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A. Sobolev Spaces

A.1. Weak derivatives and Sobolev spaces

Def: $\Omega \subseteq \mathbb{R}^n$ open
$f : \Omega \to \mathbb{R}$: locally integrable
$g : \Omega \to \mathbb{R}$

a. g is a weak j-th partial derivative of f if

$$\forall \varphi \in C_0^\infty(\Omega) : \int_\Omega g \varphi = - \int_\Omega f \, \partial_j \varphi$$

Note: We write $\partial_j f := g$

degree of a

b. $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ multiindex with $|a| := \sum_{i=1}^n a_i$

$$\partial^a \varphi := \frac{\partial^{|a|} \varphi}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}$$

g is a weak derivative of f corresponding to a if

$$\forall \varphi \in C_0^\infty(\Omega) : \int_\Omega g \varphi = (-1)^{|a|} \int_\Omega f \, \partial^a \varphi$$

Write $\partial^a f := g$

Def $k \in \mathbb{N}$, $p \in [1, \infty]$ Sobolev parameters
$\Omega \subseteq \mathbb{R}^n$ open

a. The Sobolev space $W^{k,p}(\Omega)$ consists of all $f \in L^p(\Omega)$ s.t.

$\forall a \in \mathbb{N}^n, |a| \leq k$: (1) $\partial^a f$ exists
(2) $\partial^a f \in L^p(\Omega)$

b. The Sobolev norm $\|\cdot\|_{W^{k,p}(\Omega)}$ is defined as follows:

$$\forall f \in W^{k,p}(\Omega): \quad \|f\|_{W^{k,p}(\Omega)} := \begin{cases} \sum_{|a| \leq k} \|\partial^a f\|_{L^p(\Omega)} & p < \infty \\ \max_{|a| \leq k} \{\|\partial^a f\|_{L^\infty(\Omega)}\} & p = \infty \end{cases}$$

c. $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

Rem: 1. For $k \in \mathbb{N}$, $p \in [1, \infty]$, $(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}})$ is a Banach space

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$$\underbrace{L^p(\Omega) \times \dots \times L^p(\Omega)}_{\# \{\alpha \in \mathbb{N}^n \mid |\alpha| \leq k\}} \supseteq W^{k,p}(\Omega)$$

2. For $p \in (1, \infty)$, $W^{k,p}(\Omega)$ is reflexive and separable.

Def: $\Omega \subseteq \mathbb{R}^n$ is a Lipschitz domain if it is open and if $\partial\bar{\Omega}$ can be locally represented as the graph of a Lipschitz function.

Thm A.1 (density of smooth functions)

$\Omega \subseteq \mathbb{R}^n$: bdd, Lipschitz
$p \in [1, \infty)$, $k \in \mathbb{N}$

Then $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$.

Thm A.2 (Sobolev embedding and compactness)

$\Omega \subseteq \mathbb{R}^n$: Lipschitz
$p \in [1,\infty)$, $k \in \mathbb{N}$, $kp > n$

a. For all $d \in \mathbb{N}$:

$$W^{k+d,p}(\Omega) \hookrightarrow C^d(\overline{\Omega})$$ is continuous embedding

If Ω is additionally bdd, the embedding is compact.

$\rightsquigarrow \exists C>0 \ \forall f \in W^{k+d,p}(\Omega): \ \|f\|_{C^d} \leq C \|f\|_{W^{k+d,p}}$

b. $q \in [1,\infty)$, $m \in \mathbb{N}$ s.t. $m \leq k$, $p \leq q$

(*) $k - \frac{n}{p} \geq m - \frac{n}{q}$

Then $W^{k,p}(\Omega) \hookrightarrow W^{m,q}(\Omega)$ is a cont. embedding

The embedding is compact is Ω is additionally bdd and (*) strict

$\rightsquigarrow \exists C>0 \ \forall f \in W^{k,p}(\Omega): \ \|f\|_{W^{m,q}} \leq C \|f\|_{W^{k,p}}$

Thm A.3 (product estimate)

