The Topology of 4 - Manifolds

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Contents

1 Homology and Cohomology

First we recall the Poincare duality isomorphism. Later we will use it to define the intersection form.

Theorem 1.1. *Let X be a closed orientable 4-manifold, then we have an isomorphism*

$$PD : H^i(X;\mathbb{Z}) \xrightarrow{\cong} H_{2-i}(X;\mathbb{Z}).$$

Now we denote by $PD^{-1}(x)$ the Poincare dual of any homology class $[x] \in H_{2-i}(X;\mathbb{Z})$.

Next we will show an easy but important fact about the homology and cohomology of simply-connected four manifolds.

Theorem 1.2. *Let X be a simply-connected closed oriented 4-manifold, then $H_2(X,\mathbb{Z})$ is a free abelian group.*

Proof. This is a simple computation: We have:

$$H_2(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z}).$$

Where we use Poincare duality. And also

$$H_1(X;\mathbb{Z}) = \mathrm{Ab}(\pi_1(X)) = 0.$$

Thus by the universal coefficient theorem:

$$\begin{aligned} H^2(X,\mathbb{Z}) &= \mathrm{Ext}^1_{\mathbb{Z}}(H_1(X;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}) \\ &= \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}). \end{aligned}$$

Since $H_2(X;\mathbb{Z})$ is fin. generated we have that $\mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z})$ is free. $\square$

Can we see $H^2(X;\mathbb{Z}) \cong H_2(X;\mathbb{Z})$ geometrically?

- For $\alpha \in H^2(X;\mathbb{Z})$ choose a complex line bundle L s.t. $c_1(L) = \alpha$.
- Take a generic section σ
- We have an embedded surface $\Sigma_\alpha = \sigma^{-1}(0)$
- $[\Sigma_\alpha] = PD(\alpha)$

Note: Different construction using Eilenberg-MacLean spaces in appendix of notes.
Next we will define an additional structures on $H^2(X;\mathbb{Z})$. Namely we will define the intersection form or intersection product. Please note that we do not assume the manifold to be smooth for this. But first we need to quickly recall relevant notions from algebraic topology.

2 Intersection Forms

Recall that for any homology class $a \in H_2(X;\mathbb{Z})$ and cohomology class $b \in H^2(X;\mathbb{Z})$ the *cap product* is a bilinear map

$$\frown: H_p(X;\mathbb{Z}) \times H^q(X;\mathbb{Z}) \to H_{p-q}(X;\mathbb{Z}).$$

Note the important fact that the Poincare isomorphism is given by $\alpha \mapsto [X] \frown \alpha$ for the fundamental class $[X] \in H_n(X;\mathbb{Z})$. We also have the Kronecker pairing

$$\langle \cdot, \cdot \rangle : H^p(X;G) \times H_p(X;G) \to G.$$

It descends from the evaluation of a cochain on a chain. The last thing we have is the *cup product* which is a bilinear map

$$\smile: H^i(X;\mathbb{Z}) \times H^j(X;\mathbb{Z}) \to H^{j+i}(X;\mathbb{Z}).$$

Now we have everything we need to make this definition:

Definition 2.1. Let X be a closed oriented **topological** 4-manifold. Then the bilinear map

$$Q : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

given by

$$(\alpha, \beta) \mapsto \langle \alpha \smile \beta, [X] \rangle$$

is called (cohomology) *intersection form* of X.

Here we should keep in mind that choosing a fundamental class $[X] \in H_4(X;\mathbb{Z})$ is the same as choosing an orientation of X.
This is a very algebraic definition. For a smooth four manifold we can interpret it in a more geometric way:

Theorem 2.1. *Let X be closed oriented simply-connected **smooth** 4-manifold. Let $\alpha, \beta \in H^2(X;\mathbb{Z})$ and $[\Sigma_\alpha], [\Sigma_\beta] \in H_2(X;\mathbb{Z})$ be their duals. There are closed 2-forms ω_α and ω_β representing α, β such that*

$$Q(\alpha, \beta) = \langle \alpha \smile \beta, [X] \rangle = \Sigma_\alpha \cdot \Sigma_\beta = \int_X \omega_\alpha \wedge \omega_\beta.$$

Since $H^2(X;\mathbb{Z})$ is torsion free we can go forth and back between integral and de Rahm cohomology.

