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Leaf-wise intersections and Rabinowitz Floer homology.

Albers - Frauenfelder arXiv:0810.3845 J. Top. Anal.

Outline: • Recall notation and basics from Alex's talk.

- Definition of Leaf-wise intersection and perturbed Rabinowitz action functional.
- Invariance of associated homology and relation with RFH.
- Non-vanishing $\Rightarrow$ $\exists$ leaf-wise intersection

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Notation:

- $(\tilde{V}, \tilde{\lambda})$ a Liouville domain:
  - compact manifold $2n$-dim'l mfld with boundary
  - $\tilde{\lambda}$ 1-form, $\tilde{\omega} = d\tilde{\lambda}$ symplectic form
  - $\tilde{\lambda}|_{\partial\tilde{V}} =: \tilde{\alpha}$ is a contact form
  - Y vector field defined by $d\tilde{\lambda}(Y, \cdot) = \tilde{\lambda}$ pointing outward along $\partial\tilde{V}$
- (V, λ) completion of $(\tilde{V}, \tilde{\lambda})$:

$$V = \tilde{V} \cup_{0 \times \partial\tilde{V}} \left([0,\infty) \times \partial\tilde{V}\right)$$

$$\lambda = \begin{cases} \tilde{\lambda} & \text{on } \tilde{V} \\ e^{r} \cdot \tilde{\alpha} & \text{on } [0,\infty) \times \partial\tilde{V} \end{cases}$$

- $\Sigma \subset V$ is an exact convex hypersurface:
  - Σ $2n-1$ dim'l with $\lambda|_{\Sigma} =: \alpha$ contact
  - Σ is bounding: $V \setminus \Sigma$ is two connected components one with compact closure.

Recall: A Hamiltonian G is defining for Σ if $\Sigma = G^{-1}(0)$ and $X_G|_{\Sigma} = R_\alpha$.

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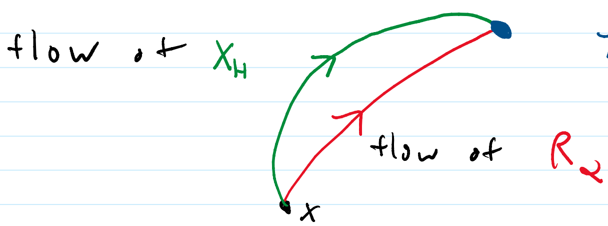
Leaf-wise intersections and theorem statements

- Σ is foliated by characteristic line bundle spanned by R_α. Denote the leaves by L_x, $x \in \Sigma$

Definition: Let $H \in C_c^\infty(V)$ and φ_H^t the associated Hamiltonian flow. We say $x \in \Sigma$ is a leaf-wise intersection point if $\varphi_H^t(x) \in L_x$ for some t.



$$\varphi_H^t(x) = \varphi_G^s(x)$$

- A "classical-type" result:

Recall the Hofer norm:

$H : S^1 \times V \to \mathbb{R}$ compactly supported

$$\|H\|_+ := \int_0^1 \max_{x \in V} H(t,x)\,dt \qquad \|H\|_- := -\int_0^1 \min_{x \in V} H(t,x)\,dt$$

$$\|H\| = \|H\|_+ + \|H\|_-$$

For $\varphi \in Ham_c(V,\omega)$

$$\|\varphi\| = \inf\{\|H\| \,/\, \varphi = \varphi_H^1\}$$

Let $\mathcal{R}(\Sigma,\alpha)$ denote the minimal period of a Reeb orbit that is contractible in V.

$\mathcal{R}(\Sigma,\alpha) = \infty$ if $\nexists$ contractible Reeb orbit.

Theorem (Albers-Frauenfelder) If $\|\varphi\| < \mathcal{R}(\Sigma,\alpha)$, then $\exists$ a leafwise intersection point for φ.

- However, we will focus on something stronger.

Recall from Alex's talk the definition of $RFH(V,\Sigma)$:

$$\mathcal{A}^F(u,\eta) = \int_0^1 \gamma^* v\,dt - \eta \int_0^1 F(v(t))\,dt$$

Recall from (Alex) talk the definition of $RFH(V, \Sigma)$.

$$\mathcal{A}^F(u, \eta) = \int_0^1 \lambda^* v\,dt - \eta \int_0^1 F(v(t))\,dt$$

gives a Morse-Bott homology on a free loop space. Differential counts (unbroken) Morse flowlines on critical submanifolds along with "cascades".

