Differential Geometry 1 (M13)
Exercise Sheet 8

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Try to solve the following problems by yourself before the tutorial on **2021-01-13**.

Definition. Let $n \in \mathbf{N}$ and $k \in \mathbf{Z}$.

1. Define the equivalence relation $\sim$ on $\left(\mathbf{C}^{n+1}\backslash\{0\}\right) \times \mathbf{C}$ by

   $$(x,z) \sim (y,w) \quad \text{if and only if} \quad x = \lambda y \text{ and } z = \lambda^k w \text{ for some } \lambda \in \mathbf{C}^\times.$$

   Set

   $$\mathscr{O}_{\mathbf{C}P^n}(k) \coloneqq ((\mathbf{C}^{n+1}\backslash\{0\}) \times \mathbf{C})/\sim .$$

2. Denote by

   $$\pi\colon\ \mathscr{O}_{\mathbf{C}P^n}(k) \to \mathbf{C}P^n$$

   the map induced by $\mathrm{pr}_1\colon\ (\mathbf{C}^{n+1}\backslash\{0\}) \times \mathbf{C} \to \mathbf{C}^{n+1}\backslash\{0\}$.

3. For every $x \in \mathbf{C}^{n+1}\backslash\{0\}$ the map $\phi_x\colon\ \mathbf{C} \to \pi^{-1}[x]$ defined

   $$\phi_x(z) \coloneqq [x; z]$$

   is bijective. Since $\phi_{\lambda x}^{-1} \circ \phi_x(z) = \lambda^k z$, there is a unique structure of a $\mathbf{C}$–vector space on $\pi^{-1}[x]$ such that the maps ϕ_x are isomorphisms. •

Problem 1. Prove that $\mathscr{O}_{\mathbf{C}P^n}(k) \xrightarrow{\pi} \mathbf{C}P^n$ is a vector bundle (that is: construct local trivializations). ◇

Problem 2. Let $p \in \mathbf{C}[z_0, \ldots, z_n]$ be a homogeneous polynomial of degree k. Prove that there is a smooth map $\mathbf{p}\colon\ \mathbf{C}P^n \to \mathscr{O}_{\mathbf{C}P^n}(k)$ satisfying

$$\mathbf{p}([z_0 : \cdots : z_n]) = [z_0 : \cdots : z_n; p(z_0 : \cdots : z_n)] \tag{1.1}$$

and $\pi \circ \mathbf{p} = \mathrm{id}_{\mathbf{C}P^n}$. ◇

Problem 3. Prove that the Möbius bundle

$$M := \left\{([\theta], z) \in S^1 \times \mathbf{C} : \operatorname{Im}(e^{i\theta/2} z) = 0\right\} \xrightarrow{\mathrm{pr}_1} S^1$$

is not trivial; that is: it is not isomorphic to the trivial bundle $S^1 \times \mathbf{R}$. (Hint: Prove that if M were isomorphiC to $S^1 \times \mathbf{R}$, then it would have a section $s \in \Gamma(M)$ with $s(x) \neq 0$ for every $x \in S^1$. Prove that the latter is impossible.) ◇

Problem 4. Let X be a compact smooth manifold. Let $E \xrightarrow{\pi} X$ be a vector bundle of rank k.

1. Prove that there are $N \in \mathbf{N}_0$, and an morphism of vector bundles

$$\Lambda\colon\ E \longrightarrow X \times \mathbf{R}^N$$

   over X such that for every $x \in X$ the map $\Lambda_x\colon\ E_x \longrightarrow \mathbf{R}^N$ is injective. (Hint: contemplate the proof of Whitney's embedding theorem.)

2. Construct a map $f\colon\ X \longrightarrow \mathrm{Gr}_k(\mathbf{R}^N)$ and an isomorphism of vector bundles

$$E \cong f^* \gamma_k(\mathbf{R}^N).$$

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