Proof. First we notice:

$$
\begin{aligned}
Q(\alpha, \beta) &= \langle \alpha \smile \beta, [X] \rangle = \langle \alpha, [X] \frown \beta \rangle \\
&= \langle \alpha, PD(\beta) \rangle = \langle \alpha, [\Sigma_\beta] \rangle
\end{aligned}
$$

Switching to de Rahm cohomology:

$$
\langle \alpha, [\Sigma_\alpha] \rangle = \int_{\Sigma_\beta} \omega_\alpha
$$

Now we have to show:

$$
\int_{\Sigma_\beta} \omega_\alpha = \Sigma_\alpha \cdot \Sigma_\beta
$$

Choose $\Sigma_\alpha \pitchfork \Sigma_\beta$. Then we have a finite number of intersection points. Since ω_α vanishes away from Σ_α it is enough to compute the integral at the intersection points.
Around any intersection point choose $\mathfrak{U}$ and oriented local coordinates x_1, x_2, x_3, x_4 s.t.

$$
\mathfrak{U} \cap \Sigma_\alpha = \{x_3 = x_4 = 0\} \qquad\qquad \mathfrak{U} \cap \Sigma_\beta = \{x_1 = x_2 = 0\}
$$

and $\mathfrak{U} \cap \Sigma_\alpha$ is oriented by $dx_1 \wedge dx_2$. Then

$$
\omega_\alpha = f(x_3, x_4) dx_3 \wedge dx_4
$$

for a bump function $f : \mathbb{R}^2 \to \mathbb{R}$. Then

$$
\int_{\mathfrak{U} \cap \Sigma_\beta} f(x_3, x_4) dx_3 \wedge dx_4 = \pm 1
$$

depending on orientation.
By summing over all intersection points we get:

$$
\int_{\Sigma_\beta} \omega_\alpha = \Sigma_\alpha \cdot \Sigma_\beta.
$$

For the last equality we have:

$$
\langle \omega, [N] \rangle = \int_N \omega
$$

$$
[\omega_1 \wedge \omega_2] = [\omega_1] \smile [\omega_2]
$$

Giving us

$$
Q(\alpha, \beta) = \int_X \omega_\alpha \wedge \omega_\beta.
$$

□

Theorem 2.2. *Let X be closed oriented simply-connected 4-manifold. Then Q_X is unimodular, i.e. $a \to Q(\cdot, a)$ and $b \to Q(b, \cdot)$ are isomorphisms.*

Proof. By the universal coefficient theorem

$$
\begin{aligned}
H^2(X; \mathbb{Z}) &\to \mathrm{Hom}(H_2(X; \mathbb{Z})) \\
\alpha &\mapsto \langle \alpha, \cdot \rangle
\end{aligned}
$$

is an isomorphism. This suffices, as

$$
Q(\alpha, \beta) = \langle \alpha, PD(\beta) \rangle
$$

and Q is symmetric. Here using the cohomology intersection form really comes in handy, cutting the proof by half. □

This simple example might show how useful our geometric interpretation is when it comes to concrete computations:

Example 2.1. Consider

$$X = S^2 \times S^2.$$

Then

$$H^2(X;\mathbb{Z}) = \langle PD^{-1}([\{\mathrm{pt}\} \times S^2]), PD^{-1}([\{\mathrm{pt}\} \times S^2])\rangle.$$

And

$$Q \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

What happens if $H^2(X;\mathbb{Z})$ is not free?
Let $\alpha \in H^2(X;\mathbb{Z})$ s.t. $n \cdot \alpha = 0$, then

$$nQ(\alpha, \beta) = Q(n \cdot \alpha, \beta) = Q(0, \beta) = 0.$$

So we can define

$$\tilde{Q} : \left(H^2(X;\mathbb{Z}) \big/ \mathrm{Ext}^1_{\mathbb{Z}}(H_1(X;\mathbb{Z}), \mathbb{Z}) \right)^2 \longrightarrow \mathbb{Z}$$

and use the arguments there. Sometimes this is also given as the definition of the intersection form. With this definition we also get unimodularity without the assumption of a simply-connected manifold. Next we look at some invariants of intersection forms:

- **Parity:**
  If $Q(\alpha, \alpha) \in 2\mathbb{Z} \forall \alpha \in H^2(X;\mathbb{Z})$ we call Q **even**. Otherwise it is called **odd**.

