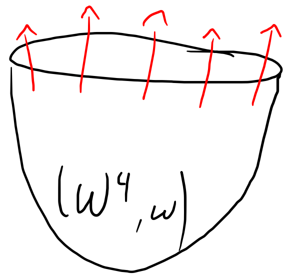
spinal open books & symp. fillings

(j. w/ S. Lisi & J. Van Horn-Morris)



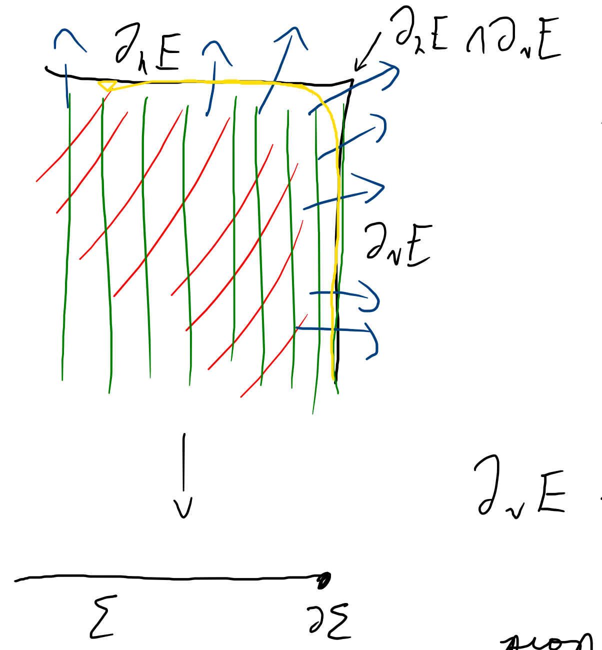
(M^3, ξ) = contact mfd ($\xi = \ker\alpha$ for $\alpha \in \Omega^1(M)$ s.t. $\alpha \wedge d\alpha > 0$)

$\partial W = M$: (W, ω) is a (strong) symp. filling of (M, ξ) if near ∂W, $\omega = d\lambda$ for some λ s.t. $\ker(\lambda|_{T(\partial W)}) = \xi$

$\underline{Q}$: Given (M, ξ), what are all symp. fillings of (M, ξ) up to sympl. deformation equivalence?

bordered Lefschetz fibrations

E = cpct oriented 4-mfd w/ bdry & corners s.t. $\partial E = \partial_h E \cup \partial_v E$ has 2 smooth faces ($\partial_h E \cap \partial_v E$ = "corner")



Σ := cpct oriented surface w/ bdry

$\Pi : E \longrightarrow \Sigma$ a submersion except at fin.-many interior crit. pts. modelled by $\Pi(z_1, z_2) = z_1^2 + z_2^2$ in local cpx coords compat. w/ orientations (equiv: $\Pi(z_1, z_2) = z_1 z_2$)

$\partial_v E = \Pi^{-1}(\partial\Sigma)$, $\quad \partial_h E = \bigcup_{z \in \Sigma} \partial(\Pi^{-1}(z))$

__prop__ (cf. Thurston + Gompf):

If $\partial\Sigma \neq \emptyset$ & $\partial_h E \neq 0$, then $\exists!$ (up to deformation) sympl. form ω on E s.t.

(1) $\omega = d\lambda$ near ∂E for a 1-form λ which is ctct on $\partial_h E$ & $\partial_v E$

(2) $\omega|_{\text{fiber}} > 0$

(3) Reeb vec. fld of $\lambda|_{\partial_h E}$ is tangent to $\partial(\text{fiber})$, respecting orientation. ☒

__cor__: $\Pi : E \longrightarrow \Sigma$ determines (after smoothing corner) a symp. filling of a ctct mfd ! up to def. equiv. ☒

structure at bdry:

$\Pi|_{\partial_v E} : \partial_v E \longrightarrow \partial\Sigma = S^1 \amalg \ldots \amalg S^1$ disjoint union of surface fibration over S^1

$\Pi|_{\partial_h E} : \partial_h E \longrightarrow \Sigma$ fibration w/ fibers $\partial(\Pi^{-1}(*)) \cong S^1 \amalg \ldots \amalg S^1$

$$\begin{tikzcd} \partial_h E \arrow[r] \arrow[dr] & \Sigma \\ & \Sigma' \arrow[u, "\text{cover}"'] \end{tikzcd}$$

$\Rightarrow$ factors thru some covering map $\Sigma' \longrightarrow \Sigma$ to give an S^1-fibr. $\partial_h E \longrightarrow \Sigma'$.

