boundary gives the **connected sum**

$$X_1 \# X_2. \tag{12.13}$$

If X_1 and X_2 are oriented, then $X_1 \# X_2$ does depend on the choice of e_i. In this case one demands e_i to be orientation preserving.

Exercise 12.14. Prove that $X \# S^m \cong X$.

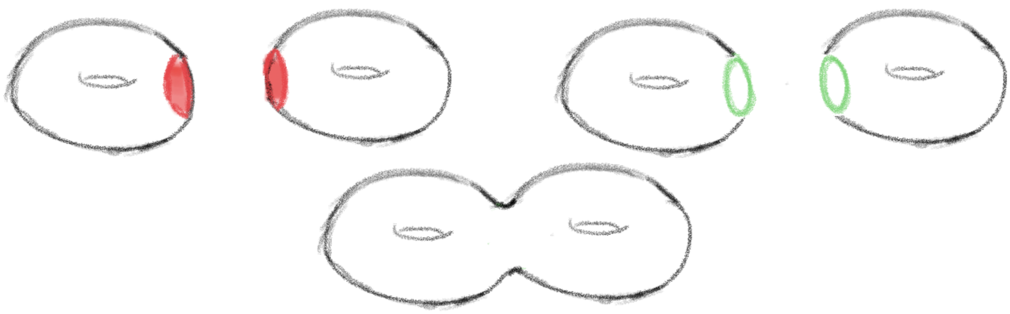

Figure 12.2: Connected sum $T^2 \# T^2$.

Theorem 12.15 (Classification of closed 2–manifolds). *Every closed smooth manifold of dimension two is either a connected sum tori or a connected sum of* $\mathbf{R}P^2$*s.*

Example 12.16. The Klein bottle is $\mathbf{R}P^2\#\mathbf{R}P^2\#\mathbf{R}P^2$ or, equivalently, $\mathbf{R}P^2\#T^2$. (However: $\mathbf{R}P^2\#\mathbf{R}P^2 \neq T^2$). ♠

Definition 12.17. Let $m \in \mathbf{N}_0$ Let X_0, X_1 be two closed smooth manifolds of dimension m. X_0 and X_1 are **cobordant** if there is a compact manifold W of dimension $m+1$ with boundary satisfying

$$\partial W \cong X_0 \amalg X_1. \tag{12.18}$$

Here $\cong$ signifies being diffeomorphic. W is said to be a **cobordism** between X_0 and X_1. •

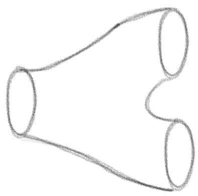

Figure 12.3: Pair of pants cobordism.

Proposition 12.19.

1. *Being cobordant defines an equivalence relation on the set of diffeomorphism classes closed smooth manifolds.*
2. *Let $m \in \mathbf{N}_0$. Denote by Ω_m^{O} the set of cobordism classes of closed smooth manifolds of dimension m. Disjoint union*

   $$[X_1] + [X_2] := [X_1 \amalg X_2] \tag{12.20}$$

   defines the structure of an abelian group on Ω_m^{O} with

   $$0 = [\varnothing]. \tag{12.21}$$

3. *Set*

   $$\Omega_\bullet := \bigoplus_{m \in \mathbf{N}_0} \Omega_m^{\mathrm{O}}. \tag{12.22}$$

   Taking products

   $$[X_1] \cdot [X_2] := [X_1 \times X_2] \tag{12.23}$$

   defines the structure of an $\mathbf{N}_0$–graded unital commutative ring on $\Omega_\bullet^{\mathrm{O}}$ with

   $$1 = [\{*\}]. \tag{12.24}$$

Proof. Omitted/Exercise. ■

Remark 12.25. There is no *set* containing every closed smooth manifold of dimension m. However, by Theorem 10.1, diffeomorphism classes closed smooth manifolds of dimension m do form a set. ♣

Definition 12.26. $\Omega_\bullet^{\mathrm{O}}$ is the **unoriented cobordism ring**. •

Remark 12.27. There is also an *oriented* cobordism ring: $\Omega_\bullet^{\mathrm{SO}}$. ♣

Exercise 12.28. Prove that

$$\Omega_0^{\mathrm{O}} \cong \mathbf{Z}/2\mathbf{Z} \quad \text{and} \quad \Omega_1^{\mathrm{O}} = 0. \tag{12.29}$$

Exercise 12.30. Prove that $\Omega_\bullet^{\mathrm{O}}$ is 2−torsion; that is: for every $x \in \Omega_\bullet$

$$x + x = 0. \tag{12.31}$$

Theorem 12.32 (Thom [Tho54]). *$\Omega_\bullet^{\mathrm{O}}$ is a polynomial algebra over $\mathbf{Z}/2\mathbf{Z}$ with one generator $[X_m] \in \Omega_m^{\mathrm{O}}$ for every m not of the form $2^k - 1$. If m is even, then X_m can be taken to be $\mathbf{R}P^m$.*

Remark 12.33. $\Omega_\bullet^{\mathrm{SO}}$ is also understood. In particular, $\Omega_0^{\mathrm{SO}} = \mathbf{Z}[\{*\}]$, $\Omega_1^{\mathrm{SO}} = \Omega_2^{\mathrm{SO}} = \Omega_3^{\mathrm{SO}} = 0$, $\Omega_4^{\mathrm{SO}} = \mathbf{Z}[\mathbf{C}P^2]$. ♣

Proposition 12.34. *Let X and Y be smooth manifolds without boundary. Suppose that Y is connected Let $f_i\colon X \to Y$ be a proper smooth map and $y_i \in Y$ a regular value of f_i ($i = 0, 1$). If f_0 is properly homotopic to f_1, then*

$$[f_0^{-1}(y_0)] = [f_1^{-1}(y_1)] \in \Omega_\bullet^{\mathrm{O}}. \tag{12.35}$$

Proof. The proper homotopy F can be chosen smooth and transverse to a smooth path $\gamma = (y_t)_{t\in[0,1]}$ joining y_0 and y_1. The desired cobordism is $F^{-1}(\gamma)$. ■

This defines a map $[X, Y] \to \Omega_\bullet$. If X and Y are of the same dimension, then this map is the **mod 2 degree** $\deg_2\colon [X, Y] \to \Omega_0 = \mathbf{Z}/2\mathbf{Z}$.

Evidently,

$$\deg_2 \mathrm{id}_X = 1 \quad \text{and} \quad \deg_2 \text{const.} = 0. \tag{12.36}$$

Therefore, id_X cannot be properly homotopic to a constant map. Since $[0,1] \times S^{m-1} \to D^m := \bar{B}_1^m(0), (r, x) \mapsto r \cdot x$ is smooth, it follows that there cannot be smooth map $f\colon D^m \to S^{m-1}$ with $f|_{S^{m-1}} = \mathrm{id}_{S^{m-1}}$. This then implies Brouwer's fixed-point theorem by the usual construction.

Remark 12.37. A classic reference for cobordism theory is Stong [Sto68]. There is also a survey by **Milnor1962**. I believe that this is also discussed in [MS74]. ♣

13 Surgery and handle decomposition

Definition 13.1. Let $m, k \in \mathbb{N}_0$ Let X be a smooth manifold of dimension m with boundary. Let $\eta\colon S^k \times D^{m-k} \to X^\circ$ be an embedding. Set $\phi := \eta|_{S^k \times S^{m-k-1}}$ and

$$X' := (X \setminus \operatorname{im} \eta^\circ) \cup_\phi (D^{k+1} \times S^{m-k-1}). \tag{13.2}$$

A smooth manifold with boundary is said to be **obtained from X by surgery along η** if it is diffeomorphic to X'. •

Figure 13.1

Figure 13.2

Figure 13.3

Proposition 13.3. *In the situation of Definition 13.1, X is obtained from X' by surgery along the canonical inclusion $\eta'\colon S^{m-k-1} \times D^{k+1} \hookrightarrow X'$.*

Proof. This is evident from the definition. ∎

∼

Figure 13.4: Smoothing $[0,\infty)^2$.

Let us discuss how to smooth the corners of products of smooth manifolds with boundary.

Proposition 13.4. *Let X_1 and X_2 be smooth manifolds with boundary.*

1. *Let κ_i be a collar of X_i. Denote by $\iota_1\colon X_1^\circ \times X_2 \to X_1\times X_2$ and $\iota_2\colon X_1\times X_2^\circ \to X_1\times X_2$ the canonical inclusions. Define $\kappa\colon [0,\infty)\times \mathbf{R}\times\partial X_1\times\partial X_2 \to X_1\times X_2$ by*

   $$\kappa(s,t,x,y) := (\kappa_1(u,x),\kappa_2(v,y)) \tag{13.5}$$

   with

   $$(u,v) = \sigma(s,t) \quad \text{and} \quad \sigma(re^{i\theta}) := re^{i(\theta/2+\pi/4)}. \tag{13.6}$$

   There is a unique smooth structure on $X_1\times X_2$ with respect to which ι_i and κ are embeddings.

2. *Up to diffeomorphism, the smooth structure in (1) is independent of κ_i.*

3. *The boundary of $X_1\times X_2$ is*

   $$\partial(X_1\times X_2) = \partial X_1\times X_2 \amalg X_1\times\partial X_2; \tag{13.7}$$

   moreover, the inclusion maps

   $$\jmath_1\colon \partial X_1\times X_2\to X_1\times X_2 \quad \text{and} \quad \jmath_2\colon X_1\times\partial X_2\to X_1\times X_2 \tag{13.8}$$

   are embeddings.

Proof. The proof of (1) is analogous to the proof of Proposition 12.5 (2).

The proof of (2) is omitted.

The assertion about $\partial(X_1\times X_2)$ is obvious. The maps $\jmath_i$ are homeomorphisms onto their image. Therefore, it suffices to prove that they are smooth. They key observation is that

$$\jmath_1(x,\kappa_2(t,y)) = \kappa(t,0,x,y). \tag{13.9}$$

for $t\geqslant 0$. ∎

Remark 13.10. Another way of thinking about this is that we smooth the corners of the manifolds with corners $X_1 \times X_2$. This smoothing procedure is canonical only up to diffeomorphism. ♣

Definition 13.11. Define $\sigma_1\colon [0,\infty)^2 \to [0,\infty) \times \mathbf{R}$ by

$$\sigma_1(re^{i\theta}) := re^{i(-2\theta+\pi/2)} \tag{13.12}$$

and $\sigma_2\colon [0,\infty) \times (-\infty, 0] \to [0,\infty) \times \mathbf{R}$ by

$$\sigma_2(re^{i\theta}) := re^{i(2\theta+\pi/2)}. \tag{13.13}$$

•

These maps are homeomorphisms. Their restrictions to the interiors are diffeomorphisms. Moreover:

$$\sigma_1(u,0) = (0,u) = \sigma_2(u,0). \tag{13.14}$$

[TBD: picture of how this allows one to glue to copies of $[0,\infty)\times\mathbf{R}$ along $\{0\}\times[0,\infty)$.]

Proposition 13.15. *Let X_1 and X_2 be smooth manifolds with boundary. Let $Y_i \subset \partial X_i$ be a submanifold of codimension zero with boundary. Let $\phi\colon Y_1 \to Y_2$ be a diffeomorphism.*

1. *The topological space*

$$X := X_1 \cup_\phi X_2 \tag{13.16}$$

is Hausdorff and paracompact.

2. *Let κ_i be a collar of X_i. Let $\tau_i\colon \mathbf{R} \times \partial Y_i \to \partial X_i$ be an embedding whose restriction to $[0,1) \times \partial Y_i$ defines a collar of Y_i and such that*

$$\tau_2(t,\phi(x)) = \phi \circ \tau_1(t,x) \tag{13.17}$$

for every $t \geqslant 0$ and $x \in \partial Y_1$. Define $\kappa\colon \mathbf{R} \times Y_1^\circ \to X$ and $\lambda\colon [0,\infty) \times \mathbf{R} \times \partial Y_1 \to X$ by

$$\kappa(t,x) := \begin{cases} \kappa_1(-t,x) & \text{if } t \leqslant 0, \\ \kappa_2(t,\phi(x)) & \text{if } t \geqslant 0, \end{cases} \tag{13.18}$$

and

$$\lambda(u,v,x) := \begin{cases} \lambda_1(\sigma_1(u,v),x) & \text{if } v \leqslant 0, \\ \lambda_2(\sigma_2(u,v),\phi(x)) & \text{if } v \geqslant 0 \end{cases} \tag{13.19}$$

with

$$\lambda_i(s,t,x) := \kappa_i(s,\tau_i(t,x)). \tag{13.20}$$

Denote by $\iota_i\colon X_i \backslash Y_i \to X$ the canonical inclusion. There is a unique smooth structure on X such that ι_i, κ, λ are embeddings.

3. *Up to diffeomorphism, the smooth structure in (2) depends only on (the isotopy class of) ϕ.*

4. *The boundary ∂X is diffeomorphic to the smooth manifold obtained by gluing $\partial X_1 \backslash Y_1^\circ$ and $\partial X_2 \backslash Y_2^\circ$ along the boundary via $\phi|_{\partial Y_1}$.*

Proof. The proof is similar to that of Proposition 12.5. ■

Definition 13.21. In the situation of Proposition 13.15, X is said to be obtained by **gluing X_1 and X_2 along Y_1 and Y_2 via ϕ**. (This is an abuse of notation.) •

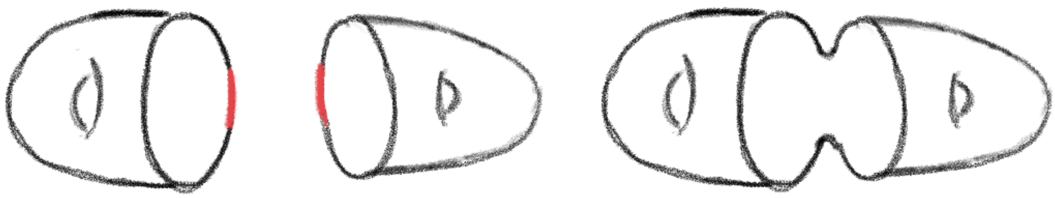

Figure 13.5: Gluing along part of the boundary.

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Let X_i be smooth manifolds of dimension m with boundary. Let $e_i \colon D \hookrightarrow X_i$ be embeddings. Applying Proposition 13.15 with $Y_i = e_i(D)$ yields the **boundary connected sum**

$$X_1 \#_b X_2. \tag{13.22}$$

By Proposition 13.15(4), the boundary of the boundary connected sum is the connected sum of the boundaries.

14 Vector bundles

Definition 14.1. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Let X be a C^k manifold. A **C^k $\mathbf{K}$–vector bundle over** X consists of:

1. a C^k manifold E, the **total space**,

2. a smooth map $\pi \colon E \to X$, and

3. for every $x \in X$ the structure of a finite-dimensional $\mathbf{K}$–vector space on

$$E_x := \pi^{-1}(x), \tag{14.2}$$

the **fiber of $E \xrightarrow{\pi} X$ over** x, such that for every $x \in X$ there are an open neighborhood U and a **local trivialization of** $E \xrightarrow{\pi} X$ **over** U; that is: a finite-dimensional $\mathbf{K}$–vector space V, and a C^k diffeomorphism

$$\phi\colon\ E|_U := \pi^{-1}(U) \to U \times V \tag{14.3}$$

such that:

4. $\mathrm{pr}_1 \circ \phi = \pi$, and
5. for every $y \in U$ the map $\phi_y := \mathrm{pr}_V \circ \phi|_{E_y}\colon\ E_y \to V$ is an isomorphism of $\mathbf{K}$–vector spaces. •

Notation 14.4. 1. If $\mathbf{K}$ is not specified, then $\mathbf{K} = \mathbf{R}$ is assumed.

2. A C^k $\mathbf{K}$–vector bundle over X is denoted by $E \xrightarrow{\pi} X$; that is: the $\mathbf{K}$–vector space structures are suppressed. Often, $E \xrightarrow{\pi} X$ is further abbreviated to E. ∘

Remark 14.5. $\mathbf{K}$ can be replaced by any finite-dimensional $\mathbf{R}$–algebra. ♣

Definition 14.6. Assume the situation of Definition 14.1. The **rank of** E is the map $\mathrm{rk}.\, E\colon\ X \to \mathbf{N}_0$ defined by $\mathrm{rk}_x\, E := \dim_{\mathbf{K}} E_x$. •

Proposition 14.7. *Assume the situation of Definition 14.1. The map* $\mathrm{rk}\, E\colon\ X \to \mathbf{N}_0$ *is locally constant.* ■

Example 14.8. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a C^k manifold and let V be a finite-dimensional vector space. The **trivial bundle over** X **with fiber** V is

$$\mathrm{pr}_1\colon\ X \times V \to X. \tag{14.9}$$

This vector bundle is often denoted by $\underline{V}$. ♠

Example 14.10. Consider S^1 as $\mathbf{R}/2\pi\mathbf{Z}$. The subset

$$M := \left\{([\theta], z) \in S^1 \times \mathbf{C} : \mathrm{Im}(e^{i\theta/2} z) = 0\right\} \tag{14.11}$$

is a submanifold. Denote the restriction of the canonical projection by $\pi\colon\ M \to S^1$. Evidently, for every $[\theta] \in S^1$, $\pi^{-1}([\theta])$ inherits the structure of a real vector space from $\mathbf{C}$. Set $U_+ := \{[\theta] : \theta \in (-\pi, \pi)\}$ and $U_- := \{[\theta] : \theta \in (0, 2\pi)\}$. The maps $\phi_\pm\colon\ \pi^{-1}(U_\pm) \to U_\pm \times \mathbf{R}$ defined by by

$$\phi_\pm([\theta], z) := ([\theta], e^{i\theta/2}\lambda). \tag{14.12}$$

are local trivializations. Therefore, $M \xrightarrow{\pi} S^1$ is a vector bundle. This is called the **Möbius bundle**. ♠

Example 14.13. Let $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Let $k, n \in \mathbf{N}$. The Grassmannian of k–planes in $\mathbf{K}^n$ is the quotient $\mathrm{St}_k(\mathbf{K}^n)/\mathrm{GL}_k(\mathbf{K})$ and has been equipped with the structure of a C^ω manifold in Example 1.48. The subset

$$\gamma_k(\mathbf{K}^n) := \{([T], v) \in \mathrm{Gr}_k(\mathbf{K}^n) \times \mathbf{K}^n : v \in \operatorname{im} T\} \tag{14.14}$$

is a C^ω submanifold. The map $\pi\colon \gamma_k \to \mathrm{Gr}_k(\mathbf{K}^n)$ induced by pr_1 is smooth. The **tautological bundle over** $\mathrm{Gr}_k(\mathbf{K}^n)$ is the C^ω vector bundle $\gamma_k \xrightarrow{\pi} \mathrm{Gr}_k(\mathbf{K}^n)$. ♠

Exercise 14.15. Find local trivializations for $\gamma_k(\mathbf{K}^n)$.