- **Definiteness:**
  If $Q(\alpha, \alpha) > 0 \forall \alpha \in H^2(X;\mathbb{Z})$ we call Q **positive-definite.** If $Q(\alpha, \alpha) < 0 \forall \alpha \in H^2(X;\mathbb{Z})$ we call Q **negative-definite.** Otherwise it is called **indefinite**.

- **Rank:**
  The second Betti number $b_2(X)$ is called the **rank** of Q.

- **Signature:**
  Over $\mathbb{R}$ Q has b_2^+ positive and b_2^- negative eigenvalues. We call

  $$\operatorname{sign} Q = b_2^+ - b_2^-$$

  the **signature** of Q.

Theorem 2.3 (Hasse-Minkowski)**.** *Let H be a free $\mathbb{Z}$ module. If $Q : H \times H \to \mathbb{Z}$ is an odd indefinite bilinear form then*

$$Q \cong l(1) \oplus m(-1)$$

with $l, m \in \mathbb{N}_0$. If $Q : H \times H \to \mathbb{Z}$ is an even indefinite bilinear form then

$$Q \cong l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus mE_8$$

with $l, m \in \mathbb{N}_0$.

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

What about definite forms?

- No easy classification
- Many exotic forms
- Number of unique even definite forms of some ranks:

Rank	8	16	24
#	1	2	5

These numbers grow rapidly with rank and contain no structure (known to me).

Warning: Any intersection form is diagonalizable over $\mathbb{Q}$ but might not be over $\mathbb{Z}$.

Exercise. Show that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is not diagonalizable over $\mathbb{Z}$.

3 Homotopy Type

We will now find a direct link between homotopy type and intersection form of four manifolds.

Theorem 3.1 (Milnor (1958))**.** *The oriented homotopy type of a simply-connected closed oriented 4-manifold is determined by its intersection form.*

We need to use an oriented homotopy type here because the intersection form depends on orientation up to sign.

The proof is not given here in detail but rather only sketched. There is a reference to the full proof included below.

Proof. Define $X' = X \setminus B^4$. Then

$$H_k(X';\mathbb{Z}) = \begin{cases} H_2(X) & k = 2 \\ 0 & k = 1,3,4 \end{cases}.$$

By Hurewicz's theorem:

$$f : S^2 \vee \ldots \vee S^2 \to X'$$

represents $\pi_2(X) \cong H_2(X';\mathbb{Z})$. This induces an isomorphism

$$H_k(S^2 \vee \ldots \vee S^2;\mathbb{Z}) \cong H_k(X';\mathbb{Z})$$

for every k.
Thus

$$X \simeq (S^2 \vee ... \vee S^2) \cup_h e^4$$

with $[h] \in \pi_3(S^2 \vee ... \vee S^2)$. Left to show: $[h]$ depends only on Q. Complete proof can be found in: [1, p.141ff] Sketch:

- $[X] \in H_4(X;\mathbb{Z})$ corresponds to $[e^4] \in H_4((S^2 \vee ... \vee S^2) \cup_h e^4;\mathbb{Z})$
- $S^2 \vee \cdots \vee S^2 = \mathbb{CP}^1 \vee \cdots \vee \mathbb{CP}^1 \subset \mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty$
- Long exact sequence on relative homotopy groups:

$$\begin{aligned} \pi_4(\times_m \mathbb{CP}^\infty) \longrightarrow \pi_4(\times_m \mathbb{CP}^\infty, \vee_m S^2) &\longrightarrow \pi_3(\vee_m S^2) \\ &\longrightarrow \pi_3(\times_m \mathbb{CP}^\infty) \end{aligned}$$

