HIGH DIMENSIONAL APPLICATIONS OF SPINAL OPEN BOOK DECOMPOSITIONS

j.w. Jonathan Bowden, Agustin Moreno

1) Intro / Motivation

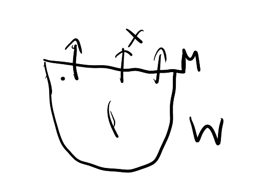
- Contact structures on M^{2n+1}:

  hyperplane fields $\xi = \ker(\alpha)$, $\alpha \in \Omega^1(M)$ s.t. $\alpha \wedge d\alpha^n > 0$

  Strong symplectic filling (W^{2n}, ω) of $(M = \partial W, \xi)$

  if, near M, $\omega = d\lambda$, $\xi = \ker(\lambda|_M)$

  ($\Longleftrightarrow$ $\exists X$ Liouville ($\mathcal{L}_X \omega = \omega$) near M, transverse to M outwards pointing)



  Q1 Given (M,ξ), is it fillable?
  Q2 If so, "how many" fillings?

- Explicit construction of high dim contact manifolds:

  Bourgeois '02: (M^{2n-1}, ξ) with supporting open book decomposition $\rightsquigarrow$ η contact on $M \times T^2$

  Massot-Niederkrüger-Wendl '13: (M,ξ) weakly fillable $\overset{\forall OBD}{\Longrightarrow}$ $(M\times T^2, \eta)$ weakly fillable

  Lisi-Marinković-Niederkrüger '19: • (M,ξ) subcr. Stein fillable $\overset{\exists OBD}{\Longrightarrow}$ $(M\times T^2, \eta)$ Stein fillable
  • examples of (M,ξ) OT with $(M\times T^2, \eta)$ tight (for some OBD)

Thm (Bowden-G-Moreno) $\dim M = 3$
- $(M\times T^2, \eta)$ is tight (independently of ξ, OBD)
- page Σ^2 planar ($g(\Sigma) = 0$), $(M\times T^2, \eta)$ strongly fillable $\Longrightarrow$ monodromy ϕ is in commutator subgroup of $MCG(\Sigma, \partial)$

Same techniques give:

Thm (BGM)
- Any sympl aspherical strong filling of (S^*T^n, ξ_{std}) is diffeo to D^*T^n } independently proven by Geiges-Kwon-Zehmisch
- If open book $(\underset{\text{page}}{D^*S^n}, \underset{\text{monodromy}}{\tau = \text{Dehn Seidel Twist}})$ for (S^{2n+1}, ξ_{std}) then $(S^{2n+1}\times T^2, \eta)$ is not strongly fillable (but it is weakly fillable)

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2) Bourgeois construction and spinal open book dec

- Abstract contact open book decomposition

  $(\Sigma^{2n}, \omega = d\lambda)$ Liouville domain: page

  $\phi \in Sympl(\Sigma, \partial)$ exact ($\phi^*\lambda - \lambda$ exact): monodromy

  $(B = \partial\Sigma, \alpha_B = \lambda|_B)$ binding

$$OBD(\Sigma,\phi) = \Big(\underbrace{B}_{(B,\alpha_B)} \times \underset{r,\theta}{D^2}, \alpha_B + r^2 d\theta\Big) \bigcup_{B\times S^1} \Big(\Sigma_\phi, \pi^* d\theta + \lambda_{fib}\Big)$$

$\Sigma_\phi = \Sigma \times \mathbb{R} / (p,\theta) \sim (\phi^{-1}(p), \theta+1)$, with fiber (Σ, λ)

$$\begin{array}{ccc} B\times D^2 & & \Sigma_\phi \\ \downarrow p & & \downarrow \pi \\ (D^2, r^2 d\theta) & & (\underset{\theta}{S^1}, d\theta) \end{array}$$

ctct form denoted by $\alpha_{\Sigma,\phi}$

Giroux '02: $\forall (M^{2n+1}, \xi)$ $\exists (\Sigma^{2n}, \phi)$ s.t. $(M,\xi) \overset{ctct}{\simeq} OBD(\Sigma,\phi)$

Notice: $\exists \psi: OBD(\Sigma,\phi) \longrightarrow D^2 \subset \mathbb{R}^2$ s.t.
- $\psi = p$ on $B\times D^2$
- $\psi = \pi$ on Σ_ϕ (we see $S^1 \subset D^2$)

Bourgeois '02: If $(M,\xi) = OBD(\Sigma^{2n}, \phi)$, the following is contact on $M\times T^2$:

$$\beta = \alpha_{\Sigma,\phi} + \psi_1 dx - \psi_2 dy$$

with $(x,y) \in T^2$, $\psi = (\psi_1, \psi_2)$

Denote $BO(\Sigma,\phi) = (M\times T^2, \beta)$.

