Gauge theory seminar

5.1.21

Taubes' gluing theorem

$\mathcal{A}$ - space of $L^2_{\ell-1}$-connections on $E \to X$, $\ell \geq 2$

$\mathcal{G}$ - gauge group

$\mathcal{B} = \mathcal{A}/\mathcal{G}$

$M_E = \{A \in \mathcal{B} \mid F^+_A = 0\}$ space of ASD connections

$T_{A,\varepsilon} = \{a \in \Omega^1(\mathfrak{g}_E) \mid d_A^* a = 0, \ \|a\|_{L^2_{\ell-1}} < \varepsilon\}$

- Projection map from $\mathcal{A}$ to $\mathcal{B}$ induces homeomorphism h from $T_{A,\varepsilon}/\Gamma_A$ to nbhd of A in $\mathcal{B}$ (for small ε).

$\psi : T_{A,\varepsilon} \to \Omega^+(\mathfrak{g}_E)$

$\psi(a) = d_A^+ a + (a \wedge a)^+$

- Then h induces homeomorphism from the quotient $\psi^{-1}(0)/\Gamma_A$ to nbhd of $[A]$ in M.

  - Differential operator $d_A^* \oplus d_A^+ : \Omega^1 \to \Omega^0 \oplus \Omega^+$ is elliptic $\Rightarrow$ Fredholm.

- If A is regular (d_A^+ surjective), then by implicit function theorem $\psi^{-1}(0)$ is finite-dimensional manifold whose dimension depends only on topological data.

- $\boxed{\text{Thm (transversality):}}$ $\exists$ subset $\mathcal{C}' \subset \mathcal{C}$ of conformal classes on X of second category s.t. for $g \in \mathcal{C}'$ and any $SU(2)$ bundle E over X, any irreducible connection is regular.

$d_A^+ \circ d_A = F_A^+ = 0$, then homology of

$$\Omega^0(g_E) \xrightarrow{d_A} \Omega^1(g_E) \xrightarrow{d_A^+} \Omega^+(g_E)$$

we denote by H_A^i

In particular, $H_A^2 = \operatorname{coker} d_A^+ = 0$ iff A regular

Details of ~~all of~~ this will be completed in one of the lectures, today we are more interested in nbhds of ~~M~~ points in $\overline{M_E} \setminus M_E$ in M_E.

- Recall the definition of ideal connections on $\overbrace{E \to X}^{SU(2) \text{ bundle}}$ of Chern class k

$$IM_k = M_k \cup (M_{k-1} \times X) \cup (M_{k-2} \times s^2(X)) \cup \ldots$$

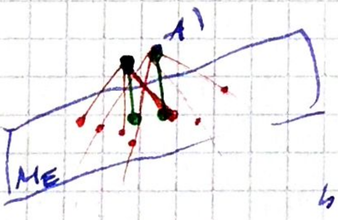
- We have proven last time that $\overline{M_E}$ is compact.

* **Goals of today's lecture:**

- understand a bit better nbhds of points $\overline{M}_E \setminus M_E$ in M_E. Complete proof will be given in a lecture on collar theorem,
- see how we can „glue" two ASD connections on X_1, X_2 into new ASD connection on $X_1 \# X_2$,
- prove the existence of ASD connections on $SU(2)$ bundles over manifolds more complicated than S^4 (and recall PBST instantons) where explicit construction is not possible.

The main points are

- construction of „almost" ASD connections by cutting and gluing ASD connections over two manifolds
- perturbing this „almost" ASD connection into ASD connection



As we have seen, $\dim M_E$ can be anything and there can be many possible directions in which we can perturb „almost" ASD connection A'. We can expect that along the way we need to make some choices (which ~~we~~ need to ~~make~~ work for ~~modular~~ problem as ~~well~~). For us most of it will be given by particular choice of right inverses of $d_{A'}^{+}$. ↯

F

Perturbation of „almost" ASD connection

- Let A be connection on $E \to X$ s.t.

  $\| F^+(A) \| \le \varepsilon$ and such that

  d_A^+ has bounded right inverse $P : \Omega^+(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$

  $\left(\| P\eta \|_{L^4} \le C \| \eta \|_{L^2} \right)$.

- We are looking for $a \in \Omega^1(\mathfrak{g}_E)$ s.t.

  $$\boxed{0 = F^+(A+a) = F^+A + d_A^+ a + (a \wedge a)^+}$$

  We set $a = P\xi$, then $\uparrow$ is equivalent to

  $$\boxed{\xi + (P\xi \wedge P\xi)^+ = -F^+(A).}$$

  Denote $q(\eta) = (P\eta \wedge P\eta)^+ : \Omega^+ \to \Omega^+$

  From Hölder we get

  $$\|(a \wedge b)^+\|_{L^2} \le \|a\|_{L^4} \|b\|_{L^4},$$

  so

  $$\| q(\eta) \|_{L^2} \le \| P\eta \|_{L^4}^2 \le C^2 \|\eta\|_{L^2}^2,$$

  and

- $$\| q(\eta_1) - q(\eta_2) \|_{L^2} = \| (P\eta_1 \wedge P\eta_1 - P\eta_2 \wedge P\eta_2)^+ \| =$$

  $$= \| ((P\eta_1 - P\eta_2) \wedge (P\eta_1 + P\eta_2))^+ \| \le$$

  $$\le C^2 \| \eta_1 - \eta_2 \|_{L^2} \left(\|\eta_1\|_{L^2} + \|\eta_2\|_{L^2} \right)$$

