Analysis of reducibles and Donaldson's theorem

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Outline

Review: 4-manifolds and intersection forms

- *always*: X closed oriented simply connected 4-manifold
- $H_i(X)$ trivial for $i \neq 2 \to$ focus on $H_2(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z})$
- $H^2(X;\mathbb{Z})$ corresponds to **intersection form**

$$Q_X \colon H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to \mathbb{Z}, (\alpha, \beta) \mapsto \langle a \cup b, [X] \rangle,$$

Q_X is symmetric, $\mathbb{Z}$-bilinear and unimodular

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  Q_X is symmetric, $\mathbb{Z}$-bilinear and unimodular
- three basic invariants
  - **parity**: Q is *even* iff $\mathrm{im}(Q) \subset 2\mathbb{Z}$, otherwise *odd*
  - **rank**: $\mathrm{rk}(Q) = b_2(X) = \dim_{\mathbb{Q}} H_2(X;\mathbb{Q})$
  - **signature**: $\mathrm{sign}\, Q = b_2^+ - b_2^-$

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- Freedman '82: for topological manifolds,
  $X \mapsto Q_X$ is surjective and at most two-to-one.
- Rohlin '52: X smooth with Q_X even $\Rightarrow \mathrm{sign}\, Q_X \in 16\mathbb{Z}$

Review: intersection forms and Donaldson's theorem

- indefinite unimodular forms are classified by the rank, signature and parity
- Hasse-Minkowski theorem: all indefinite unimodular forms are $l(1) \oplus m(-1)$ (odd type) or $l\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus mE_8$ (even type).
- definite forms: many exotic examples
- diagonalisable over $\mathbb{Q}$, but not necessarily over $\mathbb{Z}$

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Donaldson's theorem (1983)

X oriented closed simply connected smooth 4-manifold with Q_X definite. Then Q_X is diagonalisable.

Recall: setup and reducible connections

- G compact Lie group: $G = \mathrm{SO}(3)$ or $G = \mathrm{SU}(2)$,
  X compact simply connected oriented Riemannian 4-manifold,
  $E \to X$ G-principal bundle.
- $\mathcal{A} = \{\text{connection 1-forms on } E\}$,
  $\mathcal{B} = \mathcal{A}/\mathcal{G}$ quotient by gauge group $\mathcal{G}$ of E,
  $\mathcal{M} = \{[A] \in \mathcal{B} : F_A^+ = 0\}$ moduli space of ASD instantons.

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  $\mathcal{M} = \{[A] \in \mathcal{B} : F_A^+ = 0\}$ moduli space of ASD instantons.
- Each $A \in \mathcal{A}$ has
  - a holonomy group $H_A \overset{\text{Lie}}{\leq} \mathrm{Aut}(E_x) \cong G$ and
  - an isotropy group $\Gamma_A = \{u \in \mathcal{G} : u(A) = A\}$.
- For X connected, Γ_A is isomorphic to the centraliser of H_A.
- A is reducible $\Leftrightarrow H_A \lneqq G$ is a proper subgroup
  $\Leftrightarrow Z(G) \lneqq \Gamma_A$ is a proper subgroup.

Classification of reducible connections

Proposition

If $G = SU(2)$ or $SO(3)$ and $H \leqslant G$ is a closed connected Lie subgroup, then $H = \{\mathrm{id}\}$, $H = G$ or $H \cong \mathbb{S}^1$.

$H_A = \{\mathrm{id}\}$ means E is trivial and A is the product connection.
For SU(2)-bundles,

$$H_A \cong \mathbb{S}^1 \Leftrightarrow E \cong L \oplus L^{-1} \Leftrightarrow c_2(E) = -c_1(L)^2$$

for a complex line bundle L; for SO(3)-bundles

$$H_A \cong \mathbb{S}^1 \Leftrightarrow E \cong \mathbb{R} \oplus L \Leftrightarrow p_1(E) = c_1(L)^2.$$

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Fact

A line bundle L over X admits an ASD connection
iff $c_1(L)$ is represented by an ASD 2-form,
and the connection is unique up to gauge equivalence.

