2021-01-26: The collar theorem

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Throughout: • X 1-conn, or., closed 4-mfd w/ $b^+ = 0$

• $P \to X$ $SU(2)$-bundle w/ $c_2(P) = 1$

What can we say about M, the moduli space of ASD instantons on P?

(1) Atiyah-Hitchin-Singer: $\delta_A = (d_A^+, d_A^*) : \Omega^1 \to \Omega^+ \oplus \Omega^0$

$$\text{v dim } M = \text{index } \delta_A = 8 \cdot 1 + 3 \cdot (\cancel{b_1}^{0} - \cancel{b^+}^{0} - \cancel{b^0}^{1}) = 5$$

(2) Freed-Uhlenbeck: for a generic metric g, every irred. ASD instanton A is unobstructed ($H^2_A = 0$, δ_A surj.)

$\rightsquigarrow M^*$ 5-mfd

(3) Taubes' gluing theorem: M^* contains ASD instantons concentrated at $x \in X$, obtained by gluing in BPST instantons at scale $\lambda \ll 1$ into the product conn. Θ

(4) Uhlenbeck compactness / removable sing.: Given $([A_i]) \in M^{\mathbb{N}}$, the following holds up to gauge transformations and passing to a subsequence:

either (a) $A_i \xrightarrow{C^\infty} A$ ← ASD on P

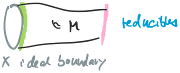
or (b) • $|F_{A_i}|^2 \to 8\pi^2 \delta_x$

• $A_i|_{X \setminus \{x\}} \xrightarrow{C^\infty_{loc}} B$ w/ $F_B = 0$ $\overset{\pi_1 X = \{1\}}{\rightsquigarrow}$ $B = \Theta$

• rescaling A_i at x $\rightsquigarrow$ charge 1 $SU(2)$-instanton on $\mathbb{R}^4$

= the BPST instanton (up to scale, translation)

naive expectation: M should look like this



The collar theorem: There is an open subset $U \subset M^*$ s.th.:

(1) $M \setminus U$ is cpct

$0 < \lambda \ll 1$

(2) there is a diffeomorphism $p : \mathcal{U} \to X \times (0, \Lambda)$.

Plan:

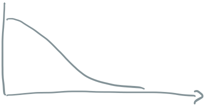
(1) concentrated connections ⇝ def. of $\mathcal{U}$ and p

(2) Taubes projection

(3) p is a covering map

(4) p is single-sheeted ⇝ p is a diffeo.

§1.



BPST instanton on $\mathbb{R}^4 = \mathbb{H}$

$$I = \frac{\operatorname{Im}(x^* dx)}{1+|x|^2} \rightsquigarrow |F_I|^2 = \frac{48}{(1+|x|^2)^4}$$

Scaling $s_\lambda(x) = x/\lambda$ and translation $t_y(x) = x-y$ act as

$$I_{\lambda,y} = t_y^* s_\lambda^* I \rightsquigarrow |F_{I_{\lambda,y}}|^2 = \frac{48\lambda^2}{(\lambda^2+|x-y|^2)^4}$$

Prop 1: $\nu_{\lambda,y} := |F_{I_{\lambda,y}}|^2 \operatorname{vol}$ satisfies

- $\int \nu_{\lambda,y} = 8\pi^2$ • $y = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \eta \cdot \nu_{\lambda,y}(\eta)$

- $\lambda = \min\{ r>0 : \exists x \in \mathbb{R}^4 : \int_{B_r(x)} \nu_{\lambda,y} = 4\pi^2 \}$

  ↑ optimal choice is y

Transfer to X as follows: Choose $\varkappa \in C^\infty([0,\infty),[0,1])$ with

$$\varkappa(r) = \begin{cases} 1 & r \le 1-\varepsilon \\ 0 & r \ge 1+\varepsilon \end{cases} \quad \text{and} \quad \varkappa' \le 0.$$

Define $\mu : \mathcal{A} \times X \times (0, \overset{\Lambda \ll 1}{\Lambda}) \to [0,\infty)$ by

$$\mu(A; y, r) := \int_X \varkappa\left(\frac{d(x,y)}{r}\right) \cdot |F_A|^2 \operatorname{vol}_g \sim \int_{B_r(y)} |F_A|^2 \operatorname{vol}_g$$

Rmk: $\Lambda \ll 1$ ⇝ after rescaling, g can be assumed arbitrarily close to g_0.