From Banach fixed point theorem, we get the following lemma:

- **Lemma:** $S : B \to B$ ~~[illegible]~~ smooth map on Banach space with $S(0) = 0$ and $\| S\eta_1 - S\eta_2 \| \le K (\|\eta_1\| + \|\eta_2\|) \|\eta_1 - \eta_2\|$ $\forall \eta_1, \eta_2 \in B$. Then for all η, $\|\eta\| < \frac{1}{10K}$, there exists unique ξ with $\|\xi\| \le \frac{1}{5K}$ such that

  $$\xi + S(\xi) = \eta.$$

A Let $T\xi = \eta - S\xi$

for $\|\eta\| < \frac{1}{10K}$ and all $\xi, \|\xi\| < \frac{1}{5K}$, we have

$$\|T\xi\| \le \|\eta\| + \|S\xi\| = \|\eta\| + \|S\xi - S0\| \le \frac{1}{10K} + K\cdot\left(\frac{1}{5K}\right)^2 < \frac{1}{5K}$$

and $\|T\xi_1 - T\xi_2\| = \|S\xi_2 - S\xi_1\| \le K(\|\xi_1\| + \|\xi_2\|)\|\xi_1 - \xi_2\| \le$

$\le \frac{2}{5}\|\xi_1 - \xi_2\|$

$\Rightarrow$ T has unique fixed point ξ in $B_{\frac{1}{5K}}$

that satisfies $\xi + S\xi = \eta$ □

- Now apply lemma to

$S = 2$, $\eta = -F^+(A)$ for ε small enough

and $\|F^+(A)\|_{L^2} < \varepsilon$,

to get unique ξ, $\|\xi\|_{L^2} < \delta$

such that $A + P\xi$ is ASD connection.

A bit about right inverse of d_A^+ for regular ASD connection

denote $\mathcal{D} = d_A^+$, this map is surjective by assumption.

assume that $\mathcal{D}^*\xi = 0$ for formal adjoint of $\mathcal{D}$.

Then $\xi = \mathcal{D}\eta$ and

$$\mathcal{D}^*\mathcal{D}\eta = 0$$

this gives us $\langle \mathcal{D}^*\mathcal{D}\eta, \eta\rangle_{L^2} = 0$ $\Rightarrow$

$\Rightarrow$ $\|\xi\| = \|\mathcal{D}\eta\| = 0$

$\Rightarrow$ $\mathcal{D}^*$ is injective

Map $\mathcal{D}\mathcal{D}^*$ is elliptic self-adjoint operator (Laplacian)

assume $\mathcal{D}\mathcal{D}^*$ is not surjective

then $\exists \xi \neq 0$ s.t. $\langle \mathcal{D}\mathcal{D}^*\eta, \xi\rangle_{L^2} = 0$ $\forall \eta$

$\Rightarrow$ $\underset{\text{contradiction}}{\mathcal{D}^*\xi = 0}$ $\Rightarrow$ $\xi = 0$

Since $\mathcal{D}\mathcal{D}^*$ is self-adjoint Fredholm op., it also must be injective.

Now, let $P = \mathcal{D}^* \circ (\mathcal{D}\mathcal{D}^*)^{-1}$

Connected sum

$i \in \{1,2\}$

X_i compact 4-mfd, $x_i \in X_i$

$E_i \to X_i$ G-bundle

Let A_i be ASD connections on E_i

- We want to perform cutting and gluing operation that gives us „almost" ASD connection A' on $X = X_1 \# X_2$, and we also construct right inverse that we needed (under assumption that $H^2_{A_i} = 0$ of course, we will see what can be done in irregular case).

- $\lambda > 0$ small and fixed

  also fix orientation reversing linear isometry

  $\sigma : T_{x_1}X_1 \to T_{x_2}X_2$ that induces inversion map

  $f_\lambda : T_{x_1}X_1 \setminus \{0\} \to T_{x_2}X_2 \setminus \{0\}$

  $f_\lambda(\xi) = \dfrac{\lambda\,\sigma(\xi)}{\|\xi\|^2}$

(We can assume metric is flat in nbhds of x_i, changing the metric in such way otherwise produces small error term that can be ignored.)

Also fix N s.t. $N\lambda^{\frac12} \ll 1$.

$\Omega_i \subset X_i$ annulus around x_i with inner radius $N^{-1}\lambda^{\frac12}$ and outer radius $N\lambda^{\frac12}$. Then f_λ induces orientation reversing diffeomorphism

$f_\lambda : \Omega_1 \to \Omega_2$

Define $X_i' = X_i \setminus B(x_i, N^{-1}\lambda^{\frac12})$, $X_i'' = X_i \setminus B(x_i, \frac12 \lambda^{\frac12})$

and $\underline{\underline{X = X_1' \cup_{f_\lambda} X_2'}}$

$(X_i'' \subset X$ and $X_1'' \cup X_2'' = X)$.

equivalently, define cylindrical ends on X_i around x_i by composing neighborhoods $B_{N\lambda^{\frac12}}$ with ~~map~~ map

$\mathbb{R} \times S^3 \to \mathbb{R}^4 \setminus \{0\}$

$(t, \omega) \to e^t \omega$

then annuli Ω_i map to tubes

$\left(\frac12 \log\lambda - T, \frac12 \log\lambda + T\right) \times S^3$ where $T = \log N$

(length of the „neck" $= 2T$)

and glue cylindrical ends by reflection to obtain X (see images).

f_λ is conformal map (flat nbhds), so $[g_1], [g_2]$ on X_1, X_2 give us unique conformal class on X.

