Compactness Theorem

Recall :

Let (X, g) be a Oriented Riemannian n-manifold and let $E \longrightarrow X$ be a G-bundle over X.

$\longrightarrow$ $G = SU(2)$ unless otherwise stated.

$\longrightarrow$ $\langle \cdot , \cdot \rangle$ natural tensor product metric on $\Lambda^2 T^* M \otimes \mathfrak{g}_E$ induced by g and the Ad_G invariant metric $-\mathrm{tr}(XY)$ on $\mathfrak{g}$

$\longrightarrow$ we get $A \longmapsto |F_A|$ invariant under the action of $\mathcal{G}(E)$.

Yang mills Functional :

$$YM(A) = \int_M |F_A|^2$$

The Euler Lagrange equations give the following equations.

$$d_A F_A = 0 = d_A^* F_A$$

$\longrightarrow$ $YM \geq 8\pi^2 k$ if $k>0$ with equality iff $F_- = 0$ SD instanton

$\longrightarrow$ $YM \geq -8\pi^2 k$ if $k<0$ with equality iff $F_+ = 0$ ASD instanton

where k is $c_2(E)$.

if A is self-dual or Anti-Self dual then, A automatically satisfy the Yang mills equation.

Uhlenbeck's Gauge Fixing theorem.

There are constants $\varepsilon, M > 0$ s.t any connection A on the trivial bundle over $\bar{B}^4$ with $\|F_A\|_{L^2} < \varepsilon$ is gauge equivalent to a connection $\tilde{A}$ over B^4 with

$$d^* \tilde{A} = 0$$

$$\lim_{|x| \to 1} \tilde{A}_r = 0$$

$$\| \tilde{A} \|_{W^{1,2}} \leq M \| F_{\tilde{A}} \|_{L^2}$$

Moreover, the constants ε, M, the connection $\tilde{A}$ is uniquely determined by these properties upto the gauge transformation $\tilde{A} \longrightarrow u_0 \tilde{A} u_0^{-1}$ for a constant u_0.

Removable Singularity theorem :

Let A be a Connection over a punctured ball $B^4 \setminus \{0\}$ which is ASD w.r.t a smooth metric over B^4. If

$$\int_{B^4 \setminus \{0\}} |F(A)|^2 < \infty$$

then $\exists$ connection A' on a bundle E' over B^4 & a bundle map $p : E \longrightarrow E'|_{B^4 \setminus \{0\}}$ with $p^*(A') = A$.

$\mathcal{A}$ - space of all Connections on E

$\mathcal{G}$ - gauge group

define $\mathcal{B} = \mathcal{A}/\mathcal{G}$

let $l \geq 2$,

$\Longrightarrow$ $W^{l-1,2}$ be the Connections which differ from a smooth Connection by a $W^{l-1,2}$ section of $T^*_X \otimes \mathfrak{g}_E$

$\Longrightarrow$ $W^{l,2}$ gauge transformation acts on them.

Define $\mathcal{B}(l) := \mathcal{A}(l)/\mathcal{G}(l)$

where $\mathcal{A}(l) = W^{l-1,2}$ Connections on E

$\mathcal{G}(l) = W^{l,2}$ gauge transformation

Define L^2 metric on $\mathcal{A}(l)$

$$\|A - B\| = \left(\int_X |A - B|^2 \, d\mu \right)^{1/2}$$

$$A - B \in W^{l-1,2}(X, T^*_X \otimes \mathfrak{g}_E)$$

This L^2 metric is gauge invariant, so descends to a distance function on $\mathcal{B}(l)$.

$$d([A],[B]) = \inf_{g \in \mathcal{G}} \|A - g(B)\|$$

Lemma: d is a metric on $\mathcal{B}(\ell)$

Proof:

we have to show, $d([A],[B]) = 0 \Rightarrow [A] = [B]$.

Let B_α be a sequence in $\mathcal{A}$, gauge equivalent to B Converging in L^2 to A. We have to Show A is gauge equivalent to B

we have, $B_\alpha = u_\alpha B u_\alpha^{-1} - d_B u_\alpha u_\alpha^{-1}$

$$\Rightarrow \quad d_B u_\alpha = u_\alpha B - B_\alpha u_\alpha \qquad \text{---} \qquad (*)$$

The u_α are uniformly bounded since G is Compact & $\ell > 2$.

