Uhlenbeck's Removable Singularity Theorem

Ref - Freed–Uhlenbeck Appendix D

Donaldson–Kronheimer Thm 4.4.12

<u>Thm</u> :- Fix $x \in M^4$ and suppose D is a self-dual connection on a bundle $E \longrightarrow M \setminus \{x\}$ w/ $\int_M |F_D|^2 < \infty$. Then for some $s \in \mathcal{G}$, $s^*(E)$ extends to a smooth bundle $\overline{E} \longrightarrow M$ and D extends to a smooth ASD connection $\overline{D}$ on $\overline{E}$.

We'll prove

<u>Thm</u> (RST) Let D be an anti self-dual instanton in $B^4 - \{0\}$ w/ $\int_{B^4} |F_D|^2 < \infty$, $D = d + A$ and $A \in L^2_{1,loc}(B^4 - \{0\})$. Then D is gauge equivalent to a connection $\tilde{D}$ which extends smoothly across the singularity, to a smooth connection.

We first deform the metric conformally so it is approx.

cylindrical. (Similar idea – Taubes' Thm & Collar Thm).

If δ is the flat metric in B^4 and if r is the distance from the origin, then consider the conformal metric $\frac{\delta}{r^2}$ on $B^4 - \{0\}$. In polar coordinates $r, \theta^1, \theta^2, \theta^3$, $\left(B^4 - \{0\}, \frac{\delta}{r^2}\right)$ is a cylinder with

$$\frac{\delta}{r^2} = \frac{dr^2}{r^2} + d\theta^2$$

metric on S^3.

Substitute $r = e^{-\tau}$ so that we have the standard product metric on the cylinder

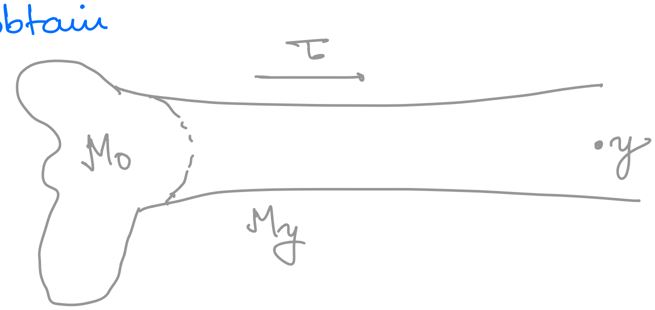
$$\tilde{\delta} = d\tau^2 + d\theta^2.$$

Now, if g is an arbitrary metric on B^4, then we obtain a metric which is approx. $\tilde{\delta}$. Use geodesic polar coord. to get

$$|\tilde{g} - \tilde{\delta}| = O(e^{-2\tau})$$

and so all derivatives of $\tilde{g}$ vanish exponentially fast as $\tau \to \infty$.

Rem:- For any M^4, we can do this to a nbd of $y \in M$ to obtain



$\forall \ \epsilon \quad \exists \ \tau_1(\epsilon_1)$ s.t

$$\left| W^-_{g_y}(\tau,\theta) \right| < \epsilon \qquad \text{if } \tau \geq \tau_1$$

$W = W^+ + W^-$

Weyl curvature

$$\left| R(\tau,\theta) - 6 \right| < \epsilon \qquad \text{if } \tau \geq \tau_1$$

as for cylinder in 4d, $R = (4-2)(4-1) = 6$

which is showing that the cylinder is conformally flat.

Standard results for compact manifolds carry over to M_y.

__Prop 1__ Let $f \in C^2(M_y)$.

i) If $f, df \in L^1(M_y)$ then $\int_{M_y} \Delta f = 0$.

ii) $\|f\|_{L^4} \leq C \|f\|_{L^2_1}$

Use cut-off functions in the usual proof.

__Prop 2__ Suppose $u(\tau,\theta)$ is a non-negative function on the region $\{\tau_l \leq \tau \leq \tau_r\} \subset M_y$ s.t. [Maximum principle]

i) $\Delta u + \gamma'^2 u \leq 0$

ii) $u(\tau_l,\theta) \leq a_l$

iii) $u(\tau_r,\theta) \leq a_r$. Then

$$u(\tau,\theta) \leq a_l e^{-\gamma(\tau-\tau_l)} + a_r e^{-\gamma(\tau_r-\tau)}$$

for $\gamma = \gamma(\tau_l) < \gamma'$ with $\gamma(\tau_l) \longrightarrow \gamma'$ as $\tau_l \longrightarrow \infty$.