Now, to glue the rest of the data, we perturb A_i to connections that are trivial on Ω_i.

Fix $b > 8N\lambda^{\frac{1}{2}}$ and trivializations of E_i over $B(x_i, b)$. Using bump functions perturb A_i to a new connection A_i' that is trivial ~~xxxxxxx~~ over $B(x_i, b)$

such that $\| F^+(A_i') \|_{L^2} \le \tilde{C}_1 b^2$

for const C_1 indep of b

(b^2 appears because of measure of $B(x, b)$ $\sim O(b^4)$)

and $\| A_i - A_i' \|_{L^4} \le \tilde{C}_2 b$

- After choosing $g : (E_1)_{x_1} \xrightarrow{G} (E_2)_{x_2}$, together with fixed trivializations from above, we get bundle $\underline{E(g) \to X}$ by identifying fibers over $\Omega_1 \overset{\text{tr}}{\sim} \Omega_2$ using g, and $\underline{\text{connection}}$ $\underline{A'(g)}$ on $E(g)$, with small $\| F^+ A' \|$.

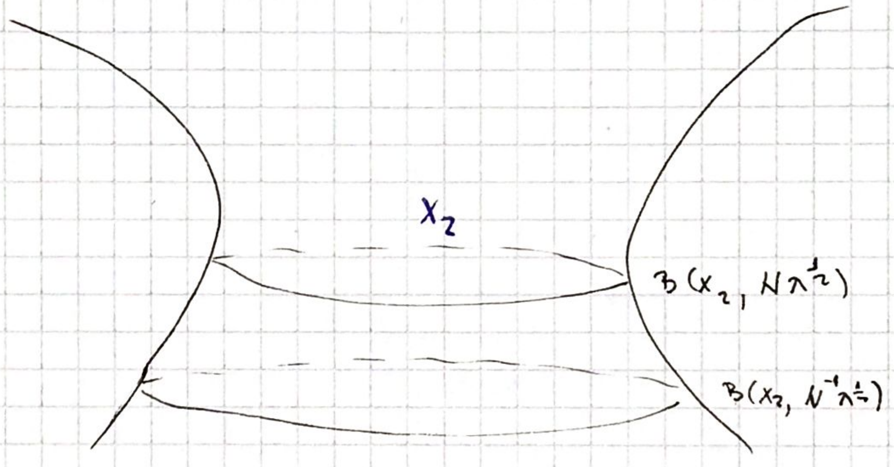
(Note that bundles $E(g)$ for different g are equivalent.)

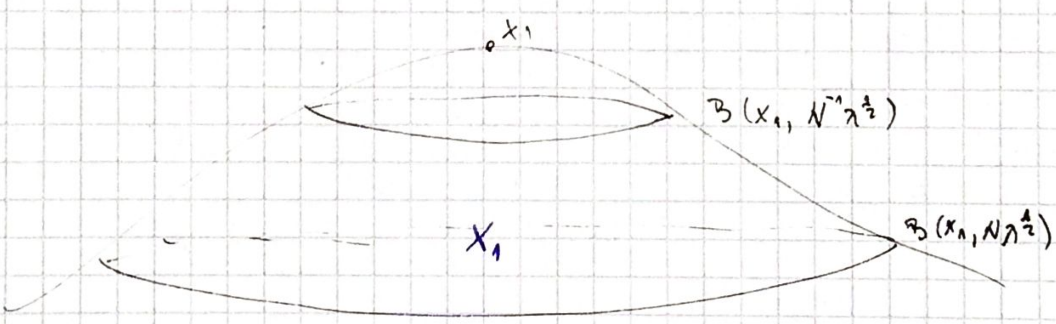
Denote $Gl = Hom_G((E_1)_{x_1}, (E_2)_{x_2}) \cong G$

