Donaldson's proof of Narasimhan-Seshadri theorem;

- Holomorphic vector bundles & stability
- Proof the NS theorem modulo lemmas
- proof of the lemmas
- Riemann Hilbert Correspondence.

Holomorphic vector bundles and Connections

Σ: Complex mfld.

$\begin{matrix} E \\ \downarrow \\ \Sigma \end{matrix}$: Complex vector bundle

$\Lambda^k(E) :=$ E-valued k-forms

$\Lambda^{p,q}(E) = \underbrace{\Lambda^{p,q}(\Sigma)}_{(p,q)\text{-forms}} \otimes_{\mathbb{C}} E$

E.g. $\Sigma = \mathbb{C}^n$
$\Lambda^{1,0} = \langle dz_i \rangle$
$\Lambda^{0,1} = \langle d\bar{z}_i \rangle$
$\Lambda^{p,q} = (\Lambda^{1,0})^p \wedge (\Lambda^{0,1})^q$

$\Omega^k_{\mathbb{C}}(E)$: Sections of $\Lambda^k(E) \otimes \mathbb{C}$

$\Omega^{p,q}(E)$: Sections of $\Lambda^{p,q}(E)$

$\Omega^k_{\mathbb{C}}(E) = \bigoplus_{p+q=r} \Omega^{p,q}(E)$

Given a Connection A, Covariant derivative

$d_A : \Omega^k_{\mathbb{C}}(E) \longrightarrow \Omega^{k+1}_{\mathbb{C}}(E)$ (extend linearly)

Define,

$\bar{\partial}_A : \Omega^{p,q}(E) \longrightarrow \Omega^{p,q+1}(E)$

by

$$\begin{tikzcd} \Omega^{p,q}(E) \arrow[r, "d_A"] \arrow[dr, "\bar{\partial}_A"'] & \Omega^{k+1}_{\mathbb{C}}(E) \arrow[d, "\text{proj}"] \\ & \Omega^{p,q+1}(E) \end{tikzcd}$$

Def: $\begin{matrix} E \\ \downarrow \\ \Sigma \end{matrix}$: cpx vector bundle is said to be holomorphic if $\exists$ local trivializations $\varphi_i : E_{i|U_i} \to U_i \times \mathbb{C}^n$ s.t. $g_{ij} : \varphi_i \circ \varphi_j^{-1} : U_i \cap U_j \to GL(n, \mathbb{C})$ are holomorphic.

Def: Suppose, $\begin{matrix} \mathcal{E} \\ \downarrow \\ \Sigma \end{matrix}$ is holomorphic vector bundle, then $\bar{\partial}_{\mathcal{E}} : \Omega^0(E) \longrightarrow \Omega^{0,1}(E)$ is defined by

$\bar{\partial}_{\mathcal{E}} s := \bar{\partial} s^i \otimes e_i$, $s = \sum s^i e_i$, on each trivialization $U_i \times \mathbb{C}^n$

Theorem: A complex vector bundle $E \to \Sigma$ is holomorphic iff $\exists$ a connection A such that $\bar\partial_A^2 = \bar\partial_A \circ \bar\partial_A : \Omega^0(E) \to \Omega^{0,2}(E)$ vanishes i.e. $F_A^{0,2} = 0$.

proof: See [Donaldson & Kronheimer], Section 2.2.2.

Note: If Σ : Riemann surface, then $\Omega^{0,2}(\Sigma) = 0$
$\Rightarrow$ $\bar\partial_A^2 = 0$ always true.

Def: Dolbeault operator on E is a $\mathbb{C}$ linear map

$$\bar\partial_E : \Omega^0(E) \longrightarrow \Omega^{0,1}(E) \text{ s.t.}$$

$$\bar\partial_E(fs) = \bar\partial f \otimes s + f\,\bar\partial_E s \quad \forall f \in C^\infty(\Sigma, \mathbb{C}) \ \&\ s \in \Gamma(E).$$

$\mathcal{G}^{\mathbb{C}}$:= gauge transformations of E.
i.e. cpx bundle isomorphisms

$\mathcal{G}^{\mathbb{C}} \curvearrowright \mathrm{Dol}_0(E) := \{\bar\partial_E : \bar\partial_E^2 = 0\}$

$u \cdot \bar\partial_E = u(\bar\partial_E u^{-1})$.

prop: $\left\{\begin{array}{c}\text{Holomorphic}\\ \text{structures on}\\ E\end{array}\right\} \Big/ \sim \ \longrightarrow \ \mathrm{Dol}_0(E) / \mathcal{G}^{\mathbb{C}}$

$$\mathcal{E} \longmapsto \bar\partial_{\mathcal{E}}$$

is bijective.