Thm (Regularity thm) There is a constant $\epsilon_1 > 0$ s.t. if $\tilde{A}$ is any ASD connection on B^4 which satisfies the Coulomb gauge condition $d^*\tilde{A} = 0$ and $\|\tilde{A}\|_{L^4} \leq \epsilon_1$, then for any interior domain $D \subset B^4$ and $l \geq 1$, we have

$$\|\tilde{A}\|_{L^2_l(D)} \leq M_{l,D} \|F_A\|_{L^2(B^4)} \qquad \text{— (1)}$$

$$\Rightarrow \quad \|\tilde{A}\|_{C^k(D)} \leq M_{k,D} \|F_A\|_{L^2(B^4)}. \quad \text{In particular,} \qquad \text{— (2)}$$

$$\max_{|x| \leq \frac{1}{2}} |F_A| \leq M \|F_A\|_{L^2(B^4)}. \qquad \text{— (3)}$$

The proof will be given later. For now, let's return to the proof of RST. We work in the cylinder $\{\tau_0 \leq \tau\}$.

N.B. If $g \to \kappa^2 g$ then

$$|A| \longrightarrow \kappa^{-1} |A|$$

$$|F| \longmapsto \kappa^{-2} |F|$$

But the YM equations and the L^2-norms remain invariant.

Lemma 3 $\lim_{\tau \to \infty} |F(\tau, \theta)| = 0.$

Proof: :- $\lim_{\bar{\tau} \to \infty} \int_{\tau \geq \bar{\tau}} |F(\tau, \theta)|^2 = 0.$

Let $B_\sigma(y_0)$ be a ball over which $\int_{B_\sigma(y_0)} |F(y)|^2 dy < \epsilon$.

ϵ depends on the geometry of the ball but for τ large enough, it can be fixed as the geometry of the cylinder is uniform. By ③ in Reg. thm

$$\sigma^2 |F(y_1)|^2 \leq M \int_{B_\sigma(y_0)} |F(y)|^2 dy$$

if $\quad \text{dist}(y_0, y_1) < \frac{\sigma}{2}$.

Choose $\bar{\tau}$ sufficiently large, so that $\int_{\tau \geq \bar{\tau}} |F(\tau, \theta)|^2 < \epsilon$ and

σ is of uniform size. Hence

$$|F(\bar{\tau}, \theta)|^2 \leq \frac{M}{\sigma^2} \int_{\bar{\tau}-1 \leq \tau < \bar{\tau}+1} |F(x)|^2 dx \longrightarrow 0 \quad \text{as } \bar{\tau} \to \infty.$$

$\square$

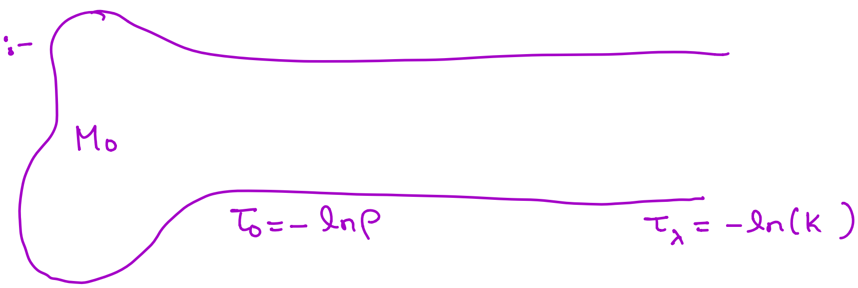
We now prove decay estimates for F.

Lemma 4 (Decay estimates)

For any $\gamma < \sqrt{2}$, we can choose $\bar{\tau}$ such that for $\tau \geq \bar{\tau}$,

$$|F(\tau,\theta)| \leq \max_{\theta} |F(\bar{\tau},\theta)| \, e^{\gamma(\bar{\tau}-\tau)} \qquad \text{— (4)}$$

Fact:-



We have $\max_{\tau \leq \tau_\lambda} |F_A| \leq \frac{\nu}{2}$ where $\nu = \frac{c}{K^2}$

Proof Recall the Weitzenböck formula for anti self-dual 2-forms

$$2 D_- D_-^* = \nabla^* \nabla - 2W^-(\cdot) + \frac{R}{3} + [P_- F, \cdot]$$

where $D_- = P_- D$

W^- = anti self-dual part of the Weyl tensor

$[\,,\,] : \Omega^2_- \otimes \Omega^2_- \longrightarrow \Omega^2_-$ is the Lie bracket.