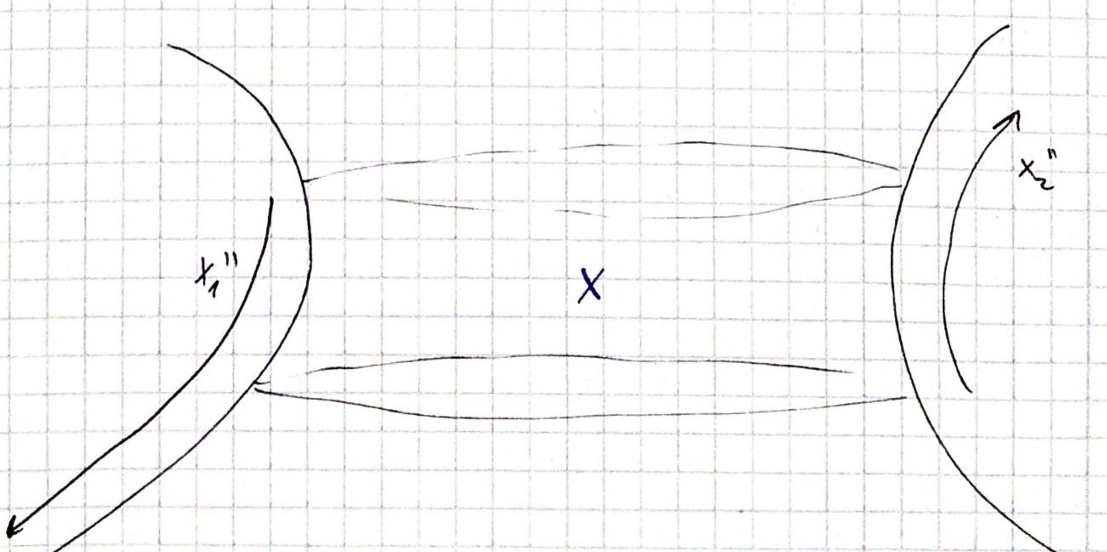
- Let $P_i : L^2(\mathfrak{g}_E \oplus \Lambda^2_+) \to L^4(\mathfrak{g}_E \oplus \Lambda^1)$ be right inverses of $d^+_{A_i}$ constructed before (we assume $H^2_{A_i} = 0$).

- We need to choose cut-off functions β_i, γ_i with specific properties for the purpose of gluing P_i

$$\begin{cases} \beta_i \equiv 1 \text{ on } B(x_i, N\lambda^{\frac{1}{2}})^c \\ \beta_i \equiv 0 \text{ on } B(x_i, N^{-1}\lambda^{\frac{1}{2}}) \\ \| \nabla \beta_i \|_{L^4} \le K (\underbrace{\log N}_{=:T})^{-\frac{3}{4}} \\ \quad \text{where } K \text{ is independent of } \lambda \text{ and } T \end{cases}$$

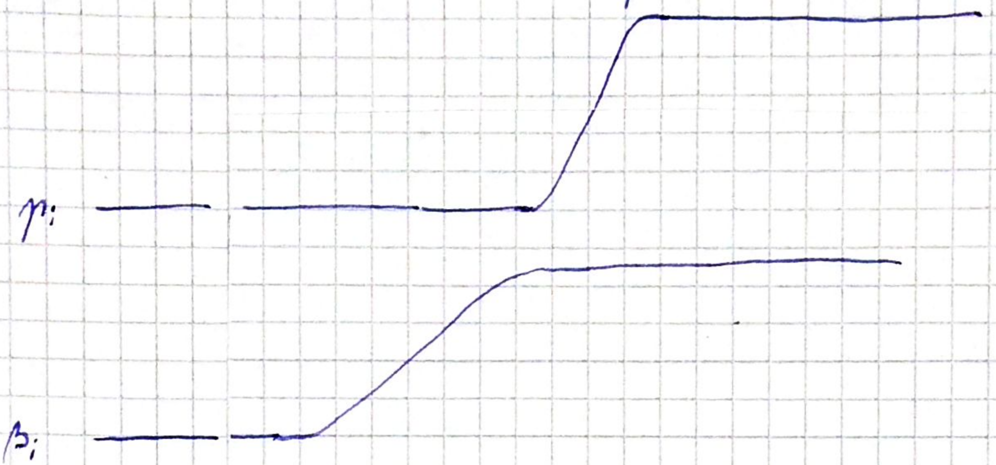

X_2
$B(x_2, N\lambda^{\frac{1}{2}})$
$B(x_2, N^{-1}\lambda^{\frac{1}{2}})$

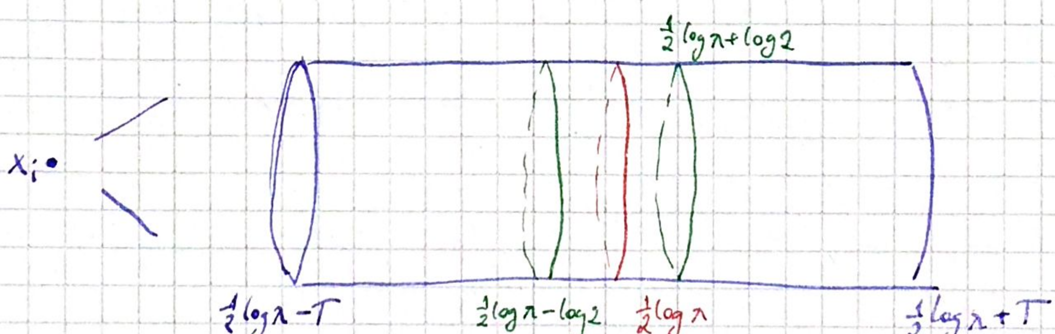

x_1
$B(x_1, N^{-1}\lambda^{\frac{1}{2}})$
X_1
$B(x_1, N\lambda^{\frac{1}{2}})$


x_1''
X
x_2''

$$\begin{cases} \mu_i \text{ supported in } X_i'' = X_i \setminus B(x_i, \tfrac{1}{2}\lambda^{\frac{1}{2}}) \\ \beta_i \equiv 1 \text{ on } X_i \setminus B(x_i, 2\lambda^{\frac{1}{2}}) \\ \nabla\mu_i \text{ supported on part where } \beta_i \equiv 1 \quad \left(\text{so that } \mu_i \cdot \beta_i = \mu_i\right) \end{cases}$$