$(*)$ shows $u_\alpha \in End(E)$ has a subsequence that Converge weakly in $W^{1,2}$ and strongly in L^2 to a limit u and u satisfies the equation.

$$d_B u = uB - Au$$

If φ is any test function on End E we have,

$$\langle d_B u, \varphi \rangle = \lim \langle d_B u_\alpha, \varphi \rangle = \lim \langle u_\alpha B - B_\alpha u_\alpha, \varphi \rangle$$

$$= \langle uB - Au, \varphi \rangle \quad \text{as} \quad B_\alpha u_\alpha \longrightarrow Au.$$

This equation u is an overdetermined elliptic equation with $W^{l-1,2}$ coefficients. So by usual elliptic bootstrapping, $u \in W^{l,2}$.

clearly u is unitary section in End E.

Moduli space :

Let the moduli space of ASD instantons $F^+(A) = 0$ be denoted $\mathcal{M}(l)$, $l > 2$.

clearly $\mathcal{M}(l) \subset \mathcal{B}(l)$ of $W^{l-1,2}$ ASD instantons modulo $W^{l,2}$ gauge transformation.

Proposition :

$W^{l,2}$ Connections

The natural inclusion of $\mathcal{M}(l+1) \hookrightarrow \mathcal{M}(l)$ is a homeomorphism.

$W^{l-1,2}$ Connect

Proof :

We know from the gauge fixing theorem, $\exists\ \varepsilon > 0$ s.t $\forall$ $W^{l-1,2}$ Connection with $\| A - B \|_{W^{l-1,2}} < \varepsilon$

$\exists$ $u \in W^{l,2}$ gauge transformation with

$$d^*_A (u^{-1}(B) - A) = 0 \quad (u^{-1}(B) \text{ Coulomb w.r.t } A)$$

By symmetry, A is also coulomb gauge relative to $u^{-1}(B)$ i.e., $d^*_B (u(A) - B) = 0$

By writing, $A' = u(A) = B + a$;

$$d_B^* a = 0$$

Since the smooth connections are dense, we can choose B to be smooth. The difference 1-form "a" also satisfies

$$d_B^+ a + (a \wedge a)^+ = -F_B^+ \quad \text{(ASD equation for } A' \text{)}$$

$$F_{A+\alpha}^+ = F_A^+ + d_A \alpha^+ + a \wedge a^+$$

Thus $(d_B^* \oplus d_B^+)\, a$ lies in $W^{l-1,2}$ because

F_B is smooth & $(a \wedge a)^+ \in W^{l-1,2}$

Recall : $d_B^* \oplus d_B^+$ is elliptic

So by basic elliptic regularity results gives

$a \in W^{l,2}$.

This shows the natural map is surjective & its clearly injective.

∎

The Compactification theorem :

Defn: An ideal ASD connection over X of chern class k is a pair

$$([A] , (x_1, \ldots, x_\ell))$$

where $[A] \in M(k-\ell)$ and $(x_1, \ldots, x_\ell)$ is a multiset of unordered ℓ-tuple of points in X. The curvature density of $([A], (x_1, \ldots, x_\ell))$ is the measure

$$|F(A)|^2 + 8\pi^2 \sum_{r=1}^{\ell} \delta_{x_r}$$

Defn: (weak convergence)

Let A_α, $\alpha \in \mathbb{N}$ be a sequence of connections of chern class k. We say that $[A_\alpha]$ of gauge equivalence classes converge weakly to a limiting ideal ASD connection $([A], (x_1, \ldots, x_\ell))$ if

1. The curvature densities converge as a measure, ie $\forall f \in C(X)$

$$\int_X |F(A_\alpha)|^2 f \, d\mu \longrightarrow \int_X |F(A)|^2 f \, d\mu + 8\pi^2 \sum_{r=1}^{\ell} f(x_r)$$

2. There are bundle maps

$$f_\alpha : P_1|_{X \setminus \{x_1, \dots x_\ell\}} \longrightarrow P_\alpha|_{X \setminus \{x_1, \dots x_\ell\}}$$

such that $f_\alpha^*(A_\alpha)$ converges to A, C^∞ on compact subsets.