$\Omega \subseteq \mathbb{R}^n$: open, bdd., Lipschitz

$p, q \in [1, \infty)$, $k, m \in \mathbb{N}$ s.t. $kp > n$, $k \geq m$

$(*)$ $k - \frac{n}{p} \geq m - \frac{n}{q}$

Then
$$\begin{aligned} W^{k,p}(\Omega) \times W^{m,q}(\Omega) &\longrightarrow W^{m,q}(\Omega) \\ (f, g) &\longmapsto fg \end{aligned}$$
is bilinear and continuous

$\rightsquigarrow$ $\exists C > 0 \; \forall f \in W^{k,p}, g \in W^{m,q}$: $\|fg\|_{W^{m,q}} \leq C \|f\|_{W^{k,p}} \|g\|_{W^{m,q}}$

Cor: In particular, $W^{k,p}(\Omega)$ is a Banach algebra.

A.2. Sobolev spaces on sections of vector bundles

Def. M: smooth, compact n-mfld

$\pi : E \longrightarrow M$: smooth VB of rank k

A section $\sigma : M \to E$ is of class $W^{k,p}$ if all its local coordinate representations are in $W^{k,p}$

$W^{k,p}(E)$: Sobolev space of sections of class $W^{k,p}$

Rem: This definition is independent of the choice of coordinates.

Rem: Taking the sum of the $W^{k,p}$- norms over finitely charts covering M defines a norm on $W^{k,p}(E)$

$(U_j)_{j=1,\dots,m}$: finite open cover of M

s.t. $\forall j \in \{1,\dots,m\}$ $\exists$ $\varphi_j : U_j \longrightarrow V_j \subseteq \mathbb{R}^k$: smooth chart

$\phi_j : E|_{U_j} \longrightarrow U_j \times \mathbb{R}^k$: trivialization

$\text{"}\pi^{-1}(U_j)\text{"}$

Let $\{\rho_j : M \to [0,1]\}$: partition of unity subordinate to $(U_j)_{j=1,\dots,m}$

Define for a section $\sigma : M \longrightarrow E$:

Define for a section $\sigma : M \longrightarrow E$:

$$\|\sigma\|_{W^{k,p}(E)} := \sum_{j=1}^{m} \| pr_2 \circ \phi_j \circ (\rho_j \cdot \sigma) \circ \varphi_j^{-1} \|_{W^{k,p}(V_j)}$$

$$\begin{tikzcd} V_j \arrow[r, "\varphi_j^{-1}"] \arrow[d, phantom, "\cap" description] & U_j \arrow[r, "\rho_j \cdot \sigma"] & E|_{U_j} \arrow[r, "\phi_j"] \arrow[d, phantom, "\parallel" description] & U_j \times \mathbb{R}^k \arrow[r, "pr_2"] & \mathbb{R}^k \\ \mathbb{R}^n \text{ open} & & \pi^{-1}(U_j) & & \end{tikzcd}$$

<u>Note</u>: This norm is not canonically defined but the topology resulting from it is.

B.1. The Laplace Operator

Def. $\Omega \subseteq \mathbb{R}^n$: open

The Laplace operator is a differential operator given by

$$\Delta : C^k(\Omega) \longrightarrow C^{k-2}(\Omega)$$
$$u \longmapsto \nabla \cdot \nabla u = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u$$

Rem: Δ is a second-order differential operator.

Laplace's equation: $\Delta u = 0$

Def. $u \in C^2(\Omega)$ is called harmonic if $\Delta u = 0$.

Rem: The fundamental solution to Δ is the function

$$\left.\begin{aligned} K : \Omega &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} (2\pi)^{-1} \log|x| & n = 2 \\ (2-n)^{-1} \omega_n^{-1} |x|^{2-n} & n \geq 3 \end{cases} \end{aligned}\right\} (*)$$

derivatives: $K_j = \frac{\partial K}{\partial x_j} \rightsquigarrow K_j(x) = \frac{x_j}{w_n |x|^n}$

$K_{jk} = \frac{\partial^2 K}{\partial x_j \partial x_k} \rightsquigarrow K_{jk}(x) = \frac{-n x_j x_k}{w_n |x|^{n+2}}$

In particular: K is locally integrable

Lemma B.1

a. $u \in C_0^2(\mathbb{R}^n)$: $u = K * \Delta u$ (Poisson's identity)