It's not hard to come up with such functions





(~0 otherwise)

$\ulcorner$ $\|\nabla\beta_i\| \sim \frac{2}{T}$ on measure $\frac{4}{3}\pi\frac{T}{2}$

$\Rightarrow \|\nabla\beta_i\|_{L^4} \sim \left(\left(\frac{2}{T}\right)^4 \cdot \frac{4}{3}\pi\frac{T}{2}\right)^{\frac{1}{4}} \sim K T^{-\frac{3}{4}}$

$\|\nabla\mu_i\|_{L^4}$ indep. of $\lambda, T \ldots$

$\llcorner$

We can also choose $\mu_2 = 1 - \mu_1$

Now let $Q_i\xi = \beta_i P_i(\mu_i\xi)$

- $\boxed{\text{Lemma:}}$ $\exists\, c_1(N,b)$ s.t. $c_1 \to 0$ when $N \to \infty, b \to 0$ and $\forall\lambda$ with $4N\lambda^{\frac{1}{2}} \leq b$

$\left(\forall \xi \in L^2(\mathfrak{g}_E \otimes \Lambda^2_+)\right)$ $\|\mu_i\xi - d^+_{A_i} Q_i\xi\|_{L^2} \leq c_1\|\xi\|_{L^2}$.

Δ $A_i' = A_i + a_i$, $\|a_i\|_{L^4} \leq \tilde{c}_2 b$

$$d^+_{A_i'} = d^+_{A_i} + [a_i, \ ]$$

$$d^+_{A_i'}(Q_i \xi) = \left(d^+_{A_i} + [a_i, \ ]\right)\left(\beta_i P_i(\rho_i \xi)\right) =$$

$$= \beta_i d^+_{A_i} P_i(\rho_i \xi) + d\beta_i P_i(\rho_i \xi) + [a_i, \beta_i P_i(\rho_i \xi)] =$$

$$\overset{\beta_i \rho_i = \rho_i}{=} \rho_i \xi + d\beta_i P_i(\rho_i \xi) + [a_i, \beta_i P_i(\rho_i \xi)]$$

The statement follows from

$$\|\nabla \beta_i\|_{L^4} \leq K(\log N)^{-\frac{3}{4}}$$

$$\|P_i(\rho_i \xi)\|_{L^4} \leq C\|\xi\|_{L^2}$$

$\|a_i\|_{L^4} \leq \tilde{c}_2 b$ and Hölder ineq. $\square$

- Now, we glue Q_i into operator on E with

$$Q : \Omega^+(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$$

$$\underline{\underline{Q\xi = Q_1 \rho_1 \xi + Q_2 \rho_2 \xi}}$$

this is well def. because supp $\rho_i \subset X_i''$ and $Q_i \rho_i \xi$ supported in $X_i'' \cup \Omega$, so

$$d^+_{A'} Q\xi - \xi = d_{A_1'} Q_1 \rho_1 \xi + d_{A_2'} Q_2 \rho_2 \xi - \rho_1 \xi - \rho_2 \xi$$

$$\Rightarrow \quad d^+_{A'} Q = \mathbb{1} + R$$

for $\|R\| \leq c_1(N,b) + c_2(N,b)$

We can choose N, b st.

$$c_i < \frac{1}{3}$$

Lemma If $\|1 - A\| < 1$, then A is invertible.

$$\left(A^{-1} = (1-(1-A))^{-1} = \sum_{n=0}^{\infty} (1-A)^n \leftarrow \text{well def.}_{\|1-A\|<1} \right)$$

- Now, $P = Q \circ (1+R)^{-1}$ is right inverse for $d^+_{A'}$.

- This is enough to proceed as before and solve

$$F^+_{A'} + d^+_{A'} a + (a \wedge a)^+ = 0$$

for $a = P\xi$.

This gives us unique choice of $a_g \in \Omega^1(g_E)$

$\|a\|_{L^4} \le \delta C$, s.t. $A'(g) + a_g$ is ASD

connection on $E \to X$.

Obstruction map

- Now we explain a bit about case $H^2_{A_i} \neq 0$

$\operatorname{Im}(d^+_{A_i})(d^+_{A_i})^* \subset \operatorname{Im} d^+_{A_i}$, so $\operatorname{Im} d^+_{A_i}$ is cofinite-dim.

Choose its complement

$$\sigma_i : H^2_{A_i} \longrightarrow \Omega^+(g_{E_i})$$

(we can also assume that forms in $\operatorname{Im}\sigma_i$ are supp. in X_i' by mult. w/ ~~smooth~~ cut-off functions)

Then the map $d^+_{A_i} \oplus \sigma_i (\eta, N) = d^+_{A_i}\eta + \sigma_i N$

is surjective

and we can find inverse

$P_i \oplus \pi_i$, where $\pi_i : \Omega^+ \to H^2_{A_i}$ is

σ_i^{-1} composed with projection in Ω^+ along $\operatorname{Im} d^+_{A_i}$ to $\operatorname{Im}\sigma_i$.