Example 14.16. Let $n \in \mathbf{N}$ and $k \in \mathbf{Z}$. Define the equivalence relation $\sim$ on $\big(\mathbf{C}^{n+1}\backslash\{0\}\big) \times \mathbf{C}$ by

$$(x, z) \sim (y, w) \quad \text{if and only if} \quad x = \lambda y \text{ and } z = \lambda^k w \text{ for some } \lambda \in \mathbf{C}^\times. \tag{14.17}$$

Set

$$\mathscr{O}_{\mathbf{C}P^n}(k) := ((\mathbf{C}^{n+1}\backslash\{0\}) \times \mathbf{C})/\!\sim. \tag{14.18}$$

Denote by $\pi\colon \mathscr{O}_{\mathbf{C}P^n}(k) \to \mathbf{C}P^n$ the map induced by $\mathrm{pr}_1\colon (\mathbf{C}^{n+1}\backslash\{0\}) \times \mathbf{C} \to \mathbf{C}^{n+1}\backslash\{0\}$. For every $x \in \mathbf{C}^{n+1}\backslash\{0\}$ the map $\phi_x\colon \mathbf{C} \to \pi^{-1}[x]$ defined

$$\phi_x(z) := [x, z] \tag{14.19}$$

is bijective. Since $\phi_{\lambda x}^{-1} \circ \phi_x(z) = \lambda^k z$, there is a unique structure of a $\mathbf{C}$–vector space on $\pi^{-1}[x]$ such that the maps ϕ_x are isomorphisms. $\mathscr{O}_{\mathbf{C}P^n}(k) \xrightarrow{\pi} \mathbf{C}P^n$ is a $\mathbf{C}$–vector bundle. ♠

Remark 14.20. The preceding example can be defined with $\mathbf{R}$ or $\mathbf{H}$ instead of $\mathbf{C}$ as well. ♣

Example 14.21. Section 15 introduces the tangent bundle TX: a vector bundle over X with fibers T_xX. ♠

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Definition 14.22. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Let X be a C^k manifold with boundary. Let $E \xrightarrow{\pi} X$ be a C^k $\mathbf{K}$–vector bundle over X. A C^k **section** of E is a C^k map $s\colon X \to E$ satisfying

$$\pi \circ s = \mathrm{id}_X. \tag{14.23}$$

The set of C^k sections is denoted by $C^k\Gamma(X, E)$ (or $C^k\Gamma(E)$). •

Notation 14.24. For $k = \infty$ the C^k is omitted. ○

Proposition 14.25. *Assume the situation of Definition 14.22. $C^k\Gamma(X, E)$ admits a unique structure of a* $\mathbf{K}$*–vector space such that for every $s, t \in C^k\Gamma(X, E)$ and $\lambda \in \mathbf{K}$*

$$(s + \lambda t)(x) = s(x) + \lambda t(x). \tag{14.26}$$

Example 14.27. Assume the situation of Definition 14.22. The **zero section** is the map $x \mapsto 0 \in E_x$. ♠

Example 14.28. A section of the trivial bundle $\underline{V}$ is nothing but a C^k map $f\colon X \to V$. ♠

Example 14.29. Let $n \in \mathbf{N}$ and $k \in \mathbf{N}_0$. Let $\mathcal{O}_{\mathbf{C}P^n}(k) \xrightarrow{\pi} \mathbf{C}P^n$ be as in Example 14.16. Let $p \in \mathbf{C}[z_0, \dots, z_n]$ be a homogeneous polynomial of degree k. The map $\mathbf{p}\colon \mathbf{C}P^n \to \mathcal{O}_{\mathbf{C}P^n}(k)$ defined by

$$\mathbf{p}([z_0 : \dots : z_n]) = [z_0 : \dots : z_n; p(z_0 : \dots : z_n)] \tag{14.30}$$

is a section of $\mathcal{O}_{\mathbf{C}P^n}(k)$. ♠

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Definition 14.31. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Let X and Y be C^k manifold. Let $E \xrightarrow{\pi} X$ and $F \xrightarrow{\rho} Y$ be C^k $\mathbf{K}$–vector bundles.

1. A **morphism of C^k K–vector bundles** from E to F is a pair (Λ, λ) consisting of a C^k map $\Lambda\colon E \to F$ and a C^k map $\lambda\colon X \to Y$ such that:
   (a) the diagram

$$\begin{tikzcd} E \arrow[r, "\Lambda"] \arrow[d, "\pi"] & F \arrow[d, "\rho"] \\ X \arrow[r, "\lambda"] & Y \end{tikzcd} \tag{14.32}$$

   commutes, and
   (b) for every $x \in X$ the map

$$\Lambda_x\colon E_x \to F_{\lambda(x)} \tag{14.33}$$

   induced by Λ is $\mathbf{K}$–linear.

2. Suppose that $X = Y$. A **morphism of C^k K–vector bundles over** X from E to F is a C^k map $\Lambda\colon E \to F$ such that (Λ, id_X) is a morphism of C^k $\mathbf{K}$–vector bundles. •

Example 14.34. Define $\tilde{\Lambda}\colon \mathcal{O}_{\mathbf{C}P^n}(-1) \to \mathbf{C}^{n+1}\backslash\{0\}$ by

$$c[(x_1, \ldots, x_{n+1}; z)] \coloneqq z \cdot (x_1, \ldots, x_{n+1}). \tag{14.35}$$

Evidently, $\mathbf{C}P^n = \mathrm{Gr}_1(\mathbf{C}^{n+1})$. The map $\tilde{\Lambda}$ induces a (iso)morphism

$$\begin{tikzcd} \mathcal{O}_{\mathbf{C}P^n}(-1) \arrow[r, "\Lambda"] \arrow[d, "\pi"] & \gamma_1(\mathbf{C}^{n+1}) \arrow[d, "\rho"] \\ \mathbf{C}P^n \arrow[r, equal] & \mathrm{Gr}_1(\mathbf{C}^{n+1}). \end{tikzcd} \tag{14.36}$$

♠

Definition 14.37. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Let $E \xrightarrow{\pi} X$ be a C^k $\mathbf{K}$–vector bundle. A C^k submanifold $F \subset E$ is a C^k **subbundle** if, for every $x \in X$, $F \cap E_x$ is a $\mathbf{K}$–linear subspace of E_x. •

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Proposition 14.38. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Let X, Y be C^k manifolds with boundary. Let $E \xrightarrow{\pi} Y$ be a C^k $\mathbf{K}$–vector bundle. Let $f\colon X \to Y$ be a C^k map.*

1. *The fiber product*

$$f^*E \coloneqq Y \times_f E. \tag{14.39}$$

is a C^k submanifold.

2. *Denote the canonical projection by $\rho\colon f^*E \to X$. For every $x \in X$, $\rho^{-1}(x) = \{x\} \times E_{f(x)}$ and thus inherits the structure of a $\mathbf{K}$–vector space. This makes $f^*E \xrightarrow{\rho} X$ into a C^k $\mathbf{K}$–vector bundle satisfying*

$$\mathrm{rk}_x f^*E = \mathrm{rk}_{f(x)} E. \tag{14.40}$$

3. *Define $\iota\colon f^*E \to E$ by*

$$\iota(x, v) \coloneqq v. \tag{14.41}$$

The pair (f^*E, ι) has the following universal property. If F and (Λ, f) is a morphism, then there is a unique morphism $\tilde{\Lambda}\colon F \to f^*E$ over X such that

$$\Lambda = \iota \circ \tilde{\Lambda}. \tag{14.42}$$

Proof. The maps $\pi\colon E \to X$ and $\partial\pi\colon \partial E \to \partial X$ are submersions. Therefore, by Proposition 8.52, f^*E is a C^k submanifold of $Y \times E$. This proves (2).

If $\phi\colon E|_U \to U \times V$ is a local trivialization if E, then the map $\psi\colon (f^*E)|_{f^{-1}(U)} \to f^{-1}(U) \times V$ defined by

$$\psi(x, v) := (x, \phi_{f(x)}(v)) \tag{14.43}$$

is a local trivialization of f^*E. This proves (2).

The proof of (3) is an exercise. ∎

Definition 14.44. Assume the situation of Proposition 14.38. The **pullback of** E **by** f is the C^k $\mathbf{K}$–vector vector bundle $f^*E \xrightarrow{\rho} X$. •

Theorem 14.45. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. Let X be a connected C^k manifold with boundary.*

1. *For every C^k $\mathbf{K}$–vector bundle $E \xrightarrow{\pi} X$ of rank r there are an $N \in \mathbf{N}$ and a C^k map $f\colon X \to \mathrm{Gr}_r(\mathbf{K}^N)$ and an isomorphism*

$$E \cong f^*\gamma_r(\mathbf{K}^N). \tag{14.46}$$

2. *Let $f, g\colon X \to \mathrm{Gr}_r(\mathbf{K}^N)$ be C^k maps. The pullbacks $f^*\gamma_r(\mathbf{K}^N)$ and $g^*\gamma_r(\mathbf{K}^N)$ are isomorphic if and only of f and g are C^k homotopic.*

15 The tangent bundle

It will be convenient for the next couple of sections to make the following observation.

Lemma 15.1. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a set. Let $\mathscr{A} = \{\phi_\alpha\colon U_\alpha \to \tilde{U}_\alpha : \alpha \in A\}$ be a set of bijective maps such that with $\tilde{U}_\alpha \subset \mathbf{M}_\alpha$ open and $\mathbf{M}_\alpha$ denoting either $\mathbf{R}^{m_\alpha}$ or $[0, \infty) \times \mathbf{R}^{m_\alpha - 1}$. Suppose that:*

1. *$X = \bigcup_{\alpha \in A} U_\alpha$.*

2. *For every for every $\alpha, \beta \in A$, $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbf{M}_\alpha$ and $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbf{M}_\beta$ are open, and the map $\tau^\alpha_\beta\colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ defined by*

$$\tau^\alpha_\beta := \phi_\beta \circ \phi_\alpha^{-1} \tag{15.2}$$

   is C^k.

3. *For every $x \neq y \in U$ there is an $\alpha \in A$ with $x, y \in U_\alpha$ or there are $\alpha \neq \beta \in A$ with $x \in U_\alpha$, $y \in U_\beta$, and $U_\alpha \cap U_\beta = \varnothing$.*

4. *Denote by $\sim$ the equivalence relation on A generated imposing that $\alpha \sim \beta$ whenever $U_\alpha \cap U_\beta = 0$. Partition A into its equivalence classes $A = \coprod_{i\in I} A_i$. For every $i \in I$ there is a countable subset $A_i^\# \subset A_i$ such that*

$$\bigcup_{\alpha\in A_i^\#} U_\alpha = \bigcup_{\alpha\in A_i} U_\alpha. \tag{15.3}$$

In this case there is a unique Hausdorff, paracompact topology on X such that $\mathscr{A}$ is a C^k atlas.

Proof. The set $\mathscr{B} := \{\phi_\alpha^{-1}(\tilde{V}) : \alpha \in A, \tilde{V} \subset \tilde{U}_\alpha \text{ open}\}$ is a basis of a topology. To see this, observe the following. By (1) every $x \in X$ is contained in a subset of the form $\phi_\alpha^{-1}(\tilde{V})$. Moreover, by (2),

$$\phi_\alpha^{-1}(\tilde{V}) \cap \phi_\beta^{-1}(\tilde{W}) = \phi_\beta^{-1}\big(\tau_\beta^\alpha(\tilde{V}) \cap \tilde{W}\big) \in \mathscr{B}.$$

Denote by $\mathscr{O}$ the topology associated with $\mathscr{B}$; that is: declare a subset of X to be open if it is an union of elements of $\mathscr{B}$. This is the unique topology with respect to which the maps ϕ_α are homeomorphisms.

By (2), $\mathscr{A}$ is a C^k atlas on X. By (3) and (4), X is Hausdorff and paracompact. ■

Proposition 15.4. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X be a C^k manifold with boundary. Set*

$$TX := \coprod_{x\in X} T_x X. \tag{15.5}$$

1. *There is a unique structure of a C^{k-1} manifold with boundary on TX such that if $\phi\colon U \to \tilde{U}$ is an admissible chart of X, then the map $\Phi\colon TU \to \tilde{U} \times \mathbf{R}^{m_\alpha}$ defined by*

   $$\Phi(x, [\psi, \tilde{v}]) := (\phi(x), \mathrm{d}_{\psi(x)}(\phi \circ \psi^{-1})\tilde{v}). \tag{15.6}$$

   is an admissible chart of TX.

2. *The canonical projection by $\pi\colon TX \to X$ is C^{k-1} with respect to this C^{k-1} structure, and $TX \xrightarrow{\pi} X$ is C^{k-1} $\mathbf{R}$–vector bundle.*

3. *Let Y be a C^k manifold with boundary. Let $f\colon X \to Y$ be C^k. The map $Tf\colon TX \to TY$ defined by*

   $$Tf(v) = T_x f \quad \text{for} \quad v \in T_x X. \tag{15.7}$$

   Moreover, $(Tf, f)\colon TX \to TY$ is a morphism of C^{k-1} vector bundles.

Definition 15.8. In the situation of Proposition 15.4, TX is called the **tangent bundle** of X, and Tf is called the **derivative of** f. •

Proof of Proposition 15.4. Denote by $\mathscr{A} \coloneqq \{\phi_\alpha \colon U_\alpha \to \tilde{U}_\alpha : \alpha \in A\}$ the C^k structure of X. For every $\alpha \in A$ define Φ_α as above. The maps $\Phi_\beta \circ \Phi_\alpha^{-1}$ satisfy

$$\Phi_\beta \circ \Phi_\alpha^{-1}(x, v) = (\phi_\beta \circ \phi_\alpha^{-1}(x), \mathrm{d}_x(\phi_\beta \circ \phi_\alpha^{-1})v); \tag{15.9}$$

hence, they are C^{k-1} diffeomorphisms. Therefore, Lemma 15.1 implies (1).

The map π is C^{k-1} because

$$\phi_\alpha \circ \pi \circ \Phi_\alpha^{-1}(x, v) = x. \tag{15.10}$$

The maps $TU_\alpha \to U_\alpha \times \mathbf{R}^{m_\alpha}$ given by

$$(x, [\psi, \tilde{v}]) = (x, \mathrm{d}_{\psi(x)}(\phi \circ \psi^{-1})\tilde{v}) \tag{15.11}$$

are local trivializations of TX. This proves (2).

The (straight-forward) proof of (3) is an exercise. ∎

16 Vector fields

Definition 16.1. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a C^{k+1} manifold with boundary A C^k **vector field on** X is a C^k section $v \colon X \to TX$ The set of all C^k vector field on M is denoted by $C^k\mathrm{Vect}(X)$. •

Notation 16.2. For $k = \infty$ the C^k is omitted. ∘

Proposition 16.3. *Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$ Let X be a C^{k+1} manifold. If $v \in \mathrm{Vect}(X)$ and $f \in C^k(X)$, then the map $vf \colon M \to \mathbf{R}$ defined by*

$$(vf)(x) = v(f)(x) \coloneqq v(x)f = \mathrm{d}_x f(v(x)) \tag{16.4}$$

is C^k.

Proof. Let $\phi \colon U \to \tilde{U}$ be an admissble chart on X. Denote by Φ the corresponding admissible chart of TX. Set $\tilde{f} \coloneqq f \circ \phi^{-1} \colon U \to \mathbf{R}$ and define $\tilde{v} \colon \tilde{U} \to \mathbf{R}^m$ by

$$\Phi \circ v \circ \phi^{-1}(x) = (x, \tilde{v}(x)). \tag{16.5}$$

By definition,

$$(vf) \circ \phi^{-1}(x)\mathrm{d}_x\tilde{f}(\tilde{v}(x)) \tag{16.6}$$

is C^k. ∎

Proposition 16.7. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X be a C^{k+1} manifold with boundary. If $v, w \in C^k\mathrm{Vect}(M)$, then there exists a unique $[v,w] \in C^{k-1}\mathrm{Vect}(M)$ such that for every $f \in C^k(M)$*

$$[v,w](f) = v(w(f)) - w(v(f)). \tag{16.8}$$

Definition 16.9. In the situation of Proposition 16.7, $[v,w]$ is called the **Lie bracket of v and w.** •

Proof of Proposition 16.7. It suffices to prove this for $X = U \subset \mathbf{M}$ open. For

$$v = \sum_{i=1}^{m} v_i \partial_i \quad \text{and} \quad w = \sum_{i=1}^{m} w_i \partial_i \tag{16.10}$$

a direct computation shows that

$$v(w(f)) - w(v(f)) = [v,w]f \quad \text{with} \quad [v,w] \coloneqq \sum_{j=1}^{m} \left(\sum_{i=1}^{m} v_i \partial_i w_j - w_i \partial_i v_j \right) \partial_j. \tag{16.11}$$

■

Proposition 16.12. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X be a C^{k+1} manifold with boundary. The Lie bracket $[\cdot,\cdot] \colon C^k\mathrm{Vect}(M) \times C^k\mathrm{Vect}(M) \to C^{k-1}\mathrm{Vect}(M)$ is bilinear, it is alternating: that is:*

$$[v,v] = 0, \tag{16.13}$$

and for $k \geqslant 2$ it satisfies the Jacobi identity

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0. \tag{16.14}$$

Proof. This is straight-forward. ■

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Proposition 16.15. *Let X be a smooth manifold with boundary. The map $\Upsilon \colon \mathrm{Vect}(X) \to \mathrm{Der}(C^\infty(X))$ defined by*

$$\Upsilon(v)(f) \coloneqq vf \tag{16.16}$$

is an isomorphism of $C^\infty(X)$–modules.

Proof. The following diagram commutes

$$
\begin{tikzcd}
\mathrm{Vect}(X) \arrow[r, "\Upsilon"] \arrow[dd, hook, "\mathrm{ev}"] & \mathrm{Der}(C^\infty(X)) \arrow[d, hook] \\
& \mathrm{Der}(C^\infty(X), \mathrm{Map}(X, \mathbf{R})) \arrow[d, "\cong"] \\
\prod_{x \in X} T_x X \arrow[r, "\prod_{x\in X} \Upsilon_x", "\cong"'] & \prod_{x \in X} \mathrm{Der}(C^\infty(X), \underline{\mathbf{R}}_x).
\end{tikzcd}
\tag{16.17}
$$

This implies that Υ is injective.

To prove that it is surjective, let $\delta \in \mathrm{Der}(C^\infty(X))$. For every $x \in X$ there is a $v_x \in T_x X$ such that $\delta(f)(x) = v_x(f)$. It remains to prove that the corresponding map $v \in \mathrm{Map}(X, TX)$ is smooth. This is a consequence of the fact that for every $f \in C^\infty(X)$ the map $\delta(f)$ is smooth. To see this, let $\phi\colon U \to \tilde{U}$ be a chart of X and denote by Φ the corresponding chart of TX. There exists a unique map $\tilde{v}\colon \tilde{U} \to \mathbf{R}^n$ such that

$$\Phi \circ v \circ \phi^{-1}(x) = (x, \tilde{v}(x)). \tag{16.18}$$

For every $f \in C^\infty(X)$ the map

$$\delta(f) \circ \phi^{-1} = \sum_{i=1}^{n} \tilde{v}^i \cdot \frac{\partial(f \circ \phi^{-1})}{\partial x^i} \tag{16.19}$$

is smooth. For every $y \in \tilde{U}$ there is an $f \in C^\infty(X)$ such that $f \circ \phi^{-1}$ equals x^i in an neighborhood of y shows that $\tilde{v}^i$ is smooth in a neighborhood of y. Therefore, $\tilde{v}$ is smooth; hence: v is smooth. ■

Remark 16.20. Let A be an $\mathbf{R}$–algebra. A brief computation shows that if $\delta, \varepsilon \in \mathrm{Der}(A)$ then $[\delta, \varepsilon] := \delta\varepsilon - \varepsilon\delta \in \mathrm{Der}(A)$. ♣

Definition 16.21. Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X and Y be C^{k+1} manifolds. Let $f\colon X \to Y$ be C^{k+1}. Two vector fields $v \in C^k\mathrm{Vect}(X)$ and $w \in C^k Vect(Y)$ are f–**related** if for every $x \in X$

$$T_x f(v(x)) = w(f(x)) \tag{16.22}$$

•

Proposition 16.23. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X and Y be C^{k+1} manifolds. Let $f\colon X \to Y$ be a C^{k+1} map. Let $v, w \in C^k\mathrm{Vect}(X)$ and $\tilde{v}, \tilde{w} \in C^k\mathrm{Vect}(Y)$.*

1. *The vector field v and $\tilde{v}$ are f–related if and only if for every $g \in C^k(Y)$*

$$(\tilde{v}g) \circ f = v(g \circ f). \tag{16.24}$$

2. *If v and $\tilde{v}$ are f–related, and w and $\tilde{w}$ are f-related, then $[v, w]$ and $[\tilde{v}, \tilde{w}]$ are f-related.*

Proof. This is immediate from the definitions. ■

Proposition 16.25. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X and Y be C^{k+1}–manifolds. For every C^{k+1} diffeomorphism $f\colon X \to Y$ and every $v \in C^k\mathrm{Vect}(X)$ there is a unique $f_* v \in C^k\mathrm{Vect}(Y)$ which is f–related to v.*

Definition 16.26. In the situation of Proposition 16.25, $f_* v$ is called the **pushforward of v via f**. •

Proof of Proposition 16.25. This is self-evident. ■

17 The flow of a vector field

Definition 17.1. Let $k \in \mathbf{N}_0 \cup \{\infty, \omega\}$. Let X be a C^{k+1} manifold with boundary. Let $v \in C^k\mathrm{Vect}(X)$.