- $\mathbb{CP}^\infty$ is $K(\mathbb{Z}, 2) \implies$

$$\pi_3(\vee_m S^2) \cong \pi_4(\times_m \mathbb{CP}^\infty, \vee_m S^2) \cong H_4(\times_m \mathbb{CP}^\infty, \vee_m S^2)$$

- $\pi_3(\vee_m S^2) \cong \pi_4(\times_m \mathbb{CP}^\infty, \vee_m S^2) \cong H_4(\times_m \mathbb{CP}^\infty, \vee_m S^2)$
- Oriented manifold: $[h]$ is determined by $\alpha_k(h_*([e^4]))$ for $\alpha_1, ..., \alpha_l$ basis of $H^4(\times_m \mathbb{CP}^\infty)$.
- Basis is given by cupping $PD^{-1}([S_i^2])$. Since

$$H^2(\times_m \mathbb{CP}^\infty) \cong H^2(\vee_m S^2) \cong H^2(X';\mathbb{Z}) = H^2(X;\mathbb{Z})$$

  these classes can be seen in X.
- We are done since $\langle PD^{-1}([S_i^2]) \smile PD^{-1}([S_j^2]), h_*([e^4])\rangle = \langle \omega_i, \omega_j, [X]\rangle = Q(\omega_i, \omega_j)$

$\square$

Exercise. Fill in the gaps in the proof sketch.

4 The "Big" Structure Theorems

Theorem 4.1 (Freedman)**.** *Let Q be an quadratic (i.e. unimodular symmetric bilinear) form over $\mathbb{Z}$, then there exists a **topological** 4-manifold M s.t. Q is (up to isomorphism) the intersection form of M. If Q is even, then M is unique.*

Theorem 4.2 (Rohlin)**.** *Let X be a simply-connected closed oriented **smooth** 4-manifold with $w_2(X) = 0$. Then*

$$\operatorname{sign} Q_X \in 16\mathbb{Z}.$$

The original proof by Rohlin is very involved. Simpler proof due to Atiyah and Singer using the Atiyah-Singer index theorem. Reference: [2, Theorem 29.9] Such a manifold is also called a *spin* manifold and it is not by accident that the above proof can be found in lecture notes on "spin geometry". The spin condition is actually equivalent to the intersection form being even so that we get this corollary:

Corollary 4.2.1. *Let X be a simply-connected closed oriented **smooth** 4-manifold with even intersection form Q_X. Then*

$$\operatorname{sign} Q_X \in 16\mathbb{Z}.$$

Another exciting result we get directly is the existence of four manifolds without any smooth structures.

Corollary 4.2.2. *There exists a simply-connected closed 4-manifold E_8 with intersection form E_8 that has no smooth structure.*

Proof. E_8 is a negative definite even form with signature -8. The existence is given by Freedman's theorem. ☐

Now we have arrived at the theorem that is the holy grail we are chasing in this seminar:

Theorem 4.3 (Donaldson)**.** *Let X be a simply-connected closed* ***smooth*** *4-manifold. If Q is definite, Q is diagonalizable over $\mathbb{Z}$.*

Corollary 4.3.1. *Let X be a simply-connected closed* ***smooth*** *4-manifold. If Q is positive-definite then*

$$X \cong \#_k \mathbb{CP}^2$$

as topological manifolds.

5 Whitney Disks and the Failure of the h-Cobordism Principle in Dimension Four

In this last section I will briefly go over the intuition of why dimension four is so different from dimension five and up.

Definition 5.1. Let M and N be closed simply-connected manifolds and W be a cobordism between them (i.e. $\partial W = M \cup \bar{N}$). If the inclusions $M \to W$ and $N \to W$ are homotopy equivalences, then M and N are called *h-cobordant*.