This is done similar as before, by finding inverse of $d_A^+ \circ (d_A^+)^*$ restricted to complements of ker and ~~[illegible]~~ Im.

- This means $d_{A_i}^+ \circ P_i \xi = \xi - \sigma_i \circ \pi_i \xi$

  Then we would (if $F^+A_i \neq 0$) look for solutions $(\xi, h) \in \Omega^+ \times H^2_{A_i}$ of

$$\boxed{\begin{aligned} &F^+(A_i + P_i\xi) - \sigma_i h = 0 \\ &F^+A_i + q(\xi) + \xi - \sigma_i \pi_i \xi - \sigma_i h = 0 \end{aligned}}$$

  so we set $h = \pi_i \xi$

  and find unique solution of

$$\xi + q(\xi) = -F^+A_i$$

  as before

- The difference now is that perturbation $A_i + P\xi$ is ASD iff $\pi_i \xi = 0$.

- We glue the data same as before and find inverse of $d_{A'}^+ \oplus \sigma$ $P \oplus \pi'$.

$$\pi' : \Omega^+(g_E) \to H^2 = H^2_{A_1} \oplus H^2_{A_2}$$

with $A'_S + P\xi_S$ is ASD iff $\pi' \xi_S = 0$.

- This gives us map $\phi : Gl \to H^2$ called <u>obstruction map</u> s.t. set $\phi^{-1}(0)$ corresponds to different gluings of $A_{1,2}$ wrt $\xi \in \phi^{-1}(0)$.

- It's possible to investigate obstruction map ψ and that its leading term as $\lambda \to 0$ is given by pairing the curvature of one of the connections A_i at x_i with harmonic forms representing $H^2_{A_j}$ at x_j using the identification σ of tangent spaces at x_1, x_2, ~~[illegible]~~ ~~[illegible]~~ see ~~[illegible]~~1984 this should give us dimension of $\psi^{-1}(0)$ (for more see Taubes 1984, „Self-dual connections on mfds with indefinite intersection matrix" and Donaldson 1986, „Connections, cohomology and the intersection forms of four mfds")

- We can allow connections over X_i to vary and construct gluing map $I :$ ~~[illegible]~~ $T \to M_E$ (by perturbing pregluing map $J : T \to B_E$, $(\hat{A}_1, \hat{A}_2, \rho) \mapsto A'_{\rho}(\hat{A}_1, \hat{A}_2)$ by $\xi_\rho(\hat{A}_1, \hat{A}_2)$) for $T = \{ (\hat{A}_1, \hat{A}_2, \rho) \mid \hat{A}_i$: connection in some nbhd C_i of A_i in $\psi_i^{-1}(0)$, $\rho \in Gl \}$ $\Gamma_{\hat{A}_1} \times \Gamma_{\hat{A}_2}$

Theorem: For small λ and small nbhds C_i, ~~the~~ map I induces homeomorphism from T to an open set in M_E.

If A_i are not regular, then ~~t~~ domain of I is not T, but subset given by $\phi^{-1}(0)$ for ϕ obstruction map.

This is generalization of collar theorem that describes open nbhds of points $\overline{M}_E \setminus M_E$ in M_E, which we will prove in a lecture later.

ASD connections over S^4

standard metric on S^4

- $S^7 = \{ (q^1, q^2) \in \mathbb{H}^2 \mid \|q^1\|^2 + \|q^2\|^2 = 1 \}$

  $SU(2) \cong$ group of unit quaternions $\cong S^3$

  we have free action of $SU(2)$ on S^7.

  $$q \cdot a = (q^1 a, q^2 a)$$

  and $SU(2)$-bundle over

  $$S^7 / SU(2) \cong \mathbb{H}P^1 \cong S^4$$

  of Chern class $k = 1$, denote it by $\tilde{E}$ (this is one of the Hopf fibrations).

- Real orthogonal complement of vertical space gives canonical connection 1-form

  $$\theta = \mathrm{Im}(\bar{q}^1 dq^1 + \bar{q}^2 dq^2)$$

  $$\bar{x} = x^1 - x^2 i - x^3 j - x^4 k$$
  $$\mathrm{Im}\, x = x^2 i + x^3 j + x^4 k$$
  $$dx = dx^1 + i\,dx^2 + j\,dx^3 + k\,dx^4$$
  $$d\bar{x} = dx^1 - i\,dx^2 - j\,dx^3 - k\,dx^4$$

  After stereographic projection to $\mathbb{H}$, this becomes

  $$A = \mathrm{Im}\left(\frac{\bar{q}}{1 + |q|^2} dq \right)$$, and its curvature is

  $$F = dA + A \wedge A = \frac{1}{(1+|q|^2)^2} d\bar{q} \wedge dq$$

$$d\bar{q} \wedge dq = 2\big((dq^1 \wedge dq^2 - dq^3 \wedge dq^4) i + (dq^1 \wedge dq^3 + dq^2 \wedge dq^4) j + (dq^1 \wedge dq^4 - dq^2 \wedge dq^3) k \big)$$

[illegible] is ~~[illegible]~~ in Λ^2_+

A is ASD connection on $\tilde{E}$.