This notion of convergence endows the set of all ideal ASD connections of fixed chern class k with a topology.

$$IM_k = M_k \cup M_{k-1} \times X \cup M_{k-1} \times S^2(X) \cup \dots$$

The ordinary moduli space M_k is embedded as a open set into IM_k.

$\longrightarrow$ $\overline{M}_k$ denotes its closure.

$M_k \hookrightarrow IM_k$

Theorem:

The space $\overline{M}_k$ is compact.

Enough to prove that any infinite sequence in M_k has a weakly convergent subsequence with a limit point in $\overline{M}_k$.

Patching arguements.

In the following by Convergence we mean C^∞ Convergence over Compact sets.

Lemma 1:

Suppose A_α is a sequence of Connections on a bundle E over a base manifold Ω (possibly non Compact) and let $\tilde{\Omega} \subseteq \Omega$ be an interior domain. Suppose $\exists\ u_\alpha \in \text{Aut } E$ & $\tilde{u}_\alpha \in \text{Aut } E|_{\tilde{\Omega}}$ s.t. $u_\alpha(A_\alpha)$ converges over Ω and $\tilde{u}_\alpha(A_\alpha)$ Converges over $\tilde{\Omega}$. Then for any Compact set $K \subset \tilde{\Omega}$ we can find a subsequence $\{\alpha'\} \subset \{\alpha\}$ and gauge transformation $w_{\alpha'} \in \text{Aut } E$ s.t $w_{\alpha'} = \tilde{u}_{\alpha'}$ in a nbhd of K & the Connections $w_{\alpha'}(A_{\alpha'})$ Converge over Ω.

$K \subseteq \tilde{\Omega} \subseteq \Omega$

Proof:

WLOG assume u_α's are identity. So over $\tilde{\Omega}$ both A_α and $\tilde{u}_\alpha(A_\alpha)$ are Convergent sequence of Connections.

→ take a subsequence $\tilde{u}_{\alpha'}$ which converges over $\tilde{\Omega}$ to $\tilde{u}$.

→ Fix a precompact nbhd N s.t $K \subseteq N \subseteq \tilde{\Omega}$

→ extend $\tilde{u}|_N$ to Ω arbitrarily to u^*

$\rightarrow$ Also over N write

$$\tilde{u}_{\alpha'} = \exp(\xi_{\alpha'})\,\tilde{u}$$

$\tilde{u} = u^*$ on N

for sections $\xi_{\alpha'}$ of g_E which converges to 0.

$\rightarrow$ Let ψ be a function s.t $\mathrm{supp}(\psi) \subseteq N$ &

$$\psi|_K \equiv 1.$$

$\rightarrow$ define $\quad w_{\alpha}' = \exp(\psi\,\xi_{\alpha'})\,u^*$

$w_{\alpha}' = \tilde{u}_{\alpha'}$ on N

Then $w_{\alpha'}(A_{\alpha'})$ are convergent on Ω because $\xi_{\alpha'}$ converge over the $\mathrm{supp}(\psi)$ & $w_{\alpha'} = \tilde{u}_{\alpha'}$ on a nbhd of K.

Lemma 2:

Suppose that Ω is exhausted by an increasing sequence of precompact open sets.

$$\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \ldots \ldots \subseteq \Omega \qquad \bigcup_{n=1}^{\infty} \mathcal{U}_n = \Omega$$

Suppose A_α is a sequence of connections over Ω and for each n there is a subsequence $\{\alpha'\}$ & gauge transformation $u_{\alpha'} \in \mathrm{Aut}\,E|_{\mathcal{U}_n}$ s.t $u_{\alpha'}(A_{\alpha'})$ converges over $\mathcal{U}_n$. Then $\exists$ is a subsequence & a sequence of gauge transformation s.t the transformed connection converge over all Ω.

Lemma 3 :

Suppose Ω is a union of domains $\Omega = \Omega_1 \cup \Omega_2$ and A_α is a sequence of connections on a Bundle E over Ω. If there are sequences of gauge transformations $v_\alpha \in \operatorname{Aut} E|_{\Omega_1}$ & $w_\alpha \in \operatorname{Aut} E|_{\Omega_2}$ s.t $v_\alpha(A_\alpha)$ and $w_\alpha(A_\alpha)$ Converge over Ω_1 & Ω_2, then $\exists$ a subsequence $\{\alpha'\}$ and gauge transformations $u_{\alpha'}$ over Ω s.t $u_{\alpha'}(A_{\alpha'})$ Converges over Ω.