- Supporting spinal open book decomposition (SOBD)

  Lisi-Van Horn-Morris-Wendl '15 in dim 3

  Moreno '18 in dim $2n+1$

$$BO(\Sigma,\phi) = \Big(\underbrace{B}_{(B,\alpha_B)} \times \underbrace{D^*T^2}_{D^*T^2}, \alpha_B + \lambda_{D^*T^2}\Big) \cup \Big(\underbrace{\Sigma_\phi}_{(\Sigma,\lambda)} \times T^2, \lambda_{fib} + \cos\theta dx - \sin\theta dy\Big)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ (D^*T^2, \lambda_{D^*T^2}) & & (\underset{\theta}{S^1} \times \underset{x,y}{T^2}, \cos\theta dx - \sin\theta dy) \\ & & = \\ & & (S^*T^2, \alpha_{std}) \end{array}$$

$B\times D^*T^2$ spine — $\Sigma_\phi \times T^2$ paper

D^*T^2 vertebrae — Σ pages

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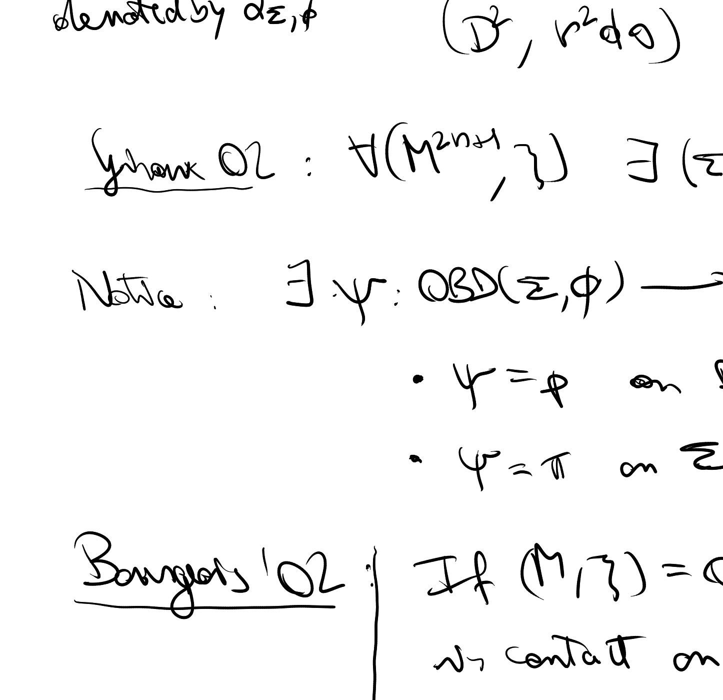
3) Pf of "Σ^2 planar, $BO(\Sigma,\phi)$ str fill $\Longrightarrow$ ϕ in commutator subgp"

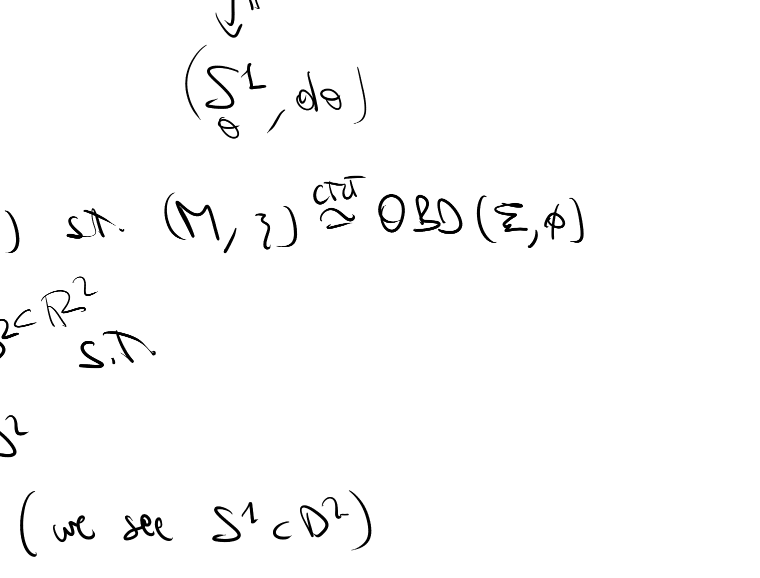
(W^6, ω) str filling of $BO(\Sigma,\phi)^5$

for well chosen J, SOBD on $BO(\Sigma,\phi)$ gives a hol fol by punctured spheres on $\mathbb{R} \times BO(\Sigma,\phi)$ = symplectization



induces $\rightsquigarrow$

extend the J generically in W, get a mod space $\mathcal{M}$ of hol copies of Σ in $\hat{W}$ = completion of W