Theorem 5.1 (Wall)**.** *Two simply-connected four-manifolds with isomorphic intersection form are h-cobordant.*

Theorem 5.2 (Smale (1961))**.** *Let M and N be cobordant smooth n-manifolds with $n > 4$. Then M and N are diffeomorphic.*

Warning: This theorem only holds for $n \geq 5$.

Why does the (smooth) h-cobordism principle fail in dimension four?

- Short answer:

  The statement

  $$2 + 2 < 4$$

  is optimistic but sadly wrong!

- That is not so helpful, we are looking for a longer answer.

Strategy of the proof in higher dimensions:

- Goal: Show that $W \cong M \times [0, 1]$

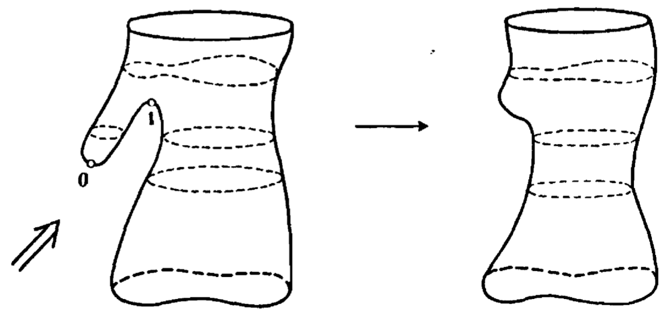

Figure 1: Cancelling an index 0 critical point with an index 1 critical point. From [3]

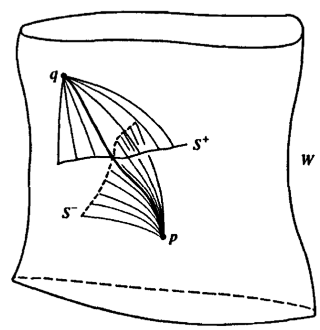

Figure 2: Analogy in dimension three showing S_+ and S_- intersection transversely and the resulting flow line. From [3]

- Choose a Morse function $f : W \to [0,1]$ with $f(M) = 0$ and $f(N) = 1$
- If f has no critical values we are done!
- Idea: Modify f s.t. all critical values disappear

Removing critical points of index $0, 1, 4, 5$ works in dimension four. But: Canceling critical points of index 3 and 2 does not work (with this method).

- Suppose f has two critical points: p of index 2 and q of index 3
- Let p and q be separated by $Z_{1/2} = f^{-1}(\frac{1}{2})$
- Fact: p and q can be canceled if there is exactly one flow line from p to q

We define

$$S_+ = \{x \in Z_{1/2} \mid x \text{ flows to } p \text{ as } t \to \infty\}$$
$$S_- = \{x \in Z_{1/2} \mid x \text{ flows to } q \text{ as } t \to -\infty\}.$$

These are embedded spheres. If $S_- \pitchfork S_+$ is a single point we can glue the flow lines and are done. The algebraic intersection number is 1 because W is h-cobordism. Problem: The geometric intersection number might not agree! We need an isotopy to correct this.
Usual procedure:

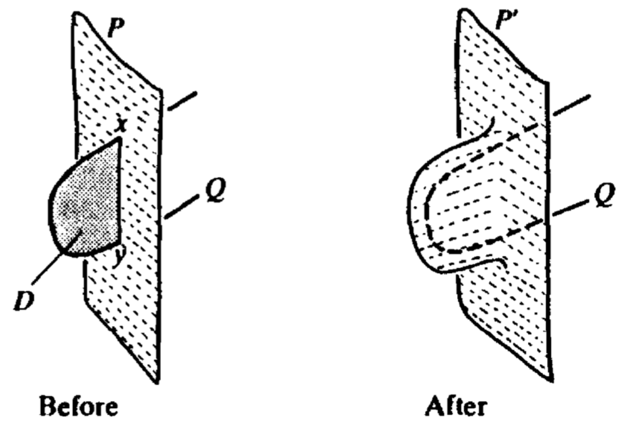

Figure 3: Removing intersection points in pairs. From [3]

- Choose intersection points with opposite signs, e.g. x and y
- Find path $\alpha \subset S_+$ and $\beta \subset S_-$ joining them
- W simply-connected $\implies$ $\alpha \cup \beta$ inessential
- There is a disk $D \subset W$ with $\partial D = \alpha \cup \beta$
- If the disk lies outside S_+ and S_- we get an isotopy removing the intersection points

In dimension $n \geq 5$:

- D is generically embedded
- D generically does not intersect S_+ and S_- in any interior points

In dimension four on the other hand both is not true! The intersection form makes this clear.