1. An **integral curve of** v is map $\gamma\colon I \to X$ from an interval $I \subset \mathbf{R}$ satisfying

$$\dot{\gamma} = v \circ \gamma \quad \text{with} \quad \dot{\gamma}(t) \coloneqq T_t\gamma(\partial_t). \tag{17.2}$$

2. An integral curve $\gamma\colon I \to X$ of v is **maximal** if for every integral curve $\delta\colon J \to X$ of v the following holds: if there is a $t \in I \cap J$ with $\delta(t) = \gamma(t)$, then $J \subset I$.

3. The vector field v is **complete** if every maximal integral curve of v is defined on $\mathbf{R}$. •

Theorem 17.3. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X be a C^{k+1} manifold without boundary. Let $v \in C^k\mathrm{Vect}(X)$. The following hold:*

1. *For every $x \in X$ there is a unique maximal integral curve $\gamma_x^v\colon I_x^v \to X$ with $\gamma_x^v(0) = x$.*

2. *The subset $\mathscr{I}_v \coloneqq \{(t, x) \in \mathbf{R} \times X : t \in I_x^v\}$ is open.*

3. *The map $\mathrm{flow}_v\colon \mathscr{I}_v \to X$ defined by*

$$\mathrm{flow}_v(t, x) = \mathrm{flow}_v^t(x) \coloneqq \gamma_x^v(t) \tag{17.4}$$

is C^k; moreover, $\partial_t \mathrm{flow}_v$ is C^k.

4. *If $(t, x) \in \mathscr{I}_v$ and $(s, \mathrm{flow}_v^t(x)) \in \mathscr{I}_v$, then $(s + t, x) \in \mathscr{I}_v$ and*

$$\mathrm{flow}_v^{s+t}(x) = \mathrm{flow}_v^s \circ \mathrm{flow}_v^t(x). \tag{17.5}$$

5. *If $\lambda \in \mathbf{R}$ and $(\lambda t, x) \in \mathscr{I}_v$, then $(t, x) \in \mathscr{I}_{\lambda v}$ and*

$$\mathrm{flow}_v^{\lambda t}(x) = \mathrm{flow}_{\lambda v}^t(x). \tag{17.6}$$

6. *Let Y be a C^{k+1} manifold without boundary. Let $f\colon X \to Y$ be a C^{k+1} map. If v and $w \in C^k\mathrm{Vect}(Y)$ are f–related, then $(\mathrm{id}_{\mathbf{R}} \times f)(\mathscr{I}_v) \subset \mathscr{I}_w$ and for every $(t, x) \in \mathscr{I}_v$.*

$$\mathrm{flow}_w^t \circ f(x) = f \circ \mathrm{flow}_v^t(x). \tag{17.7}$$

7. *If*

$$\mathrm{supp}(v) \coloneqq \overline{\{x \in X : v(x) \neq 0\}} \tag{17.8}$$

is compact, then v is complete; that is: $\mathscr{I}_v = \mathbf{R} \times X$.

Definition 17.9. In the situation of Theorem 17.3, flow_v is called the **flow of** v. •

Proof of Theorem 17.3. The Picard–Lindelöf Theorem proves (1), (2), and (3) for $X = U \subset \mathbf{R}^n$ open.

By the Picard–Lindelöf Theorem, for every $x \in X$ there is an integral curve $\gamma\colon I \to X$ of v with $\gamma(0) = x$. Let $\gamma_i\colon I_i \to X$ ($i = 1, 2$) be two integral curves of v with $\gamma_i(0) = x$. Set

$$J \coloneqq \{t \in I_1 \cap I_2 : \gamma_1(t) = \gamma_2(t)\}. \tag{17.10}$$

Evidently, $0 \in J$. Since X is Hausdorff, $\Delta_X \subset X \times X$ is closed. Therefore,

$$J = (\gamma_1, \gamma_2)^{-1}(\Delta_X) \tag{17.11}$$

is closed. For every $t \in J$, by Picard–Lindelöf Theorem, there is an $\varepsilon > 0$ such that γ_1 and γ_2 agree on $(t - \varepsilon, t + \varepsilon)$. Therefore, J is open. Since $I_1 \cap I_2$ is connected, $J = I_1 \cap I_2$. This implies (1).

(2) and (3) follow from the assertion for $X = U \subset \mathbf{M}$ open.

Let $\gamma\colon I \to X$ be an integral curve of v. For every $s \in I$ the map $\delta\colon I - s \to X$ defined by

$$\delta(t) \coloneqq \gamma(s + t) \tag{17.12}$$

is a integral curve of v with with $\delta(0) = \gamma(s)$. This proves (4). Similarly, for every $\lambda \in \mathbf{R}$ the map $\delta\colon \lambda^{-1}I \to X$ defined by

$$\delta(t) \coloneqq \gamma(\lambda t) \tag{17.13}$$

is an integral curve of v. This proves (5).

To prove (6), observe that for every $t \in I^v_x$.

$$T_t(f \circ \gamma^v_x)(\partial_t) = T_{\gamma^v_x(t)} f \circ T_t\gamma^v_x(\partial_t) = T_{\gamma^v_x(t)} f(v(\gamma^v_x(t))) = w(f \circ \gamma^v_x(t)). \tag{17.14}$$

It remains to prove (7). If $x \notin \operatorname{supp}(v)$, then $I^v_x = \mathbf{R}$ and γ^v_x is constant. Therefore, if $x \in \operatorname{supp}(v)$ and $t \in I^v_x$, then $\gamma_x(t) \in \operatorname{supp}(v)$. If $\operatorname{supp}(v)$ is compact, then there is an $\varepsilon > 0$ such that

$$[-\varepsilon, \varepsilon] \times \operatorname{supp}(v) \subset \mathscr{I}_v. \tag{17.15}$$

Therefore, if $x \in \operatorname{supp}(v)$, $t \in I^v_x$, then $t + [\varepsilon, \varepsilon] \subset I^v_x$. Consequently, $I^v_x = \mathbf{R}$. ■

Remark 17.16. Theorem 17.3 holds with some modifications if X has boundary. The key issue that the integral curve γ of ∂_t on $[0, \infty)$ is defined only for $t \in [0, \infty)$. This affects (2) and (7). The proper formulation of Theorem 17.3 for manifolds with boundary replaces $\mathscr{I}_v$ with an open subset of a manifold with corners. ♣

Exercise 17.17. Compute the flow of $y\partial_x - x\partial_y$ on $\mathbf{R}^2$.

Exercise 17.18. Compute the flow of $x^2\partial_x$ on $\mathbf{R}$.

Proposition 17.19. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X be a C^{k+1} manifold. Let $v \in C^k\mathrm{Vect}(X)$ be a complete. For every $t \in \mathbf{R}$, the map $\mathrm{flow}^t_v\colon X \to X$ is a C^k diffeomorphism of X.*

Proof. By Theorem 17.3, the map flow^{-t}_v is the inverse of flow^t_v. ■

Proposition 17.19 is useful for constructing diffeomorphism.

Proposition 17.20. *Let X be a connected smooth manifold with $\dim X \geqslant 2$. Let $x_1, \dots, x_k$ be distinct points of X. Let $y_1, \dots, y_k$ be distinct points of X. There is a diffeomorphism $f \in \mathrm{Diff}(X)$ with $f(x_i) = y_i$ for every $i \in \{1, \dots, k\}$.*

Proposition 17.21. *Let $k \in \mathbf{N}_0 \cup \{\infty\}$. Let X be a C^k manifold with boundary. Let $\pi\colon E \to X$ be a C^k vector bundle. Let $Z \subset X$ be a submanifold which is closed as a subset. For every $s \in C^k\Gamma(E|_Z)$ there is a $\tilde{s} \in C^k\Gamma(E)$ satisfying*

$$\tilde{s}|_Z = s. \tag{17.22}$$

Proof. The proof is analogous to that of Proposition 9.24. Choose a cover $\mathscr{U} = \{U_0\} \cup \{U_i : i \in I\}$ of X with $U_0 \coloneqq X \backslash Z$ and such that for every $i \in$ there is an admissible chart $\phi_i\colon U_i \to \tilde{U}_i$ of X as in Definition 6.1 and a local trivialization $\psi_i\colon E_{U_i} \to U_i \times V_i$ of E. For every $i \in I$ define $f_i \in C^k(U_i, V_i)$ by

$$f_i \coloneqq \mathrm{pr}_{V_i} \circ \psi_i \circ s_i \tag{17.23}$$

and define $\tilde{s}_i \in C^k\Gamma(E|_{U_i})$ by

$$\tilde{s}_i(x) := \psi_i^{-1}(f_i \circ \phi \circ \mathrm{pr}_1 \circ \phi^{-1}(x), x) \tag{17.24}$$

with $\mathrm{pr}_1 \colon \mathbf{R}^m = \mathbf{R}^n \times \mathbf{R}^{m-n} \to \mathbf{R}^n$ denoting the canonical projection satisfies. By construction,

$$\tilde{s}_i|_Z = s. \tag{17.25}$$

Choose a C^k partition of unity $\{\rho_i : i \in \{0\} \cup I\}$ subordinate to $\mathscr{U}$. Since $U_0 = X \backslash Z$,

$$\sum_{i \in I} \rho_i|_Z = 1. \tag{17.26}$$

Therefore, the C^k section $\tilde{s} \in \Gamma(E|_{U_i})$ defined by

$$\tilde{s} := \sum_{i \in I} \rho_i \cdot \tilde{s}_i \tag{17.27}$$

satisfies

$$\tilde{s}|_Z := \sum_{i \in I} \rho_i \cdot \tilde{s}_i|_Z = \sum_{i \in I} \rho_i \cdot s = s. \tag{17.28}$$ ■

Proof of Proposition 17.20. Since X is connected, for every $i \in \{1, \ldots, k\}$ is an embedding $\gamma_i \colon [0,1] \hookrightarrow X$ with

$$\gamma_i(0) = x_i \quad \text{and} \quad \gamma_i(1) = y_i, \tag{17.29}$$

Since $\dim X \geqslant 2$, these embeddings can be arranged to have pairwise disjoint images. By Proposition 16.25 and Proposition 17.21, there is a vector field v on which is γ_i–related to ∂_t for every $i \in \{1, \ldots, k\}$. By Theorem 17.3 (6), $\mathrm{flow}_v^1(x_i) = y_i$. ■

Lemma 17.30. *Let $k \in (2 + \mathbf{N}_0) \cup \{\infty, \omega\}$. Let X be a C^{k+1} manifold without boundary. Let $v, w \in C^k\mathrm{Vect}(X)$.*

1. *For every $x \in X$*

$$[v, w](x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_{\mathrm{flow}_v^t(x)}\mathrm{flow}_v^{-t}(w(\mathrm{flow}_v^t(x))). \tag{17.31}$$

2. *If $[v, w] = 0$, then*

$$\mathrm{flow}_v^s \circ \mathrm{flow}_w^t(x) = \mathrm{flow}_w^t \circ \mathrm{flow}_v^t(x). \tag{17.32}$$

Remark 17.33. If v is complete, then

$$(\mathrm{flow}_v^t)_* w(x) = T_{\mathrm{flow}_v^t(x)}\mathrm{flow}_v^{-t}(w(\mathrm{flow}_v^t(x))). \tag{17.34}$$ ♣

Proof of Lemma 17.30. It suffices to prove (1) for $X = U \subset \mathbf{R}^m$ open. If

$$v = \sum_{i=1}^m v_i\partial_i \quad \text{and} \quad w = \sum_{i=1}^m w_i\partial_i, \tag{17.35}$$

then

$$\mathrm{flow}_v^t(x) = x + tv(x) + O(t^2) \quad \text{and} \quad \mathrm{d}_x\mathrm{flow}_v^{-t} = \mathbf{1} - t\mathrm{d}_x v + O(t^2). \tag{17.36}$$

Therefore,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} T_{\mathrm{flow}_v^t(x)}\mathrm{flow}_v^{-t}(w(\mathrm{flow}_v^t(x))) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\mathbf{1} - t\mathrm{d}_x v)w(x + tv(x)) \\ &= -\mathrm{d}_x v \cdot w(x) + \mathrm{d}_x w \cdot v(x) = [v, w](x). \end{aligned} \tag{17.37}$$

For every $(-s, x) \in \mathscr{I}_v$ set

$$w_s(x) := \big((\mathrm{flow}_v^s)_* w\big)(x). \tag{17.38}$$

By (1),

$$\partial_s w_s(x) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\mathrm{flow}_v^{s+t})_* w(x) = (\mathrm{flow}_v^s)_*\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} ((\mathrm{flow}_v^t)_* w)(x)\right) = 0. \tag{17.39}$$

Therefore, $(\mathrm{flow}_v^s)_* w = w$. By Theorem 17.3 (6),

$$\mathrm{flow}_w^t \circ \mathrm{flow}_v^s(x) = \mathrm{flow}_v^s \circ \mathrm{flow}_w^t(x). \tag{17.40}$$ ■

18 The Ehresmann fibration theorem

Definition 18.1. Let X be a manifold. A **fiber bundle over** X is a manifold F together with a smooth map $\pi\colon F \to X$ such that for every $x \in X$ there is an neighborhood U of x and a diffeomorphism

$$\phi\colon F|_U := \pi^{-1}(U) \to U \times F_x \quad \text{with} \quad F_x := \pi^{-1}(x) \tag{18.2}$$

satisfying

$$\pi = \mathrm{pr}_1 \circ \phi. \tag{18.3}$$ •

[pictures]

Theorem 18.4 (Ehresmann fibration theorem). *Let F and X be smooth manifolds. If $\pi\colon F \to X$ is a proper submersion, then $F \xrightarrow{\pi} X$ is a fiber bundle.*

The proof requires the following two observations.

Proposition 18.5. *Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X and Y be a C^{k+1} manifold without boundary. Let $f\colon X \to Y$ be a proper C^{k+1} map. Let $v \in C^k\mathrm{Vect}(X)$ and $w \in C^k\mathrm{Vect}(Y)$. If v is f–related to w, then*

$$(\mathrm{id}_{\mathbf{R}} \times f)^{-1}\mathscr{I}_w \subset \mathscr{I}_v. \tag{18.6}$$

Proof. Let $y \in Y$. If $I \subset I_y^w$ is a compact interval, then $f^{-1}(\gamma(I))$ is compact. Therefore, there is an $\varepsilon > 0$ such that, for every $x \in f^{-1}(\gamma_y^w(I))$, $[-\varepsilon, \varepsilon] \subset I_x^v$.

Let $x \in \pi^{-1}(y)$. For every $t \in I_x^w$, $f \circ \gamma_x^v(t) = \gamma_y^w(t)$. In particular, if $I \subset I_y^w$ is compact, then $I \subset I_x^v$; hence: $I_y^w \subset I_x^v$. ■

Proposition 18.7. *Let X and Y be smooth manifolds. Let $f\colon X \to Y$ be a submersion. For every $x \in X$ there is an open neighborhood U such that:*

1. *$V := \pi(U) \subset Y$ is open.*
2. *For every $v \in \mathrm{Vect}(V)$ there is a $\tilde{v} \in \mathrm{Vect}(U)$ which is f–related to v.*

Proof. This is obvious for $\mathrm{pr}_1\colon \mathbf{R}^k \times \mathbf{R}^{n-k} \to \mathbf{R}^k$. By Theorem 5.40, f is locally equivalent to pr_1. ■

Proof of Theorem 18.4. Without loss of generality, $X = (-1,1)^m \subset \mathbf{R}^m$. Choose a cover $\mathscr{U} = \{U_j : j \in J\}$ of F with U_j in Proposition 18.7. For $i \in \{1, \dots, m\}$ and $j \in J$ choose $v_{i,j} \in \mathrm{Vect}(U_i)$ π–related to ∂_i. Choose a partition of unity $\{\rho_j : j \in J\}$ subordinate to $\mathscr{U}$. The vector fields $v_i \in \mathrm{Vect}(F)$ defined by

$$v_i := \sum_{j \in J} \rho_j \cdot v_{i,j} \tag{18.8}$$

are π–related to ∂_i. Define $\psi\colon (-1,1)^m \times F_0 \to F$ by

$$\psi(t_1, \dots, t_m, x) := \mathrm{flow}_{v_1}^{t_1} \circ \dots \circ \mathrm{flow}_{v_m}^{t_m}(x). \tag{18.9}$$

By Proposition 18.5, this is well-defined. By Theorem 17.3 (3) and Theorem 17.3 (4), ψ is a C^k diffeomorphism. By Theorem 17.3 (6),

$$\pi \circ \psi(t_1, \dots, t_m) = \mathrm{flow}_{\partial_1}^{t_1} \circ \dots \circ \mathrm{flow}_{\partial_m}^{t_m}(0, \dots, 0) = (t_1, \dots, t_m). \tag{18.10}$$

Therefore, $\phi = \psi^{-1}$ is as required by Definition 18.1. ■

Remark 18.11. p.21 has an apparently different proof of the Ehresmann fibration theorem. Closer inspection shows that these proofs are essentially the same.

Another exposition of this proof can be found in . ♣

19 Frobenius' Theorem

Definition 19.1. Let X be a smooth manifold.

1. A **distribution on** X is a subbundle D of the tangent bundle.

Let D be a distribution on X.

2. D is **involutive** for every $v, w \in \Gamma(D)$,

   $$[v, w] \in \Gamma(D). \tag{19.2}$$

3. A **integral submanifold of** D is a connected submanifold $Y \subset X$ such that for every $x \in Y$

   $$T_x Y = D_x. \tag{19.3}$$

4. D is **integrable** if for every $x \in X$ there is an integral submanifold of D containing x.

5. D is **completely integrable** if every $x \in X$ there is an admissible chart $\phi\colon U \to \tilde{U}$ with $x \in U$ and $\phi(x) = 0$ such that

   $$\phi_* D = \langle \partial_1, \ldots, \partial_r \rangle \tag{19.4}$$

   with $r := \mathrm{rk}_x D$. •

Example 19.5. Let X be a smooth manifold. Let $v \in \mathrm{Vect}(X)$ be a nowhere vanishing vector field. $D := \langle v \rangle$ is a distribution of X. By Theorem 17.3, D is integrable. ♠

Example 19.6. Define the distribution D on $\mathbf{R}^3$ by

$$D = \langle v_1, v_2 \rangle \quad \text{with} \quad v_1 := \partial_1 \quad \text{and} \quad v_2 := \partial_2 + x_1 \partial_3. \tag{19.7}$$

D does not have any integral submanifolds because

$$[v_1, v_2] =: v_3 \partial_3 \tag{19.8}$$

is nowhere vanishing. Indeed, if Y where an integral submanifold of D, then

$$[v_1|_D, v_2|_D] = [v_1, v_2]|_D; \tag{19.9}$$

however, $-\partial_3 \notin D$. ♠

Remark 19.10. Identifying $\mathbf{R}^3$ with the **Heisenberg group**

$$H := \left\{ \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbf{R} \right\} \tag{19.11}$$

the vector fields v_1, v_2, v_3 can be see to be left-invariant. ♣

Exercise 19.12. Equip $\mathbf{R}^3$ with the above distribution. Let $x \in \mathbf{R}^3$. Prove that there is a smooth path $\gamma\colon [0,1] \to \mathbf{R}^3$ with $\gamma(0) = 0$, $\gamma(1) = x$, and such that for every $t \in [0,1]$

$$\dot\gamma(t) \in D_{\gamma(t)}. \tag{19.13}$$

Writing $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ and

$$\dot\gamma = \lambda_1 v_1 + \lambda_2 v_2. \tag{19.14}$$

we have

There is more general principle behind this. An application of this can be used to analyze the car parking problem; see `https://www.math.wisc.edu/~robbin/parking_a_car.pdf` for a careful discussion.