The map μ is smooth and $\mathcal{G}$-invariant. Moreover,

$$\frac{\partial \mu}{\partial r} = -\int \varkappa'\left(\frac{d(x,y)}{r}\right)\frac{d(x,y)}{r^2} |F_A|^2 \ge 0 \quad (>0 \text{ if } A \text{ non-flat ASD})$$

Set

$\tilde V := \{ A \in \mathcal{A} : \exists y \in X, r \in (0,\Lambda) :$ $|F_A|$ nowhere vanishing and $\mu(A,y,r) > 4\pi^2 \}$ open

The map

$$\tilde\lambda : \tilde V \times X \to (0,\Lambda)$$

implicitly def. by

$$\mu(A, x, \tilde\lambda(A,x)) = 4\pi^2$$

is smooth and $\mathcal{G}$-invariant.

Set

$$\lambda(A) := \min_{x \in X} \tilde\lambda(A,x).$$

Prop 2: If $([A_n]) \in M^{\mathbb{N}}$ has no convergent subsequence, then $A_n \in \tilde V$ for $n \gg 1$ and $\lambda(A_n) \to 0$.

Pf: If not then no subsequence of $|F_{A_n}|^2 \mathrm{vol}$ can converge to $8\pi^2 \cdot \delta_x$ ■

Cor/Ex 3: For $\Lambda \ll 1$, $M \cap \lambda^{-1}(0,\Lambda)$ contains no reducibles.

λ defines one part of p. The remaining part should be $y(A) \in X$, the $y \in X$ s.th. $\lambda(A) = \tilde\lambda(A,y)$, the minimum of $x \mapsto \tilde\lambda(A,x)$.

Why is this possible?

Prop 4: Let $([A_n]) \in M^{\mathbb{N}}$ w/ $\lambda_n := \lambda(A_n) \to 0$. Choose y_n with $\lambda_n = \tilde\lambda(A_n, y_n)$ ($\exists$, but possibly not unique).

Set

$$\tilde B_n := s^*_{\lambda_n^{-1}} \exp^*_{y_n} A_n \quad \text{(up to gauge)}$$

The following hold:

BPST instanton

(1) $\tilde B_n \longrightarrow I$ in $C^k(B_R(0))$ for every $k \in \mathbb{N}$ and $R > 0$.

(2) $|F_{\tilde B_n}|(x) \leq \dfrac{c}{(1+|x|)^{4-\delta}}$ on $B_{\lambda_n^{-1}\cdot R}(0)$ with $\delta = \delta(R) \to 0$ $(R \to 0)$

Pf: By construction, $|F_{\tilde B_n}|$ cannot concentrate. By Uhlenbeck compactness, $\tilde B_n \to B$ as in (1) with B an ASD instanton on $\mathbb{R}^4$

satisfying

$$\bullet \int_{\mathbb{R}^4} |F_{\mathbb{B}}|^2 = 8\pi^2 \qquad \underbrace{\bullet\ \lambda(\mathbb{B}) = 1 \qquad \bullet\ y(\mathbb{B}) = 0}$$

$\Downarrow$

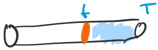
$\mathbb{B} = I_{\lambda, y}$ (upto gauge) $\qquad \lambda = 1,\ y = 0$ (if we choose x carefully). //(1)

