Differential Geometry 1 (M13)
Exercise Sheet 6

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Try to solve the following problems by yourself before the tutorial on **2020-12-09**.

Throughout this exercise sheet, let $N \in \mathbf{N}_0$ and let $X \subset \mathbf{R}^N$ be a smooth submanifold without boundary.

Problem 1. The **normal bundle** of X in $\mathbf{R}^N$ is the subset

$$NX \coloneqq \{(x, v) \in X \times \mathbf{R}^N : v \perp T_x X\}.$$

Prove that NX is a smooth submanifold of $\mathbf{R}^N \times \mathbf{R}^N$. ◇

Problem 2. Define $\Phi\colon NX \to \mathbf{R}^N$ by

$$\Phi(x, v) \coloneqq x + v.$$

An open subset $U \subset \mathbf{R}^N$ is a **tubular neighborhood of** X if there is a continuous function $\varepsilon\colon X \to (0, \infty)$ such that the restriction of Φ to

$$V \coloneqq \{(x, v) \in NX : |v| < \varepsilon(x)\}.$$

is a diffeomorphism onto U.

Prove that X admits a tubular neighborhood. ◇

Problem 3. Prove that there is an open subset $U \subset \mathbf{R}^N$ with $X \subset U$ and a smooth submersion $r\colon U \to X$ with $r|_X = \mathrm{id}_X$. ◇

Problem 4. Let Y be a compact smooth manifold and let $f\colon Y \to X$ be smooth. Prove that there is an $M \in \mathbf{N}$ and a smooth submersion $F\colon B_1^M(0) \times Y \to X$ with $F(0, \cdot) = f$. ◇