[This needs to be move to later. It is not important before one talks about principal bundles and connections anyway.]

Proposition 19.15. *Let X be a smooth manifold. Let D be a distribution on X. There is a unique tensor field $F \in \mathrm{Hom}(\Lambda^2 D, TX/D)$ such that for every $v, w \in D$ and $x \in X$*

$$F(v(x), w(x)) = -[v, w](x) \mod D_x. \tag{19.16}$$

Exercise 19.17. Proof this!

Definition 19.18. The tensor field F is the **curvature of** D. •

Theorem 19.19 (Frobenius' Theorem). *Let X be a smooth manifold. Let D be a distribution on X. The following are equivalent:*

1. *D is involutive.*
2. *D is integrable.*
3. *D is completely integrable.*

Proof. If D is completely integrable and ϕ is as in Definition 19.1 (5), then

$$\phi^{-1}(\tilde{U} \cap (\mathbf{R}^r \times \{0\})) \tag{19.20}$$

is an integral submanifold of D containing x. Therefore, (3) implies (2).

If D is completely integrable, $v, w \in \Gamma(D)$, $x \in X$, and Y is an integral submanifold of D containing x, then

$$[v, w](x) \in T_x Y = D_x. \tag{19.21}$$

Therefore, (2) implies (1).

It remains to prove that (1) implies (3). Let $x \in X$. Choose an admissible chart $\phi\colon U \to \tilde{U}$ with $x \in U$ and $\phi(x) = 0$ such that

$$\phi_* D = \langle v_1, \dots, v_r \rangle \quad \text{with} \quad v_i(x) = \partial_i + \sum_{j=1}^{m} a_i^j(x)\partial_j \tag{19.22}$$

and $a_i^j(0) = 0$. After shrinking U, for every $x \in \tilde{U}$

$$\mathbf{1} + A(x) \quad \text{with} \quad A \coloneqq (a_i^j(x))_{i,j=1}^r$$

is invertible. Therefore, after redefining v_i,

$$v_i(x) = \partial_i + \sum_{j=r+1}^{m} a_i^j(x)\partial_j. \tag{19.23}$$

A brief computation shows that

$$[v_i, v_j] = \sum_{k=r+1}^{n} (\partial_i a_j^k - \partial_j a_i^k)\partial_k + \sum_{k,\ell=r+1}^{n} (a_i^\ell \partial_\ell a_j^k - a_j^\ell \partial_\ell a_i^k)\partial_k. \tag{19.24}$$

The salient point is that the expression on the right-hand side has vanishing components along $\partial_1, \dots, \partial_r$. Since $[v_i, v_j] \in \Gamma(D)$,

$$[v_i, v_j] = 0. \tag{19.25}$$

Therefore, by Lemma 17.30

$$\mathrm{flow}_{v_i}^{t_i} \circ \mathrm{flow}_{v_j}^{t_j} = \mathrm{flow}_{v_j}^{t_j} \circ \mathrm{flow}_{v_i}^{t_i}. \tag{19.26}$$

Set

$$\psi(t_1, \dots, t_m) \coloneqq \mathrm{flow}_{v_1}^{t_1} \circ \dots \circ \mathrm{flow}_{v_r}^{t_r}(0, \dots, 0, t_{r+1}, \dots, t_m). \tag{19.27}$$

Evidently,

$$\psi(0) = 0. \tag{19.28}$$

A brief computation shows that

$$\begin{aligned}\partial_i\psi(t) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathrm{flow}_i^{t_i+\varepsilon} \circ \mathrm{flow}_{v_1}^{t_1} \circ \widehat{\mathrm{flow}_{v_i}^{t_i}} \cdots \circ \mathrm{flow}_{v_r}^{t_r}(0,\dots,0,t_{r+1},\dots,t_m) \\ &= v_i \circ \mathrm{flow}_i^{t_i} \circ \mathrm{flow}_{v_1}^{t_1} \circ \widehat{\mathrm{flow}_{v_i}^{t_i}} \cdots \circ \mathrm{flow}_{v_r}^{t_r}(0,\dots,0,t_{r+1},\dots,t_m) \\ &= v_i \circ \psi(t).\end{aligned} \tag{19.29}$$

Therefore, $\mathrm{d}_0\psi = \mathbf{1}$, and ∂_i and v_i are ψ–related. Hence, the admissible chart $\psi^{-1} \circ \phi$ is as required by (5). ∎

Here is a consequence which is sometimes important.

Proposition 19.30. *If $TX = D \oplus E$ and both D and E are involutive, then each $x \in M$ has a neighbourhood U and coordinates x_i such that*

$$D = \langle \partial_1, \dots, \partial_r \rangle \quad \textit{and} \quad E = \langle \partial_{r+1}, \dots, \partial_m \rangle.$$

Proof. By Theorem 19.19 there are charts $\phi_D \colon U \to \mathbf{R}^n$ and $\phi_E \colon U \to \mathbf{R}^n$ such that

$$(\phi_D)_* D = \langle \partial_1, \dots, \partial_r \rangle \quad \text{and} \quad (\phi_E)_* E = \langle \partial_{r+1}, \dots, \partial_m \rangle. \tag{19.31}$$

If we denote by π_1 and π_2 the projection onto the first and second factors of $\mathbf{R}^m = \mathbf{R}^r \oplus \mathbf{R}^{m-r}$, then $\phi \colon U \to \mathbf{R}^m$ defined by

$$\phi \coloneqq (\pi_1 \circ \phi_E, \pi_2 \circ \phi_D)$$

satisfies

$$\phi_* D = \langle \partial_1, \dots, \partial_r \rangle \quad \text{and} \quad \phi_* E = \langle \partial_{r+1}, \dots, \partial_m \rangle.$$

If U is sufficiently small, then ϕ is a diffeomorphism onto its image. ∎

[POSSIBLE TODO: discussion of foliations]

20 Review of multi-linear algebra

Here we briefly review the multi-linear algebra underlying differential forms.[1]

[1] For us, every vector space is a vector space over the real numbers $\mathbf{R}$.

20.1 The tensor product

Definition 20.1. Let $V_1, \dots, V_k$ and W be vector spaces. A map $M\colon V_1 \times \cdots \times V_k \to W$ is called **multi-linear** if for each $i = 1, \dots, k$ and each $(v_1, \dots, \dots, v_k) \in V_1 \times \cdots \times V_k$ the map $V_i \to W$ defined by

$$v \mapsto M(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$$

is linear.

Denote by $\mathrm{Mult}(V_1 \times \cdots \times V_k, W)$ the vector space of multi-linear maps $V_1 \times \cdots \times V_k \to W$ •

Proposition 20.2 (Universal property of the tensor product). *Let $V_1, \dots, V_k$ be vector spaces. There exists a vector space $V_1 \otimes \cdots \otimes V_k$ and a multi-linear map $\mu\colon V_1 \times \cdots \times V_k \to V_1 \otimes \cdots \otimes V_k$ such that the following holds: If $M\colon V_1 \times \cdots \times V_k \to W$ is a multi-linear map, then there exists a unique linear map $\tilde{M}\colon V_1 \otimes \cdots \otimes V_k \to W$ such that*

$$M = \tilde{M} \circ \mu.$$

Moreover,

$$\dim V_1 \otimes \cdots \otimes V_k = \dim V_1 \cdots \dim V_k. \tag{20.3}$$

Remark 20.4. Proposition 20.2 is often expressed by the following diagram:

$$\begin{tikzcd}
V_1 \times \cdots \times V_k \arrow[r, "M"] \arrow[d, "\mu"'] & W \\
V_1 \otimes \cdots \otimes V_k \arrow[ur, dashed, "\exists! \tilde{M}"'] &
\end{tikzcd}$$

or by saying that the map

$$\mathrm{Hom}(V_1 \otimes \cdots \otimes V_k, W) \to \mathrm{Mult}(V_1 \times \cdots \times V_k, W)$$

defined by

$$\tilde{M} \mapsto \tilde{M} \circ \mu$$

is bijective. ♣

Definition 20.5. The vector space $V_1 \otimes \cdots \otimes V_k$ together with the multi-linear μ is called the **tensor product** of $V_1, \dots, V_k$. We write

$$v_1 \otimes \cdots \otimes v_k := \mu(v_1, \cdots, v_k).$$

•

Remark 20.6. Everything about the tensor product can be proved using Proposition 20.2! For most purposes this is the best way to proceed.

♣

Remark 20.7. Since μ is multi-linear,

$$\begin{aligned} & v_1 \otimes \cdots \otimes v_{i-1} \otimes (v_i' + \lambda v_i'') \otimes v_{i+1} \otimes \cdots \otimes v_k \\ & = v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i' \otimes v_{i+1} \otimes \cdots \otimes v_k \\ & + \lambda \cdot v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i'' \otimes v_{i+1} \otimes \cdots \otimes v_k. \end{aligned}$$

This is the key relation one needs to know for computations. ♣

Proposition 20.8. *Let $V_1, \ldots, V_k$, $W_1, \ldots, W_k$ be vector spaces and $A_i \colon V_i \to W_i$ be linear maps. There exists a unique linear map $A_1 \otimes \cdots \otimes A_k \colon V_1 \otimes \cdots \otimes V_k \to W_1 \otimes \cdots \otimes W_k$ such that*

$$(A_1 \otimes \cdots \otimes A_k)(v_1 \otimes \ldots \otimes v_k) = A_1 v_1 \otimes \ldots \otimes A_k v_k$$

Proof.

$$\begin{tikzcd} V_1 \times \cdots \times V_k \arrow[r, "A_1 \times \cdots \times A_k"] \arrow[d, "\mu"] & W_1 \times \cdots \times W_k \arrow[d, "\mu"] \\ V_1 \otimes \cdots \otimes V_k \arrow[r, "\exists!"] & W_1 \otimes \cdots \otimes W_k \end{tikzcd}$$

■

Proposition 20.9. *There exists a unique linear map*

$$m \colon (V_1 \otimes \cdots \otimes V_k) \otimes (V_{k+1} \otimes \cdots \otimes V_{k+\ell}) \to V_1 \otimes \cdots \otimes V_{k+\ell}.$$

such that

$$m((v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_{k+\ell})) = v_1 \otimes \cdots \otimes v_k \otimes v_{k+1} \otimes \cdots \otimes v_{k+\ell}.$$

Proof. The domain of m is generated by vector of the form $(v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_{k+\ell})$, so one can verify this proposition defining m by the above formula on a basis and checking the everything is well-defined. One can proceed like this, but one should not. (Choosing a basis of an abstract vector space is barbaric and checking something is well-defined is not fun.)

So let's practice some abstract non-sense instead. By Proposition 20.2, a map m is equivalent to a bilinear map

$$M \colon (V_1 \otimes \cdots \otimes V_k) \times (V_{k+1} \otimes \cdots \otimes V_{k+\ell}) \to V_1 \otimes \cdots \otimes V_{k+\ell};$$

but a bilinear such a bilinear map is equivalent to a multi-linear map

$$(V_1 \times \cdots \times V_k) \times (V_{k+1} \times \cdots \times V_{k+\ell}) = V_1 \times \cdots \times V_{k+\ell} \to V_1 \otimes \cdots \otimes V_{k+\ell}.$$

Of course, $\mu \colon V_1 \times \cdots \times V_{k+\ell} \to V_1 \otimes \cdots \otimes V_{k+\ell}$ is the map we were looking for. In principle, this completes the proof, but it is instructive draw a diagram and explicitly chase down the identifications underlying the above argument.

Diagrammatic representation. This construction is visualized in the following diagram:

$$
\begin{tikzcd}
(V_1 \times \cdots \times V_{k+\ell}) \arrow[r, "\mu"] \arrow[d, "="] & V_1 \otimes \cdots \otimes V_{k+\ell} \\
(V_1 \times \cdots \times V_k) \times (V_{k+1} \times \cdots \times V_{k+\ell}) \arrow[d, "\mu\times\mu"] & \\
(V_1 \otimes \cdots \otimes V_k) \times (V_{k+1} \otimes \cdots \otimes V_{k+\ell}) \arrow[d] \arrow[uur, dashed, bend right] & \\
(V_1 \otimes \cdots \otimes V_k) \otimes (V_{k+1} \otimes \cdots \otimes V_{k+\ell}) \arrow[uuur, dotted, bend right] &
\end{tikzcd}
$$

The dashed arrow represents the bilinear map obtained by applying Proposition 20.2 with the multi-linear maps

$$V_1 \times \cdots \times V_k \to V_1 \otimes \cdots \otimes V_{k+\ell}, \quad (v_1, \ldots, v_k) \mapsto \mu(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell})$$

for fixed $v_{k+1}, \ldots, v_{k+\ell}$ and

$$V_{k+1} \times \cdots \times V_{k+\ell} \to V_1 \otimes \cdots \otimes V_{k+\ell}, \quad (v_{k+1}, \ldots, v_{k+\ell}) \mapsto \mu(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell})$$

for fixed $(v_1, \ldots, v_k)$.

Chasing identifications. Here is a more formal way of stating the argument. We observe that the map

$$\begin{aligned} &\mathrm{Bil}((V_1 \otimes \cdots \otimes V_k) \times (V_{k+1} \otimes \cdots \otimes V_{k+\ell}), V_1 \otimes \cdots \otimes V_{k+\ell}) \\ &\qquad \to \mathrm{Mult}((V_1 \times \cdots \times V_k) \times (V_{k+1} \times \cdots \times V_{k+\ell}), V_1 \otimes \cdots \otimes V_{k+\ell}) \end{aligned}$$

defined by

$$\tilde{B} \mapsto \tilde{B} \circ (\mu \times \mu)$$

is bijective. The multiplication map is then the map in

$$m \in \mathrm{Hom}((V_1 \otimes \cdots \otimes V_k) \otimes (V_{k+1} \otimes \cdots \otimes V_{k+\ell}), V_1 \otimes \cdots \otimes V_{k+\ell})$$

which corresponds to

$$\mu \in \mathrm{Mult}(V_1 \times \cdots \times V_{k+\ell}, V_1 \otimes \cdots \otimes V_{k+\ell})$$

under the chain of bijections

$$\begin{aligned} &\mathrm{Mult}(V_1 \times \cdots \times V_{k+\ell}, V_1 \otimes \cdots \otimes V_{k+\ell}) \\ &\quad = \mathrm{Mult}((V_1 \times \cdots \times V_k) \times (V_{k+1} \times \cdots \times V_{k+\ell}), V_1 \otimes \cdots \otimes V_{k+\ell}) \\ &\quad \cong \mathrm{Bil}((V_1 \otimes \cdots \otimes V_k) \times (V_{k+1} \otimes \cdots \otimes V_{k+\ell}), V_1 \otimes \cdots \otimes V_{k+\ell}) \\ &\qquad \cong \mathrm{Hom}((V_1 \otimes \cdots \otimes V_k) \otimes (V_{k+1} \otimes \cdots \otimes V_{k+\ell}), V_1 \otimes \cdots \otimes V_{k+\ell}). \end{aligned}$$

■

Definition 20.10. The **tensor algebra on** V is the vector space

$$TV := \bigoplus_{k=0}^{\infty} V^{\otimes k}.$$

equipped with the multiplication operation defined above. •

Remark 20.11. TV is an associative algebra; that is, the multiplication is associative. TV also has a grading given by the summands $V^{\otimes k}$. Multiplication respects this grading:

$$V^{\otimes k} \cdot V^{\otimes \ell} \subset V^{\otimes (k+\ell)}.$$

♣

Corollary 20.12. *Define a multi-linear map* $(V^*)^k \to \mathrm{Mult}(V^k, \mathbf{R})$ *by*

$$(v_1^*, \ldots, v_k^*) \mapsto ((v_1, \ldots, v_k) \mapsto \prod_{i=1}^{k} v_i^*(v_i)).$$

There is a unique linear map $(V^*)^{\otimes k} \to \mathrm{Mult}(V^k, \mathbf{R})$ *such that the diagram*

$$\begin{tikzcd}
(V^*)^k \arrow[r] \arrow[d] & \mathrm{Mult}(V^k, \mathbf{R}) \\
(V^*)^{\otimes k} \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

This map is injective. If V is finite-dimensional, this map is an isomorphism. ■

Remark 20.13. If V is finite-dimensional, we can treat elements of $(V^*)^{\otimes k}$ as multi-linear maps $V^k \to \mathbf{R}$. ♣

Remark 20.14. If V is finite-dimensional, you could identify $V^{\otimes k}$ with $\mathrm{Mult}((V^*)^k, \mathbf{R})$ (in principle). But it is actually rather more awkward to define even such simple things the tensor algebra, and your algebraist friends will be mad at you. ♣

20.2 Alternating tensor product

Definition 20.15. A multi-linear map $M\colon V^k \to W$ is called **alternating** if

$$M(v_1, \ldots, v_k) = 0$$

whenever there is an $i = 1, \ldots, k-1$ such that $v_i = v_{i+1}$. We write $\mathrm{Alt}^k(V)$ for the space of alternating multi-linear maps $V^k \to W$. •

Remark 20.16. Over the $\mathbf{R}$, alternating is the same as

$$M(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) = -M(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k).$$

Number theorist and algebraists will mad at you if you define alternating like this in general. ♣

Proposition 20.17 (Universal property of the alternating tensor product). *Let V be a vector space and $k \in \mathbf{N}$. There exists a vector space $\Lambda^k V$ and an alternating multi-linear map $\mu\colon V^k \to \Lambda^k V$ such that the following holds: If $M\colon V^k \to W$ is an alternating map, then there exists a unique linear map $\tilde{M}\colon \Lambda^k V \to W$ such that*

$$M = \tilde{M} \circ \mu.$$

Moreover,

$$\dim \Lambda^k V = \binom{\dim V}{k}. \tag{20.18}$$

Remark 20.19. Proposition 20.17 is often expressed by the following diagram:

$$\begin{tikzcd} V^k \arrow[r, "M"] \arrow[d, "\mu"'] & W \\ \Lambda^k V \arrow[ur, dashed, "\exists!\tilde{M}"'] & \end{tikzcd}$$

or by saying that the map

$$\mathrm{Hom}(\Lambda^k V, W) \to \mathrm{Alt}^k(V, W)$$

defined by

$$\tilde{M} \mapsto \tilde{M} \circ \mu$$

is bijective. ♣

Definition 20.20. The vector space $\Lambda^k V$ together with the multi-linear μ is called the **k-th exterior power of V**. We write

$$v_1 \wedge \cdots \wedge v_k := \mu(v_1, \cdots, v_k).$$

●

Remark 20.21. If $\sigma \in S_k$ is a permutation of $\{1, \ldots, k\}$, then

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \mathrm{sign}(\sigma) v_1 \wedge \cdots \wedge v_k.$$

Moreover, one has the same relations as in the tensor product. ♣

Proposition 20.22. *Let V, W be vector spaces and $A\colon\ V \to W$ be a linear map. There exists a unique linear map $\Lambda^k A\colon\ \Lambda^k V \to \Lambda^k W$ such that*

$$(\Lambda^k A)(v_1 \wedge \ldots \wedge v_k) = Av_1 \wedge \ldots \wedge Av_k$$

The following is the key reason why we care about the alternating tensor product.

Proposition 20.23. *Let $k \in \mathbf{N}_0$ and $A \in \operatorname{End}(\mathbf{R}^k)$, $\Lambda^k A$ agrees with multiplication by* $\det(A)$.