Proof :

By Lemma 2, it suffices to consider a Compact Subset of Ω Covered by precompact sets $\Omega_1' \subseteq \Omega_1$ & $\Omega_2' \subseteq \Omega_2$. $\underline{\Omega_1' \cup \Omega_2'}$

$\longrightarrow$ choose K Compact s.t $K \subseteq \tilde{\Omega} \subseteq \Omega$

$$\Omega_1' \cap \Omega_2' \subseteq K \subseteq \Omega_1 \cap \Omega_2 \subseteq \Omega_1$$

After modifying v_α and taking a subsequence, we may assume $v_\alpha = w_\alpha$ on $\Omega_1' \cap \Omega_2'$. Then these sequences glue together to define u_α over the union $\Omega_1' \cup \Omega_2'$. ∎

Corollary:

Suppose A_α is a sequence of connections on a bundle E over Ω s.t $\forall x \in \Omega$ $\exists$ open nbhd D_x of x, a subsequence $\{\alpha'\}$ & gauge transformations $v_{\alpha'}$ defined over D_x s.t $v_{\alpha'}(A_{\alpha'})$ converges over D_x.

Then $\exists$ a subsequence $\{\alpha''\}$ and gauge transformations $u_{\alpha''}$ defined over all of Ω s.t $u_{\alpha''}(A_{\alpha''})$ converges over all Ω.

Proof:

Again by lemma 2, we restrict to a precompact subset of Ω, also assume this set is a finite union of nbhds $D_1 \ldots D_M$ s.t D_i's satisfy the hypothesis.

Then by induction, $\exists$ a subsequence & gauge transformations s.t the transformed connections converge over $\Omega_{M-1} := D_1 \cup \ldots \cup D_{M-1}$ & by lemma 3 applied to the pair Ω_{M-1}, D_M gives the result

$\blacksquare$

From the ASD equation and Uhlenbecks theorem we get,

Theorem :

Let Ω be an oriented Riemannian 4 manifold.

True for Yang-Mills.

Suppose A_α is a sequence of ASD connections on E over Ω with the property as follows.

$\forall\ x \in \Omega\ \exists$ geodesic ball D_x s.t $\forall\ \alpha >> 0$

$$\int_{D_x} |F(A_\alpha)|^2 d\mu \leq \varepsilon^2$$

where $\varepsilon > 0$ is the constant from the gauge fixing theorem. Then $\exists$ a subsequence $\{\alpha'\}$ & gauge transformation $u_{\alpha'}$ s.t $u_{\alpha'}(A_{\alpha'})$ converges over Ω.

Uhlenbeck's theorem :

For any sequence of ASD connections A_α over $\overline{B}^4$ with $\|F(A_\alpha)\|_{L^2} \leq \varepsilon$ $\exists$ subsequence α' & gauge equivalent connections $\tilde{A}_{\alpha'}$ which converge in C^∞ on the open ball.

Proof of Compactness theorem :

The proof follows from two pieces of general theory.

1. We shall consider the curvature density of an ASD connection as a measure.

By Reisz Representation theorem

Thm:

Let X be a Compact hausdorff space.
then $C(X)^* \simeq$ Complex measures with total variation norm $\mathcal{M}(X)$.

$$\varphi : \mathcal{M}(X) \longrightarrow C(X)^*$$
$$\mu \longmapsto \varphi(\mu) : f \longmapsto \int_X f \, d\mu$$

For any sequence of positive measures on X with $\int_X d\nu_\alpha$ bounded then by Banach - Alaoglu theorem $\exists$ subsequence $\{\alpha'\}$ Converging to a limiting measure ν in the sense, $\forall f$ continuous on X,

$$\int_X f \, d\nu_{\alpha'} \longrightarrow \int_X f \, d\nu$$

weak * Convergence

2. The second piece of theory involves interpreting the curvature density of ASD connection A as a topological invariant.

$$\int_X |F(A)|^2 = -\int Tr(F(A)^2) = -8\pi^2 k(E).$$

The main role of this is to give a L^2 bound of the curvature of ASD connections.

Chern - Simon invariant

cf: Floer Homology groups in Yang - Mills theory by S. K. Donaldson.