Def: A Hermitian vector bundle over Σ is a pair (E, h), where E is a cpx vector bundle and h is a hermitian metric on E.

A connection A on E is said to be unitary connection if it is compatible with $h = \langle \cdot, \cdot \rangle$ i.e.

$$d\langle s, t\rangle = \langle \nabla_A s, t\rangle + \langle s, \nabla_A t\rangle$$

$\forall s, t$, sections of E.

$\mathcal{G}^{\mathbb{C}} \curvearrowright \mathcal{A}(E, h)$ =: unitary connection, by $u \cdot \bar\partial_E = (u\,\bar\partial_E u^{-1})^*$

prop: $\left\{\begin{array}{c}\text{Unitary connections}\\ \text{on } (E,h)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Dolbeault}\\ \text{operators}\\ \text{on } E\end{array}\right\}$.

Conclusion: $\left\{ \begin{array}{l} \text{Holomorphic} \\ \text{strs on } E \end{array} \right\} \Big/ \sim \quad \longleftrightarrow \quad \left\{ \begin{array}{l} \text{Unitary Conections} \\ \text{on } (E, h) \end{array} \right\} \Big/ \mathcal{G}^{\mathbb{C}}$

Thm: Let, (E, h) be a hermitian, holomorphic vector bundle. Then $\exists$ unique unitary connection A s.t. $\nabla_A^{0,1} = \bar{\partial}_E$. (This connection is called chern connection).

Idea of proof:

- It is enough to prove locally
- Assume $E = \Sigma \times \mathbb{C}^r$ &

$$\underbrace{\nabla_A = d + A}_{\text{type } (1,0)} \quad \& \quad \underbrace{\begin{array}{l} H : \Sigma \longrightarrow M_{\mathbb{C}}(r) \\ x \longmapsto (H_{ij}(x)) \end{array}}_{\text{Hermitian metric}}.$$

define, $A := \bar{H}^{-1} \partial(\bar{H})$.

i.e. $\nabla_A = \nabla_A^{1,0} + \bar{\partial}_E$ & F_A $\boxtimes$

Remark: If (E, h) is a hermitian holomorphic line bundle, then

$$\nabla_A = d + \partial \log h \quad \&$$

$$F_A = d(\partial \log h) + \underbrace{\partial \log h \wedge \partial \log h}_{\to 0}$$

$$\Rightarrow \boxed{F_A = \bar{\partial} \partial \log h} \quad (\text{as } F_A^{0,2} = 0)$$

Assume Σ is a Riemann Surface.

$$i * F_A \in C^\infty(\Sigma, \mathbb{R})$$

$$i * F_A = \lambda + \Delta f$$

$$= \lambda + 2\partial^* \partial f$$

define, $\tilde{h} = e^{-2f} h$

$$F_{\tilde{A}} = \bar{\partial}\partial \log \tilde{h}$$

$$= \bar{\partial}\partial(-2f) + \bar{\partial}\partial \log h$$

$$= -2\bar{\partial}\partial f + F_A$$

$$\Rightarrow i * F_{\tilde{A}} = -2 \underline{i * \bar{\partial}} \partial f + i * F_A$$

$$= -2 \underline{\partial^*} \partial f + i * F_A$$

$$= \lambda$$

Corollary: Let, $\mathcal{L} \to \Sigma$ be a holomorphic. Then $\exists$ a hermitian metric h & a unitary connection A s.t

$$\frac{i}{2\pi} * F_A = \mu \quad \& \quad \bar{\partial}_A = \bar{\partial}_{\mathcal{L}}.$$

Note: $\mu = \int_\Sigma c_1(\mathcal{L}) \; (= \deg \mathcal{L})$

Narasimhan - Seshadri thm:

Suppose $\mathcal{E} \to \Sigma$ is an indecomposable, holomorphic vector bundle, where Σ is a Riemann surface. Then

$\mathcal{E}$ is stable $\Longleftrightarrow$ $\exists$ a hermitian metric and a unitary connection A s.t. $\bar{\partial}_A = \bar{\partial}_{\mathcal{E}}$

$$\frac{i}{2\pi} * F_A = \mu(\mathcal{E}).$$

Moreover A is unique upto unitary gauge transformations.