Proof. $\Lambda^k \mathbf{R}^k$ is one-dimensional. It is spanned by

$$e_1 \wedge \cdots \wedge e_k$$

Therefore, $\Lambda^k A$ must be multiplication with some real number. To determine which, it suffices to compute

$$\Lambda^k A(e_1 \wedge \cdots \wedge e_k)$$

If

$$Ae_i = \sum_{j=1}^{k} a_i^j e_j,$$

then

$$\begin{aligned}\Lambda^k A(e_1 \wedge \cdots \wedge e_k) &= \left(\sum_{j_1=1}^{k} a_1^{j_1} e_{j_1}\right) \wedge \cdots \wedge \left(\sum_{j_k=1}^{k} a_k^{j_k} e_{j_k}\right) \\ &= \sum_{j_1=1}^{k} \cdots \sum_{j_k=1}^{k} a_1^{j_1} \cdots a_k^{j_k} e_{j_1} \wedge \cdots \wedge e_{j_k}.\end{aligned}$$

The term in the sum on the right-hand side is non-zero if and only there is a permutation $\sigma \in S_k$ with $j_i = \sigma(i)$. Therefore,

$$\begin{aligned}\Lambda^k A(e_1 \wedge \cdots \wedge e_k) &= \sum_{\sigma \in S_k} \prod_{i=1}^{k} a_1^{\sigma(i)} e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)} \\ &= \sum_{\sigma \in S_k} \left(\operatorname{sign} \sigma \prod_{i=1}^{k} a_1^{\sigma(i)}\right) \cdot e_1 \wedge \cdots \wedge e_k.\end{aligned}$$

This is precisely Leibniz' formula for the determinant. ∎

Proposition 20.24. *Denote by* $\pi\colon V^{\otimes k} \to \Lambda^k V$ *the unique linear map such that the diagram*

$$\begin{tikzcd} V^k \arrow[r, "\mu"] \arrow[dr, "\mu"'] & V^{\otimes k} \arrow[d, "\pi"] \\ & \Lambda^k V \end{tikzcd}$$

commutes; see Proposition 20.2.

1. *The map is surjective and its kernel* $R = \ker \pi$ *is generated by vectors of the form*

$$v_1 \otimes \cdots \otimes v_k$$

   with $v_i = v_{i+1}$ *for some* $i = 1, \ldots, k-1$.

2. *There is a unique linear map* $i\colon \Lambda^k V \to V^{\otimes k}$ *such that*

$$\pi \circ i = \mathrm{id}_{\Lambda^k V}$$

   and

$$i(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) v_1 \otimes \cdots \otimes v_k.$$

In particular, $\Lambda^k V = V^{\otimes k}/R$ *and* $i\colon \Lambda^k V \to V^{\otimes k}$ *is an injection.*

Proposition 20.25. *There exists a unique linear map*

$$m\colon \Lambda^k V \otimes \Lambda^\ell V \to \Lambda^{k+l} V$$

such that

$$m((v_1 \wedge \cdots \wedge v_k) \otimes (v_{k+1} \wedge \cdots \wedge v_{k+\ell})) = v_1 \wedge \cdots \wedge v_k \wedge v_{k+1} \wedge \cdots \wedge v_{k+\ell}.$$

Remark 20.26. If $\alpha \in \Lambda^k V$ and $\beta \in \Lambda^\ell V$, then

$$\alpha \wedge \beta = \pi(i(\alpha) \otimes i(\beta)).$$

♣

Definition 20.27. This operation is called the **wedge product**. It is denoted by $\wedge$. •

Definition 20.28. The **exterior algebra on** V is the vector space

$$\Lambda^\bullet V = \Lambda V := \bigoplus_{k=0}^{\infty} \Lambda^k V$$

equipped with the multiplication operation defined above. •

Remark 20.29. ΛV is an associative algebra; that is, the multiplication is associative. ΛV also has a grading given by the summands $\Lambda^k V$. Multiplication respects this grading:

$$\Lambda^k V \cdot \Lambda^\ell V \subset \Lambda^{k+\ell} V.$$

The multiplication is also **graded commutative**, that is if $\alpha \in \Lambda^k V$ and $\beta \in \Lambda^\ell V$, then

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha$$

♣

Corollary 20.30. *Define an alternating map* $(V^*)^k \longrightarrow \mathrm{Alt}^k(V, \mathbf{R})$ *by*

$$(v_1^*, \ldots, v_k^*) \mapsto \Big((v_1, \ldots, v_k) \mapsto \det\big((v_i^*(v_j))_{i,j=1}^k\big)\Big).$$

There is a unique linear map $\Lambda^k V^* \longrightarrow \mathrm{Alt}^k(V, \mathbf{R})$ *such that the diagram*

$$\begin{tikzcd}
(V^*)^k \arrow[r] \arrow[d] & \mathrm{Alt}^k(V,\mathbf{R}) \\
\Lambda^k V^* \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

This map is injective. If V is finite-dimensional, this map is an isomorphism. ■

Remark 20.31. The map $\Lambda^k V^* \longrightarrow \mathrm{Alt}^k(V, \mathbf{R})$ also make the following diagram commute:

$$\begin{tikzcd}
\Lambda^k V^* \arrow[r] \arrow[d, "i"] & \mathrm{Alt}^k(V,\mathbf{R}) \arrow[d, "\subset"] \\
(V^*)^k \arrow[r] & \mathrm{Mult}(V^k, \mathbf{R})
\end{tikzcd}$$

This is how we did it in class. ♣

Proposition 20.32. *There exists a unique linear map* $V \otimes \Lambda^k V^* \longrightarrow \Lambda^{k-1} V^*$

$$v \otimes \alpha \mapsto i(v)\alpha$$

such that

$$v \otimes (v_1^* \wedge \cdots \wedge v_k^*) \mapsto i(v) \sum_{i=1}^k (-1)^{i+1} v_i^*(v) \cdot v_1^* \wedge \cdots v_{i-1}^* \wedge v_{i+1}^* \cdots \wedge v_k^*$$

Thinking of α and $i(v)\alpha$ as elements of $\mathrm{Alt}^k(V, \mathbf{R})$ *and* $\mathrm{Alt}^{k-1}(V, \mathbf{R})$ *respectively, we have*

$$(i(v)\alpha)(v_1, \ldots, v_{k-1}) = \alpha(v, v_1, \ldots, v_{k-1}).$$

Definition 20.33. This operation is called **contraction**. •

21 Vector bundles (continued)

Proposition 21.1. *Let $k \in \mathbb{N}_0 \cup \{\infty, \omega\}$. Let $p, q \in \mathbb{N}_0$. Let $\mathfrak{F}\colon \mathbf{Vect}_{\mathbb{K}}^p \times \left(\mathbf{Vect}_{\mathbb{K}}^{\mathrm{op}}\right)^q \to \mathbf{Vect}_{\mathbb{K}}$ be a functor such that the maps*

(21.2)
$$\mathfrak{F}\colon \prod_{i=1}^{p} \mathrm{Hom}_{\mathbb{K}}(V_i, W_i) \times \prod_{j=p+1}^{q} \mathrm{Hom}_{\mathbb{K}}(W_j, V_j) \to \mathrm{Hom}_{\mathbb{K}}(\mathfrak{F}(V_1, \dots, V_{p+q}), \mathfrak{F}(W_1, \dots, W_{p+q}))$$

are C^k. For every C^k manifold X the following hold:

1. *Let $\left(E_i \xrightarrow{\pi_i} X\right)_{i=1}^{p+q}$ be a $(p+q)$–tuple of C^k vector bundles over X. Set*

   $$\mathfrak{F}(E_1, \dots, E_{p+q}) := \coprod_{x \in X} \mathfrak{F}((E_1)_x, \dots, (E_{p+q})_x) \tag{21.3}$$

   and denote by $\rho\colon \mathfrak{F}(E_1, \dots, E_{p+q}) \to X$ the canonical projection. There is a unique topology and C^k structure on $\mathfrak{F}(E_1, \dots, E_{p+q})$ such that $\mathfrak{F}(E_1, \dots, E_{p+q}) \xrightarrow{\rho} X$ is a C^k vector bundle and the following holds: if $\left(\phi_i\colon U \times V_i \to \pi_i^{-1}(U)\right)_{i=1}^{p+q}$ is a $(p+q)$–tuple of local trivializations, then the map $\psi\colon U \times \mathfrak{F}(V_1, \dots, V_{p+q}) \to \rho^{-1}(U)$ defined by

   $$\psi(x, \cdot) = \mathfrak{F}\left((\phi_1)_x, \dots, (\phi_p)_x, (\phi_{p+1})_x^{-1}, \dots, (\phi_{p+q})_x^{-1}\right) \tag{21.4}$$

   is a local trivialization.

2. *Let $\left(F_i \xrightarrow{\rho_i} X\right)_{i=1}^{p+q}$ be a $(p+q)$–tuple of C^k vector bundles over X. Let $(f_i\colon E_i \to F_i)_{i=1}^{p}$ and $(f_{p+j}\colon F_{p+j} \to E_{p+j})_{j=1}^{q}$ be a p–tuple and a q–tuple of C^k vector bundle morphisms respectively. There is a unique C^k vector bundle morphism*

   $$\mathfrak{F}(f_1, \dots, f_{p+q})\colon \mathfrak{F}(E_1, \dots, E_{p+q}) \to \mathfrak{F}(F_1, \dots, F_{p+q}) \tag{21.5}$$

   such that

   $$\mathfrak{F}(f_1, \dots, f_{p+q})_x = \mathfrak{F}\left((f_1)_x, \dots, (f_{p+q})_x\right) \tag{21.6}$$

   for every $x \in X$.

3. *The above constructions define a functor*

   $$\mathfrak{F}\colon C^k\mathbf{VectBun}_{\mathbb{K},X}^p \times (C^k\mathbf{VectBun}_{\mathbb{K},X}^{\mathrm{op}})^q \to C^k\mathbf{VectBun}_{\mathbb{K},X}. \tag{21.7}$$

Proof. Denote by $\{((\phi_{i,\alpha}\colon U_\alpha \times V_{\alpha,i} \to \pi^{-1}(U_\alpha))_{i=1}^{p+q} : \alpha \in A\}$ the set of $(p+q)$–tuples of local trivializations. For every $\alpha \in A$ define $\psi_\alpha\colon U_\alpha \times \mathfrak{F}(V_{\alpha,1},\dots,V_{\alpha,p+q}) \to \rho^{-1}(U_\alpha)$ by (21.4). These are bijections and

$$\begin{aligned}\psi_\beta^{-1}\circ\psi_\alpha(x,v) = \big(x, \mathfrak{F}\big(&(\phi_{\beta,1})_x^{-1}\circ(\phi_{\alpha,1})_x,\dots,(\phi_{\beta,p})_x^{-1}\circ(\phi_{\alpha,p})_x,\\ &(\phi_{\beta,p+1})_x\circ(\phi_{\alpha,p+1})_x^{-1},\dots,(\phi_{\beta,p+q})_x\circ(\phi_{\alpha,p+q})_x^{-1}\big)(v)\big).\end{aligned} \tag{21.8}$$

By hypothesis, these maps are C^k. The assertion thus follows from Lemma 15.1.

(2) is evident.

(3) is a consequence of $\mathfrak{F}$ being a functor. ∎

22 Tensor bundles and Lie derivatives

Definition 22.1. Let $k \in \mathbf{N}_0 \cup \{\infty,\omega\}$. Let X be a C^{k+1} manifold.

1. A **tensor bundle** is a vector bundle obtain from TX via Proposition 21.1. A **tensor field** is a section of a tensor bundle.
2. The **cotangent bundle of** X is the vector bundle

$$T^*X := (TX)^*. \tag{22.2}$$

3. Let $p, q \in \mathbf{N}_0$. The **bundle of** (p,q) **tensors** is

$$TX^{\otimes p} \otimes T^*X^{\otimes q}. \tag{22.3}$$

•

Most geometric objects and structures on manifolds come to us in the form of tensor fields.

Example 22.4. Every vector field is a tensor field. ♠

Example 22.5. Every **poly-vector field** is a section of $\Lambda^k TX$. ♠

Example 22.6. A **Riemannian metric on** X is a section $g \in \Gamma(\mathrm{Hom}(S^2TX,\mathbf{R}))$ with $g(v,v) \geqslant 0$ and $g(v,v) = 0$ if and only if $v = 0$. ♠

Example 22.7. An **almost complex structure on** X is a section $I \in \Gamma(\mathrm{End}(TX))$ satisfying

$$I^2 = -\mathbf{1} \in \mathrm{End}(TX). \tag{22.8}$$

♠

Example 22.9. An **differential** k**–form on** X is a section $\alpha \in \Gamma(\Lambda^k T^*X))$. ♠

Proposition 22.10. *If* $\mathrm{S} \in \mathrm{Hom}(\mathrm{Vect}(X)^{\otimes p}, E)$ *satisfies*

$$\mathrm{S}(fT) = f\mathrm{S}(T) \tag{22.11}$$

for every $f \in C^\infty(X)$*, then there is a tensor field* $S \in \mathrm{Hom}(TX^{\otimes p}, TX^{\otimes p}$ *with*

$$\mathrm{S}(T) = S(T). \tag{22.12}$$

Definition 22.13. Let $\mathfrak{F}$ be as in Proposition 21.1. Set $E := \mathfrak{F}(TX, \dots, TX)$.

$$(\mathscr{L}_v S)(x) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{pull}_{x,t} S(\mathrm{flow}_v^t(x)). \tag{22.14}$$

with $\mathrm{pull}_{x,t} \colon E_{\mathrm{flow}_v^t(x)} \to E_x$ defined by

$$\mathfrak{F}_{x,t} := (T_{\mathrm{flow}_v^t(x)}\mathrm{flow}_v^{-t}, \dots, T_{\mathrm{flow}_v^t(x)}\mathrm{flow}_v^{-t}, T_x\mathrm{flow}_v^t, \dots, T_x\mathrm{flow}_v^t). \tag{22.15}$$ •

Proposition 22.16. *1.*

$$\mathscr{L}_v f \tag{22.17}$$

2.

$$\mathscr{L}_v w = [v, w] \tag{22.18}$$

3.

$$\mathscr{L}_v(S \otimes T) = (\mathscr{L}_v S) \otimes T + S \otimes (\mathscr{L}_v T). \tag{22.19}$$

4.

$$\mathscr{L}_v \beta(w) = (\mathscr{L}_v \beta)(w) + \beta(\mathscr{L}_v w). \tag{22.20}$$

Proof. This is an easy exercise. ■

23 Differential forms

Definition 23.1. Let X be a smooth manifold with boundary.

1. A **differential form of degree** k **(**k**–form) on** X is a section $\alpha \in \Gamma(\Lambda^k T^*X)$.
2. The vector space of differential forms of degree k on X is denoted by

$$\Omega^k(X) := \Gamma(\Lambda^k T^*X). \tag{23.2}$$

3. The graded vector space of differential forms on X is denoted by

$$\Omega^\bullet(X) := \bigoplus_{k=0}^{\infty} \Omega^k(X). \tag{23.3}$$

•

Notation 23.4. For $\alpha \in \Omega^k(X)$ set

$$\deg(\alpha) := k. \tag{23.5}$$

∘

Notation 23.6. For an open subset $U \subset \mathbf{R}^n$ with its standard smooth structure there is a preferred chart: $\phi = \mathrm{id}_U$. This induces a preferred isomorphism $(\varpi_\phi^{-1})^* \colon (\mathbf{R}^n)^* \cong T_x^* U$. It is customary to identify

$$(\mathbf{R}^n)^* = T_x^* U \tag{23.7}$$

via $(\varpi^{-1})^* = (\varpi_\phi^{-1})^*$ and set

$$\mathrm{d}x^i := \varpi(e_i^*). \tag{23.8}$$

If the coordinates of $\mathbf{R}^n$ have been labeled in some other way (as it is sometimes convenient), then this notation is adjusted correspondingly. For example, it is customary to write $\mathrm{d}t$ for the image of $1 \in \mathbf{R}$ under $(\varpi^{-1})^*$. ∘

Definition 23.9. The **wedge product** is the map $\cdot \wedge \cdot \colon \Omega^\bullet(X) \otimes \Omega^\bullet(X) \to \Omega^\bullet(X)$ induced by the wedge product on $\Lambda^\bullet T^* X$. •

The wedge product on $\Lambda^\bullet T^* X$ makes $\Omega^\bullet(X)$ into a graded commutative algebra; that is:

$$\alpha \wedge \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \wedge \alpha. \tag{23.10}$$

Exercise 23.11. Prove that

$$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta. \tag{23.12}$$

With this in mind, in a chart ϕ every k–form α can be expressed as

$$\tilde{\alpha} := (\phi^{-1})^* \alpha = \sum_{i_1,\ldots,i_k} \alpha_{i_1,\ldots,i_k} \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}.$$

~

Definition 23.13. Let X be a smooth manifold with boundary. Let $v \in \mathrm{Vect}(X)$. The **contraction with** v is the linear map $i_v \colon \Omega^\bullet(X) \to \Omega^{\bullet-1}(X)$ defined by

$$i_v \alpha := \alpha(v \wedge \cdot). \tag{23.14}$$

•

Exercise 23.15. 1. Prove that $i_v^2 = 0$.

2. Prove that $i_v i_w + i_v i_w = 0$.

3. Prove that

$$i_v(\alpha \wedge \beta) = (i_v\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge i_v\beta. \tag{23.16}$$

∼

Proposition 23.17. *There is a unique linear map of degree one* $\mathrm{d}\colon \Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$ *satisfying the following:*

1. *For every* $f \in \Omega^0(X)$, $\mathrm{d}f$ *agrees with the derivative of* f.

2. $\mathrm{d} \circ \mathrm{d} = 0$.

3. $\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge \mathrm{d}\beta$.

This map satisfies

$$\mathrm{d} \circ f^* = f^* \circ d. \tag{23.18}$$

Moreover,

$$\mathrm{d}\alpha_{i_1,\ldots,i_k} \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} = \sum_{j=1}^{m} \partial_j \alpha_{i_1,\ldots,i_k} \mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}. \tag{23.19}$$

Definition 23.20. The map $d\colon \Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$ is the **exterior derivative**. •

Remark 23.21. This makes $\Omega^\bullet(M)$ in to a differential graded commutative algebra (DGA). ♣

Proof of Proposition 23.17. The proof has three steps.

Step 1. *Proof for* $X = U \subset \mathbf{R}^n$.

Every $\alpha \in \Omega^k(U)$ can be uniquely writen as

$$\alpha = \sum_{0 \leqslant i_1 < \cdots < i_k \leqslant k} \alpha_{i_1 \ldots i_k} \,\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}. \tag{23.22}$$

Set

$$\mathrm{d}\alpha := \sum_{j=1}^{m} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant k} \partial_j \alpha_{i_1 \ldots i_k} \,\mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} \tag{23.23}$$

and observe that this definition is forced by the hypothesis.

Evidently, (1) and (3) hold. To establish (2), compute

$$\begin{aligned} \mathrm{d} \circ \mathrm{d}f &= \sum_{i,j=1}^{m} \partial_j \partial_i f \,\mathrm{d}x^i \wedge \mathrm{d}x^j \\ &= \sum_{i<j} (\partial_j \partial_i f - \partial_i \partial_j f)\,\mathrm{d}x^i \wedge \mathrm{d}x^j = 0. \end{aligned} \tag{23.24}$$

This proves that $\mathrm{d} \circ \mathrm{d} = 0$ on $\Omega^0(X)$. This implies (2) in general in light of (3).

Step 2. *If $U \subset \mathbf{R}^m$ and $V \subset \mathbf{R}^n$ are open, and $f\colon U \to V$ is smooth, then*

$$\mathrm{d}f^*\alpha = f^*\mathrm{d}\alpha$$

For $g \in C^\infty(V)$

$$\mathrm{d}(f^*g) = \mathrm{d}(g \circ f) = \mathrm{d}g \circ \mathrm{d}f = f^*\mathrm{d}g.$$

This proves the assertion on $\Omega^0(V)$. Since pullback commutes with the wedge product, the assertion holds.

Step 3. *The global case.*

Let α be a differential form on X. If $\phi\colon U \to \tilde{U}$ and $\psi\colon V \to \tilde{V}$ is are admissible charts on X, then on $U \cap V$

$$\begin{aligned} \psi^*(\mathrm{d}(\psi^{-1})^*\alpha) &= \psi^*(\mathrm{d}[(\psi \circ \phi^{-1})^{-1}]^*(\phi^{-1})^*\alpha) \\ &= \psi^*(\psi^{-1})^*\phi^*(\mathrm{d}(\phi^{-1})^*\alpha) = \phi^*(\mathrm{d}(\phi^{-1})^*\alpha). \end{aligned} \tag{23.25}$$

This implies that there is a unique $\mathrm{d}\alpha$ such that

$$(\mathrm{d}\alpha)|_U = \phi^*(\mathrm{d}(\phi^{-1})^*\alpha). \tag{23.26}$$

for every admissible chart ϕ.

