Differential Geometry 1 (M13)
Exercise Sheet 1

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Try to solve the following six problems by yourself before the tutorial on **2020-11-11**.

Problem 1.

1. Construct a smooth function $\rho\colon \mathbf{R} \to \mathbf{R}$ such that $\rho(t) = 0$ for $t \leqslant 0$ and $\rho(t) > 0$ for $t > 0$.
2. Use ρ to construct a smooth function $\chi\colon \mathbf{R} \to [0,1]$ such that $\chi(t) = 0$ for $|t| \geqslant 2$ and $\chi(t) = 1$ for $|t| \leqslant 1$. ◇

Definition. Let X be a topological space. A C^0–atlas $\mathcal{A} = \{\phi_\alpha : \alpha \in A\}$ on X is **affine** if for every $\alpha, \beta \in A$ the transition function τ_β^α is affine (that is: it is of the form: $\tau_\beta^\alpha(x) = Ax + b$ with $A \in \mathrm{GL}(\mathbf{R}^n)$ and $b \in \mathbf{R}^n$). •

Problem 2. Let $n \in \mathbf{N}$. Construct an affine atlas on

$$T^n \coloneqq \mathbf{R}^n/\mathbf{Z}^n.$$ ◇

Remark. It is an interesting question to ask which manifolds admit affine atlases; e.g.: does S^2 admit an affine atlas? (Of course, this question is out of reach in week one of this course.) ♣

The next two problems construct spaces which admit smooth atlases but fail to be Hausdorff or second-countable.

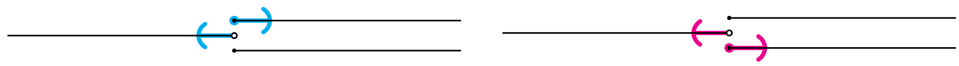
Problem 3. Define the equivalence relation $\sim$ on $\mathbf{R} \times \{+1, -1\}$ by

$$(x, i) \sim (y, j) \quad \text{if and only if} \quad x = y \text{ and } (i = j \text{ or } x < 0).$$

The **branching line** is the quotient space

$$\Lambda \coloneqq (\mathbf{R} \times \{+1, -1\})/\sim .$$

Here is an attempt to illustrate Λ and some of its open subsets:



Prove that Λ not Hausdorff but second-countable and admits a smooth atlas. $\diamond$

Remark. The previous problem shows that if you glue two copies of the intervals $(-1, 1)$ along $(-1, 0)$ in an orientation-preserving way, then the resulting space is not Hausdorff. If you glue in an orientation-reversing way, however, then the resulting space is Hausdorff. This observation lies at the heart of an elementary proof of the fact that every connected manifold of dimension one is either diffeomorphic to S^1 or an interval. ♣

The next problem requires a little preparation.

Definition. Let S be a set. A relation $\preceq$ on S is a **total order** if for every $x, y, z \in S$:

1. if $x \preceq y$ and $y \preceq x$, then $x = y$,
2. if $x \preceq y$ and $y \preceq z$, then $x \preceq z$, and
3. $x \preceq y$ or $y \preceq x$.

A total order $\preceq$ is a **well-order** if for every $\varnothing \neq T \subset S$ has a **least element** $\min T$ satisfying $\min T \preceq x$ for every $x \in T$.

A **ordered set** is a pair $(S, \preceq)$ consisting of a set S and a total order $\preceq$ on S. A **well-ordered set** is a pair $(S, \preceq)$ consisting of a set S and a well-order $\preceq$ on S. •

Remark. If $(S, \preceq)$ is a well-ordered set and $x \in S$, then either x is the greatest element or there is a unique least element greater than x. In the later case, set

$$x + 1 := \min\{y \in S : x \prec y\}.$$

Here $x \prec y$ if and only if $x \preceq y$ and $x \neq y$. ♣

Definition. Let $(S_1, \preceq_1)$ and $(S_2, \preceq_2)$ be ordered sets. The **lexicographic order** $\preceq$ on $S_1 \times S_2$ induces by $\preceq_1$ and $\preceq_2$ is the total order defined by

$$(x_1, x_2) \preceq (y_1, y_2) \quad \text{if and only if} \quad x_1 \preceq_1 y_1 \text{ and } (x_1 \neq y_1 \text{ or } y_1 \preceq_2 y_2). \qquad \bullet$$

Definition. If $(S, \preceq)$ is an ordered set, then the **order topology** $\mathcal{O}_\preceq$ is the coarsest topology on S with respect to which for every $a, b \in S$ the subset

$$(a, \infty) := \{x \in S : a \prec x\} \quad \text{and} \quad (-\infty, b) := \{x \in S : x \prec b\} \tag{1.1}$$

are open. •

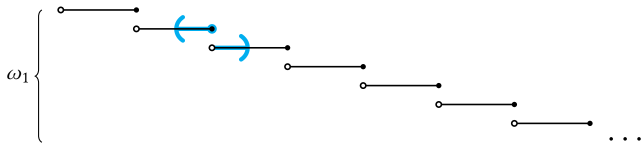
Example. The order topology on $(\mathbf{R}, \leqslant)$ agrees with the topology induces by the metric $d(x, y) := |x - y|$. ♠

Problem 4. Let $(\omega_1, \preceq)$ be an uncountable well-ordered set. It is a fact of set theory that these exist. In fact, the axiom of choice is equivalent to the well-ordering theorem which asserts that every set admits a well-order.

The **long line** is

$$\mathbf{L} := \omega_1 \times (0, 1]$$

equipped with the order topology induced by the lexicographic order. Here is an attempt to illustrate $\mathbf{L}$ and one of its open subsets:



Prove that $\mathbf{L}$ not second-countable but Hausdorff and admits a smooth atlas. ◇

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Problem 5. Let $n \in \mathbf{N}$. Prove that the map $\Pi\colon \mathbf{R}P^n \to \mathrm{End}(\mathbf{R}^{n+1})$ defined by

$$\Pi([x])v := \frac{\langle v, x\rangle x}{|x|^2}$$

is smooth. ◇

Problem 6. Let $A \in \mathbf{R}^{3\times 3}$ be an orthogonal matrix; that is: $A^t A = \mathbf{1}$. Prove that the map $f\colon S^2 \to S^2$ defined by

$$f(x) := Ax$$

is smooth. ◇