(2) follows from the proof of removable singularities. ∎

⌈

$|F_A| \leq c \cdot |x|^{-\gamma} \rightsquigarrow$ cyl. model $\quad |F_A| \leq c \cdot e^{-\gamma t}$

consider



$[0,T] \times S^3$

$$K(t) = \int_{[t,T] \times S^3} |F_A|^2$$

ε-reg. $\rightsquigarrow$ enough to bound $K(t) \leq C e^{-\lambda t}$.

$$J(t) := \int_{\{t\} \times S^3} CS(A) \qquad CS(A) = \operatorname{tr}(A \wedge dA + \tfrac{2}{3} A \wedge [A \wedge A]) + C_0$$

$\int_{S^3} \omega \wedge d\omega \leq \frac{1}{2} \int_{S^3} |d\omega|^2$ $\leq \frac{1}{2} \int_{\{t\} \times S^3} |dA|^2 +$ err. $\qquad CS(A|_{T \times S^3}) \overset{(*)}{\geq} 0$

$$d\, CS(A) = F_A \wedge F_A = |F_A|^2.$$

$$K(t) \overset{\text{Stokes}}{=} J(t) - J(T)$$

$$\leq J(t)$$

$$J'(t) = -2 \int_{t \times S^3} |\overset{dA + \frac{1}{2}[A \wedge A]}{F_A}|^2 = -2 \int_{t \times S^3} |dA|^2 + \text{err.}$$

$\rightsquigarrow J \leq -4 J' +$ err. term.

ignoring error term & integrating gives

$$J \leq c \cdot e^{-4t}.$$

$$\Rightarrow \int_{[t,T] \times S^3} |F_A|^2 \leq c \cdot e^{-4t}.$$

$$\Rightarrow |F_A| \leq C \cdot e^{-2t}.$$

⌊

Prop 5: For $\lambda << 1$ the following holds for

$\mathcal{U} := \{[A] \in \mathcal{M} : \lambda(A) < \Lambda\}$.

(1) For every $[A] \in \mathcal{U}$ there is a unique

$y = y[A]$ w/ $\lambda[A] = \tilde{\lambda}(A, y)$.

(2) The map $p : \mathcal{U} \to (0, \Lambda) \times X$ def. by

$$p[A] = (\lambda[A], y[A])$$

is smooth and $\mathcal{G}$-invariant.

(3) $|T_{[A]} p(a)| \leq C \cdot \lambda[A] \cdot \|d_A a\|_{L^2}$

Pf: If there are y_1, y_2 w/ $\hat{\lambda}(A, y_1) = \hat{\lambda}(A, y_2) = \lambda(A) =: \lambda$,

then $d(y_1, y_2) \leq 2\lambda$ (o/w: ↯ $\int |F_A|^2 = 8\pi^2$).

For the BPST instanton I,

$$y \mapsto \int \varkappa(|x - y|)\, |F_I|^2$$

has a unique max. at $y = 0$. By Prop 4, the same is true for $\tilde{B}$ (rescaling of A) $\Lambda \ll 1$ and thus the analogous statement holds for A. $\Rightarrow y_1 = y_2$. // (1)

(2) This is the IFT applied to

$$(*) \quad \mu(A, \lambda, y) = 4\pi^2, \quad \frac{\partial \mu}{\partial x}(A, \lambda, y) = 0 \quad //$$

(3) Differentiate $(*)$ along A_t, λ_t, y_t to get

$$0 = 2 \int \underbrace{\varkappa\left(\frac{d(x, y_0)}{\lambda_0}\right) \langle F_{A_0}}_{\|\cdot\|_{L^2} \leq C}, \frac{d}{dt}\Big|_{t=0} \overset{\swarrow}{F_{A_t}} \rangle \quad \llcorner \underline{d_{A_0} a} \quad \text{w/ } \frac{d}{dt}\Big|_{t=0} A_t = a$$

$$- \lambda_0^{-1} \cdot \left(\frac{d}{dt}\Big|_{t=0} \lambda_t\right) \cdot \underbrace{\int \varkappa'\left(\frac{d(x, y_0)}{\lambda_0}\right) \frac{d(x, y_0)}{\lambda_0} |F_{A_0}|^2}_{\text{scale inv't}, \; \geq \frac{1}{C}}$$

$$\Rightarrow \left|\frac{d}{dt}\Big|_{t=0} \lambda_t\right| \leq C \lambda_0 \cdot \|d_{A_0} a\|_{L^2}.$$