- By translations on the base space $x \to x - b$ and dilations $x \to \frac{x}{\lambda}$, for $b \in \mathbb{R}^4$, $\lambda \in (0,+\infty)$,

We obtain a family of ASD 1-forms

$$A_{\lambda,b} = \operatorname{Im}\left(\frac{\bar{q} - \bar{b}}{\lambda^2 + |q-b|^2}\, dq \right)$$

with curvature

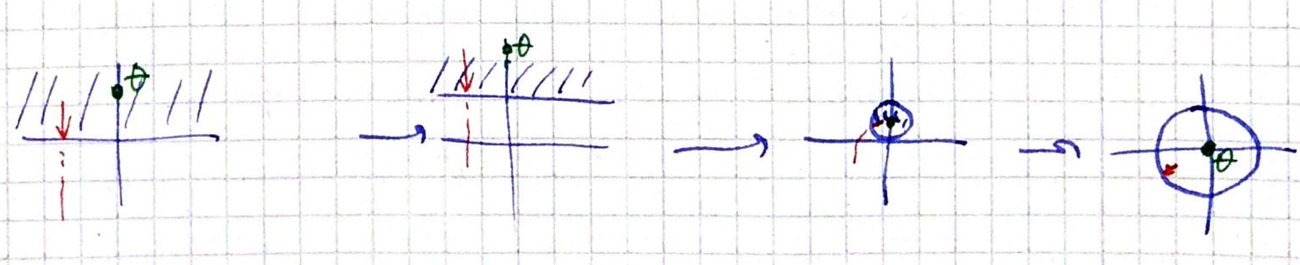
$$F_{\lambda,b} = \frac{\lambda^2}{(\lambda^2 + |q-b|^2)^2}\, d\bar{q} \wedge dq$$

$\frac{\lambda^2}{\lambda^2+|q-b|^2}$ has ~~maximum~~ maximum $\frac{1}{\lambda^2}$ at b, so the curvature becomes more and more concentrated around b as $\lambda \to 0$.

- Moduli space $M_{\tilde{E}}$ contains ~~this~~ this $\mathbb{R}^+ \times \mathbb{R}^4$ family (BPST instantons), Atiyah, Hitchin, Singer (1978) showed that this is entire moduli space ($\dim M_{\tilde{E}} = 5$).

  $M_{\tilde{E}}$ corresponds to compactification of $\mathbb{R}^+ \times \mathbb{R}^4$ by S^4 by adding points $\{0\} \times \mathbb{R}^4$ and one at infinity.

- There is another $\underbrace{\text{description}}_{M_{\tilde{E}} \cong B_5 = \{x \in \mathbb{R}^5 \mid |x|<1\}}$ by looking more at S^4 instead of $S^4 \setminus \{0\}$, which corresponds to composition w/ conformal map $(0,\infty) \times \mathbb{R}^4 \to B_5$ given by inversion.



θ is represented by the center now, and ~~[illegible]~~ since restriction of this map to $\{0\}\times\mathbb{R}^4$ is actually stereographic projection to $S^4 = \partial B_5$, points on ∂B_5 represent ideal connections in $\overline{M_{\tilde{E}}} \setminus M_{\tilde{E}}$ whose curvature is concentrated at that point. Therefore, standard compactification of B_5 corresponds to compactification $\overline{M_{\tilde{E}}}$.

- Atiyah, Drinfeld, Hitchin, Manin gave construction of ASD conn. over $SU(2)$-bundles on S^4 of any Chern class.

$M_E \cong B_5 \cong \{x \in \mathbb{R}^5 \mid \|x\| < 1\}$

$\overline{M_E} \cong D_5 \cong \{x \in \mathbb{R}^5 \mid \|x\| \leq 1\}$

- Because of the nature of compactification and us representing pts in $\overline{M}_E \setminus M_E$ by $(A, x_1 - x_\ell) \in M_{k-\ell} \times s^\ell(X)$ (ℓ means we need to glue ℓ S^4's, gluing construction is easily generalized to a case of connected sum of more than 2 manifolds, as $\ell+1$ here), instead of taking nbhd of θ in $M_{\tilde{E}}$ in the domain of gluing, we take parameter ~~[illegible]~~ λ that replaces „dilation freedom", and choice of gluing point $x \in X$ ~~[illegible]~~ replaces „translation freedom".
This is why it is enough to give just ℓ points together with connection on $E_{k-\ell} \to X$ (after local reparametrization centered at local curvature center and with radius that ~~[illegible]~~ encircles curvature $4\pi^2$, standard instanton θ is going to „bubble off!").

- In case we are dealing with structure group other than $SU(2)$, we might need more information than just λ and x, this depends on symmetries of the moduli space of ASD instantons on G-bundles over S^4.

- We can now imagine what collar of points in $\overline{M}_E \setminus M_E$ is supposed to look like.
  For example, on $k=1$ $SU(2)$ bundle $E \to X$ it will be $X \times (0, \lambda_0]$.

Existence of ASD connections

~~We return now to a proble~~
~~Explicit construction as on S^4 is usually hard for most 4-mflds X.~~

- We return now to the problem of existence of ASD connections.
- IFT gave us dimension of mfd of regular ASD connections on E, that is, if it is not empty. For our constructions later, we want to know ~~that~~ whether there exist ASD conn.
- Explicit construction as on S^4 is usually not possible.