Stable bundle

$\mathcal{E} \downarrow \Sigma$ holo vect. bundle, Σ: cpt Riemann surface

Indecomposable :$\equiv$ Not a direct sum of holo sub bundles

Def: Slope of $\mathcal{E}$, $\mu(\mathcal{E})$ is defined by

$$\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\operatorname{rank} \mathcal{E}}$$

where $\deg \mathcal{E} = \int_\Sigma c_1(\mathcal{E})$

Def: $\mathcal{E}$ is called stable (semi stable) $\forall$ nonzero proper subbundle $\mathcal{F}$ of $\mathcal{E}$, we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$ $(\leq)$ or equivalently $\mu(\mathcal{E}/\mathcal{F}) > \mu(\mathcal{E})$.

Exercise: If $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{H} \to 0$ a exact seq of holo bundles, then $\mu(\mathcal{F}) < \mu(\mathcal{E}) \Longleftrightarrow \mu(\mathcal{H}) > \mu(\mathcal{E})$.

Hint: deg and rank are additive.

The following follows as $\mathcal{O}_\Sigma$ is PID.

Lemma: Let, $\alpha \neq 0 \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$. Then $\exists$ a factorizations

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{P} \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, "\alpha"] & \mathcal{Q} \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 & \mathcal{S} \arrow[l] & \mathcal{F} \arrow[l] & \mathcal{R} \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

s.t. $\operatorname{rank} \beta = \operatorname{rank}$ of $\mathcal{E}$ and

$\operatorname{rank} \mathcal{Q} = \operatorname{rank} \mathcal{R}$ and

$\deg \mathcal{Q} \leq \deg \mathcal{R}$.

Harder - Narsimhan filtration:

Any holo vect. bundle $\mathcal{E} \to \Sigma$ admits a canonical filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$$

s.t. $\mathcal{H}_i := \mathcal{E}_i / \mathcal{E}_{i-1}$ is semistable

and $\mu(\mathcal{H}_{i+1}) < \mu(\mathcal{H}_i)$ $\forall\, 1 \leq i \leq k$.

Jordan - Hölder filtration: Any semistable bundle $\mathcal{E}$ has a filtration

$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$ s.t. $\mathcal{E}_i / \mathcal{E}_{i-1} =: \mathcal{H}_i$ stable & $\mu(\mathcal{E}) = \mu(\mathcal{H}_i)$ $\forall i$.

M: $n \times n$ Hermitian matrix

Define norms $\nu(M) := \operatorname{tr}\sqrt{MM^*} = \operatorname{tr}\sqrt{M^2}$

and $|M| := \sqrt{\operatorname{tr}(MM^*)}$ (standard!)

Exercises:

(i) $|M| \leq \nu(M) \leq n|M|$

(ii) If $M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$, then

$$\nu(M) \geq |\operatorname{tr} A| + |\operatorname{tr} D|$$

$\mathcal{A}^{1,2}$:= space of $W^{1,2}$ unitary connections

$\mathcal{G}_{\mathbb{C}}^{2,2}$:= group of $W^{2,2}$ gauge transformations i.e. complex automorphisms.

Note: $\mathcal{A}^{1,2}/\mathcal{G}^{2,2} = \mathcal{A}/\mathcal{G}$.

$$J(A) := \left\| \nu\left(\frac{i}{2\pi} * F_A - \mu(\mathcal{E}) I\right) \right\|_{L^2(\Sigma)}, \quad A \in \mathcal{A}^{1,2}$$

Lemma 1: If $0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{N} \to 0$ is an exact sequence holo bundles and $\mu(\mathcal{M}) \geq \mu(\mathcal{E})$, then $\forall A$ unitary conn.

$$J(A) \geq \underbrace{\text{rank}\,\mathcal{M}\,(\mu(\mathcal{M}) - \mu(\mathcal{E})) + \text{rank}\,\mathcal{N}\,(\mu(\mathcal{E}) - \mu(\mathcal{N}))}_{=: J_0}$$

& equality occurs only if the sequence splits.