The desired properties hold and uniqueness follows from uniqueness for $U \subset \mathbf{R}^m$. ■

Corollary 23.27. *For every $v \in \mathrm{Vect}(X)$*

$$\mathscr{L}_v\mathrm{d} = \mathrm{d}\mathscr{L}_v. \tag{23.28}$$

■

Proposition 23.29 (Cartan's magic formula). *Let X be a smooth manifold with boundary. For every $v \in \mathrm{Vect}(X)$ and $\alpha \in \Omega^\bullet(X)$*

$$\mathscr{L}_v\alpha = \mathrm{d}i_v\alpha + i_v\mathrm{d}\alpha. \tag{23.30}$$

Proof. Define $\tilde{\mathscr{L}}_v \colon \Omega^\bullet(X) \to \Omega^\bullet(X)$ by

$$\tilde{\mathscr{L}}_v := \mathrm{d}i_v + i_v\mathrm{d}. \tag{23.31}$$

The map $\tilde{\mathscr{L}}_v$ satisfies the following:

1. $\tilde{\mathscr{L}}_v f = \mathrm{d}f(v) = \mathscr{L}_v f$,
2. $\mathrm{d}\tilde{\mathscr{L}}_v = \tilde{\mathscr{L}}_v\mathrm{d}$, and
3. $\tilde{\mathscr{L}}_v(\alpha \wedge \beta) = \tilde{\mathscr{L}}_v\alpha \wedge \beta + \alpha \wedge \tilde{\mathscr{L}}_v\beta$.

This first two are obvious and the last follows from a short computation:

$$\begin{aligned}
\tilde{\mathscr{L}}_v(\alpha \wedge \beta) &= \mathrm{d}(i_v\alpha \wedge \beta + (-1)^{\deg\alpha}\alpha \wedge i_v\beta) \\
&\quad + i_v(\mathrm{d}\alpha \wedge \beta + (-1)^{\deg\alpha}\alpha \wedge \mathrm{d}\beta) \\
&= (\tilde{\mathscr{L}}_v\alpha) \wedge \beta + \alpha \wedge \tilde{\mathscr{L}}_v\beta \\
&\quad - (-1)^{\deg\alpha} i_v\alpha \wedge \mathrm{d}\beta + (-1)^{\deg\alpha}\mathrm{d}\alpha \wedge i_v\beta) \\
&\quad - (-1)^{\deg\alpha}\mathrm{d}\alpha \wedge i_v\beta) + (-1)^{\deg\alpha} i_v\alpha \wedge \mathrm{d}\beta \\
&= \tilde{\mathscr{L}}_v\alpha \wedge \beta + \alpha \wedge \tilde{\mathscr{L}}_v\beta.
\end{aligned}$$

From these observation the proof follows shortly. Indeed, the statement is local; hence, it suffices to prove it for open subset $U \subset \mathbf{R}^m$ and $\alpha = f\mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}$:

$$\begin{aligned}
\tilde{\mathscr{L}}_v\alpha &= (\tilde{\mathscr{L}}_v f)\mathrm{d}(\tilde{\mathscr{L}}_v x_{i_1}) \wedge \ldots \wedge \mathrm{d}(\tilde{\mathscr{L}}_v x_{i_1}) \\
&= (\mathscr{L}_v f)\mathrm{d}(\mathscr{L}_v x_1) \wedge \ldots \wedge \mathrm{d}(\mathscr{L}_v x_k) \\
&= \mathscr{L}_v\alpha.
\end{aligned}$$

■

~

The following observation is useful for computation where a choice of coordinate is not at hand or would be unnatural.

Proposition 23.32 (Invariant formula for the exterior derivative). *For $\alpha \in \Omega^k(X)$ and $v_1, \ldots, v_{k+1} \in \mathrm{Vect}(X)$*

$$\begin{aligned}
(\mathrm{d}\alpha)(v_1, \ldots, v_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\mathscr{L}_{v_i}(\alpha(v_1, \ldots, \widehat{v_i}, \ldots, v_{k+1})) \\
&\quad + \sum_{i<j=1}^{k+1}(-1)^{i+j}\alpha([v_i, v_j], v_1, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_{k+1}).
\end{aligned} \tag{23.33}$$

Proof. Exercise. ■

Exercise 23.34. Let X be a n–dimensional manifold and let $D \subset TM$ be a distribution of rank k. Define $\mathrm{Ann}(D) \subset \Lambda^\bullet T^*X$ by

$$\mathrm{Ann}(D) := \{\alpha \in \Lambda^\bullet T^*M : \alpha|_D = 0\}.$$

Find a characterisation of the involutivity of D in terms of $\mathrm{Ann}(D)$.

24 Integrating differential forms

Theorem 24.1. *Let $m \in \mathbf{N}_0$. Let $U, V \subset \mathbf{R}^m$ be open. Let $\phi\colon U \to V$ be a C^1 diffeomorphism. Let $f \in \mathrm{Map}(V, \mathbf{R})$. The function f is integrable if and only if $f \circ \phi \cdot |\det(\mathrm{d}\phi)|$ is; moreover:*

$$\int_V f\,\mathrm{d}x^1 \cdots \mathrm{d}x^m = \int_U f \circ \phi \cdot |\det(\mathrm{d}\phi)|\,\mathrm{d}x^1 \cdots \mathrm{d}x^m \tag{24.2}$$

Let X be a smooth manifold of dimension m. Suppose that $\Omega^n(X)$ has compact support. If ϕ is an admissible chart, then

$$\phi_*\alpha = f \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n.$$

Therefore, it is tempting to integrate α. If ψ is further admissible chart and $\tau = \phi \circ \psi^{-1}$, then

$$\psi_*\alpha = f \circ \tau \cdot \det(\mathrm{d}\tau) \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n.$$

This is *almost* (24.2): $\det(\mathrm{d}\tau)$ vs. $|\det(\mathrm{d}\tau)|$. Therefore, one can *almost* integrate differential forms. To get out of this sign quagmire, one introduced the following concept.

Definition 24.3. Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Let X be a C^k manifold with boundary of dimension m.

1. A **volume form on** X is a nowhere-vanishing m–form.
2. An **orientation on** X is a equivalence class of volume forms on X with respect to the relation $\nu \sim \mu$ if and only if there is a positive C^k function f with $\nu = f\mu$.
3. The **standard orientation on $\mathbf{R}^m$** is the orientation $[\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m]$.
4. An oriented C^k manifold with boundary is a C^k manifold with boundary together with an orientation $[\mu]$.
5. A chart ϕ on an oriented C^k manifold is **orientation preserving** if $[\phi_*\mu]$ is the standard orientation on $\mathbf{R}^m$. It is **orientation reversing** if $[-\phi_*\mu]$ is the standard orientation on $\mathbf{R}^m$.

6. A basis $(e_1, \ldots, e_m)$ is positive if for any μ in the class determining the orientation $\mu(e_1, \ldots, e_n) > 0$. ●

Remark 24.4. If $\dim X = 0$, then an orientation is nothing but a choice of sign $\varepsilon(x) = \pm 1$ for every $x \in X$. ♣

Proposition 24.5. *Let X be an oriented C^k manifold with boundary of dimension m. There is a unique linear map*

$$\int_X : \Omega_c^m(X) \to \mathbf{R} \tag{24.6}$$

with the following property: if $\phi\colon U \to \tilde{U}$ is a preserving/reversing chart and $f \in C_c^\infty(\tilde{U})$, then

$$\int_X \phi^*(f \cdot \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m) = \pm \int_{\tilde{U}} f \,\mathrm{d}x^1 \cdots \mathrm{d}x^m. \tag{24.7}$$

Proof. For open subsets $U \subset \mathbf{R}^m$ define

$$\int_U f \cdot \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m := \int_U f \,\mathrm{d}x^1 \cdots \mathrm{d}x^m. \tag{24.8}$$

This evidently has the desired properties.

Choose an atlas $\{\phi_\alpha : U_\alpha \to \tilde{U}_\alpha\}$ consisting of charts which are either orientation preserving or orientation reversing; set $\varepsilon_\alpha := \pm 1$ accordingly. Let $\{\rho_\alpha : \alpha \in A\}$ be a partition of unity subordinate to $\{U_\alpha : \alpha \in A\}$. For $\alpha \in \Omega_c^m(X)$ set

$$\int_X \alpha := \sum_{\alpha \in A} \varepsilon_\alpha \cdot \int_{\tilde{U}_\alpha} (\phi_\alpha)_*(\rho_\alpha \cdot \alpha). \tag{24.9}$$

(This is a finite sum since $\operatorname{supp} \alpha$ is compact.) This definition is forced upon us by linearity. ■

Remark 24.10. If $\dim X = 0$, then

$$\int_X f = \sum_{x \in X} \varepsilon(x) f(x). \tag{24.11}$$

♣

Exercise 24.12. Prove that the existence of an orientation is equivalent to the existence of a **volume form**, that is, a nowhere vanishing m–form.

Exercise 24.13. Show that complex manifolds have canonical orientations.

Exercise 24.14. Show that $\mathbf{R}P^2$ is not orientable.

Notation 24.15. Let X, Y be C^k manifolds with boundary. Suppose that $\dim Y = m$. If $\alpha \in \Omega_c^m(X)$ and $\iota\colon Y \hookrightarrow X$ is an immersion, then

$$\int_N \alpha := \int_N \iota^* \alpha. \tag{24.16}$$

∘

Example 24.17. On $\mathbf{C}$ consider the $\mathbf{C}$–valued 1–form

$$\mathrm{d}z := \mathrm{d}x + i\mathrm{d}y. \tag{24.18}$$

For $f\colon \mathbf{C} \to \mathbf{C}$ and $\gamma \subset \mathbf{C}$ a closed curve

$$\int_\gamma f\, \mathrm{d}z \tag{24.19}$$

is the line integral which you are familiar with from complex analysis. ♠

25 Stokes' Theorem

Definition 25.1. Let X be an oriented C^k manifold with boundary. The **induced orientation** on X is the unique orientation such that if $\phi\colon U \to \tilde{U}$ is a chart with $\tilde{U} \subset [0, \infty) \times \mathbf{R}^{m-1}$, then the chart $\psi \cap U \cap \partial X \to U \cap \{0\} \times \mathbf{R}^{m-1}$ is orientation reversing/preserving if and only if ϕ is orientation preserving/reversing. •

Theorem 25.2 (Stokes' Theorem). *Let X be an oriented C^k manifold with boundary of dimension m. For every $\alpha \in \Omega_c^{m-1}(X)$*

$$\int_X \mathrm{d}\alpha = \int_{\partial X} \alpha. \tag{25.3}$$

Proof. Choose an atlas of X consisting of orientation preserving/reversing charts $\{\phi_\beta\colon U_\beta \to \tilde{U}_\beta : \beta \in A\}$ with $\tilde{U}_\beta$ either $(0,1)^m$ or $[0,1) \times (0,1)^{m-1}$. Choose a partition of unity $\{\rho_\beta : \beta \in A\}$ subordinate to $\{U_\beta : \beta \in A\}$. Since

$$\int_X \mathrm{d}\alpha = \sum_{\beta \in A} \varepsilon_\beta \int_X \mathrm{d}(\rho_\beta \alpha) \quad \text{and} \quad \int_{\partial X} \beta = \sum_{\beta \in A} \varepsilon_\beta \int_{\partial X} \rho_\beta \alpha, \tag{25.4}$$

it suffices to prove the assertion for $X = (0,1)^m$ and $X = [0,1) \times (0,1)^{m-1}$.

In either case, every $\alpha \in \Omega_c^{m-1}(X)$ decomposes as

$$\alpha = \sum_{i=1}^{m} f_i \cdot \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^m. \tag{25.5}$$

Therefore,

$$\mathrm{d}\alpha = \sum_{i=1}^{m} (-1)^{i+1} \partial_i f_i \cdot \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m. \tag{25.6}$$

By definition,

$$\int_X \partial_i f_i \cdot \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m = \int_0^1 \cdots \int_0^1 (\partial_i f_i)(x_1, \ldots, x_m) \cdot \mathrm{d}x^1 \cdots \mathrm{d}x^m. \tag{25.7}$$

Using Fubini's theorem and the fundamental theorem of calculus, the latter can be evaluated to be

$$\int_0^1 \cdots \int_0^1 (f_i(x_1, \ldots, x_i = 1, \ldots, x_m) - f_i(x_1, \ldots, x_i = 0, \ldots, x_m)) \cdot \mathrm{d}x^1 \cdots \widehat{\mathrm{d}x^i} \cdots \mathrm{d}x^m. \tag{25.8}$$

These expressions vanish unless $i = 1$ and $X = [0, 1) \times (0, 1)^{m-1}$; in which case it simplifies to

$$-\int_0^1 \cdots \int_0^1 f_1(0, x_2, \ldots, x_m) \cdot \mathrm{d}x^2 \cdots \mathrm{d}x^m = \int_{\partial X} \alpha. \tag{25.9}$$

■

26 Riemannian volume, divergence, Hodge ∗−operator

Definition 26.1. A **Riemannian manifold** is a smooth manifold X together with a Riemannian metric g. •

Definition 26.2. Let (X, g) be an oriented Riemannian manifold. The **Riemannian volume form** is the unique volume form $\mathrm{vol}_g \in \Omega^n(X)$ with the property that

$$|\mathrm{vol}_g(e_1, \ldots, e_n)| = 1$$

whenever $(e_1, \ldots, e_n)$ is a positive orthonormal basis of T_xX. •

If (X, g) an oriented Riemannian manifold and $f \in C^\infty(M)$, we write

$$\int_X f := \int_X f \,\mathrm{vol}_g.$$

Exercise 26.3. Suppose $(x_1, \ldots, x_n)$ is a positive local coordinate system and the metric is given by $g_{ij} := g(\partial_{x_i}, \partial_{x_j})$. Show that the volume form is given by

$$\mathrm{vol}_g := \sqrt{\det(g_{ij})} \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$$

Definition 26.4. Suppose $v \in \mathrm{Vect}(X)$. The **divergence** of v is the function $\mathrm{div}(v) = \mathrm{div}_g(v) \in C^\infty(M)$ defined by

$$\mathscr{L}_v \mathrm{vol}_g = \mathrm{div}_g(v) \cdot \mathrm{vol}_g. \tag{26.5}$$

•

Exercise 26.6. In the situation of Exercise 26.3 if

$$v = \sum_{i=1}^{n} v^i \partial_{x_i}, \tag{26.7}$$

then

$$\mathrm{div}(v) = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{k=1}^{m} \partial_{x_k} \left(\sqrt{\det(g_{ij})} \cdot v^k \right). \tag{26.8}$$

Exercise 26.9. Let (X, g) be a oriented Riemannian manifold with boundary. For $h, k \in C^\infty(M)$ and $v \in \mathrm{Vect}(M)$, prove the integration by parts formula

$$\int_M \Big((\mathscr{L}_v k) \cdot h + k \cdot (\mathscr{L}_v h) + k \cdot h \cdot \mathrm{div}(v) \Big) \mathrm{vol}_g = \int_{\partial M} k \cdot h \ i(v)\mathrm{vol}_g. \tag{26.10}$$

∼

Definition 26.11. Let (X, g) be an Riemannian manifold with boundary. Given $f \in C^\infty(X)$, the **gradient** of f is the vector field $\nabla f = \nabla_g f \in \mathrm{Vect}(X)$ defined by

$$\mathrm{d}f(v) = g(\nabla f, v) \tag{26.12}$$

and the **Laplacian** of f is the function $\Delta f \in C^\infty(X)$ defined by

$$\Delta f := -\,\mathrm{div}(\nabla f). \tag{26.13}$$

The **outward pointing unit normal** is the vector field $n \in \Gamma(TX|_{\partial X})$ characterised by the following conditions: (a) $|n|_g = 1$, (b) $n(x) \perp T_x\partial X$, (c) if $(e_2, \ldots, e_m)$ is a positive basis of $T_x\partial X$, then $(-n, e_2, \ldots, e_m)$ is a positive basis of T_xX. •

Exercise 26.14. 1. Prove **Green's identities**

$$\int_X h\Delta k \,\mathrm{vol}_{X,g} = \int_X \langle \nabla h, \nabla k\rangle \,\mathrm{vol}_{X,g} - \int_{\partial X} h\mathscr{L}_n k \,\mathrm{vol}_{\partial X,g} \tag{26.15}$$

and

$$\int_X (h\Delta k - k\Delta h) \,\mathrm{vol}_{X,g} x = \int_{\partial X} (k\mathscr{L}_n h - h\mathscr{L}_n k) \,\mathrm{vol}_{\partial X,g} \tag{26.16}$$

with n denoting the outward-pointing unit normal.

2. Show that if $\partial X = \varnothing$, then $\Delta h = 0$ implies that h is constant.

3. Show that if $\partial X \neq \varnothing$, then $\Delta h = \Delta k = 0$ and $h|_{\partial X} = k|_{\partial X}$ implies that $h = k$.

∼

Definition 26.17. Let (V, g) be an m–dimensional Euclidean vector space. The **Hodge** $*$**–operator** is the linear map $*\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$ defined by

$$\alpha \wedge *\beta \coloneqq \langle \alpha, \beta\rangle_g \mathrm{vol}_g \tag{26.18}$$ •

Remark 26.19. Note that

$$* f = f \,\mathrm{vol}_g. \tag{26.20}$$ ♣

This operator naturally carries over to an operator on the space of differential forms on a Riemannian manifold.

Exercise 26.21. Show that in dimension m for α a k–form

$$* * \alpha = \begin{cases} \alpha & m \text{ odd} \\ (-1)^k & m \text{ even} \end{cases} \tag{26.22}$$

and thus $*^{-1}\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$ satisfies

$$*^{-1} = (-1)^{k(n-k)} * . \tag{26.23}$$

Remark 26.24. In dimension 2 and on 1–forms, $** = -\mathbf{1}$. Therefore, $*$ is (equivalent to) a complex structure. ♣

Remark 26.25. In dimension $4k$ and on $2k$–forms, $** = \mathbf{1}$. Therefore, $*\colon \Omega^{2k}(X) \to \Omega^{2k}(X)$ has eigenvalues ± 1. ♣

Exercise 26.26. Let $U \subset \mathbf{R}^m$ be an open and with smooth boundary. Denote by n the outward-pointing unit normal along ∂U. For $v = \sum_{i=1}^m v_i \partial_i$ a vector field define the 1–form $v^\flat$ by

$$v^\flat := \sum_{i=1}^m v_i \mathrm{d}x^i. \tag{26.27}$$

Conversely, for $\alpha = \sum_{i=1}^m a_i \mathrm{d}x^i$ a 1–form define the vector field $\alpha^\sharp$ by

$$v^\sharp := \sum_{i=1}^m a_i \partial_i i. \tag{26.28}$$

1. Prove that:
   (a) $\mathrm{d} * v^\flat = * \operatorname{div} v$,
   (b) $(*v^\flat)|_{\partial U} = \langle v, n \rangle \operatorname{vol}_{\partial U}$, and
   (c) $\int_U \operatorname{div} v \operatorname{vol}_U = \int_{\partial U} \langle v, n \rangle \operatorname{vol}_{\partial U}$.
2. Suppose that $m = 3$. Recall that, $\operatorname{curl} v = \nabla \times v$. Prove that
   (a) $(\operatorname{curl} v)^\flat = *\mathrm{d}v^\flat$,
   (b) $\operatorname{curl} \operatorname{div} = 0$.

   Let S be a surface with boundary in $\mathbf{R}^3$ oriented by a unit normal n. Denote by t the unit tangent vector field along ∂S such that, n, t, and the outward pointing unit normal along ∂S are positive. Prove that

   (c) $\int_S \langle \operatorname{curl}(v), n \rangle \operatorname{vol}_S = \int_{\partial S} \langle v, t \rangle \operatorname{vol}_{\partial S}$.