Also note that for a function ψ

$$\langle \nabla^* \nabla \psi, \psi \rangle = \frac{1}{2} \Delta |\psi|^2 + |\nabla \psi|^2 \geq \frac{1}{2} \Delta |\psi|^2$$

We also have the Kato inequality

$$|\nabla \psi| \geq |\nabla |\psi|| \implies \langle \nabla^* \nabla \psi, \psi \rangle \geq |\psi| \Delta |\psi|$$

$\therefore$ for $|F|$, we get

$$|F|\Delta|F| \leq \langle \nabla^*\nabla F, F\rangle$$

$$= \langle 2D_-D_-^* F + 2W^-(F) - \frac{R}{3}F + [P_-F, F], F\rangle$$

$$|F|\Delta|F| + 2|F|^2 \leq (\epsilon + |F|)|F|$$

as $\tau \to \infty$, $K \to \infty \Rightarrow |F| \to 0$

$\Rightarrow$ we get

$$\Delta|F| + \underbrace{\overline{\gamma}}_{<2}|F| \leq 0 \qquad \text{for } \tau \geq \overline{\tau}.$$

By the maximum principle, for $\overline{\tau} \leq \tau \leq \tau_n$ and $\gamma < \sqrt{\overline{\gamma}}$, we get

$$|F(\tau,\theta)| \leq \max_\theta |F(\overline{\tau},\theta)| e^{\gamma(\overline{\tau}-\tau)} + \underbrace{\max_\theta |F(\tau_n,\theta)| e^{\gamma(\tau-\tau_n)}}_{\to 0 \text{ as } \tau_n \to \infty}$$

so,

$$|F(\tau,\theta)| \leq \max_\theta |F(\overline{\tau},\theta)| e^{\gamma(\overline{\tau}-\tau)}$$

$\blacksquare$

We also have the following estimate on A, which we show (later) using exponential gauges.

Lemma 5 If $\overline{\tau}$ is sufficiently large, then $\exists$ a gauge on $\{\tau \geq \overline{\tau}\}$ such that

$$|A(\tau,\theta)| \leq c \cdot e^{\gamma(\overline{\tau}-\tau)} \qquad — \quad (5)$$

We now transfer back to the ball B^4. In B^4, (4) and (5) become

$$|A(x)| \leq c|x|^{\gamma-1}$$

$$|F(x)| \leq c|x|^{\gamma-2}$$

$\left\{ \begin{array}{l} \text{recall} \\ r = |x| = e^{-\tau} \end{array} \right\}$

$\Rightarrow$ $A \in L^2_1$ and F is bounded in L^2 $\Rightarrow$ $\exists$ Coulomb gauge, i.e., A is gauge equivalent to $\tilde{A} = s^{-1}ds + s^{-1}As$ w/ $d^*\tilde{A} = 0$ and $\tilde{A} \in L^2_1(B^4)$.

$\therefore$

$$d^*\tilde{A} = 0$$

$$d^*d\tilde{A} + d^*(\tilde{A}\wedge\tilde{A}) + (\tilde{A} \lrcorner d\tilde{A}) + (\tilde{A} \lrcorner (\tilde{A}\wedge\tilde{A})) = 0$$

can be made into a single elliptic equation

$$\Delta\tilde{A} + \text{lower order terms} = 0$$

$\Rightarrow$ from the regularity theorem, $\exists$ smooth $\tilde{A}$ which is the desired extension on B^4.

So we have to prove the following results.