Proposition

Reducible ASD connection 1-forms with holonomy group $\cong \mathbb{S}^1$
$\leftrightarrow$ *pairs* $\{c, -c\}$ *where* $c \neq 0 \in H^2(X; \mathbb{Z})$ *satisfies*
$c^2 = -c_2(E)$ *(for* $G = SU(2)$*) resp.* $c^2 = p_1(E)$ *(for* $G = SO(3)$*).*

Classification of reducible connections (cont.)

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Corollary

If Q_X is definite, there are only finitely many reducible connections (up to gauge equivalence).

Locals models of the moduli space $\mathcal{M}$

Recall that

- $\mathcal{G}$ is an infinite-dimensional Banach Lie group, the action $\mathcal{G} \times \mathcal{A} \to \mathcal{A}$ is a smooth map of Banach manifolds.
- Its differential in $\mathcal{G}$ at $A \in \mathcal{A}$ is $-d_A \colon \Omega^0(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$, with ASD part is $d_A^+ \colon \ker d_A^* \to \Omega_X^+(\mathfrak{g}_E)$.
- $\delta_A := d_A^+ \oplus d_A^*$ is elliptic, hence Fredholm.

Proposition

If A is an ASD connection over X, a neighbourhood of $[A]$ in $\mathcal{M}$ is modelled on a quotient $f^{-1}(0)/\Gamma_A$, where $f \colon \ker \delta_A \to \operatorname{coker} d_A^+$ is a Γ_A-equivariant map. □

Local models near reducible connections (cont.)

Let A be a reducible connection, $G = \mathrm{SU}(2)$ or $G = \mathrm{SO}(3)$.

- **Case 1**: $H_A \cong \mathbb{S}^1$
  Since $\mathbb{S}^1$ is abelian, $C_G(H_A) = H_A$, hence $\Gamma_A \cong \mathbb{S}^1$.
  Near a reducible connection with $H_A \cong \mathbb{S}^1$, $\mathcal{M}$ is modeled on a quotient $\mathbb{R}^n/\mathbb{S}^1$, a cone over projective space.

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- **Case 2**: H_A is trivial. In this case, $\Gamma_A \cong C_G(H_A) = G$.
  $\mathcal{B}$ has a stratification $\mathcal{B} = \bigsqcup_{[\Gamma]\in C} \mathcal{B}^\Gamma$ with strata

$$\mathcal{B}^\Gamma := \{[A] \in \mathcal{B} : \Gamma_A \cong_{\mathrm{conj}} \Gamma\},$$

where $C = \{\Gamma \leqslant G : \text{closed subgroup}\}/\text{conjugation}$.
Near a reducible connection with $H_A = \{\mathrm{id}\}$,
$\mathcal{M}$ is modeled on a cone over a singular space.

Donaldson's theorem: proof outline

Recall: Donaldson's theorem

X oriented closed simply connected smooth 4-manifold with Q_X definite.
Then Q_X is diagonalisable.

Proof outline

- Take a suitable SU(2)-bundle $E \to X$;
  consider $\mathcal{M} = \{\text{ASD connections}\}$.
- Collar theorem: $\mathcal{M}^*$ smooth manifold with ideal boundary X.
- Truncate: cut a neighbourhood of each reducible connection
  $\Rightarrow$ cobordism between X and disjoint union of $\mathbb{CP}^2$'s.
- Use cobordism invariance of signature and a small computation.

Proof of Donaldson's theorem

- Suppose w.l.o.g. Q_X is negative definite, i.e. $b^+ = 0$.
- Let $E \to X$ be a smooth SU(2)-bundle with $c_2(E) = 1$, consider $\mathcal{M} := \{\text{smooth ASD connections on } E\}$.
- Choose a generic metric on X, then $\mathcal{M}^*$ is a smooth manifold of dimension $8 \cdot 1 + 3 \cdot (b_1 - b^+ - b^0) = 5$.
- Finitely many reducibles $[A_e]$, correspond to $\{e, -e\} \subset H^2(X;\mathbb{Z})$ with $e^2 = -c_2(E) = -1$.
- Near each $[A_e]$, $\mathcal{M}$ is modelled as $\mathbb{C}^3/\mathbb{S}^1$, a cone over $\mathbb{CP}^2$.
- Choose a conical neighbourhood U_e of each $[A_e]$, denote $P_e := \partial U_e \cong \mathbb{CP}^2$. Let $\mathcal{M}' := \mathcal{M} \setminus (U \cup \bigcup_e U_e)$.