$\partial_j u = K_j * \Delta u$

$f \in C_0^\infty(\mathbb{R}^n)$: $\Delta(K * f) = f$

$\Delta(K_j * f) = \partial_j f$

b. $\Omega \subseteq \mathbb{R}^n$: open

$u, f \in L^1_{loc}(\Omega)$

u is called a weak solution of $\Delta u = f$ if

Poisson's equation

$$\forall \varphi \in C_0^\infty(\Omega): \int_\Omega u \, \Delta \varphi \, dx = \int_\Omega f \varphi \, dx$$

There holds:

- u weak solution of $\Delta u = f \iff u = K * f$
- u weak solution of $\Delta u = \partial_j f \iff u = K_j * f$

Lemma B.2 (Weyl's lemma)

Every weak solution $u \in L^1_{loc}(\Omega)$ of $\Delta u = 0$ is harmonic.

Thm B.3 (Calderón - Zygmund)

K: fundamental solution to Δ given by (*)
$p \in (1, \infty)$

There is a constant $c = c(n,p) > 0$ s.t.

$$\forall f \in C_0^\infty(\mathbb{R}^n), j \in \{1, \dots, n\}: \quad \|\nabla (K_j * f)\|_{L^p} \leq c \|f\|_{L^p}$$

Rem: This is the fundamental estimate for the L^p-theory of elliptic operators.

Corollary B.4 (elliptic estimate)

For $n \in \mathbb{N}$, $p > 1$:

$$\exists C > 0 \ \forall u \in C_0^\infty(\mathbb{R}^n): \sum_{j,k=1}^{n} \|\partial_j \partial_k u\|_{L^p} \leq C \|\Delta u\|_{L^p}$$

Thm B.5 (interior regularity)

$p \in (1, \infty)$, $k \in \mathbb{N}$
$\Omega \subseteq \mathbb{R}^n$: open domain
$f \in W^{k,p}_{loc}(\Omega)$ s.t. $u \in L^1_{loc}(\Omega)$ is a weak solution of $\Delta u = f$

Then: a. $u \in W^{k+2,p}_{loc}(\Omega)$

b. For every $\Omega' \subseteq \mathbb{R}^n$ bdd. s.t. $\overline{\Omega'} \subseteq \Omega$ we find $c = c(k, p, n, \Omega', \Omega) > 0$ s.t.

$$\forall u \in C^\infty(\bar{\Omega}): \|u\|_{W^{k+2,p}(\Omega')} \leq C \left(\|\Delta u\|_{W^{k,p}(\Omega)} + \|u\|_{L^p(\Omega)} \right)$$

Proof of a. Let Ω' as in b.

Let $U \supseteq \overline{\Omega'}$ open, bdd nbhd s.t. $\overline{U} \subseteq \Omega$

$\beta \in C_0^\infty(\Omega)$ smooth cutoff fct s.t. $\beta|_U \equiv 1$

Set $v := K * \beta f$

Claim: $v \in W^{k+2,p}(\Omega)$

$\ulcorner$ Let $(f_i) \subseteq C_0^\infty(\Omega)$ s.t.

$$\lim_{i\to\infty} \|f_i - \beta f\|_{W^{k,p}(\Omega)} = 0$$

Obs: $v_i := K * f_i$ is smooth for all $i \in \mathbb{N}$

Then:

$$\begin{aligned}
&\lim_{i\to\infty} \|v_i - v\|_{L^p} \\
&= \lim_{i\to\infty} \|(K * f_i) - (K * \beta f)\|_{L^p} \\
&= \lim_{i\to\infty} \|K * (f_i - \beta f)\|_{L^p} \\
&\le \lim_{i\to\infty} \Big(\underbrace{\|K\|_{L^1}}_{<\infty} \underbrace{\|f_i - \beta f\|_{L^p}}_{\to 0} \Big) \\
&= 0
\end{aligned}$$

(Young's inequality)

$u \in L^p$, $v \in L^q$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$

$\Rightarrow \|u * v\|_{L^r} \le \|u\|_{L^p} \|v\|_{L^q}$

Recall: $\|v_i\|_{W^{k+2,p}(\Omega)} = \sum_{|a|\le k+2} \|\partial^a v_i\|_{L^p}$