Let $E \to X$ be $SU(2)$

- Grafting procedure:

  As we have seen, ~~gluing procedure allows us~~ it is enough to construct a connection ~~that~~ s.t. $\|F^+A\| < \varepsilon$ is small enough, we can do this by grafting ∂ „into" trivial connection.

  What we mean by this is the following:

  Let X be 4mfd and $x \in X$ a point, consider normal coordinates around x and fix ball $B(x, r)$ in it.

  ~~(r small enough)~~ Then we can define maps $\tilde{p}_{\lambda,x} : M \to S^4$ of degree 1 given by

$\lambda \in (0, \frac{r}{2})$

$$p_{\lambda,x}(z) = \begin{cases} s\left(\frac{z}{\mu\left(\frac{|z|}{\lambda}\right)}\right), & z \in B(x,r) \\ s(\infty), & \text{otherwise} \end{cases}$$

$s : \mathbb{R}^4 \cup (\infty) \to S^4$ stereogr. proj.

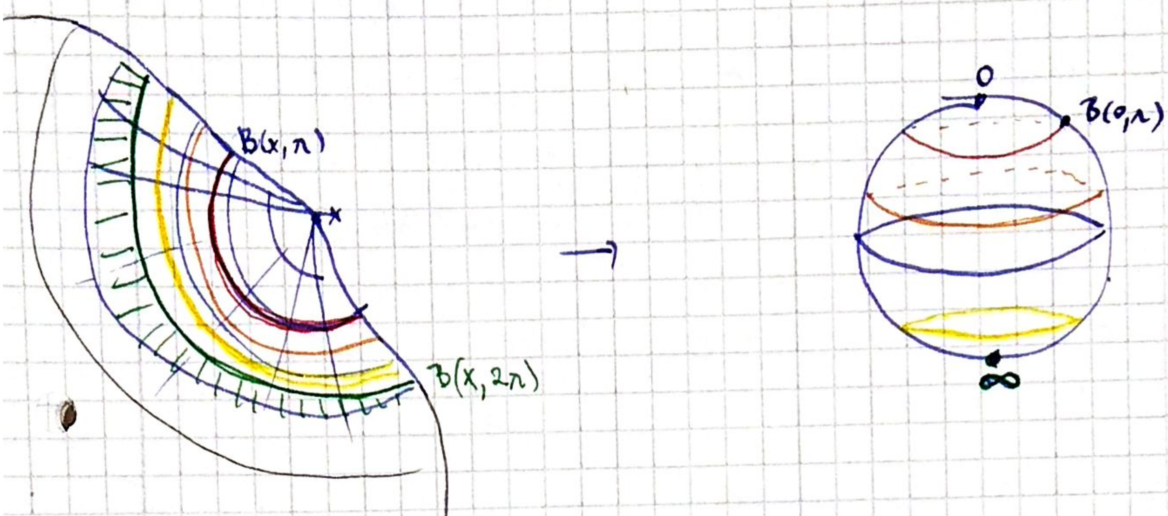
for ~~$\beta(\lambda)[\theta(\lambda)]$~~

$\beta : \mathbb{R} \to \mathbb{R}$ bump func. w/

$\beta \equiv 1$ on $r \leq 1$

$\beta \equiv 0$ on $r \geq 2$

and $\|\beta'\| \leq 2$



Then $p_{\lambda,x}$ is map of degree 1

$E = p_{\lambda,x}^*(\check{E})$ (easily generalized by choosing k points and constr deg k map)

is Chern class 1 SU(2) v. bundle ~~map~~ on X

(~~indep of λ because~~ ~~$\check{E}$ is invar. under dilations~~)

and $p_{\lambda,x}^* \theta$ is connection over X

s.t. $\|F^+(p_{\lambda,x}^* \theta)\| < \epsilon$.

- We can describe these in the following way as well:

Outside of $B(x,2\lambda)$, E is trivializable, let 1 be trivial connection on $X \times SU(2)$

Then by using ~~structure~~ identification given by $p_{\lambda,x}$

we perform connected sum of 1 and θ at points x and ∞ s.t. annulus ident. (previously denoted by f_λ) ~~are~~ σ and ξ induced by $p_{\lambda,x}$ as we constructed it. This way we produce ~~the same~~

connected sum connection A' equal to $\phi^*_{\lambda x}\theta$ and $\|F^+(\phi^*_{\lambda,x}\theta)\|$ is small.
~~Since trivial conn. has $F=0$ and~~

(One remark: ~~metric~~ we also constructed metric on connected sum, here it is not needed, we take fixed metric on X. Note that this adds another λ-factor to approx of F^+A, but working on small normal nbhd ~~of small enough~~ λ of x makes this factor small)

- We can perturb it to ASD connection on $k=1$ $SU(2)$-bundle $E \to X$.

now by grafting more θ connections into obtained ~~bundle~~ ASD connection on E, or by const. map $\phi_{x_1,\dots,x_k,\lambda}$ of degree k by choosing k points instead of one, we get the existence result on any $SU(2)$ bundle over X!

$\boxed{\text{Thm:}}$ $\exists$ ASD connection on any $SU(2)$-bundle $E \to X$.