Differential Geometry 1 (M13)
Exercise Sheet 12

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Try to solve the following problems by yourself before the tutorial on **2021-02-17**.

Problem 1. Define $\alpha \in \Omega^2(\mathbf{R}^4)$ by

$$\alpha = x_1 x_2 x_3 \mathrm{d}x_2 \wedge \mathrm{d}x_4.$$

Set

$$T^2 \coloneqq \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : x_1^2 + x_2^2 = 1 = x_3^2 + x_4^2\}.$$

Compute

$$\int_{T^2} \alpha.$$

◇

Problem 2. Let (X, g) be a Riemannian manifold. Let $Y \subset X$ be a submanifold. Denote by h the Riemannian metric on Y induced by g. Denote by ∇^g and ∇^h the Levi-Civita connection of g and h resp. Prove that for every $x \in Y$, $v \in T_x Y$, and $w \in \mathrm{Vect}(X)$

$$\nabla^h_v w = \nabla^g_v w - (\nabla^g_v w)^\perp.$$

Here $(\cdot)^\perp$ denotes the orthogonal projection $T_x X \to N_x Y$. ◇

Problem 3. Let g be a Riemannian metric on $\mathbf{R}^m$. Set $g_{ij} \coloneqq g(\partial_i, \partial_j)$. Define Γ^k_{ij} by

$$\nabla^g_{\partial_i} \partial_j = \sum_{k=1}^m \Gamma^k_{ij} \partial_k.$$

Derive a formula for Γ^k_{ij} in terms of (g_{ij}) and its derivatives. ◇

Problem 4. Consider $\mathbf{R}^{m+1}$ with the symmetric tensor

$$h \coloneqq -\mathrm{d}x_1 \otimes \mathrm{d}x_1 + \sum_{i=2}^{m+1} \mathrm{d}x_i \otimes \mathrm{d}x_i.$$

Set

$$H^n \coloneqq \left\{x \in \mathbf{R}^{m+1} : h(x, x) = -1, x_1 > 0\right\}.$$

1. Prove that H^n is a submanifold.
2. Prove that the restriction of h to H^n defines a Riemannian metric g on H^n.
3. Compute the Levi-Civita connection g of H^n.
4. Compute the Riemann curvature tensor of g. ◇