By Calderón - Zygmund

$(v_i)_{i\in\mathbb{N}}$ is a Cauchy-sequence in $W^{k+2,p}(\Omega)$

$\Rightarrow$ $v \in W^{k+2,p}(\Omega)$

We have $v = K * \beta f$

$\overset{\text{B.1.b}}{\Rightarrow}$ v weak solution of $\Delta v = \beta f$

$\Rightarrow$ $(u-v)|_U$ weak sol. of $\Delta(u-v) = 0$

$\overset{\text{Weyl}}{\Rightarrow}$ $u-v$ is real analytic

$\Rightarrow$ $u \in W^{k+2,p}(\Omega')$ $\square$

B.2 The Cauchy-Riemann operator

<u>Def.</u> The standard Cauchy-Riemann operator is

$$\bar\partial := \partial_s + i\partial_t \qquad \text{for } z = s+it \in \mathbb{C}$$
$$\bar\partial u := \partial_s u + i\partial_t u$$

We can also consider

$$\partial := \partial_s - i\partial_t$$

<u>Rem</u>: • ∂ and $\bar\partial$ are linear first-order differential operators

• The fundamental solution to $\bar\partial$ is

$$K : \mathbb{C} \longrightarrow \mathbb{C} \qquad \in L^1_{loc}(\mathbb{C},\mathbb{C})$$
$$z \longmapsto \frac{1}{2\pi z}$$

$$\forall f \in C_0^\infty(\mathbb{C}) : \bar\partial(K*f) = f$$

Thm B.6 (Calderón-Zygmund for $\bar\partial$)

$p \in (1,\infty)$

$\exists\, C > 0 \;\; \forall f \in C_0^\infty(\mathbb{C},\mathbb{C}) : \| \partial(K * f)\|_{L^p(\mathbb{C})} \leq C \|f\|_{L^p(\mathbb{C})}$

Proof (for $p = 2$)

Let $u := K * f$

Dirac delta

K satisfies $\bar\partial K = \delta$ in the sense of distributions

$\Rightarrow$ K is a tempered distribution

Thus

$$(*)\; \begin{cases} \widehat{\bar\partial K} = \hat\delta \\ \quad \| \qquad\quad \| \\ 2\pi i \xi \hat K(\xi) = 1 \end{cases}$$

(Fourier transform on both sides)

$\Rightarrow$ $u = K * f$ is also a tempered distribution and

$$\widehat{\bar\partial u}(\xi) = 2\pi i \xi \hat u(\xi) \underset{\hat u = \hat K \hat f}{=} \underbrace{2\pi i \xi \hat K(\xi)}_{= 1 \; (*)} \hat f(\xi) = 1 \cdot \hat f(\xi) \quad (\dagger)$$

Thus: $\|\partial(K * f)\|_{L^2}^2 = \|\partial u\|_{L^2}^2$

$$= \int_{\mathbb{C}} |\partial u|^2 \, d\mu(\zeta)$$

$$= \int_{\mathbb{C}} |\widehat{\partial u}|^2 \, d\mu(\zeta) \qquad \text{(Plancherel's thm)}$$

$$= \int_{\mathbb{C}} |2\pi i \bar{\zeta} \hat{u}(\zeta)|^2 \, d\mu(\zeta)$$

$$= \int_{\mathbb{C}} \left|\frac{\bar{\zeta}}{\zeta} 2\pi i \zeta \hat{u}(\zeta)\right|^2 d\mu(\zeta)$$

$$\overset{(+)}{=} \int_{\mathbb{C}} |\hat{f}(\zeta)|^2 \, d\mu(\zeta)$$

$$= \int_{\mathbb{C}} |f(\zeta)|^2 \, d\mu(\zeta) \qquad \text{(Plancherel)}$$

$$= \|f\|_{L^2}^2$$

$\square$

Thm B.7. (fundamental elliptic estimate for $\bar{\partial}$)

$p \in (1, \infty)$, $k \in \mathbb{N}$

$\exists c = c(p,k) > 0 \ \forall u \in W_0^{k,p}(\mathbb{B}): \ \|u\|_{W^{k,p}} \leq c \|\bar{\partial} u\|_{W^{k-1,p}}$

<u>Rem.</u>: Thus $f \mapsto \partial(K * f)$ for $f \in C_0^\infty(\mathbb{C})$ extends to a bdd. linear operator on $L^p(\mathbb{C})$