Proof of Donaldson's theorem (cont.)

Truncated moduli space: $\mathcal{M}' := \mathcal{M} \setminus (U \cup \bigcup_e U_e)$.

- Denote $n(Q_X) := \#$reducibles:
  $\mathcal{M}'$ is a cobordism between X and $\sqcup_e U_e \cong \bigsqcup_{k=1}^{n(Q_X)} \mathbb{CP}^2 =: Y$.

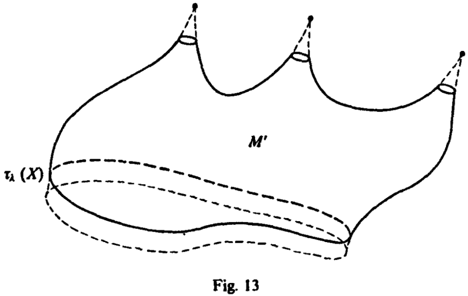

Figure: Sketch of the moduli space $\mathcal{M}'$. Figure taken from Donaldson-Kronheimer, The Geometry of four-manifolds, 1990.

Proof of Donaldson's theorem (cont.)

X is cobordant to $Y := \bigsqcup_{k=1}^{n(Q_X)} \mathbb{CP}^2$.

Lemma (Algebraic fact)

$n(Q_X) \leq \mathrm{rk}(Q_X)$ *with equality iff* $Q_X \cong n(-1)$.

Proposition

If W *is an oriented cobordism between closed simply connected 4-manifolds,* $\mathrm{sign}(Q_X) = \mathrm{sign}(Q_Y)$.

Q_X is negative definite: $\mathrm{sign}(Q_X) = -rk(Q_X)$, thus

$$\mathrm{rk}(Q_X) = |\mathrm{sign}(Q_X)| = |\mathrm{sign}\, Q_Y| \leq n(Q_X) \underbrace{\mathrm{sign}(\mathbb{CP}^2)}_{=+1} = n(Q_X),$$

thus sign $Q_X = n(Q)$ and Q is diagonalisable. □

Proof of Donaldson's theorem (concluded)

Lemma (Algebraic fact)

Let Q be a negative definite quadratic form over $\mathbb{Z}$. Denoting $n(Q) := \#\{\{\alpha, -\alpha\} : Q(\alpha, \alpha) = -1\}$, we have $n(Q) \leq \mathrm{rk}(Q)$ with equality iff Q is diagonalisable, i.e. $Q \cong n(-1)$.

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Proof sketch.

- Induction over $r = \mathrm{rk}(Q)$. Base case is clear.
- If α satisfies $Q(\alpha, \alpha) = -1$, get a splitting

$$\mathbb{Z}^r = \mathbb{Z}\alpha \oplus \alpha^\perp, \beta \mapsto \langle \beta, \alpha \rangle \alpha \oplus (\alpha - \langle \beta, \alpha \rangle \alpha).$$

- Since Q is definite, $n(Q) = 1 + n(Q|_{\alpha^\perp})$ and $\mathrm{rk}(Q|_{\alpha^\perp}) + 1$. □

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About cobordism-invariance of the signature:

- Chern-Weil theory implies that $p_1(TX)$ is cobordism-invariant.
- Hirzebruch signature theorem relates $p_1(TX)$ and $\mathrm{sign}(Q_X)$.

Fintushel-Stern's proof of Donaldson's theorem

Theorem

There is no smooth, oriented simply connected closed four-manifold X with intersection form $Q_X \cong -E_8 \oplus -E_8$.