With the existence of non-smooth manifolds one the one hand and the failure of the h-cobordism on the other hand, we see that the topology and geometry of four-manifolds is quite unique.

6 Appendix

6.1 Dual Surfaces via Eilenberg–MacLane Spaces

The concept from algebraic topology we need is that of an Eilenberg–MacLane space.

Definition 6.1. Let $n > 0$ and G be an abelian group then a space $K(G, n)$ is called Eilenberg-McLane space if it is a CW-complex and

$$\pi_k(K(G,n)) = \begin{cases} G & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

holds.

Remark. The space $K(G,n)$ is unique up to homotopy.

Now we can make an important connection between the cohomology groups of a manifold (any CW-complex will work actually) and the homotopy classes of maps from the space into an Eilenberg-MacLane space. Indeed we will see that for the correct Eilenberg-McLane space they are isomorphic. And it is an isomorphism that is concrete enough so we can actually understand it. One important thing to remember in order to define this is that a continuous map $f : X \to Y$ induces a homomorphism $f^* : H^\bullet(Y;G) \to H^\bullet(X;G)$ on cohomology. First we notice the following connection:

$$H^n(K(G,n);G) \cong \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(K(G,n)),G) \oplus \mathrm{Hom}(H_n(K(G,n)),G) \cong \mathrm{Hom}(G,G).$$

So we can choose an element in $H^n(K(G,n);G)$ representing $\mathrm{id} \in \mathrm{Hom}(G,G)$. This element is called the *canonical cohomology class* and denoted ι_n. Now we can see this more powerful connection:

Theorem 6.1. *Let X be a topological space and G an abelian group. Then define a map*

$$[X : K(G,k)] \longrightarrow H^k(X;G)$$

by

$$[w] \mapsto w^*(\iota_k).$$

This map is well-defined and an isomorphism if X is a CW-complex.

Using this isomorphism we can find embedded surfaces S_α dual to cohomology class $\alpha \in H^2(X;\mathbb{Z})$. In the following let X be a smooth closed oriented 4-manifold.

Strategy for $\alpha \in H^2(X;\mathbb{Z})$:

- Observe

$$H^2(X;\mathbb{Z}) \cong [X : K(\mathbb{Z},2)] \cong [X : \mathbb{CP}^\infty] \cong [X : \mathbb{CP}^2].$$

- Identify $PD^{-1}([\mathbb{CP}^1]) \in H^2(\mathbb{CP}^2;\mathbb{Z})$ (henceforth just $[\mathbb{CP}^1]$) with $\iota_2 \in H^2(K(\mathbb{Z},2);\mathbb{Z})$.
- Choose generic $f_\alpha \pitchfork \mathbb{CP}^1$ s.t. $[f_\alpha] = \alpha$ i.e.

$$f_\alpha^*([\mathbb{CP}^1]) = f_\alpha^*(\iota_2) = \alpha.$$

- Define $S_\alpha = f_\alpha^{-1}(\mathbb{CP}^1)$. This is an embedded surface by transversality.
- With this construction $[S_\alpha] = PD(\alpha) \in H_2(X;\mathbb{Z})$.

References

[1] Alexandru Scorpan. *The wild world of 4-manifolds*. Providence, R.I., 2005.

[2] Thomas Walpuski. MTH 993 Spring 2018: Spin Geometry. `https://walpu.ski/Teaching/SpinGeometry.pdf`.

[3] Simon K Donaldson and Peter B Kronheimer. *The geometry of four-manifolds*. Oxford mathematical monographs. Oxford, 1. publ. in pbk. edition, 1997.