<u>Thm. B.8</u> (regularity)

$p \in (1,\infty)$, $k \in \mathbb{N}$
$u \in W^{1,p}(\mathbb{B})$, $f \in W^{k,p}(\mathbb{B})$ s.t. u weak sol. of $\bar{\partial} u = f$

Then: a. $\forall r \in (0,1)$: $u \in W^{k+1,p}(\mathbb{B}_r)$

b. $\forall r \in (0,1)$ $\exists c = c(p,k,r) > 0$ $\forall u \in W^{1,p}(\mathbb{B})$:

$$\|u\|_{W^{k+1,p}(\mathbb{B}_r)} \le c\left(\|u\|_{W^{1,p}(\mathbb{B})} + \|\bar\partial u\|_{W^{k,p}(\mathbb{B})}\right)$$

Proof suffices to prove the case $k=1$, rest follows by induction

Let $u, f \in W^{1,p}(\mathbb{B})$

Ad a: to show: $\forall\, r \in (0,1)$: $u \in W^{2,p}(\mathbb{B}_r)$

idea: show $\partial_s u, \partial_t u \in W^{1,p}(\mathbb{B}_r)$

Express $\partial_s u$ (and $\partial_t u$) as a limit of a difference quotient:

$$u^h(s,t) := \frac{u(s+h,t) - u(s,t)}{h} \qquad \text{for } h > 0,\ h \to 0$$

Obs:
- u^h is well-defined
- for h suff. small we have $u^h \in W^{1,p}(\mathbb{B}_r)$
- $u^h \longrightarrow \partial_s u$ in $L^p(\mathbb{B}_r)$ as $h \to 0$

Let $\beta \in C_0^\infty(\mathbb{B})$ be a smooth cutoff fct with $\beta|_{\mathbb{B}_r} \equiv 1$ for $r \in (0,1)$

$$\Rightarrow \beta u^h \in W_0^{1,p}(\mathbb{B})$$

Then: $\| u^h \|_{W^{1,p}(\mathbb{B}_r)} \leq \| \beta u^h \|_{W^{1,p}(\mathbb{B})}$

$$\leq c \| \bar\partial (\beta u^h) \|_{L^p(\mathbb{B})} \qquad \text{(B.7. elliptic estimate)}$$

$$= c \| (\bar\partial \beta) u^h + \beta (\bar\partial u^h) \|_{L^p(\mathbb{B})}$$

$$\leq c' \| u^h \|_{L^p(\mathbb{B})} + c' \| f^h \|_{L^p(\mathbb{B})}$$

bdd. for $h \to 0$ b/c $u^h \xrightarrow{L^p} \partial_s u \in L^p$
$f^h \xrightarrow[L^p]{} \partial_s f \in L^p$

Note: By Banach-Alaoglu, any sequence (u^{h_k}) with $h_k \to 0$ has a weakly convergent subsequence in $W^{1,p}(\mathbb{B}_r)$

But $u^h \to \partial_s u$ in $L^p(\mathbb{B}_r)$

$\Rightarrow u^h \rightharpoonup \partial_s u$ in $W^{1,p}(\mathbb{B}_r)$

$\Rightarrow \partial_s u \in W^{1,p}(\mathbb{B}_r)$

Ad b: Let $u \in W^{1,p}(\mathbb{B})$

Then $\forall r \in (0,1)$: $u \in W^{2,p}(\mathbb{B}_r)$ (by (a))

Choose a smooth cutoff fct $\beta \in C_0^\infty(\mathbb{B})$ with $\beta|_{\mathbb{B}_r} \equiv 1$

$\Rightarrow \beta u \in W_0^{2,p}(\mathbb{B})$

Thus: $\|u\|_{W^{2,p}(\mathbb{B}_r)} \leq \|\beta u\|_{W^{2,p}(\mathbb{B})}$

$$\leq c\,\|\bar\partial(\beta u)\|_{W^{1,p}(\mathbb{B})} \qquad \text{(B.7. elliptic estimate)}$$

$$\leq c\,\|(\bar\partial\beta)u\|_{W^{1,p}(\mathbb{B})} + c\,\|\beta\,(\bar\partial u)\|_{W^{1,p}(\mathbb{B})}$$

$$\leq c'\,\|u\|_{W^{1,p}(\mathbb{B})} + c'\,\|\bar\partial u\|_{W^{1,p}(\mathbb{B})}$$