$\boxed{\text{proof of NS theorem}}$

($\Leftarrow$) Suppose not i.e.

$\exists\ \tilde{\mathcal{E}} \subset \mathcal{E}$ s.t.

$\mu(\tilde{\mathcal{E}}) > \mu(\mathcal{E})$

Then by above lemma,

$0 = J(A) \geq J_0 \geq 0$

$\Rightarrow J_0 = 0 \Rightarrow \mathcal{E}$ is decomposable

Contradiction !!

($\Rightarrow$)

(Existence of minimizer)

$$J(A) = \| \wedge \left(\frac{i}{2\pi} * F_A - \mu \right) \|_{L^2(\Sigma)}$$

$$J_{min} := \inf_{A \in \mathcal{P} \subseteq \mathcal{A}^{1,2}} J(A).$$

Let, $J(A_i) \longrightarrow J_{min}$ with $A_i \in \mathcal{P}$

Then $\|F_{A_i}\|_{L^2}$ is bdd & hence by Uhlenbeck a weak cpt ness there exist a subseq upto gauge transformations call it again A_i s.t. $A_i \longrightarrow A$ weakly in $W^{1,2}$. This implies $F_{A_i} \to F_A$ weakly in L^2.

<u>Lemma:</u> Let, $f : \mathcal{H} \to \mathbb{R}$ be a convex function, were $\mathcal{H}$ is normed vector space. If $x_i \rightharpoonup x$ weakly in $\mathcal{H}$,

then $f(x) \leq \inf_i f(x_i)$.

Hint: Use Hahn-Banach separation theorem for convex sets.

Take $\mathcal{H} = L^2(\Omega^2(E))$. &

$f = \| \mathcal{S}(\cdot - \mu I_E) \|_{L^2}$

This will be a convex fn.

Thus, $J(A) \leq \inf_i J(A_i) = J_{min}$.

A: a minimizer.

Question: $A \in \mathcal{P}$??

We will take

$\mathcal{P} = \mathcal{G}^c$ orbit of unitary connection corresponding to $\mathcal{E}$. $= \mathcal{O}(\mathcal{E})$

By above $\exists$ A unitary connection s.t.

$$J(A) \leq \inf_{B \in \mathcal{O}(\mathcal{E})} J(B).$$

Q. $A \in \mathcal{O}(\mathcal{E})$?

Lemma 2: $\inf_{B \in \mathcal{O}(\mathcal{E})} J(B) \in \mathcal{O}(\mathcal{E})$ or $\exists$ holo bundle $\mathcal{F} \neq \mathcal{E}$ of same rank and degree as $\mathcal{E}$ and $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \neq 0$.

Lemma 3: Suppose $\mathcal{E}$ is a stable bundle s.t. $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{H} \to 0$ be an exact seq of holomorphic bundles. Assume the theorem is true for lower rank stable bundles. Then there exist a hermitian metric & unitary connection A s.t.

$$J(A) < \operatorname{rank} \mathcal{F} \, (\mu(\mathcal{E}) - \mu(\mathcal{F})) + \operatorname{rank} \mathcal{H} \, (\mu(\mathcal{H}) - \mu(\mathcal{E}))$$
$$= J_1$$

proof of ($\Rightarrow$) of the thm:

Claim 1: $\inf_{B \in \mathcal{O}(\mathcal{E})} J(B)$ is attained in $\mathcal{O}(\mathcal{E})$

Claim 2: If $A \in \mathcal{P}$ s.t. $J(A) = \inf_{B \in \mathcal{P}} J(B)$

then $J(A) = 0$.

<u>proof of claim 1:</u> Suppose not, then by lemma 2, $\exists\, \alpha \neq 0 \in \mathrm{Hom}(\mathcal{E}, \tilde{\mathcal{F}})$. Consider its factorization

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{P} \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, "\alpha"] & \mathcal{Q} \arrow[r] \arrow[d, "\beta"] & 0 \\
0 & \mathcal{S} \arrow[l] & \tilde{\mathcal{F}} \arrow[l] & \mathcal{R} \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

s.t. rank β = rank of $\mathcal{E}$ and rank $\mathcal{Q}$ = rank $\mathcal{R}$ and $\deg \mathcal{Q} \leq \deg \mathcal{R}$.