⚠ For rest of pf: W sympl aspherical

Properties: (1) uniqueness lemma for cyl end over $\Sigma_\phi \times T^2$

(2) compactness $\overline{\mathcal{M}} = \mathcal{M}$ $\longleftarrow$ Key point

$(\mathcal{M}^{tr}_*)^6$ marked mod space $\xrightarrow{ev}$ W^{tr}

$\downarrow F$ forget ↖ fibration with fiber Σ

$(\mathcal{M}^{tr})^4$

$\rightsquigarrow$

$$\begin{array}{ccc} (\partial_v \mathcal{M}^{tr}_*)^5 & \xrightarrow[\sim]{ev} & \Sigma_\phi \times T^2 \ (\subset \partial W^{tr}) \\ \downarrow F & & \downarrow \pi_{SOBD} \\ (\partial \mathcal{M}^{tr})^3 & \xrightarrow[\sim]{\text{Intersect}} & S^1 \times T^2 \ (= \partial D^*T^2) \\ \partial S = \gamma & \longrightarrow & S^1 \times \{pt\} \end{array}$$

$S \subset \mathcal{M}^{tr}$

Take $\{b\} \times D^*T^2 \subset B \times D^*T^2 \subset \partial W^{tr}$, each curve intersects it exactly once $\Rightarrow$ $I: \mathcal{M}^{tr} \longrightarrow \{b\} \times D^*T^2$ intersect map.

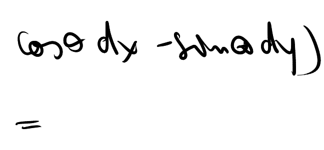
$S := I^{-1}(\{b\} \times D^2 \times \{pt\})$ surface in $\mathcal{M}^{tr}$ with bdry γ

$F|_S: F^{-1}(S) \to S$ fibr. with fiber Σ, monodromy at ∂ is ϕ

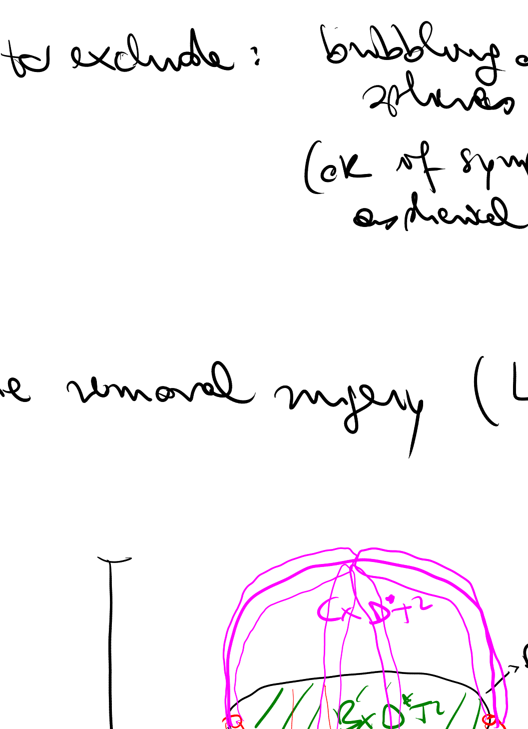
$\Longrightarrow$ ϕ in commutator subgp ∎

Idea for key pt:

need to exclude: bubbling of spheres (ok if sympl aspherical), multiple level degenerations (ok: fol property on $\mathbb{R} \times BO(\Sigma,\phi)$),



Spine removal surgery (L-VHM-W in dim 3)



$C = \bigcup_{i=1}^{|\partial\Sigma|} D^2$ cap for $B = \partial\Sigma$

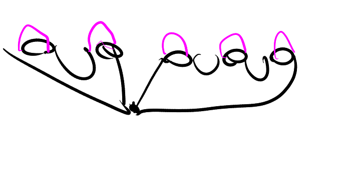
(so $\Sigma \cup C = S^2$)

$\partial W_{cap} = X$

$$\Sigma \cup C = S^2 \rightsquigarrow X = \underset{\text{sympl}}{S^2} \times \underset{\text{ctct}}{S^*T^2}$$
$$\downarrow$$
$$S^*T^2$$

Study: - $\mathcal{M}^S$ spheres from S^2 factor
- $\mathcal{M}^{cyl}$ cylinders from S^*T^2

$\Longrightarrow$ $\exists!$ homology class of S^2's in W_{cap}

$\Longrightarrow$ No  in W. ∎