Theorem (Albers-Frauenfelder): If $RFH(V, \Sigma) \neq 0$, then every $f \in Ham_c(V, \omega)$ has a leaf-wise intersection point in Σ.

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Critical points of perturbed action functional give leaf-wise intersections

- Choose a $G \in C^\infty(M)$ that is defining for Σ that is locally constant outside of
$$U_\delta := \{ x \in V \mid |G(x)| < \delta \}$$
- Fix $\rho : S^1 \to \mathbb{R}$ with $\int_0^1 \rho \, dt = 1$, $\operatorname{supp}(\rho) \subset (0, \tfrac{1}{2})$
- Set $F(t,x) = \rho(t)\, G(x)$
$$\Rightarrow \quad X_F = \rho(t)\, X_G = \rho(t)\, R_\alpha$$

Lemma:
$$\|\varphi\| = |||\varphi||| := \inf\left\{ \|H\| \;\middle|\; \varphi = \varphi_H^1,\ H(t,\cdot) = 0 \ \forall t \in [0, \tfrac{1}{2}] \right\}$$

Sketch: $\|\varphi\| \le |||\varphi|||$ obvious.

Reverse inequality: set $H^r(t,x) := r'(t)\, H(r(t), x)$ for monotone map $r : [0,1] \to [0,1]$ with $r(\tfrac{1}{2}) = 0$, $r(1) = 1$
$$\Rightarrow \quad \|H^r\| = \|H\|, \quad \varphi^1_{H^r} = \varphi^1_H$$
//

- From now on, assume $H(t,\cdot) = 0$, $t \in [0, \tfrac{1}{2}]$
- The perturbed Rabinowitz action functional is
$$\mathcal{A}_H^F(u,\eta) := \int_0^1 u^*\lambda - \int_0^1 H(t, u(t))\, dt - \eta \int_0^1 F(t, u(t))\, dt$$
where $u \in C^\infty(S^1, M)$ and $\eta \in \mathbb{R}$.
- Critical points:
$$\begin{cases} \partial_t u = X_H(t,u) + \eta\, X_F(t,u) \\ \int_0^1 F(t,u)\, dt = 0 \end{cases}$$

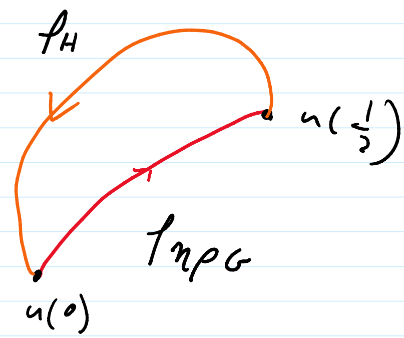
Proposition: Let $(u,\eta) \in \operatorname{Crit} \mathcal{A}_H^F$. Then $x = u(\tfrac{1}{2})$

Proposition: Let $(u,\eta)\in \mathrm{Crit}\, A_H^F$. Then $x=u(\tfrac{1}{2})$ satisfies $\phi_H(x)\in L_x$.

Proof:

Claim: $u(t)\in\Sigma \quad t\in[0,\tfrac{1}{2}]$

$$\frac{d}{dt}G(u(t)) = dG(u(t))[\partial_t u] = dG(u(t))[X_H+\eta X_F] = dG[\eta X_F] = dG[\eta\rho X_G] = 0$$



$\Rightarrow$ $G = c$ on $u([0,\tfrac{1}{2}])$

$$0 = \int_0^1 F(u(t)) = \int_0^1 \rho\, G(u(t)) = c$$

$G=0$ on $u([0,\tfrac{1}{2}])$ claim //

$\Rightarrow$ $u(0)=u(1)$, $u(\tfrac{1}{2})\in\Sigma$

- Since $X_F=\rho X_G=\rho R_2$, we have $u(0)=u(1)\in L_{u(\frac{1}{2})}$
- $X_F=0$ on $t\in(\tfrac{1}{2},1)$, so $\partial_t u = X_H$ here

$\Rightarrow$ $u(1)=\phi_H'(u(\tfrac{1}{2}))$

$\square$

Gradient flow of action functional.