Let Z be a compact oriented 4 manifold with $\partial Z = W$ & B be a connection over W.

Choose any extension of B to a connection A over Z.

$$\int_Z |F(A)|^2 \text{ is well defined mod } 8\pi^2 \mathbb{Z}$$

and depends only on the connection B over W.

Alternatively, we choose a trivilization over W to represent the connection by a connection matrix B

then

$$Tw(B) = \frac{1}{8\pi^2} \int_W Tr\left(dB \wedge B + \frac{2}{3} B \wedge B \wedge B \right) \mod \mathbb{Z}.$$

This depends on the trivilization only up to a integer.

→ The only fact we will be using is that $Tw(B)$ depends continuously on B.

Let A be a sequence of ASD connections on E with $c_2(E) = k$. We will first show that $\exists$ is a finite set $\{x_1, \dots, x_p\}$ in X s.t after taking a subsequence $X \setminus \{x_1, \dots, x_p\}$ satisfies the theorem.

choose a subsequence $\{\alpha'\}$ so that $|F(A_\alpha)|^2$ converge as a measure to ν, then we have

$$\int_X d\nu = 8\pi^2 k$$

So $\exists$ atmost $\frac{8\pi^2 k}{\varepsilon^2}$ points which have a geodesic ball of measure $> \varepsilon^2$.

i.e.,
$$\int_{D_x} d\nu > \varepsilon^2 .$$

We let these points be $\{x_1, \dots, x_p\}$.

Then by the theorem $\exists \{\alpha''\} \subset \{\alpha'\}$ & gauge transformations $u_{\alpha''}$ over $X \setminus \{x_1 \dots x_p\}$ s.t $u_{\alpha''}(A_{\alpha''})$ converges over this punctured manifold to an ASD connection A on $E|_{X \setminus \{x_1 \dots x_p\}}$.

clearly,

$$\int_{X \setminus \{x_1 \dots x_p\}} |F(A)|^2 \leq 8\pi^2 k$$

then by removable singularity theorem this extends to a connection on a bundle E' over X. $E' \not\cong E$ if $p>0$ as we have a strict inequality above if $p>0$.

The limiting measure ν is $\nu = |F(A)|^2 + \sum_{r=1}^{p} n_r \delta_{x_r}$ for some $n_r \geq \varepsilon^2$.

We need to show that $n_r \in \mathbb{Z}$. This follows from relative chern weil theory.

choose disjoint balls Z_r centered around x_r.

$$\tau_{\partial Z_r}(A) = \lim \tau_{\partial Z_r}(A''_\alpha) \in \mathbb{R}/\mathbb{Z}$$

After gauge transformation, the connections converge in C^∞ on ∂Z_r.

But we have the Convergence of measures which gives.

$$n_r = \frac{1}{8\pi^2} \lim \int_{Z_i} \mathrm{Tr}(F(A_\alpha)^2) - \mathrm{Tr}(F(A)^2)$$

using the defn of $I_{\partial Z_r}$ in terms of extension over the balls Z_r we see $n_r = 0 \mod \mathbb{Z}$.

$\blacksquare$

ε- regularity theorem.

Let (M,g) be a Riemannian Manifold. Let $inj_g(p)$ be the injectivity radius at $p \in M$.

For a fixed $p \in M$, we let $0 < r_p < inj_g(p)$ s.t

$\longrightarrow \exists$ normal nbhd $(x^1, \dots x^n)$ centered at p and

$\exists$ $c(p) > 0$ s.t g_{ij} satisfies.

1. $|g_{ij} - \delta_{ij}| \leq c(p)|x|^2$

2. $|\partial_k g_{ij}| \leq c(p)|x|$

Note: $g_{ij}(p) = \delta_{ij}$ & $\partial_k g_{ij}(p) = 0$. Taylor expansion about p gives the required nbhd.