<u>Thm (Regularity thm)</u> There is a constant $\epsilon_1 > 0$ s.t. if $\tilde{A}$ is any ASD connection on B^4 which satisfies the Coulomb gauge condition $d^*\tilde{A} = 0$ and $\|\tilde{A}\|_{L^4} \le \epsilon_1$, then for any interior domain $D \subset B^4$ and $l \ge 1$, we have

$$\|\tilde{A}\|_{L^2_l(D)} \le M_{l,D} \|F_A\|_{L^2(B^4)}$$

$$\Rightarrow \quad \|\tilde{A}\|_{C^k(D)} \le M_{k,D} \|F_A\|_{L^2(B^4)}.$$ In particular,

$$\max_{|x| \le \frac{1}{2}} |F_A| \le M \|F_A\|_{L^2(B^4)}.$$

<u>Proof</u> We have A over B^4 which is a Coulomb gauge and is ASD. Together, they can be written as

$$\delta A + (A \wedge A)^+ = 0 \qquad \text{———} \quad (6)$$

where $\delta = d^* + d^+$ is the elliptic operator from last talk. We can regard B^4 being contained in S^4 (via the stereographic projection) and for $D \Subset B^4$, let ψ be a cut-off function

$\operatorname{supp}(\psi) = B^4$, $\quad \psi = 1$ on D.

Let $\alpha = \psi A$, defined over S^4.

Then

$$\delta(\alpha) = \delta(\psi A) = \psi\,\delta(A) + (d\psi \wedge A)^+$$

$$= -\psi\,(A \wedge A)^+ + (d\psi \wedge A)^+ \quad \text{—— } ⑦$$

$$= -(A \wedge \alpha)^+ + (d\psi \wedge A)^+$$

$\because$ δ is elliptic and $H^1(S^4) = 0 \implies$

$$\|\alpha\|^2_{L_2} < \|\delta\alpha\|^2_{L_1} \quad \text{—— } ⑧$$

Also, from ⑦,

$$\|\delta\alpha\|^2_{L_1} \leq \underbrace{\|\psi A \wedge A\|^2_{L_1}}_{\sim} + \underbrace{\|d\psi \wedge A\|^2_{L_1}}_{\leq C\|A\|^2_{L_1}}$$

$\sim$

$$\nabla(\psi A \otimes A) = \{\nabla(\psi A)\} \otimes A + \psi A \otimes \nabla A$$

and

$$\psi A \otimes \nabla A = A \otimes \{\psi \nabla A\} = A \otimes \nabla(\psi A) - A \otimes \nabla\psi \otimes A$$

$\implies$

$$\nabla(\psi A \otimes A) = \nabla(\psi A) \otimes A + A \otimes \nabla(\psi A) - A \otimes \nabla\psi \otimes A$$

$\implies$

$$\|\nabla(\psi A \otimes A)\|_{L^2} \leq \|\nabla(\psi A) \otimes A + A \otimes \nabla(\psi A) - A \otimes \nabla\psi \otimes A\|_{L^2}$$

$$\leq \|\nabla(\psi A) \otimes A\|_{L^2} + \|A \otimes \nabla(\psi A)\|_{L^2} + \|A \otimes \nabla\psi \otimes A\|_{L^2}$$

$$\underset{\text{Hölders}}{\leq} C\left[\|\nabla(\psi A)\|_{L^4}\|A\|_{L^4} + \|A\|^3_{L^4}\right]$$

So, from ⑧, we get

$$\|\alpha\|_{L^2_2} \leq c\|\delta\alpha\|_{L^2_1}$$

$$\leq c\left(\|\nabla(\alpha)\|_{L^4}\|A\|_{L^4} + \|A\|^2_{L^4} + \|A\|_{L^2_1}\right)$$

$$\underset{\substack{\text{Sobolev} \\ L^2_2 \hookleftarrow L^4_1}}{\leq} c\left(\|\alpha\|_{L^2_2}\|A\|_{L^2_1} + \|A\|^2_{L^4} + \|A\|_{L^2_1}\right)$$

$\Longrightarrow$ if $\|A\|_{L^2_1}$ is sufficiently small (which is the case in hand), then rearranging gives

$$\|\alpha\|_{L^2_2} < \infty.$$

Since $\alpha = A$ on the domain, we went from L^2_1 bounds on A to L^2_2 over a smaller domain.

Iterate this argument to get an estimate on all higher derivatives. $\boxtimes$

<u>Exponential gauges</u>

We want to find a gauge in a region s.t. $d^*A = 0$ when $\|F\|_{L^\infty}$ is sufficiently small.

→ Idea is that if $|F|$ is small, then we can locally and geometrically choose a gauge w/ $|A|$ small.
(We did that w/ the Coulomb gauge by using a PDE.)

→ For connection on a tangent bundle, we can do this by using geodesic normal coordinates.