Est. on $\left|\frac{d}{dt}\Big|_{t=0} y_t\right|$ similar. ∎

UPSHOT: $p : \mathcal{U} \to (0, \Lambda) \times X$ smooth and $\mathcal{M} \setminus \mathcal{U}$ cpct

To SHOW: p diffeo. ($\Lambda \ll 1$)

§2.

Thm 6 (Taubes) For every $\nu_0, N > 0$ there are $C = C(\nu_0, N, \text{geom. of } X)$,

$\delta_0 = \delta_0(\nu_0, N, \text{geom of } X)$ s.th the following holds for every A:

if • $\|F_A^+\|_{L^3} \leq N$,

- 1st eigenvalue of $d_A^+ d_A^* : \Omega^+ \circlearrowleft$ $\geq \nu_0$, and
- $\delta = \|F_A^+\|_{L^2} + \|F_A\|_{L^4} \cdot \|F_A^+\|_{L^{4/3}} \leq \delta_0$,

then $\exists$ $a = d_A^* \omega$ $(\omega \in \Omega^+)$ s.th:

- $F_{A+a}^+ = 0$
- $\|\nabla_A a\|_{L^2} \leq C \cdot \delta$
- $\|a\|_{L^2} \leq C \cdot \|F_A^+\|_{L^{4/3}}$

Pf is (similar to) that of the gluing construction w/ delicate bookkeeping.

§3.

Prop 7: p is proper

This is really just Prop 2 again.

Prop 8: p is a submersion

Cor 9: p is a covering map

Pf: Atiyah-Hitchin-Singer $\Rightarrow$ dim $M = 5$ $\rightarrow$ p proper local diffeo. ∎

Pf of Prop 8: Let $[A] \in \mathcal{U}$. Choose normal coord. at $y[A]$.

For

$$\xi = \left(\sum v_i \partial_i, \phi \partial_\lambda\right) \in T_{y,\lambda}[X \times (0, \Lambda)]$$

def.

$$v = \chi_r \cdot \sum \left(v_0 + \frac{\phi}{\lambda} \cdot x_i\right) \partial_i$$

$\llcorner$ $\chi\left(\frac{|\cdot|}{r}\right)$

Denote by $\tilde{A}_t$ the flow of A for time t. $p[\tilde{A}_t]$ is still defined and

$$\frac{d}{dt}\Big|_{t=0} p[\tilde{A}_t] = \xi \quad (+\text{ small error from } g \text{ not being flat})$$

If only $\tilde{A}_t$ were ASD. We need

Prop 10: If $([A_n]) \in M^{\mathbb{N}}$ w/ $\lambda_n := \lambda[A_n] \to 0$, $y_n := y[A_n] \to y$, then:

(1) $A_n|_{X \setminus B_r(y)} \xrightarrow{C^\infty_{loc}} \theta$

(2) The first eigenvalue ν_n of $d^+_{A_n} d^*_{A_n} : \Omega^+ \circlearrowleft$ is unif. bounded below: $\nu_n \geq 1/c > 0$

(3) $\|F_{A_n}\|_{L^4} \leq c \cdot \lambda_n^{-1}$.

Pf: (1) follows from compactness. (3) follows from the decay est. in Prop 4 (2). Let us prove (2).

Choose $\omega_n \in \Omega^+$ s.th

$$\|\omega_n\|_{L^2} = 1 \quad \text{and} \quad \|d^+_{A_n} \omega_n\|^2_{L^2} = \nu_n^2.$$

To preserve sanity, drop n.