__defn__: A spinal open book decomp. of a closed 3-mfd M is a decomposition

$$M = \underset{\text{"paper"}}{M_P} \cup \underset{\text{"spine"}}{M_\Sigma} \quad \text{s.t.} \quad \partial M_P = \partial M_\Sigma \cong T^2 \sqcup \dots \sqcup T^2,$$

endowed with fibrations

$\pi_P : M_P \longrightarrow S^1$ (cpts of fibers = cpt oriented surfaces w/ nonempty bdry = "pages")

$\pi_\Sigma : M_\Sigma \longrightarrow \Sigma$ (cpts of Σ = " " " " = "vertebrae")

fibers $= S^1$

s.t. $\forall$ pages, $\partial(\text{page}) = \coprod (\text{fibers of } \pi_\Sigma \text{ in } \partial M_\Sigma)$

A ctct str. ξ is supported by the SOBD if $\xi = \ker\alpha$ s.t.
$d\alpha|_{\text{pages}} > 0$ & Reeb vec. fld. pos. tangent to $\pi_\Sigma^{-1}(*)$ on M_Σ.

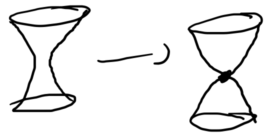
__ex__: For BLF $\pi : E \longrightarrow \mathbb{D}^2$, $\pi|_{\partial_h E}$ is trivial $\Rightarrow \partial_h E \cong \coprod (\mathbb{D}^2 \times S^1)$
& $\pi|_{\partial_v E} : \partial_v E \longrightarrow S^1$ is trivial at $\partial(\partial_v E)$ ($\Leftarrow$ monodromy is trivial at ∂(fiber))
$\Rightarrow \partial E \cong \left(\begin{array}{c}\text{surface fibr.}\\ \downarrow \\ S^1\end{array}\right) \cup \coprod (\text{solid tori})$ = open book decomp.

__big thm__: If (M^3, ξ) is supported by a "Lefschetz-amenable" SOBD containing planar (i.e. genus 0) pages, then

$$\{\text{symp. fillings of } (M,\xi)\}/_{\text{def equiv}} \xleftrightarrow{1:1} \{\text{BLFs filling the given SOBD}\}/_{\text{diffeo}}.$$

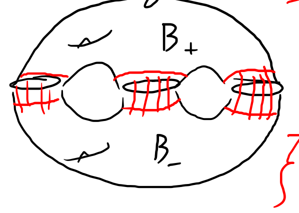
__ex 1__ (originally McDuff / Lisca): (M, ξ) supp. by an OBD w/ pages = annuli

& monodromy = (Dehn twist)^k for $k \in \mathbb{Z}$, then

- if $k < 0$, $\nexists$ fillings
- if $k \geq 0$, all fillings come from BLFs over $\mathbb{D}^2$ w/ exactly k singular fibers of form  (+ arb. many of form )

$\Rightarrow \exists$ only one filling up to sympl. def. & blowup.

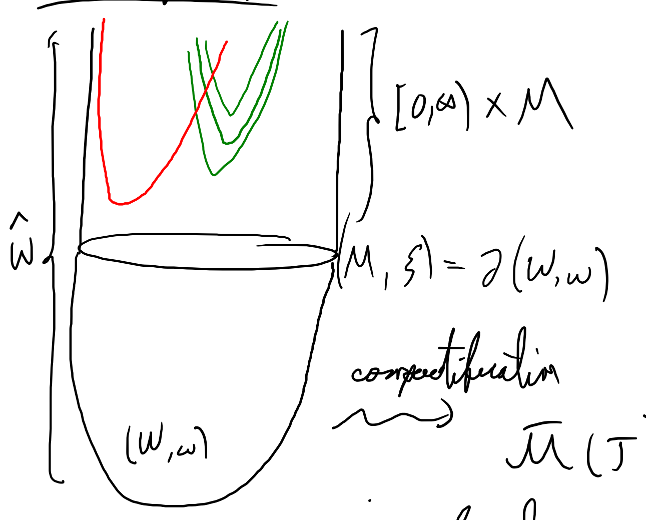
__ex 2__: $\pi : M \longrightarrow B$ a principal S^1-bndl over closed surface B, S^1-inv. ctct str. ξ_Γ det'd by a multicurve $\Gamma := \{x \in B \mid T(\pi^{-1}(x)) \subseteq \xi_\Gamma\} \subseteq B$.

e.g. $S^1 \times$ 

Γ } M_Σ
} M_P (pages $\cong S^1 \times [-1,1]$)
} $M_\Sigma \cong S^1 \times (B \setminus \text{nbhd}(\Gamma))$

__cor__: If B orientable, (M, ξ_Γ) is fillable $\Leftrightarrow B \setminus \Gamma$ has exactly 2 cpts, both homeomorphic, $e(\pi) \geq 0$. Its fillings are then ! up to def./blowup, given by BLFs over B_+ with fibers $S^1 \times [-1,1]$.

idea of pf:



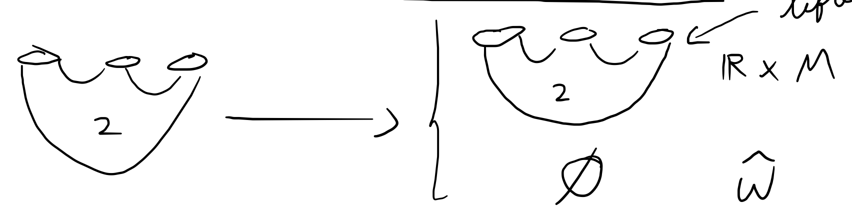
For suitable choice of compat. a.c.s J on $\widehat{W}$, planar pages M_P lift to index 2 J-hol. curves in $[0,\infty)\times M$ w/ pos. ends to Reeb orbits in M_Σ ⇝ 2-dim. moduli sp. $\mathcal{M}(J)$ of curves foliating an open dense subset of $\widehat{W}$.

compactification ⇝ $\overline{\mathcal{M}}(J)$ s.t. $\forall\, p\in\widehat{W}$, $\exists!\ u_p\in\overline{\mathcal{M}}(J)$ whose main level passes thru p ⇝ map $\Pi:\widehat{W}\longrightarrow\overline{\mathcal{M}}(J)$, $p\longmapsto u_p$.

$\exists$ also "J-hol. vertebrae" $\Sigma_1,\dots,\Sigma_r\subset\widehat{W}$ w/ pos ends, which intersect all $u\in\overline{\mathcal{M}}(J)$ positively $\Rightarrow$ $\Pi|_{\Sigma_i}:\Sigma_i\longrightarrow\overline{\mathcal{M}}(J)$ are branched covers

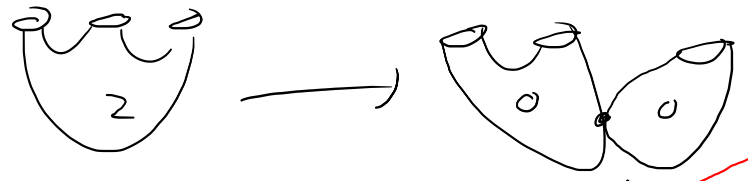
types of degeneration in $\overline{\mathcal{M}}(J)\setminus\mathcal{M}(J)$

(0)



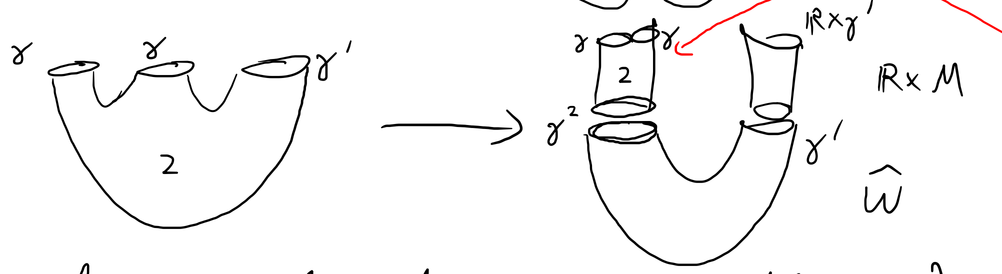
"$\partial\overline{\mathcal{M}}(J)$"

(1)



= "Lefschetz singular fibers"

(2)

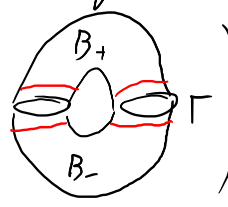


branched cover (2:1) of $\mathbb{R}\times\gamma$

"exotic fiber"

observation: Exotic fibers occur (finitely many) iff $\Sigma_i\xrightarrow{\Pi}\overline{\mathcal{M}}(J)$ has branch pts. Lefschetz-amenability = this map cannot have branch pts.

ex w/ exotic fibers: $(\mathbb{T}^3,\xi_{std})/\mathbb{Z}_2 = \Big(S^1\times$



$\Gamma\Big)$ / involution sending $B_+\xrightarrow{\text{or.-rev.}}B_-$ & reversing fibers

= nontrivial S^1-princ. bdl over Klein bottle

$M_\Sigma\cong S^1\times\mathbb{A}$ (annulus), $M_P\cong S^1\times\mathbb{A}$ $\Rightarrow$ $\partial\overline{\mathcal{M}}(J)\cong S^1$,

$\Pi|_\Sigma:\mathbb{A}\xrightarrow{2:1}\overline{\mathcal{M}}$ branched cover $\Rightarrow$ $\overline{\mathcal{M}}\cong\mathbb{D}^2$, $\exists$ 1 branch pt.

$\Rightarrow$ all fillings have completion w/ $\Pi:\widehat{W}\to\mathbb{D}^2$ having ≥ 1 exotic fiber, reg. fibers $=\mathbb{A}$.

Q: How does one classify the top. of such fibr. w/ exotic fibers?