Apply lemma 3 to top row
$\inf J|_{O(\mathcal{E})} < J_1$

Apply lemma 1 to bottom row.
$\inf\limits_{B \in O(\tilde{\mathcal{F}})} J \geq J_0$

But we have seen that $\inf\limits_{B \in O(\tilde{\mathcal{F}})} J \leq \inf J|_{O(\mathcal{E})}$ i.e. $J_0 < J_1$. But it is straightforward to check that $J_1 \leq J_0$, which is a contradiction. $\boxtimes$

<u>proof of claim 2:</u>

As, $\mathcal{E}$ is indecomposable, (Exercise)
$\ker d_A^* d_A = \{\lambda I_{\mathcal{E}} : \lambda \in \mathbb{C}\}$

Thus $\exists$ a self-adjoint section $u \in W^{2,2}$ s.t. $d_A^* d_A u = \frac{i}{2\pi} * F_A - \mu I_{\mathcal{E}}$

(by Hodge theory for $d_A^* d_A = \Delta_A$)

Define $u_t := 1 + tu \in \mathcal{G}^{\mathbb{C}}$ and $A_t := u_t \cdot A$.

Then $J(A_t) = J(A)(1-t) + O(t^2)$.

(Exercise)

$\Rightarrow$ $J(A_t)$ is minimum when $t=0$,

thus $J(A)=0$.

<u>Uniqueness</u>: Suppose the thm is true for A & $g\cdot A \in \mathcal{O}(\mathcal{E})$.

$g = pu$ $\quad p = p^*$

$\quad\quad\quad\quad u \in \mathcal{G}$

Then the thm is true for $p\cdot A$.

As $p = p^*$, p positive definite.

$F_{p\cdot A} = F_A$ implies (Exercise!)

$$\partial_A \bar\partial_A p^2 = -(\bar\partial_A p^2)p^{-1}((\bar\partial_A p^2)p^{-1})^*$$

Taking trace & denote $\tau = \mathrm{tr}(p^2)$

$\Rightarrow \Delta\tau \le 0$

$\Rightarrow \tau = 0$ (by max principle)

$\Leftrightarrow \bar\partial_A p^2 = 0$.

$\Rightarrow p = \lambda I$ as $\mathcal{E}$ is indecomposable

$\Rightarrow p\cdot A = A$ $\boxtimes$.

PROOF OF THE LEMMAS

<u>Lemma 1</u>: If $0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{N} \to 0$ is an exact sequence holo bundles and $\mu(\mathcal{M}) \geq \mu(\mathcal{E})$, then

$$J(A) \geq \underbrace{\mathrm{rank}\,\mathcal{M}\,(\mu(\mathcal{M})-\mu(\mathcal{E})) + \mathrm{rank}\,\mathcal{N}\,(\mu(\mathcal{E})-\mu(\mathcal{N}))}_{=: J_0}$$

+ uni

& equality occurs only if the sequence splits.

<u>proof</u>: $A = \begin{pmatrix} A_{\mathcal{M}} & \beta \\ -\beta^* & A_{\mathcal{N}} \end{pmatrix}$

$\Rightarrow F_A = \begin{pmatrix} F_{A_{\mathcal{M}}} - \beta\wedge\beta^* & d\beta \\ -d\beta^* & F_{A_{\mathcal{N}}} - \beta^*\wedge\beta \end{pmatrix}$

As $\mathrm{vol}(\Sigma)=1$.

$$J(A) \geq \int_\Sigma \nu\left(\frac{i}{2\pi}*F_A - \mu(\mathcal{E})I_{\mathcal{E}}\right)$$

Hölder ineq

$$\geq \left|\int_\Sigma \mathrm{Tr}\left(\frac{i}{2\pi}*F_{A_{\mathcal{M}}} - \mu(\mathcal{E})I_{\mathcal{M}}\right) - |\beta|^2\right|$$

$$+ \left|\int_\Sigma \mathrm{Tr}\left(\frac{i}{2\pi}*F_{A_{\mathcal{N}}} - \mu(\mathcal{E})I_{\mathcal{N}}\right) + |\beta|^2\right|$$

$$= -\mathrm{rank}\,\mathcal{M}\ \mu(\mathcal{M}) + \mathrm{rank}\,\mathcal{M}\,\mu(\mathcal{E})$$
$$+ \mathrm{rank}\,\mathcal{N}\ \mu(\mathcal{N}) - \mathrm{rank}\,\mathcal{N}\,\mu(\mathcal{E})$$
$$+ 2|\beta|^2$$