Theorem 1:

Let (M,g) be a Riemannian manifold with $n \geq 4$ & E be a G-bundle over M, ∇ be a Yang Mills Connection with finite L^2 energy. Given $p \in M$, $\exists$ $\varepsilon_0 > 0$ & $C \geq 0$ s.t $\forall$ ρ, $0 < \rho \leq r_p$ if

$$\varepsilon = \frac{1}{\rho^{n-4}} \int_{B_\rho(p)} |F_\nabla|^2 \, dV_g < \varepsilon_0$$

then

$$\sup_{x \in B_{\rho/4}(p)} |F_\nabla|^2(x) \leq \frac{C\varepsilon}{\rho^4}$$

Lemma 1:

Bochner type estimate

Given $p \in M$ & $0 < r < \mathrm{inj}_g(p)$ $\exists$ $c, c' > 0$ where

→ c depends on n & curvature R^g on $\overline{B_r}(p)$

→ c' depends on n & G

s.t.

$$\Delta_{\bar{g}} |F_\nabla|^2 \geq -c\, |F_\nabla|^2 - c' \, |F_\nabla|^3 \quad \text{on } B_r(p).$$

Notation: $\Delta_{\bar{g}} = -d^* d : C^\infty(M) \longrightarrow C^\infty(M)$

Lemma 2:

Harnack - Moser inequality

Let $p \in M$ & $0 < r < \frac{\mathrm{inj}_g(p)}{2}$ & $C_0 > 0$ be given.

If $u \in C^2(\overline{B_r}(p))$, $u \geq 0$ &

$$\Delta_{\bar{g}}\, u \geq -C_0\, u \quad \text{on } B_r(p)$$

then

$$\sup_{B_{\frac{r}{2}}(p)} u \leq c \int_{B_r(p)} u \, dV_g$$

c depends on n, r & R^g.

Theorem : Monotonicity formula

Let $p \in M$, r_p & $c(p)$ are as defined before.

Then $\exists$ $a = a(n, p, g) \geq O(1)\, C(p)$ s.t the following holds.

$\forall$ σ, ρ ; $0 < \sigma < \rho \leq r_p$

$$e^{a\sigma^2} \sigma^{4-n} \int_{B_\sigma(p)} |F_\nabla|^2 \, dV_g \leq e^{a\rho^2} \rho^{4-n} \int_{B_\rho(p)} |F_\nabla|^2 \, dV_g$$

Proof of main theorem :

The stated bounds won't be affected by the scaling $g \longrightarrow \lambda g$, $\rho \longrightarrow \lambda^{1/2} \rho$ for some constant $\lambda > 0$.

So, we can suppose $\rho = 1$.

$$\varepsilon = \int_{B_1(p)} |F_\nabla|^2 \, dV_g \leq \varepsilon_0$$

We have to prove $\varepsilon > 0$ sufficiently small

$$\sup_{x \in B_{\frac{1}{4}}(p)} |F_\nabla|^2 (x) \leq C \varepsilon$$

Define a function $f : [0,1] \longrightarrow [0, \infty)$

$$r \longmapsto (1-r)^2 \sup_{x \in B_r(p)} |F_\nabla| (x)$$

$$1 = \rho \leq r_p < \operatorname{inj}_g(p)$$

$$r \longmapsto \sup_{x \in B_r(p)} |F_\nabla|(x) \text{ is Continuous on } [0,1].$$

$\Longrightarrow$ f is Continuous & attains maximum Say at $r_0 \in [0,1]$

- $b := \sup_{x \in B_{r_0}(p)} |F_\nabla|(x)$

- $x_0 \in B_{r_0}(p)$ s.t $|F_\nabla|(x_0) = b$

- $\sigma := \frac{1}{2}(1-r_0)$

So, $f(r_0) = 4\sigma^2 b$.

Clearly $F_\nabla = 0$ on $B_1(p) \Leftrightarrow f=0 \Leftrightarrow b=0 \Leftrightarrow \sigma=0 \Leftrightarrow r_0=1$

If $f=0$ then we are done. The desired bound follows for any $C>0$.

If $f \neq 0$ ie $\sigma > 0$ then

$$\sup_{x \in B_\sigma(x_0)} |F_\nabla|(x) \le \sup_{x \in B_{\sigma+r_0}(p)} |F_\nabla|(x)$$

$$= \frac{1}{(1-(\sigma+r_0))^2} f(\sigma+r_0)$$

$$\le \frac{1}{(1-(\sigma+r_0))^2} f(r_0) \qquad 1-(\sigma+r_0) = \sigma > 0$$

$$= \frac{1}{\sigma^2} f(r_0) = 4b$$

Claim: $f(r_0) \le 16$ if $\varepsilon_0 = \varepsilon_0(n, p, q, G)$ is small enough.