- In what follows, $H = H_s$ for $s \in \mathbb{R}$ with
  - $H_s = H_-$ $s \leq -1$
  - $H_s = H_+$ $s \geq 1$
  - $H_s(t, \cdot) = 0$ $t \in [0, \frac{1}{2}]$
  - H_s compactly supported uniformly in s
- Recall from Alex's talk the type of a.c.s. needed:
  - $J_t(n)$, $(t,n) \in S^1 \times \mathbb{R}$
    - $\sup_n \|J_t(n)\|_{C^l} < \infty$
    - independent of (t,n) on $[R, \infty) \times \partial\tilde{V}$ $R >> 0$
  - ω compatible
  - $J \frac{\partial}{\partial r} = R_\alpha$
- Define a metric on $T_{(u,\eta)} \mathcal{L} \times \mathbb{R}$, $\mathcal{L} := C^\infty(S^1, V)_{contractible}$, via

$$g_J((v_1, \eta_1), (v_2, \eta_2)) := \int_0^1 \omega(v_1, J_t(\eta) v_2)\, dt + \eta_1 \eta_2$$

Definition: Gradient flow line for $\mathcal{A}_H^F$ is a map $w = (u, \eta) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R})$ satisfying

$$\partial_s w(s) + \nabla_s \mathcal{A}_H^F(w(s)) = 0$$

where ∇_s is with respect to the g_J.

- Gives a Floer system:

$$\begin{cases} \partial_s u + J_s(\eta)\left[\partial_t u - X_{H_s}(t,u) - \eta X_F(t,u)\right] = 0 \\ \partial_s \eta - \int_0^1 F(t,u)\, dt = 0 \end{cases}$$

- Energy: $E(w) = \int_{-\infty}^{\infty} \|\partial_s w\|_{g_J}^2 ds$
- Energy inequality:

$$E(w) \leq \mathcal{A}_{H_-}^F(w_-) - \mathcal{A}_{H_+}^F(w_+) + \int_{-\infty}^{\infty} \|\partial_s H_s\|_- ds$$

with equality if $\partial_s H = 0$

with equality if $\partial_s H = 0$

Lemma: Let w be a gradient flow line of $\nabla_s \mathcal{A}_H^F$. Then
$$|\mathcal{A}_H^F(w(s_0))| \le \max\{\mathcal{A}_{H_-}^F(w_-), -\mathcal{A}_{H_+}^F(w_+)\} + \int_{-\infty}^{\infty} \|\partial_s H_s\|_- \, ds$$
$\forall s_0 \in \mathbb{R}$.

Follows from positivity of energy.

- With this in mind, we have the following:

Theorem: Let $w_n = (u_n, \eta_n)$ be a sequence of flow lines for which there exists $a < b$ such that
$$a \le \mathcal{A}_H^F(w_n(s)) \le b \quad \forall s \in \mathbb{R}.$$
Then for every reparameterization sequence $\sigma_n \in \mathbb{R}$ the sequence $w_n(\cdot + \sigma_n)$ has a subsequence which converges in $C^\infty_{loc}(\mathbb{R}, \mathcal{L} \times \mathbb{R})$.

Sketch: Follows from L^∞ bounds on

(1) u_n (2) η_n (3) derivatives of u_n

(1) is due to convexity of V at ∞

(2) will be given below ☆

(3) follows from exactness of V $\Rightarrow$ no spheres and hence derivatives don't blow up

After obtaining bounds, one bootstraps using gradient flow equation to get C^∞_{loc} convergence. //

☆ **Proposition** (L^∞ bound on η): Given $w_-, w_+ \in \mathrm{Crit}\, \mathcal{A}_H^F$ there exists a constant $\mathcal{K} = \mathcal{K}(w_-, w_+)$ such that every gradient flow line $w = (u, \eta)$ with $\lim_{s \to \pm\infty} = w_\pm$ satisfies
$$\|\eta\|_{L^\infty(\mathbb{R})} \le \mathcal{K}.$$

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- Let $\mathcal{H} := \{ H \in C_c^\infty(S^1 \times V) \mid H(t,\cdot) = 0 \ \forall t \in [0,\tfrac{1}{2}] \}$

Recall that for the proof that a critical point of $\mathcal{A}_H^F$ gives a leaf-wise intersection, it is crucial that $H \in \mathcal{H}$.

Theorem: $\mathcal{A}_H^F$ is Morse for generic $H \in \mathcal{H}$.

Remark: To show this, we need surjectivity of $D^2 \mathcal{A}_H^F + h$ where h is a variation in $T\mathcal{H}$. Without restricting to $\mathcal{H}$, one can simply use h to get surjectivity. In this situation one also needs to use $D^2 \mathcal{A}_H^F$.

