Differential Geometry 1 (M13)
Exercise Sheet 4

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Try to solve the following six problems by yourself before the tutorial on **2020-12-02**.

Problem 1. Let X and Y be smooth manifolds. Let $p\colon X \to Y$ be a local diffeomorphism. Suppose that X and Y are compact and connected. Prove that p is a covering map. ⋄

Problem 2. Let X be a smooth manifold. Let Y be a submanifold. Prove that there is a unique smooth structure $\mathscr{A}_\subset$ on Y satisfying the following universal property:

1. The inclusion map $\iota\colon Y \hookrightarrow X$ is smooth.
2. If Z is a smooth manifold and $f\colon Z \to Y$ is a continuous map, then f is smooth with respect to $\mathscr{A}_\subset$ if and only $\iota \circ f\colon Z \to X$ is smooth. ⋄

Problem 3 (Constant rank theorem). Let X and Y be smooth manifolds without boundary. Let $f\colon X \to Y$ be a smooth map such that the map $X \to \mathbf{N}_0$, $x \mapsto \operatorname{rk} T_x f$ is locally constant. Prove that for every $y \in Y$ the level set $f^{-1}(y)$ is a submanifold. ⋄

Problem 4.

1. Prove that the special orthogonal group

$$\mathrm{SO}(n) := \{A \in \mathrm{GL}(\mathbf{R}^n) : A^t A = \mathbf{1}, \det A = 1\} \tag{1.1}$$

is a submanifold of $\mathbf{R}^{n\times n}$.

2. Prove that the special unitary group

$$\mathrm{SU}(n) := \{A \in \mathrm{GL}(\mathbf{C}^n) : A^* A = \mathbf{1}, \det A = 1\} \tag{1.2}$$

is a submanifold of $\mathbf{C}^{n\times n}$.

3. Prove that $\mathrm{SU}(2)$ is diffeomorphic to S^3.
4. Construct a local diffeomorphism $f\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$. ⋄

Problem 5. Find an embedding of $\iota\colon \mathbf{R}P^2 \to \mathbf{R}^4$ and prove that it is an embedding. ⋄

Problem 6. Let X be a smooth manifold without boundary. Let $f \in C^\infty(X)$. Prove that if $a \in \mathbf{R}$ is a regular value of f, then $f^{-1}([a,\infty))$ is a submanifold (of codimension zero). ⋄