$\square$

B.3 Elliptic operators in general

$\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$

M: smooth mfld

$\left.\begin{array}{ll} E \longrightarrow M & \text{rank } r \\ F \longrightarrow M & \text{rank } s \end{array}\right\}$ smooth $\mathbb{F}$-lin VBs

$D : \Gamma(E) \longrightarrow \Gamma(F)$: $\mathbb{F}$-lin. partial differential operator of order $m \in \mathbb{N}$

<u>Note</u>: For any choice of local trivialisations of E and F over the same coordinate nbhd of M, D can be written as

$$(Du)(x) = \sum_{|a| \le m} c_a(x)\, \partial^a u(x) \qquad (\star)$$

where: $U \subseteq \mathbb{R}^n$: image of a chosen coordinate chart on some region in M

$\left.\begin{array}{l} u : U \to \mathbb{F}^r \\ Du : U \to \mathbb{F}^s \end{array}\right\}$ sections of E, F in the chosen local trivialisations and coordinates

$a \in \mathbb{N}^n$, $|a| \le m$: multiindices

$c_a : U \longrightarrow \mathbb{F}^{s \times r}$ (s.t. $\exists\, a \in \mathbb{N}^n, |a| = m : c_a \neq 0$)

Rem: We can also consider D as an operator

$$D : W^{m,p}(E) \longrightarrow L^p(F)$$

Local setting

Localise near arbitrary $x_0 \in M$ to consider the unique operator with constant coefficients that matches ($\star$) at x_0:

$$D = \sum_{|a| \leq m} c_a \partial^a : C^\infty(\mathbb{R}^n, \mathbb{F}^r) \longrightarrow C^\infty(\mathbb{R}^n, \mathbb{F}^s)$$

with $c_a \in \mathbb{F}^{r \times s}$

Def: D: $\mathbb{F}$-linear partial differential operator with constant coefficients.

a. The symbol of D is a polynomial of degree m in $p = (p_1, \dots, p_n)$

$$\sigma^D : \mathbb{R}^n \longrightarrow \mathbb{C}^{s \times r}$$
$$p \longmapsto \sum_{|a| \leq m} (2\pi i p)^a c_a$$

6. Its principal symbol is

$$\sigma^D_m(p) := \sum_{|a|=m} p^a c_a \quad \in \mathbb{F}^{s\times r}$$

Rem: The behaviour of σ^D is determined by the principal symbol for $|p|$ large, i.e.:

$$\sigma^D(p) = (2\pi i)^m \sigma^D_m(p) + \mathcal{O}(|p|^{m-1})$$

Def: An $\mathbb{F}$-linear partial differential operator of order m w/ constant coefficients is called elliptic if its principal symbol

$$\sigma^D_m : \mathbb{R}^n \longrightarrow \mathbb{F}^{s\times r}$$

has the following property:

$$\forall p \in \mathbb{R}^n\setminus\{0\} : \sigma^D_m(p) \in \mathbb{F}^{s\times r} \quad \text{is invertible}$$

Rem: In particular: $r = s$

ex The standard CR-type operator $\bar\partial$ is a first-order operator with principal symbol

$$\sigma_1^{\bar\partial}(p_1, p_2) = (p_1 + i p_2)\,\mathrm{id} \in \mathbb{C}^{n \times n}$$

Since $\sigma_1^{\bar\partial}(p_1, p_2)$ is invertible for all $(p_1, p_2) \neq 0$, $\bar\partial$ is elliptic

C. Fredholm Operators

Def: X, Y: Banach spaces
$D: X \to Y$: bdd, linear operator

a. D is called **Fredholm** if
(1) $\ker(D)$ is finite-dimensional
(2) $\operatorname{coker}(D) = Y/\operatorname{im}(D)$ is finite-dimensional
(3) $\operatorname{im}(D)$ is closed

b. If D is Fredholm, the **Fredholm index** of D is

$$\operatorname{ind}(D) := \dim \ker(D) - \dim \operatorname{coker}(D)$$

Rem: Elliptic operators are Fredholm.