27 Covariant derivatives

Definition 27.1. Let X be a smooth manifold with boundary. Let E be a vector bundle over X. For $k \in \mathbf{N}_0$ set

$$\Omega^k(X, E) := \Gamma(\Lambda^k T^* X \otimes E). \tag{27.2}$$ •

Definition 27.3. Let X be a smooth manifold with boundary. Let E be a vector bundle over X. A **covariant derivative on** E is a linear map $\nabla \colon \Gamma(E) \to \Omega^1(X, E)$ such that for every $f \in C^\infty(X)$ and $s \in \Gamma(E)$

$$\nabla(f \cdot s) = \mathrm{d}f \otimes s + f \cdot \nabla s. \tag{27.4}$$ •

Proposition 27.5. *Let X be a smooth manifold with boundary. Let E be a vector bundle over X. The following hold:*

1. *There is a covariant derivative ∇ on E.*

2. *If ∇_0 is a covariant derivatives on E and $a \in \Omega^1(X, \mathrm{End}(E))$, then*

$$\nabla_1 := \nabla_0 + a. \tag{27.6}$$

   is a covariant derivative.

3. *If ∇_0, ∇_1 are covariant derivatives on E, then there is an $a \in \Omega^1(X, \mathrm{End}(E))$ with*

$$\nabla_1 = \nabla_0 + a. \tag{27.7}$$

In other words, the space of covariant derivatives on E is an affine space modelled on $\Omega^1(X, \mathrm{End}(E))$.

Proof. Evidently, the product vector bundle $\underline{V} = X \times V$ has a covariant derivative. Choose an open cover $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of X such that $E|_{U_\alpha}$ is trivial. Choose any connection ∇_α on $E|_{U_\alpha}$. Choose a partition of unity $\{\rho_\alpha : \alpha \in A\}$ subordinate to $\mathcal{U}$. Define ∇ by

$$\nabla s := \sum_{\alpha \in A} \rho_\alpha \nabla_\alpha s. \tag{27.8}$$

This is a covariant derivative. This proves (1).

(2) and (3) are straight-forward. ■

Definition 27.9. Let X be a smooth manifold with boundary.

1. Let E be a vector bundle over X equipped with a metric g. A covariant derivative ∇ on E is **orthogonal** if

$$\mathrm{d}g(s,t) = g(\nabla s, t) + g(s, \nabla t). \tag{27.10}$$

2. An **affine connection** is a covariant derivative on TX.

3. Let ∇ be an affine connection. The **torsion of** ∇ is the 2–form $T \in \Omega^2(X, TX)$ defined by

$$T_\nabla(v,w) := \nabla_v w - \nabla_w v - [v,w]. \tag{27.11}$$

4. An affine connection ∇ is **torsion free** if $T_\nabla = 0$. •

Theorem 27.12. *Let (X, g) be a Riemannian manifold. There is a unique orthogonal and torsion-free affine connection ∇_g^{LC} on X.*

Proof. If the Levi-Civita connection exists, then it satisfies

$$\begin{aligned}
2g(\nabla_u v, w) &= \mathscr{L}_u g(v, w) - g(\nabla_u w, v) + g(\nabla_u v, w) \\
&= \mathscr{L}_u g(v, w) - g([u, w], v) + g([u, v], w) - g(\nabla_w u, v) + g(\nabla_v u, w) \\
&= \mathscr{L}_u g(v, w) + \mathscr{L}_v g(u, w) - \mathscr{L}_w g(u, v) \\
&\quad - g([u, w], v) + g([u, v], w) + g(u, \nabla_w v) - g(u, \nabla_v w) \\
&= \mathscr{L}_u g(v, w) + \mathscr{L}_v g(w, u) - \mathscr{L}_w g(u, v) \\
&\quad + g([u, v], w) - g([v, w], u) + g([w, u], v);
\end{aligned}$$

hence,

$$\begin{aligned}
g(\nabla_u v, w) = \frac{1}{2}\big(&\mathscr{L}_u g(v, w) + \mathscr{L}_v g(w, u) - \mathscr{L}_w g(u, v) \\
&+ g([u, v], w) - g([v, w], u) + g([w, u], v)\big)
\end{aligned} \tag{27.13}$$

This is called the **Koszul formula**.

On the one hand, (27.13) shows that ∇ is determined uniquely; on the other hand, it can be checked that this formula defines a covariant derivative which is metric and torsion-free. ■

Definition 27.14. ∇_g^{LC} is the **Levi-Civita connection**. •

Example 27.15. For $\mathbf{R}^m$ with the standard Riemannian metric $g_0 := \sum_{i=1}^m \mathrm{d}x^i \otimes \mathrm{d}^i$ the Levi-Civita connection satisfies

$$\nabla^{g_0} = \mathrm{d}. \tag{27.16}$$

♠

Example 27.17. Let X be a submanifold of $\mathbf{R}^m$. Equip X with the Riemannian metric induced by the standard metric on $\mathbf{R}^m$. The Levi-Civita connection satisfies

$$\nabla_v^g w = (\nabla_v^{g_0} w)^t \tag{27.18}$$

Here $\cdot^t$ denotes the orthogonal projection $T_x\mathbf{R}^m \twoheadrightarrow T_xX$.

Define $\tilde{\mathrm{II}}\colon \mathrm{Vect}(X) \times \mathrm{Vect}(X) \to \Gamma(NX)$ by

$$\tilde{\mathrm{II}}(v, w) := -(\nabla_v^{g_0} w)^\perp. \tag{27.19}$$

with $\cdot^\perp$ denoting the orthogonal projection $T_x\mathbf{R}^m \twoheadrightarrow N_xX := T_xX^\perp$.

Evidently $\tilde{\mathrm{II}}$ is tensorial in its first argument. If n is a normal vector field to X, then

$$\langle \nabla_v^{g_0} w, n\rangle - \langle \nabla_w^{g_0} v, n\rangle = \langle [v, w], n\rangle = 0. \tag{27.20}$$

Therefore, $\tilde{\mathrm{II}}$ is symmetric. Therefore, $\tilde{\mathrm{II}}$ is tensorial. The **second fundamental form of** X is the tensor $\mathrm{II} \in \Gamma(\mathrm{Hom}(S^2TX, NX))$ associated with $\tilde{\mathrm{II}}$.

By construction,

$$\nabla^g_v w = \nabla^{g_0}_v w + \mathrm{II}(v, w). \tag{27.21}$$

♠

Remark 27.22. Let (X, g) be a Riemannian manifold. Let $x_0, x_1 \in X$. Set

$$\mathscr{P} = \mathscr{P}_{x_0,x_1} := \{\gamma \in C^\infty([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}. \tag{27.23}$$

The critical points of the functional $L\colon \mathscr{P} \to [0, \infty)$ defined by

$$E(\gamma) := \frac{1}{2}\int_0^1 |\dot\gamma(t)|^2 \,\mathrm{d}t \tag{27.24}$$

play a crucial role. Since

$$\mathrm{d}_\gamma E(\delta) = \int_0^1 \langle \dot\gamma(t), \dot\delta(t)\rangle \,\mathrm{d}t, \tag{27.25}$$

the Euler–Lagrange equation for E is

$$\nabla_t \dot\gamma = 0. \tag{27.26}$$

Solutions of this equation are called **geodesics**. ♣

Proposition 27.27. *Let X be a smooth manifold with boundary. Let E, F be a vector bundles over X. Let ∇_E be a connection on E and let ∇_F be a connection F.*

1. *There is a unique connection $\nabla_{E\otimes F}$ on $E \otimes F$ such that*

$$\nabla_{E\otimes F}(s \otimes t) = (\nabla_E s) \otimes t + s \otimes (\nabla_F t). \tag{27.28}$$

2. *There is a unique connection ∇_{E^*} on E^* such that*

$$\mathrm{d}(s^*(s)) = (\nabla_{E^*} s^*)(s) + s(\nabla_E s). \tag{27.29}$$

3. *There is a unique connection $\nabla_{\mathrm{Hom}(E,F)}$ on $\mathrm{Hom}(E, F)$ such that*

$$\nabla_F(\Lambda s) = (\nabla_{\mathrm{Hom}(E,F)}\Lambda)(s) + \Lambda(\nabla_E s). \tag{27.30}$$

Proof. It suffices to prove this for $E = X \times V$ and $F = X \times V$. After writing $\nabla_E = \mathrm{d} + a$ and $\nabla_F = \mathrm{d} + b$, the covariant derivatives are

1. $\nabla_{E\otimes F} s \otimes t = \mathrm{d}(s \otimes t) + as \otimes t + s \otimes bt$

2. $\nabla_{E^*} s^* = \mathrm{d}s^* - a^* s^* = \mathrm{d}s^* - s^* a$.

3. $\nabla_{\mathrm{Hom}(E,F)}\Lambda = \mathrm{d}\Lambda - \Lambda \circ a^* + b \circ \Lambda$. ■

Proposition 27.31. *Let X be a smooth manifold with boundary. Let E be a vector bundle over X. Let ∇ be a covariant derivative on E. There is a unique linear map $\mathrm{d}_\nabla \colon \Omega^\bullet(X, E) \to \Omega^{\bullet+1}(X, E)$ such that*

1. *For every $s \in \Gamma(E)$*

$$\mathrm{d}_\nabla s = \nabla s \tag{27.32}$$

2. *For every $\alpha \in \Omega^\bullet(X)$ and $\sigma \in \Omega^\bullet(X, E)$*

$$\mathrm{d}_\nabla(\alpha \wedge \sigma) = (\mathrm{d}\alpha) \wedge \sigma + (-1)^{\deg \alpha} \alpha \wedge (\mathrm{d}_\nabla \sigma). \tag{27.33}$$

Moreover:

3. *There is a unique $F_\nabla \in \Omega^2(X, \mathrm{End}(E))$ such that for every $\alpha \in \Omega^\bullet(X, E)$*

$$\mathrm{d}_\nabla \mathrm{d}_\nabla \alpha = F_\nabla \wedge \alpha. \tag{27.34}$$

4. *$F_\nabla \in \Omega^2(X, \mathrm{End}(E))$ satisfies*

$$F_\nabla(v, w)s = (\nabla_v \nabla_w - \nabla_w \nabla_v - \nabla_{[v,w]})s. \tag{27.35}$$

Proof. It suffices to prove this for $E = X \times V$. In this case ∇ can be written as $\mathrm{d} + a$ with $a \in \Omega^1(X, \mathrm{End}(E))$. Define $\mathrm{d}_\nabla \coloneqq \mathrm{d} + a \wedge \cdot$. This satisfies (1) and (2); and (1) and (2) force d_∇ to be defined in this way.

A computation shows that

$$(\mathrm{d} + a \wedge \cdot)(\mathrm{d} + a \wedge \cdot)\sigma = (\mathrm{d}a + a \wedge a) \wedge \sigma.$$

This proves (3) with $F_\nabla = \mathrm{d}a + a \wedge a$. ■

Definition 27.36. The 2–form F_∇ is the **curvature of ∇**. •

Proposition 27.37 (Bianchi identity). *Let X be a smooth manifold with boundary. Let E be a vector bundle over X. Let ∇ be a covariant derivative on E. The curvature F_∇ satisfies*

$$\mathrm{d}_{\nabla_{\mathrm{End}}} F_\nabla = 0. \tag{27.38}$$

Proof. This is trivial:

$$(\mathrm{d}_{\nabla_{\mathrm{End}}} F_\nabla)s = \mathrm{d}_\nabla(F_\nabla s) - F_\nabla \mathrm{d}_\nabla s = \mathrm{d}_\nabla \mathrm{d}_\nabla \mathrm{d}_\nabla s - \mathrm{d}_\nabla \mathrm{d}_\nabla \mathrm{d}_\nabla s. = 0 \tag{27.39}$$

■

Definition 27.40. The curvature of the Levi-Civita connection is the **Riemannian curvature tensor of** g and denoted by $R_g \in \Omega^2(X, \mathrm{End}(TX))$. •

Proposition 27.41. *R_g takes values in $\mathfrak{so}(TX)$: the skew-adjoint endomorphisms of TX.*

Proof. We compute

$$\begin{aligned} g(R_g(v,w)s,t) &= g(\nabla_v\nabla_w s,t) - g(\nabla_w\nabla_v s,t) - g(\nabla_{[v,w]}s,t) \\ &= \mathscr{L}_v\mathscr{L}_w g(s,t) + g(s,\nabla_w\nabla_v t) - \mathscr{L}_w\mathscr{L}_v g(s,t) - g(s,\nabla_v\nabla_w t) \\ &\quad - \mathscr{L}_{[v,w]}g(s,t) + g(\nabla_{[v,w]}s,t) \\ &= -g(s,R_g(v,w)t). \end{aligned}$$ ∎

Example 27.42. Let X be a submanifold of $\mathbf{R}^m$ with the Riemannian metric g induced by the standard Riemannian metric g_0. Denote the second fundamental form by II. We have

$$\begin{aligned} R_g(v,w)s &= \nabla_v\nabla_w s - \nabla_w\nabla_v s - \nabla_{[v,w]}s \\ &= \nabla_v(\nabla^0_w s + \mathrm{II}(w,s) - \nabla_w(\nabla^0_v s + \mathrm{II}(v,s) - (\nabla^0_{[v,w]} + \mathrm{II}([v,w],s) \\ &= \nabla^0_v\nabla^0_w s + \nabla^0_v(\mathrm{II}(w,s)) + \mathrm{II}(v,\nabla^0_w s + \mathrm{II}(w,s)) \\ &\quad - \left(\nabla^0_w\nabla^0_v s + \nabla^0_w(\mathrm{II}(v,s)) + \mathrm{II}(w,\nabla^0_v s + \mathrm{II}(v,s))\right) \\ &\quad - (\nabla^0_{[v,w]} + \mathrm{II}([v,w],s) \\ &= R_{g_0}(v,w) + (\nabla^0_v\mathrm{II})(w,s) - (\nabla^0_w\mathrm{II})(v,s). \end{aligned}$$

This is the **Codazzi–Mainardi equation**. The same line of reasoning also proves the **Gauß equation**

$$\langle R_g(v,w)s,t\rangle = \langle R_{g_0}(v,w)s,t\rangle + \langle \mathrm{II}(v,s),\mathrm{II}(w,t)\rangle - \langle \mathrm{II}(w,s),\mathrm{II}(v,t)\rangle. \tag{27.43}$$

Since $R_{g_0} = 0$, these allow to determine R_g from II algebraically! ♠

Example 27.44. Consider $S^n \subset \mathbf{R}^{n+1}$. The second fundamental form II_x at $x \in S^n$ is

$$\mathrm{II}_x(v,w) = -\langle \nabla_v w, x\rangle \cdot x = \langle v,w\rangle \cdot x \tag{27.45}$$

Therefore,

$$R_g(v,w)s = \langle w,s\rangle v - \langle v,s\rangle w. \tag{27.46}$$ ♠

Definition 27.47. Let (Σ, g) be a Riemannian manifold of dimension two.

1. The **Gauß curvature of** g is the unique function $K \in C^\infty(\Sigma)$ satisfying

   $$\kappa_g(x) = g(R_g(e_1,e_2)e_2,e_1) \tag{27.48}$$

   for every orthonormal basis e_1, e_2 of $T_x\Sigma$.

2. The **geodesic curvature of** $\partial\Sigma$ is the unique function $k_g \in C^\infty(\partial\Sigma)$ satisfying the following: if $\gamma\colon I \to \partial\Sigma$ is a positively oriented curve with $|\dot\gamma| = 1$, then

$$k_g(\gamma(t)) = g(\nabla\gamma(t), n(\gamma(t))). \tag{27.49}$$

●

Theorem 27.50 (Gauß–Bonnet). *If (Σ, g) be an oriented closed Riemannian manifold of dimension two, then*

$$\int_\Sigma K + \int_{\partial\Sigma} k_g = 2\pi\chi(\Sigma). \tag{27.51}$$

Here $\chi(\Sigma)$ denotes the Euler characteristic of Σ.

28 The degree

Definition 28.1. Let $m \in \mathbf{N}_0$. Let X and Y be oriented, smooth manifold without boundary and of dimension m. Let $f\colon X \to Y$ be a proper, smooth map. Let $y \in Y$ be a regular value of f. Set

$$\deg(f; y) := \sum_{x\in f^{-1}(y)} \operatorname{sign}(T_x f) \in \mathbf{Z}. \tag{28.2}$$

Here $\operatorname{sign} T_x f \in \{+1, -1\}$ depending on whether $T_x f$ is orientation preserving or orientation reversing. ●

Example 28.3 (PICTURES). ♠

Theorem 28.4. *Assume the situation of Definition 28.1 and that Y is connected.*

1. *The integer*

$$\deg(f) := \deg(f; y) \tag{28.5}$$

is independent of $y \in Y$.

2. *If f, g are homotopic, then*

$$\deg(f) = \deg(g). \tag{28.6}$$

The proof requires the following preparation.

Proposition 28.7. *Assume the situation of Theorem 28.4. If X is the boundary of a oriented, compact manifold W and f extends to a smooth map $F\colon W \to Y$, then* $\deg(f; y) = 0$.

Proof. Let $y \in Y$ be a regular value of f. By Proposition 5.24, there is an open neighborhood U of $y \in Y$ such that $f\colon\ f^{-1}(U) \to U$ is a covering map. A moment's thought shows that $\deg(f;z) = \deg(f;y)$ for every $z \in U$.

By Theorem 7.12, it can be assumed that y is a regular value for both f and F. By Theorem 6.12,

$$I \coloneqq F^{-1}(y). \tag{28.8}$$

is a compact 1–manifold with boundary. By Theorem 2.32, I is a finite disjoint union of embeddings $\gamma\colon\ [0,1] \to W$. It remains to prove that

$$\operatorname{sign}(T_{\gamma(0)}f) + \operatorname{sign}(T_{\gamma(1)}f) = 0. \tag{28.9}$$

The orientations of W and Y determine an orientation of I as follows. For $z \in I$ and a positive basis $(v_1, \ldots, v_{m+1})$ of T_zW with $v_1 \in T_zI$. Declare v_1 to be positive if and only if $T_zF(v_2), \ldots, T_zF(v_{m+1})$ is a positive basis of T_yf.

Without loss of generality, X is oriented as ∂W. Choose a positive vector field v_1 along $\operatorname{im}\gamma$ pointing inward at $\gamma(0)$ and outward at $\gamma(1)$. By construction

$$\operatorname{sign}(T_{\gamma(0)}f) + \operatorname{sign}(T_{\gamma(1)}f) = 0. \tag{28.10}$$

[PICTURE] ■

Proof of Theorem 28.4. If f, g are homotopic, then there exists a smooth map $H\colon\ [0,1] \times X \to Y$ with $H(0,\cdot) = f$ and $H(1,\cdot) = g$. The boundary of $[0,1] \times X$ are two copies of X oriented in opposite ways. Therefore, if y is a regular value of f and g, then

$$\deg(f;y) - \deg(g;y) = 0. \tag{28.11}$$

To see that $\deg(f;y)$ is independent of y, let z be a further regular value of f. Let ϕ be a diffeomorphism of Y with $\phi(y) = z$ and homotopic to id_Y. Evidently,

$$\deg(\phi \circ f; z) = \deg(\phi \circ f; \phi(y)) = \deg(f, y). \tag{28.12}$$

By the preceding paragraph

$$\deg(\phi \circ f; z) = \deg(f; z). \tag{28.13}$$

This finishes the proof. ■

Proposition 28.14. *Let $m \in 2\mathbf{N}_0$. The anti-podal map $a\colon\ S^m \to S^m$ is not homotopic to the identity.*

Proof. A computation shows that $\deg(a) = -1$. ■

Theorem 28.15. *S^m admits a nowhere-vanishing vector field if and only if m is odd.*

Proof. Identify $v \in \mathrm{Vect}(S^m)$ with a smooth map $v\colon S^m \to \mathbf{R}^{m+1}$ satisfying

$$\langle v(x), x\rangle = 0 \tag{28.16}$$

for every $x \in S^m$. If v is nowhere-vanishing, then we might as well assume that $|v| = 1$.