$\geq J_0$

equality $\Rightarrow \beta = 0 \Rightarrow \mathcal{E}$ is decomposable. $\boxtimes$

Lemma 2: $\inf_{A\in\mathcal{O}(\mathcal{E})} J(A) \in \mathcal{O}(\mathcal{E})$ or $\exists$ holo bundle $\mathcal{F} \not\cong \mathcal{E}$ of same rank and degree as $\mathcal{E}$ and

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \neq 0.$$

Proof: Pick a minimizing sequence $A_i \in \mathcal{O}(\mathcal{E})$ for $\inf_{A\in\mathcal{O}(\mathcal{E})} J(A)$.

$\Rightarrow \|F_{A_i}\|_{L^2}$ bdd.

$\Rightarrow A_i \rightharpoonup B$ weakly in $W^{1,2}$

and $J(B) \leq \inf J|_{\mathcal{O}(\mathcal{E})}$

It is enough to prove

$\mathrm{Hom}(\mathcal{E}, \mathcal{E}_B) \neq 0$ when

$\mathcal{E}_B$: holomorphic bundle corresponding to B.

$\mathcal{E}_B =: \mathcal{F}$ (E is the underlying C^∞ bundle.)

Suppose not

$\bar\partial_{A_i B} : \Omega^0(\mathrm{Hom}(E,E)) \longrightarrow \Omega^1(\mathrm{Hom}(E,E))$

<u>claim:</u> $\ker \bar\partial_{A_0 B} \neq 0$

Suppose not, then

$$\|\bar\partial_{A_0 B} s\|_{L^2} \gtrsim \|s\|_{W^{1,2}} \quad \forall s$$
$$\gtrsim \|s\|_{L^4}$$

as $W^{1,2} \hookrightarrow L^4$ compact,

$A_i \longrightarrow B$ in L^4.

$$\|(\bar\partial_{A_0 B} - \bar\partial_{A_0 A_i}) s\|_{L^2} \overset{\text{Hölder}}{\lesssim} \|A_i - B\|_{L^4} \|s\|_{L^4}$$

$\Rightarrow \ker \bar\partial_{A_0 A_i} = 0$ for large enough i

$\Rightarrow \mathrm{Hom}(\mathcal{E}_{A_0}, \mathcal{E}_{A_i}) = 0$ —— " ——

Contradiction as $A_0, A_i \in \mathcal{O}(\mathcal{E})$. $\boxtimes$

<u>Lemma 3:</u> Suppose $\mathcal{E}$ is a stable bundle

Let $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{H} \to 0$ be an exact seq of holomorphic bundles.

Assume the theorem is true for lower rank stable bundles. Then there exist a hermitian metric & unitary connection A s.t.

$J(A) < \operatorname{rank} \mathcal{F}(\mu(\mathcal{E}) - \mu(\mathcal{F})) + \operatorname{rank} \mathcal{H}(\mu(\mathcal{H}) - \mu(\mathcal{E}))$

<u>proof:</u> Consider Harder Narasimhan filtration of $\mathcal{F}$

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = \mathcal{F}$$

were $\mathcal{F}_i / \mathcal{F}_{i-1}$ semistable & $\mu(\mathcal{F}_i / \mathcal{F}_{i-1})$ is decreasing in i.

Then each $\mathcal{F}_i / \mathcal{F}_{i-1}$ has a Jordan Hölder filtration.

$$0 = \mathcal{S}_{i0} \subset \mathcal{S}_{i1} \subset \cdots \subset \mathcal{S}_{i\ell} = \mathcal{F}_i / \mathcal{F}_{i-1}$$

were $\mathcal{P}_{ij} / \mathcal{P}_{ij-1}$ stable & slopes are equal to $\mu\left(\mathcal{F}_i / \mathcal{F}_{i-1}\right)$.

Fact: For each exact $0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$ and a connection orbit $\mathcal{O}(\mathcal{M})$, we have a family $A_t \in \mathcal{O}(\mathcal{M})$ $(t \neq 0)$ and $A_0 \in \mathcal{O}(\mathcal{L} \oplus \mathcal{N})$.