→ For an arbitrary bundle, we translate along radial geodesics in the bundle; use geodesic normal coordinates in the total space of the bundle by using fiber metric.

Fix $p \in M$, $E \to M$ and construct a local trivialization of E by identifying $E_{\exp_p(tX)}$ with E_p via parallel translation along the radial geodesic $\{\exp_p(sX) \mid 0 \le s \le t\}$.

→ Fixing a frame at p, we still have the freedom to rotate by elements of the structure group G.

→ In geodesic polar coordinates (r, θ), $\theta \in S^{n-1}$

we have $A_r = 0$ and

$$F_{r\theta} = \frac{\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta} + [A_r, A_\theta]$$

$$= \frac{\partial A_\theta}{\partial r}$$

$$\Rightarrow \quad A_\theta(r_0, \theta_0) = \int_0^{r_0} F_{r\theta}(r, \theta_0)\, dr$$

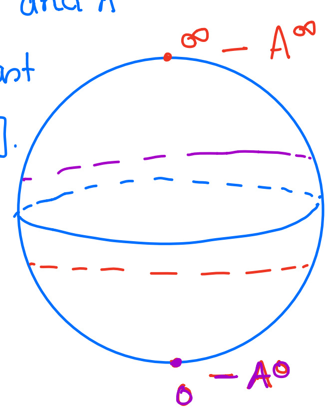
$\Rightarrow$ on region bounded away from cut locus,

$$\|A\|_{L^\infty(B_r)} \leq C\|F\|_{L^\infty(B_r)} \quad \text{—— (9)}$$

<u>Lemma</u> Let D be a connection on a bundle over S^n. If $\|F\|_{L^\infty(S^n)}$ is sufficiently small, then $\exists$ a global gauge on S^n for which $D = d + A$ w/

$$\|A\|_{L^\infty(S^n)} \leq C\|F\|_{L^\infty(S^n)}.$$

<u>Proof</u> Construct A^0 and A^∞ extending slightly past the equator [cut locus].



A^0 and A^∞ both satisfy (9).

On the intersection, $D = d^0 + A^0 = d^\infty + A^\infty$

$$\Rightarrow \quad s^{-1} d^0 s = A^0 - A^\infty$$

$\because$ A^0 and A^∞ are exponential gauges $\Rightarrow$

$$\frac{\partial s}{\partial \phi} = s\left(A^\infty_\phi - A^0_\phi\right) = 0$$ where ϕ is the polar angle.

$\Rightarrow$ $s = s(\theta)$ is only function of equatorial variables $\Rightarrow$

if s is unitary, we get

$$\|ds\|_{L^\infty} \leq 2c\,\|F\|_{L^\infty}$$

$\rightarrow$ rotate by a fixed element of $\mathcal{G}$ to set $s(\theta_0) = 1$ for some θ_0. Integrating above, we get

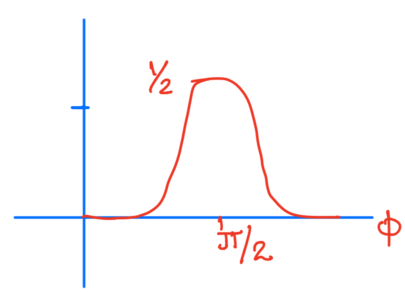
$$\|s - 1\|_{L^\infty} \leq C\pi\,\|F\|_{L^\infty}$$

$\Rightarrow$ if $\|F\|_{L^\infty}$ is sufficiently small so that

$$s(\theta) = \exp(u(\theta))$$

$$\text{for some } u : S^{n-1} \longrightarrow \mathfrak{g} = \mathrm{Lie}(G)$$

we get $\|u\|_{L^\infty_1} \leq C\,\|F\|_{L^\infty}$.



Define $s^0(\phi, \theta) = \exp\left(\beta(\phi)\, u(\theta)\right)$

$s^\infty(\phi, \theta) = \exp\left(-\beta(\phi)\, u(\theta)\right)$

we can write $D = d + A$ w/

$$A = A^0 - (s^0)^{-1}(d^0 s^0) = A^\infty - (s^\infty)^{-1}(d^\infty s^\infty)$$

$$\Longrightarrow \quad \|A\|_{L^\infty} \le C\|F\|_\infty .$$

$\boxed{\text{Vh}}$
