# Differential Geometry 1 (M13)
# Exercise Sheet 7

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Try to solve the following problems by yourself before the tutorial on **2020-01-06**.

**Problem 1.** $\mathrm{GL}(\mathbf{R}^k)$ acts on

$$\mathrm{Hom}_{\hookrightarrow}(\mathbf{R}^k, \mathbf{R}^n) := \{T \in \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^n) : T \text{ is injective}\}.$$

by right-multiplication. The **Grassmannian of $k$–planes in $\mathbf{R}^n$** is

$$\mathrm{Gr}_k(\mathbf{R}^n) := \mathrm{Hom}_{\hookrightarrow}(\mathbf{R}^k, \mathbf{R}^n)/\mathrm{GL}(\mathbf{R}^k).$$

(Since $\operatorname{im} T$ depends only on $[T] \in \mathrm{Gr}_k(\mathbf{R}^n)$ and every $k$–dimensional subspace is of this form, $\mathrm{Gr}_k(\mathbf{R}^n)$ parametrizes $k$–dimensional subspaces of $\mathbf{R}^n$.)

1. Construct a smooth structure $\mathscr{A}$ of $\mathrm{Gr}_k(\mathbf{R}^n)$. *Hint:* Every injective $T \in \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^n)$ has an invertible $k \times k$–matrix. The action of $\mathrm{GL}(\mathbf{R}^k)$ can be used to transform this matrix to $\mathbf{1}$ (in a unique way). The remaining data of $T$ is a $k \times (n-k)$–matrix. Use this to construct the chart.

2. Prove that $\mathscr{A}$ has the following universal property:

   (a) The map
   $$\pi\colon\ \mathrm{Hom}_{\hookrightarrow}(\mathbf{R}^k, \mathbf{R}^n) \to \mathrm{Gr}_k(\mathbf{R}^n)$$
   is smooth.

   (b) Let $X$ be a smooth manifold. A $f\colon\ \mathrm{Gr}_k(\mathbf{R}^n) \to X$ a map is smooth if and only if the composition $f \circ \pi\colon\ \mathrm{Hom}_{\hookrightarrow}(\mathbf{R}^k, \mathbf{R}^n) \to X$ is smooth. ◇

**Problem 2.** For every $[T] \in \mathrm{Gr}_k(\mathbf{R}^n)$ construct an isomorphism

$$T_{[T]}\,\mathrm{Gr}_k(\mathbf{R}^n) \cong \mathrm{Hom}(\operatorname{im} T, \mathbf{R}^n/\operatorname{im} T).$$ ◇

**Problem 3.** Construct a diffeomorphism

$$\phi\colon\ \mathrm{Gr}_k(\mathbf{R}^n) \to \mathrm{Gr}_{n-k}(\mathbf{R}^n).$$

*Hint:* Construct $\phi$ so that if $[S] = \phi[T]$, then $\operatorname{im} S = (\operatorname{im} T)^\perp$. ◇

**Problem 4.** Identify $\Lambda^k \mathbf{R}^n = \mathbf{R}^{\binom{n}{k}}$ and set $\mathbf{P}(\Lambda^k \mathbf{R}^n) := \mathbf{P}(\mathbf{R}^{\binom{n}{k}-1})$. The **Plücker embedding** $\iota\colon \mathrm{Gr}_k(\mathbf{R}^n) \to \mathbf{P}(\Lambda^k \mathbf{R}^n)$ is defined by

$$\iota([T]) := \mathbf{R}^\times \cdot (v_1 \wedge \ldots \wedge v_k). \tag{1.1}$$

Here $v_1, \ldots, v_k$ are the columns of $T$.

1. Convince yourself that $\iota$ is well-defined.
2. Prove that $\iota$ is smooth. *Hint:* Use the universal property.
3. Prove that $\iota$ is an embedding. ◇

**Problem 5.** Prove that the subset

$$\gamma_k := \{([T], v) \in \mathrm{Gr}_k(\mathbf{R}^n) \times \mathbf{R}^n : v \in \operatorname{im} T\}$$

is a submanifold. ◇

**Problem 6.** Prove that

$$X := \{[z_1 : z_2 : z_3 : z_4] \in \mathbf{C}P^3 : z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0\}$$

is a submanifold of $\mathbf{C}P^3$. ◇

**Problem 7.** Let $X$ be a smooth manifolds without boundary. Let $\phi \in \mathrm{Diff}(X)$ be an involution; that is: $\phi \circ \phi = \mathrm{id}_X$. Prove that the fixed-point set

$$X^\phi := \{x \in X : \phi(x) = x\}$$

is a submanifold of $X$. ◇

**Problem 8.** Let $X, Y$ be compact smooth manifolds with $\dim X = m$ and $\dim Y \geqslant 2m+1$. Let $f\colon X \to Y$ be a smooth map. Prove that are $n \in \mathbf{N}_0$ and a smooth map $F\colon B_1^n(0) \times X \to Y$ such that for almost every every $t \in B_1^n(0)$ the map $f_t := F(t, \cdot)\colon X \to Y$ is an injective immersion. ◇

**Problem 9.** Let $X$ be a smooth manifold. Let $C \subset X \times \mathbf{R}^n$ be an open subset such that for every $x \in X$ the slice $C_x := \{v \in \mathbf{R}^n : (x, v) \in C\}$ is non-empty and convex (that is: $tv + (1-t)w \in C_x$ for every $v, w \in C_x$ and $t \in [0,1]$). Prove that there is a smooth map $f\colon X \to \mathbf{R}^n$ such that

$$\mathrm{graph}(f) = \{(x, f(x)) : x \in X\} \subset C.$$

*Hint:* Use a partition of unity. ◇