Define $H\colon S^m \times [0,\pi] \times S^m$ by

$$H(x,\theta) = \cos(\theta)x + \sin(\theta)v(x). \tag{28.17}$$

(Check that $|H(x,\theta)| = 1$.) Evidently,

$$H(x,0) = x \quad \text{and} \quad H(x,\pi) = -x. \tag{28.18}$$

Therefore, m must be odd.

Indeed if $m = 2n$ is odd, then $v(x_1,\dots,x_{2n}) = (-x_2, x_1, \dots, x_{2n}, -x_{2n-1})$ is a nowhere-vanishing vector field. ■

29 The Poincaré–Hopf index theorem

Definition 29.1. Let X be a smooth manifold without boundary. Let $v \in \mathrm{Vect}(X)$.

1. A **zero** of v is a $x \in X$ with $v(x) = 0$.
2. A zero x of v is isolated, if there is an open neighborhood U of $x \in X$ such that, for every $y \in U$, $v(y) = 0$ if and only of $y = x$.
3. If $X = \mathbf{R}^m$ and 0 is an isolated zero of v, then the **index of** v **at** 0 is

   $$\mathrm{index}_0\, v := \deg(f) \tag{29.2}$$

   with $f\colon S^{m-1} \to S^{m-1}$ defined by

   $$f_\varepsilon(x) := \frac{v(\varepsilon x)}{|v(\varepsilon x)|} \tag{29.3}$$

   for $\varepsilon \ll 1$. •

The extension of the concept of index to manifolds requires the following.

Proposition 29.4. *Let $U, V \subset \mathbf{R}^m$ be open neighborhoods of 0. Let $\tau\colon U \to V$ be a diffeomorphism with $\tau(0) = 0$. Let $v \in \mathrm{Vect}(U)$ and $w \in \mathrm{Vect}(V)$ be vector fields an isolated zero at $v(0) = 0$. If v,w are τ–related and τ is orientation preserving, then*

$$\mathrm{index}_0\, v = \mathrm{index}_0\, w. \tag{29.5}$$

Proof. Suppose that $T\colon [0,1]\times U \to \mathbf{R}^m$ is a homotopy with:

1. T_t a diffeomorphism onto its image for every $t \in [0,1]$, and
2. $T_0 = \mathrm{id}$ and $T_1 = \tau$.

For $t \in [0,1]$ set $v_t := (T_t)_* v$. By construction v_t has an isolated zero at 0 for every $t \in [0,1]$, $v_0 = v$, and $v_1 = w$. Therefore, maps $f_0, f_1\colon S^{m-1} \to S^{m-1}$ are homotopic; hence: $\deg(f_0) = \deg(f_1)$.

It remains to produce T. After shrinking U, the map $T\colon [0,1]\times U \to \mathbf{R}^m$ defined by

$$T(t,x) := \begin{cases} \mathrm{d}_0\tau(x) & t = 0 \\ t^{-1}\tau(tx) & t \in (0,1] \end{cases} \tag{29.6}$$

is a homotopy of the desired kind but with $T_0 = \mathrm{d}_0\tau$. Since $\mathrm{GL}^+(\mathbf{R}^m)$ is connected, $\mathrm{d}_0\tau$ is homotopic to id through orientation-preserving isomorphisms. ■

Definition 29.7. Let X be a oriented, smooth manifold with boundary. Let $v \in \mathrm{Vect}(X)$.

1. If x is isolated zero of v, then the **index of v at x** is
$$\mathrm{index}_x\, v := \mathrm{index}_0\, \tilde{v} \tag{29.8}$$
with $\tilde{v} := \phi_* v$ for some orientation-preserving, admissible chart ϕ with $\phi(x) = 0$.
2. If v has only isolated zeros, then the **index of v**
$$\mathrm{index}\, v := \sum_{x \in v^{-1}(0)} \mathrm{index}_x\, v. \tag{29.9}$$
•

Proposition 29.10. *Let X be a smooth manifold with boundary. Let $v \in \mathrm{Vect}(X)$. Let $x \in v^{-1}(0)$.*

1. *There is a unique linear map $L = L_v \in T_xX$ such that for every chart ϕ with $\phi(x) = 0$*
$$\mathrm{d}_x\phi \circ L_v = \mathrm{d}_x\tilde{v} \tag{29.11}$$
with $\tilde{v}(y) = \mathrm{d}_y\phi(v(y))$.
2. *If L is invertible, then x is an isolated zero and* $\mathrm{index}_x\, v = \operatorname{sign}\det L$.

Proof. Omitted/Exercise. ■

Theorem 29.12 (Poincaré–Hopf index theorem). *Let X be a compact, smooth manifold without boundary. Let $v \in \mathrm{Vect}(X)$. If v has only isolated zeros, then*

$$\mathrm{index}\, v = \chi(X). \tag{29.13}$$

The right-hand side here is the Euler characteristic of X. The proof consists of two parts. The first proves the $\operatorname{index} v$ depends only on X. The second computes $\operatorname{index} v$ in a suitable case to prove its equality with $\chi(X)$. The proof presented in the following taken from [Mil97, §6].

Proposition 29.14. *Let X be a submanifold with boundary of $\mathbf{R}^m$ of codimension zero. Denote by $n \in C^\infty(\partial X, S^{m-1})$ the outward-pointing unit normal. Let $v \in \operatorname{Vect}(X)$. If v has only isolated zeros and is outward-pointing along ∂X, then*

$$\operatorname{index} v = \deg n. \tag{29.15}$$

Proof. Let $\varepsilon \ll 1$. Set

$$Y := X \backslash \bigcup \{B_\varepsilon(x) : x \in v^{-1}(0)\}. \tag{29.16}$$

Define $\tilde{v}\colon\ Y \to S^{m-1}$ by

$$\tilde{v}(x) := \frac{v(x)}{|v(x)|}. \tag{29.17}$$

By Proposition 28.7,

$$0 = \deg \tilde{v} = \deg \tilde{v}|_{\partial X} - \sum_{x \in v^{-1}(0)} \operatorname{index}_x v. \tag{29.18}$$

The sign arises because ∂B_ε has the opposite orientation in ∂Y. A moment's thought shows that $\tilde{v}|_{\partial X}$ and n are homotopic; hence, $\deg \tilde{v}|_{\partial X} = \deg n$. ■

Proposition 29.19. *In the situation of Theorem 29.12,* $\operatorname{index} v$ *depends only on X.*

Proof. To begin with one observes that it suffices to consider v is non-degenerate singularities. This is a local problem. Let $v \in C^\infty(B_2^m(0), \mathbf{R}^m)$ be a vector field with an isolated singularity at 0. Let $\chi \in C^\infty(B_2^m(0))$ be a cut-off function with compact support and $\chi|_{B_1^m(0)} = 1$. If w is a sufficiently small regular value of v, then $\tilde{v}(x) := v(x) + \chi(x)w$ has only non-degenerate zero all of which are contained in $B_1^m(0)$. Finally, it is consequence of Proposition 28.7 that $\operatorname{index} \tilde{v} = \deg(\frac{v}{|v|}\colon\ \partial B_2 \to S^{m-1}) = \operatorname{index} v$. [DRAW PICTURE FOR THIS]

To simplify By Theorem 10.1, without loss of generality X is a submanifold of $\mathbf{R}^m$. Therefore, $v \in C^\infty(X, \mathbf{R}^m)$.

Choose a tubular neighborhood U of $\mathbf{R}^m$. Denote the corresponding retraction map by $r\colon\ U \to X$.

Define $\tilde{v} \in C^\infty(U, \mathbf{R}^m)$ by

$$\tilde{v} = v \circ r + \nabla f \quad \text{with} \quad f(x) := \frac{1}{2}|x - r(x)|^2. \tag{29.20}$$

For $\varepsilon \ll 1$ set

(29.21) $$U_\varepsilon := f^{-1}((-\infty, \varepsilon]).$$

Since $\varepsilon \ll 1$, the following hold:

1. U_ε is a submanifold with boundary and of codimension zero.
2. $\tilde{v}$ is outward-pointing along ∂U_ε.
3. For every $x \in X$, $\nabla f(x) \perp v(x)$; therefore: $\tilde{v}(x) = 0$ if and only if $v(x) = 0$.
4. For every $x \in v^{-1}(x)$,

   (29.22) $$\operatorname{index}_x v = \operatorname{index}_x \tilde{v}.$$

This implies that

(29.23) $$\operatorname{index} v = \deg n$$

with n denoting the outward-pointing unit normal of ∂U_ε. The latter depends only on X (and a choice of $\varepsilon \ll 1$). ■

30 A sketch of Morse theory

Algebraic topology assigns to every manifold X and every $k \in \mathbf{N}$ an finite dimensional vector space

(30.1) $$\mathrm{H}^k(X, \mathbf{R})$$

called the cohomology of X of degree k with coefficients in $\mathbf{R}$. If X is compact, then these spaces are finite-dimensional and $\mathrm{H}^k(X, \mathbf{R}) = 0$ for $k \gg 1$. The **Euler characteristic** of X is defined as

(30.2) $$\chi(X) := \sum_{k=0}^{\infty} (-1)^k \dim \mathrm{H}^k(X, \mathbf{R}).$$

There are many ways to construct $\mathrm{H}^k(X, \mathbf{R})$, for example, using differential forms. For our purposes the following facts are important:

1. If $X = U \cup V$ is a decomposition into open subsets, then there is a long exact sequence

   (30.3) $$\cdots \to \mathrm{H}^k(X) \to \mathrm{H}^k(U) \cap \mathrm{H}^k(V) \to \mathrm{H}^k(U \cap V) \to \mathrm{H}^{k+1}(X) \to \cdots.$$

   This is the **Mayer–Vietoris sequence**. As a consequence,

   (30.4) $$\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V).$$

2. If X and Y are homotopy-equivalent, then $\mathrm{H}^k(X) \cong \mathrm{H}^k(Y)$; in particular, $\chi(X) = \chi(Y)$.

3. If X is a single point, then $\mathrm{H}^0(X) \cong \mathbf{R}$ and $\mathrm{H}^k(X) = 0$ for $k \geqslant 1$.

4. $\mathrm{H}^0(S^0) \cong \mathbf{R}^2$ and, for $m \geqslant 1$, $\mathrm{H}^0(S^m) \cong \mathbf{R} \cong \mathrm{H}^m(S^m)$ with all other cohomology groups vanishing. (This can be proved using Mayer–Vietoris.) In particular, $\chi(S^m) = 2$ if m is even and $\chi(S^m) = 0$ if m is odd.

5. If X is a compact odd-dimensional manifold, then $\chi(X) = 0$. (This is a consequence of Poincaré duality.)

The link between the the index of a vector field $v \in \mathrm{Vect}(X)$ and $\chi(X)$ can be made using **Morse theory**.

Proposition 30.5. *Let X be a smooth manifold. Let $f \in C^\infty(X)$. If $\mathrm{d}_x f = 0$, then there is a unique symmetric bilinear map* $\mathrm{Hess}\, f \in \mathrm{Hom}(S^2T_xX, \mathbf{R})$ *such that*

$$\mathrm{Hess}_x(v, w) = \mathscr{L}_w\mathscr{L}_v f(x). \tag{30.6}$$

Proof. Define $Q\colon \mathrm{Vect}(X) \times \mathrm{Vect}(X) \to \mathbf{R}$ by

$$Q(v, w) := \mathscr{L}_w\mathscr{L}_v f(x). \tag{30.7}$$

Evidently, Q is tensorial in the first variable. Since

$$Q(v, w) - Q(w, v) = \mathscr{L}_{[v,w]} f(x) = 0, \tag{30.8}$$

Q is symmetric. Therefore, it is tensorial and defines $\mathrm{Hess}_x f(v, w)$. ■

Let V be a vector space. Let $Q \in \mathrm{Hom}(S^2V, \mathbf{R})$ be a symmetric bilinear form. There is a basis of V with respect to which Q takes the form

$$Q \sim \begin{pmatrix} -\mathbf{1}_p & & \\ & \mathbf{1}_q & \\ & & 0_r \end{pmatrix}. \tag{30.9}$$

Q is non-degenerate if $r = 0$. Its index is

$$\mathrm{index}\, Q := q. \tag{30.10}$$

Definition 30.11. Let X be a smooth manifold. Let $f \in C^\infty(X)$.

1. A critical point $x \in \mathrm{crit}\, f$ is **non-degenerate** if $\mathrm{Hess}_x f$ is non-degenerate.

2. The **index of** f **at** $x \in critf$ is

$$\mathrm{index}_x f := \mathrm{index}\,\mathrm{Hess}_x f. \tag{30.12}$$

3. The function f is a **Morse function** if its critical points are non-degenerate.
4. A Morse function f is **self-indexing** if for every $x \in \mathrm{crit}(f)$

$$\mathrm{index}_x f = f(x). \tag{30.13}$$

•

A moment's thought shows that if g is a Riemannian metric on X and ∇f denotes the gradient of f, then

$$\mathrm{index}_x \nabla f = (-1)^{\mathrm{index}_x f}. \tag{30.14}$$

Theorem 30.15. *Every closed smooth manifold X admits a Morse function.*

Lemma 30.16 (Morse Lemma). *Let X be a smooth manifold. Let $f \in C^\infty(X)$. If $x \in \mathrm{crit}\, f$ is non-degenerate, then there is an admissible chart ϕ with $\phi(x) = 0$ and*

$$f \circ \phi(x_1, \dots, x_m) = f(x) - \sum_{i=1}^{p} x_i^2 + \sum_{j=p+1}^{m} x_j^2 \quad \textit{with} \quad p := \mathrm{index}_x f. \tag{30.17}$$

Proof. Without loss of generality $X = U \subset \mathbf{R}^m$, $x = 0$, and $f(x) = 0$. By Taylor expansion

$$f(x_1, \dots, x_m) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(tx_1, \dots, tx_m)\,\mathrm{d}t = \sum_{i=1}^{m} x_i g_i(x) \tag{30.18}$$

with

$$g_i(x) := \int_0^1 \partial_i f(tx_1, \dots, tx_m)\,\mathrm{d}t. \tag{30.19}$$

Since 0 is a critical point of f, $g_i(0) = 0$ for $i = 1, \dots, m$. Applying the same construction again,

$$f(x_1, \dots, x_m) = \sum_{i.j=1}^{m} x_i x_j h_{ij}(x). \tag{30.20}$$

Without loss of generality, $h_{ij} = h_{ji}$. Therefore, $h_{ij}(0)$ are the coefficients of $\mathrm{Hess}_0 f$. In particular, after shrinking U, for every $x \in U$, $\det(h_{ij}(x)) \neq 0$.

By induction it can be assumed that

$$f(x_1, \dots, x_m) = \sum_{i=1}^{r-1} \varepsilon_i x_i^2 + \sum_{i,j=r}^{m} x_i x_j h_{ij}(x) \tag{30.21}$$

and $h_{rr} \neq 0$. Define

$$\tilde{x}_r := \sqrt{|h_{rr}(x)|}\left(x_r + \sum_{i>r} x_i h_{ir}(x)/h_{rr}(x)\right).$$

Replacing x_r with $\tilde{x}_r$ rewrite f as above but with $r+1$ instead of r. ∎

Proposition 30.22. *Let X be a closed smooth manifold. Let $f \in C^\infty(X)$ be a Morse function. Let $\lambda < \mu \in \mathbf{R}$. If $[\lambda, \mu]$ does not contain any critical value of f, then*

$$X_\lambda := f^{-1}(-\infty, \lambda] \quad \text{and} \quad X_\mu := f^{-1}(-\infty, \mu] \tag{30.23}$$

are diffeomorphic.

Proof. By the Ehresmann fibration theorem, $f^{-1}[\lambda, \mu]$ is diffeomorphic to $[\lambda, \mu] \times f^{-1}(\lambda)$. [PICTURE] This implies the assertion. ∎

To understand the relation between X_λ and X_μ if $[\lambda, \mu]$ contains a critical point, consider the following:

Figure 30.1: Attaching a 1–handle passing around the critical point of $f(x, y) = x^2 - y^2$.

Definition 30.24. Let $m \in \mathbf{N}_0$ and $k \in \{0, \dots, m\}$.

1. The k–**handle** is the manifold $D^{m-k} \times D^k$ obtained from Proposition 13.4.

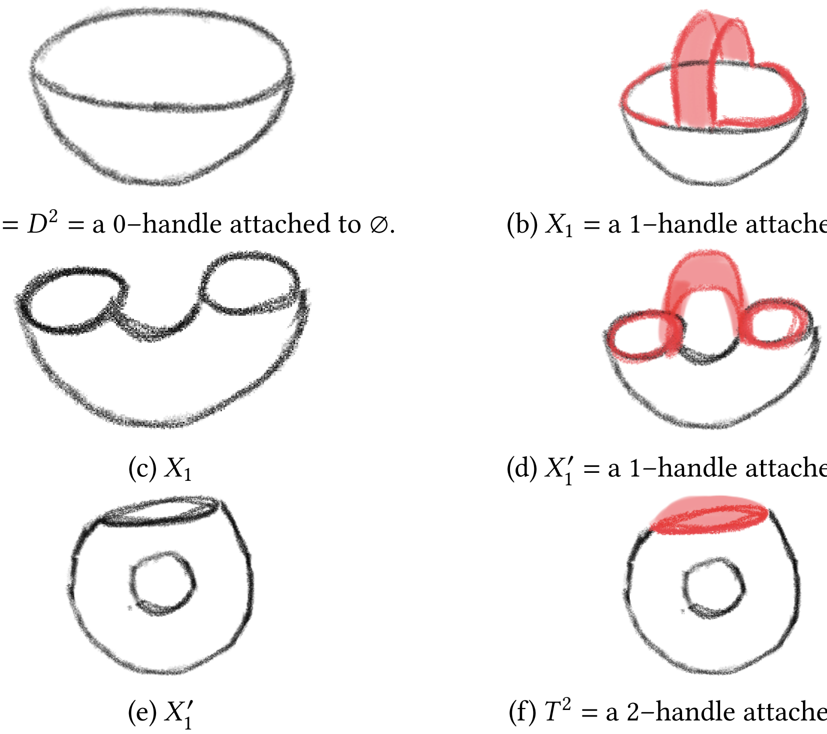
2. Let X and Y be smooth manifolds of dimension m with boundary. Let $\eta\colon S^{k-1} \times D^{m-k} \to \partial X$ be an embedding. Y is **obtained by attaching a k–handle to X along η** if it is diffeomorphic to the smooth manifold obtained by gluing X and $D^k \times D^{m-k}$ via η.

3. Let X be a closed smooth manifold of dimension m with boundary. A **handle decomposition of** X is a sequence

   (30.25) $$\varnothing =: X_{-1} \subset X_0 \subset \cdots \subset X_m = X$$

   of submanifolds such that, for every $k \in \{0, \dots, m\}$, X_k is obtained by attaching k–handles to X_{k-1}. •

Remark 30.26. Attaching a 0–handle is taking the disjoint union with D^m. ♣



(a) $X_0 = D^2$ = a 0–handle attached to $\varnothing$.

(b) X_1 = a 1–handle attached to X_0.

(c) X_1

(d) X_1' = a 1–handle attached to X_1'.

(e) X_1'

(f) T^2 = a 2–handle attached to X_1'.

Figure 30.2: Handle decomposition of T^2.

Proposition 30.27. *Let X be a closed smooth manifold. Let $f \in C^\infty(X)$ be a Morse function. Let $\lambda < \mu \in \mathbf{R}$. If $f^{-1}([\lambda, \mu])$ contains precisely one critical points of index k, then X_μ is obtained from X_λ by attaching a k–handle.*

Proof. A detailed proof is contained in Milnor [Mil63, §3]. ■

Proposition 30.28. *If X is obtained from Y by attaching n k–handles, then*

(30.29) $$\chi(X) = \chi(Y) + n \cdot (-1)^k.$$

Proof. This follows from the properties of χ mention at the top. ■

Corollary 30.30. *Let X be a closed manifold. If f is a Morse function, then*

$$\chi(X) = \sum_{x \in \operatorname{crit} f} \operatorname{index}_x f = \operatorname{index} \nabla f. \tag{30.31}$$

∎

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