Denote by $\nu_0 > 0$ the 1st eigenvalue of $d^+ d^* : \Omega^+ \circlearrowleft$

($b^+ = 0 \Rightarrow \nu_0 > 0$).

$$\rightsquigarrow \|d^*(1-\chi_r)\omega\|^2_{L^2} \geq \nu_0^2 \|(1-\chi_r)\omega\|^2_{L^2}$$

Now, $\quad d^*_A = d^* + a * \cdot$

$$\nu^2 = \|d^*_A \omega\|^2_{L^2} = \|d^*_A (1-\chi_r)\omega + d_A \chi_r \omega\|^2_{L^2}$$

$$= \|d^*_A (1-\chi_r)\omega\|^2_{L^2} \quad - \mathrm{I}$$

$$+ \|d^*_A \chi_r \omega\|^2_{L^2} \quad - \mathrm{II}$$

$$+ 2 \langle d^*_A (1-\chi_r)\omega, d^*_A \chi_r \omega \rangle_{L^2} \quad - \mathrm{III}$$

$$\mathrm{I} \geq \|d^*(1-\chi_r)\omega\|^2_{L^2} - \||a|(1-\chi_r)\omega\|^2_{L^2}$$

$$\geq (\nu_0^2 - \underbrace{\|a\|^2_{L^\infty}}_{\to 0 \ (n\to\infty)}) \|(1-\chi_r)\omega\|^2_{L^2}$$

II needs the Weitzenböck formula

$$d^+_A d^*_A = \frac{1}{2} \nabla^*_A \nabla_A + F^+_A * - + \mathrm{Riem} *$$

$$\rightsquigarrow \mathrm{II} \geq \frac{1}{4} \|\nabla_A (\chi_r \omega)\|^2_{L^2} - c\|\chi_r \omega\|^2_{L^2}$$

$$\overset{\text{Kato}}{\geq} \frac{1}{4} \|\nabla |\chi_r \omega|\|^2_{L^2} - c\|\chi_r \omega\|^2_{L^2}$$

$$\overset{\text{Sobolev}}{\geq} \frac{1}{c} \|\chi_r \omega\|^2_{L^4} - c\|\chi_r \omega\|^2_{L^2}$$

also

$$\|\chi_r \omega\|^2_{L^2} \leq c \cdot r^2 \|\chi_r \omega\|^2_{L^4}$$

so $\quad \mathrm{II} \geq \frac{1}{c} \|\chi_r \omega\|^2_{L^4} + \frac{1}{cr^2} \|\chi_r \omega\|^2_{L^2}$ $\qquad -\chi_r)\omega\|^2_{L^2}$

$$|\mathrm{III}| \overset{CS}{\leqslant} 2\,\|d_A^*(1-\overset{\swarrow}{\chi_r})\omega\|_{L^2}\left(\|d_A^*\overset{\swarrow}{\omega}\|_{L^2} + \underbrace{\||\nabla\chi_r|\,\omega\|_{L^2}}_{\leqslant C\cdot r\cdot\|\omega\|_{L^\infty} \overset{(+)}{\leqslant} C\cdot r}\right)$$

$$\leqslant 2\, I^{1/2}\cdot(\nu + c\cdot r)$$

$$\overset{CS}{\leqslant} \frac{1}{2} I + 2\nu^2 + c\,r^2$$

⌜To see (+), note that the Weitzenbock formula implies

$$\Delta|\omega| \leqslant (\nu + c)|\omega|$$

w.l.o.g. $\nu \leqslant 1$.