Proof sketch

- Suppose there were. Choose $e \in H^2(X;\mathbb{Z})$ with $e^2 = -2$.
- Consider the SO(3)-bundle $F = L \oplus \mathbb{R}$, where $c_1(L) = e$.
- Fix a regular Riemannian metric on X (are generic).
  Virtual dimension is 1, hence $\mathcal{M}_F^*$ is 1-dimensional.
- Since $\dim \mathcal{M}_F = 1$, boundary strata $\mathcal{M}_{F(r)}$ have negative dimension, hence empty $\Rightarrow \mathcal{M}_F$ compact.

Fintushel-Stern's proof of Donaldson's theorem (cont.)

- Consider the norm $\|\alpha\| = -\langle\alpha,\alpha\rangle$.
- A reducible ASD connection corresponds to $\{f,-f\}$, where $f \neq 0 \in H^2(X;\mathbb{Z})$ with $f^2 = p_1(F) = c_1(L)^2 = e^2$. Thus, $f = e (\mathrm{mod} 2)$ and $\|f\| = \|e\|$.
- By first condition, $m := \frac{e+f}{2} \in H^2(X;\mathbb{Z})$.
  By Cauchy-Schwartz $\|m^2\| \leq \|e\|^2 = 2$; equality iff $e = f = m$.

Fintushel-Stern's proof of Donaldson's theorem (cont.)

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  By Cauchy-Schwartz $\|m^2\| \leq \|e\|^2 = 2$; equality iff $e = f = m$.
- Thus, $f \neq e$ implies $\|m\| \in \{0, 1\}$. **But $E_8 \oplus E_8$ is even and doesn't contain a vector of length one.**
  Hence, $m = 0$ and $f = -e$. Thus, $\mathcal{M}_F$ contains exactly one reducible connection $[A_e]$, corresponding to $\{e, -e\}$.

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  Hence, $m = 0$ and $f = -e$. Thus, $\mathcal{M}_F$ contains exactly one reducible connection $[A_e]$, corresponding to $\{e, -e\}$.
- Local models: $[A_e]$ has neighbourhood in $\mathcal{M}_F$ modelled on a cone over $\mathbb{CP}^0 = \{pt\}$, i.e. a closed half-line.

$\Rightarrow$ $\mathcal{M}_F$ compact 1-mfd with one boundary point, contradiction!

□

Comparison of proofs

Donaldson's proof

- crucially relies on the non-compactness of $\mathcal{M}$
- uses cobordism invariance of signature
  Donaldson-Kronheimer: replace by a computation of topology of $\mathcal{B}$
- this generalises to non-definite forms as well

Fintushel-Stern's proof

- uses compactness properties: $E_8 \oplus E_8$ contains no length one vector
  not enough energy for bubbling, don't need collar theorem/gluing map
- depends on lattice
- can be adapted for general lattices, but requires Q_X definite.

Appendix

Proof: Lie subgroups of G

Proposition

If $G = SU(2)$ or $SO(3)$ and $H \leqslant G$ is a closed connected Lie subgroup, then $H = \{\mathrm{id}\}$, $H = G$ or $H \cong \mathbb{S}^1$.

Proof sketch.

Let $H \leqslant G$ be connected and closed subgroup.
Is a Lie subgroup. By the subgroups-subalgebras theorem,

$$\{\text{Lie subgroups } H \leqslant G\} \overset{1:1}{\leftrightarrow} \{\text{Lie subalgebras } \mathfrak{h} \subset \mathfrak{g}\}, H \mapsto T_{\mathrm{id}} H.$$

In our case, $\mathfrak{su}(2) \cong \mathfrak{so}(3) = \{A \in \mathbb{R}^{3\times 3} : A + A^t = 0\}$.
Choose basis and compute: $(\mathfrak{so}(3), [\cdot,\cdot]) \cong (\mathbb{R}^3, \times)$.
$=>$ $\mathfrak{h}$ cannot have dimension two, so H has dimension 0, 1 or 3.
$\Rightarrow H = \{\mathrm{id}\}$, $H = G$ or $H \cong \mathbb{S}^1$. □