Proof:

Suppose $f(r_0) > 16$

$\Rightarrow \sigma\sqrt{b} > 2$

define $\tilde{g} := bg$

So, $|F_\nabla|_{\tilde{g}} = \frac{1}{b}|F_\nabla|_g$

$$B_{\sigma\sqrt{b}}(x_0; \tilde{g}) = B_\sigma(x_0; g)$$

$$\sup_{x \in B_2(x_0; \tilde{g})} |F_\nabla|_{\tilde{g}}(x) \le \sup_{x \in B_{\sigma\sqrt{b}}(x_0; \tilde{g})} |F_\nabla|_{\tilde{g}}(x)$$

$$= \frac{1}{b} \sup_{x \in B_\sigma(x_0, g)} |F_\nabla|_g(x)$$

$$\le 4 \qquad \text{—— } (*)$$

clearly, ∇ is Yang Mills w.r.to $\tilde{g}$ too. This follows by noting $*_{\tilde{g}} = b^{\frac{n}{2}-2} *_g$ on 2-forms

& $r_p(\tilde{g}) = r_p(g)\sqrt{b}$

$$B_2(x:\tilde{g}) \subseteq B_{(\sigma + r_0)\sqrt{b}}(p; \tilde{g}) \subseteq B_{r_p(g)\sqrt{b}}(p; \tilde{g})$$

we have from, lemma 1,

$$\Delta_{\tilde{g}} |F_\nabla|^2_{\tilde{g}} \ge -c|F_\nabla|^2_{\tilde{g}} - c'|F_\nabla|^3_{\tilde{g}} \quad \text{on } B_2(x_0; \tilde{g})$$

From $(*)$ we have

$$\Delta_{\tilde{g}} |F_\nabla|^2_{\tilde{g}} \geq -(c+4c')|F_\nabla|^2_{\tilde{g}} \quad \text{on } B_2(x_0;\tilde{g})$$

From lemma 2, we obtain

$$1 = |F_\nabla|^2_{\tilde{g}}(x_0) \leq \tilde{C} \int_{B_1(x_0;\tilde{g})} |F_\nabla|^2_{\tilde{g}}\, dV_{\tilde{g}}$$

we know that $\sigma\sqrt{b} > 2$

$\sigma \leq 1$, $b > 0$ & $1 = P \leq r_p$

$\Rightarrow 0 < \frac{1}{\sqrt{b}} < \frac{1}{2} < r_p$

$$\int_{B_1(x_0,\tilde{g})} |F_\nabla|^2_{\tilde{g}}\, dV_{\tilde{g}} = \left(\frac{1}{\sqrt{b}}\right)^{4-n} \int_{B_{\frac{1}{\sqrt{b}}}(x_0,g)} |F_\nabla|^2_g\, dV_g \qquad (\tilde{g} = bg)$$

$$\leq \left(\frac{1}{\sqrt{b}}\right)^{4-n} e^{\frac{a}{b}} \int_{B_{\frac{1}{\sqrt{b}}}(x_0;g)} |F_\nabla|^2 dV_g \qquad e^{\frac{a}{b}} \geq 1$$

here "a" from montonicity

Now apply monotonicity lemma

$$\leq \left(\frac{1}{2}\right)^{4-n} e^{\frac{a}{4}} \int_{B_{\frac{1}{2}}(x_0;g)} |F_\nabla|^2_g\, dV_g$$

$$\leq 2^{n-4} e^{\frac{a}{4}} \varepsilon_0$$

So we have $1 \leq 2^{n-4} e^{\frac{a}{4}} \varepsilon_0$

If ε_0 is chosen sufficiently small

this gives a contradiction $\blacksquare$

So we have $f(r) \leq 16 \quad \forall\, r \in [0, 1]$

take $r = \frac{1}{2}$

$$\sup_{x \in B_{\frac{1}{2}}(p)} |F_\nabla|(x) \leq 64$$

Apply lemma 1 again

$$\Delta_g |F_\nabla|^2 \geq -(C + 64\, C')\, |F_\nabla|^2 \quad \text{on} \quad B_{\frac{1}{2}}(p)$$

Apply lemma 2 again to obtain the required result.