Moser iteration $\Longrightarrow$ $\|\omega\|_{L^\infty} \leqslant C\cdot\overset{=1}{\cancel{\|\omega\|_{L^2}}}$

⌞

Put everything together to get,

$$\nu^2 \geqslant \frac{1}{2}(\nu_0^2 - \|a\|_{L^\infty}^2)\,\|(1-\chi_r)\omega\|_{L^2}^2 + \frac{1}{Cr^2}\|\chi_r\,\omega_r\|_{L^2}^2$$

$$- 2\nu^2 - 2cr$$

$$\overset{r\leqslant 1}{\geqslant} \frac{1}{2}(\underbrace{\nu_0^2 - \|a\|_{L^\infty}^2}_{\to 0,\ h\to 0}) - \underbrace{2\nu^2}_{\text{rearrange}} - \underbrace{2cr}_{\text{as } r \text{ small.}}$$

$$\rightsquigarrow \nu^2 \geqslant \frac{1}{C}.$$

∎

Cont. pf of Prop 9:

Use Thm 6 to find ASD instantons

$$A_t = \hat{A}_t + a_t \quad \text{w/ } a_t = d_{A_t}^*\omega_t.$$

We have

$$\|\nabla_{A_0}\frac{d}{dt}\Big|_{t=0} a_t\|_{L^2} \leqslant C\cdot\|\underbrace{\frac{d}{dt}\Big|_{t=0}}_{\text{this arises from the variation of conformal structure}} F_{A_t}^+\|_{L^2} + \text{even smaller terms}$$

- $\left|\frac{d}{dt}\Big|_{t=0} F_{A_t}^+\right| \leqslant C\cdot|\underbrace{\pi}_{\text{proj onto } Sym_0^2}(d_\gamma g)|\,|F_A|$

- $|\pi(d_\gamma g)| \leqslant \begin{cases} d(y,\cdot)\,|\xi| & d(y,\cdot)\leqslant r \\ 0 & d(y,\cdot)\geqslant r \end{cases}$

- $|F_A| \leqslant \begin{cases} C\,\lambda^{-2} & d(y,\cdot)\leqslant\lambda \\ C\,\lambda\, d(y,\cdot)^{-3} & \lambda\leqslant d(y,\cdot)\leqslant r \end{cases}$

$$\Rightarrow \left\| \frac{d}{dt}\Big|_{t=0} F^+_{A_t} \right\|_{L^2} \leq c \cdot |\xi|$$

Finally

$$\left| \frac{d}{dt}\Big|_{t=0} p(\tilde{A}_t + a_t) - \xi \right| \leq c \cdot \lambda \cdot |\xi|$$

$\Rightarrow T_{[A_0]} p$ surj ■

§4.

Prop 11: p is a diffeomorphism

Pf$^{\vee}$ idea: Since p is a covering map and $\mathcal{M} \neq \emptyset$ (Taubes), it suffices to prove that p is single-sheeted.

If not, there are A_n, A_n' with

$$\underset{\to 0}{\Lambda_n} = \lambda[A_n] = \lambda[A_n'] \quad \text{and} \quad y_n = y[A_n] = y[A_n'] \to y.$$

but $[A_n] \neq [A_n']$.

As $n \to \infty$:

- $A_n, A_n' \to \Theta$ away from y.
- rescalings $z_n, \tilde{z}_n \to I$

FACT: $a_n := A_n' - A_n$ satisfies

$$\|a_n\|_{L^2} \leq c \cdot \Lambda_n^{3/2}, \quad \|\nabla_{A_n} a_n\|_{L^2} \leq c \cdot \Lambda_n^{1/2}.$$

Fix $n \gg 1$ and drop n.

Consider

$$\tilde{A}_t := A + t a \qquad (\tilde{A}_0 = A, \ \tilde{A}_1 = A')$$

almost ASD.

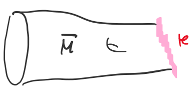
Use Thm 6 to construct

$$A_t = \tilde{A}_t + d^*_{\tilde{A}_t} \omega_t \quad \text{ASD.} \qquad (A_0 = A, \ A_1 = A')$$

The path $[A_t]$ stays in $\mathcal{M}$. Thus $[A_0]$, $[A_1]$ are in the same conn. comp. of $\mathcal{M}$.

$\overset{\pi_1(X)=1}{\Longrightarrow} [A_0] = [A_1]$. ■

UPSHOT:



reducible → next time.

$\partial \overline{M} = X$