Hint: $g_t = \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix}$, $A_t = g_t \cdot A$.

By induction, for each $\mathcal{P}_{ij}$ thm is true hence there are $A_{ij} \in \mathcal{O}(\mathcal{P}_{ij})$. Following this construction carefully we can construct

$$A^t_{\mathcal{F}} \longrightarrow A^0_{\mathcal{F}} \in \mathcal{O}(\oplus \mathcal{P}_{ij}) \ \& \ \frac{i}{2\pi} * F_{A^0_{\mathcal{F}}} = \begin{pmatrix} \mu_{11} & & \\ & \ddots & \\ & & \mu_{ij} \end{pmatrix}$$

Similarly for $\mathcal{H}$, $A^t_{\mathcal{H}}$.

To construct β_t s.t.

$$[\beta] = [\beta_t] \in H^1(\mathcal{H}^* \otimes \mathcal{F})$$

Pick β_t on Harmonic representation of β. Corresponding to $d_{A^t_{\mathcal{H}}, A^t_{\mathcal{F}}}$.

& $\|\beta_t\|_{L^2} = 1$.

Then one can prove that

$$J(A^t_{\mathcal{F}}, A^t_{\mathcal{H}}, s\beta_t) \longrightarrow J_1 \text{ as } s.t \to 0.$$

and

$$J(s,t)^2 = \int_{\Sigma} \left(J_1^2 - 2s^2 |\beta_t|^2 + o(t)\right)^2$$

for small s, t we will have

$$J(s,t) < J_1 \qquad \boxtimes.$$

(For more details see Donaldson's paper.) Lemma 3

Riemann Hilbert Correspondence

Theorem: Σ: manifold

$$\frac{\left\{ (P,A) : P \text{ is a principal } G\text{-bundle and } A \text{ is a flat connection} \right\}}{\sim} \longleftrightarrow \frac{\left\{ \rho : \pi_1(\Sigma) \to G \text{ reps} \right\}}{\sim (\text{Conjugation})}$$

Sketch of the proof:

($\Rightarrow$) Given (P, A) as above, define

$$\rho : \pi_1(\Sigma) \longrightarrow G$$
$$\gamma \longmapsto Hol_\gamma(A)$$

This is well defined as A is a flat connection & / Conjugation.

($\Leftarrow$) Given ρ as above,

define, $P = \tilde{\Sigma} \times_\rho G$, $\tilde{\Sigma} \to \Sigma$ Universal cover.

A: induced Connection of the trivial Connection on $\tilde{\Sigma} \times G$.

Examples: (1) $G = GL(n, \mathbb{C})$

$$\frac{\left\{ \text{Complex vector bundles with flat connections} \right\}}{\sim} \longleftrightarrow \frac{\left\{ \text{Reps } \rho : \pi_1(\Sigma) \to GL(r, \mathbb{C}) \right\}}{\sim}$$

(2) $G = U(n)$

$$\frac{\left\{ \text{Hermitian vector bundle with flat unitary connections} \right\}}{\sim} \longleftrightarrow \frac{\left\{ \text{Reps } \rho : \pi_1(\Sigma) \to U(n) \right\}}{\sim}$$

(3) $G = \mathbb{P}U(n) = \dfrac{U(n)}{U(1)}$

$$\frac{\left\{ \mathbb{P}U(n) \text{ bundles with flat } PU(n) \text{ connections} \right\}}{\sim} \longleftrightarrow \frac{\left\{ \text{Reps } \rho : \pi_1(\Sigma) \to \mathbb{P}U(n) \right\}}{\sim}$$ projective reps

Exercise: Given a Hermitian vector bundle E i.e. a $U(n)$ bundle, the induced $PU(n)$ bundle $E_{\mathbb{P}}$ has a flat $\mathbb{P}U(n)$ Connection iff E has a Unitary Connection A such that $F_A = \alpha I_E$ for some 2-form α.

Hint: $\mathrm{Lie}(PU(n)) = \mathfrak{u}(n)_0 =$ trace free skew Hermitian matrices.

$$\mathfrak{u}(n) \longrightarrow \mathfrak{u}(n)_0$$
$$B \longmapsto B - \frac{\mathrm{tr}(B)}{n} I$$