Differential Geometry 1 (M13)
Exercise Sheet 10

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Try to solve the following problems by yourself before the tutorial on **2021-02-03**.

Problem 1. Let G be a Lie group. Set $\mathfrak{g} := \mathrm{Lie}(G)$

1. Let $\xi \in \mathfrak{g}$. Prove that ξ is complete.

The **exponential map** $\exp\colon \mathfrak{g} \to G$ is defined by

$$\exp(\xi) := \mathrm{flow}_\xi^1(\mathbf{1}).$$

2. Prove that there are open neighborhoods U of $0 \in \mathfrak{g}$ and V of $\mathbf{1} \in G$ such that $\exp|_U\colon U \to V$ is a diffeomorphism.
3. Prove that $\exp(k\xi) = \exp(\xi)^k$.
4. Prove that if $[\xi, \eta] = 0$, then $\exp(\xi + \eta) = \exp(\xi)\exp(\eta)$. ◇

Problem 2. The matrix exponential $\mathrm{EXP}\colon \mathbf{R}^{n\times n} \to \mathbf{R}^{n\times n}$ is defined by

$$\mathrm{EXP}(A) := \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

For the Lie group $G = \mathrm{SO}(n)$ prove that the exponential map agrees with the matrix exponential. ◇

Problem 3. Let X be a smooth manifold. Let $f \in C^\infty(X)$. Prove *in detail* that $\mathrm{d}f$ defines a section of the cotangent bundle T^*X. ◇

Problem 4. Let X be a smooth manifold. Let $A\colon \mathrm{Vect}(X) \to C^\infty(X)$ be a linear map. Suppose that for every $f \in C^\infty(X)$, $v \in \mathrm{Vect}(X)$, and $x \in X$

$$A(fv)_x = f(x) \cdot A(v)_x.$$

Prove that there is an $\alpha \in \Gamma(T^*X)$ such that

$$A(v)_x = \alpha_x(v_x).$$ ◇