- Let
$$CF(\mathcal{A}_H^F) := \Big\{ \xi = \sum_{c \in \operatorname{crit} \mathcal{A}_H^F} \xi_c c \;\Big|\; \xi_c \in \mathbb{Z}_2, \; \# \{ c \mid \xi_c \neq 0, \ \mathcal{A}_H^c(c) \geq \kappa \} < \infty \Big\}$$

- Form moduli space
$$\mathcal{M}(c_-, c_+) := \Big\{ w \mid w \text{ is a gradient flow of } \mathcal{A}_H^F \; ; \; \lim_{s \to \pm\infty} w(s) = \pm c \Big\} \Big/ \mathbb{R}$$

- Let H be such that $\mathcal{A}_H^F$ is Morse.

- As in Alex's talk, transversality can be achieved for generic t and η-dependent J's.

- $\partial : CF(\mathcal{A}_H^F) \circlearrowleft \qquad \partial(c) = \sum_d n(c,d)\, d$

  where $n(c,d) = \# \mathcal{M}(c,d)_0$

- $\partial^2 = 0$ by compactness (as above) and usual gluing arguments

  set $HF(\mathcal{A}_H^F) := H(CF(\mathcal{A}_H^F), \partial)$

Theorem: If H is generic,

$HF(\mathcal{A}^F) \cong$ [illegible]

Theorem: If H is generic,

$$HF(\mathcal{A}_H^F) \cong RFH(V, \xi).$$

Corollary: If $RFH(V, \xi) \neq 0$, then there always exists a leaf-wise intersection point for (V, ξ).

Proof of Corollary: There must be a critical point of $\mathcal{A}_H^F$ by the above isomorphism with $HF(\mathcal{A}_H^F)$. Critical points are leaf-wise intersections. $\square$

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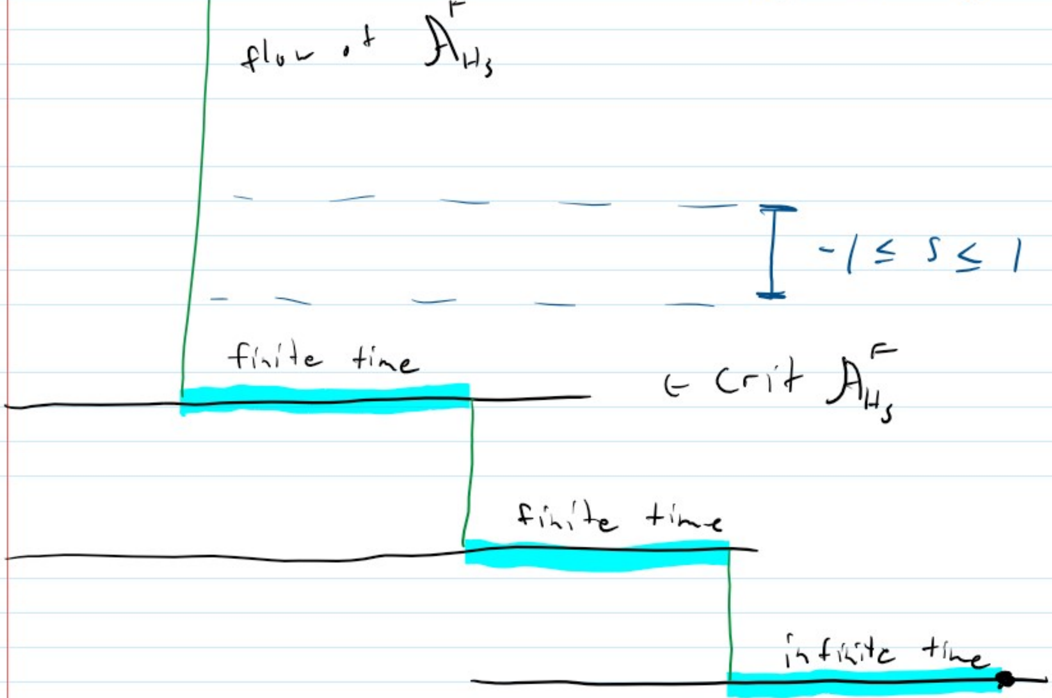
Sketch of isomorphism $HF(\mathcal{A}_H^F) \cong RFH(V, \Sigma)$.

Recall that $RFH(V, \Sigma)$ is defined as the Morse-Bott homology of $\mathcal{A}_0^F$:

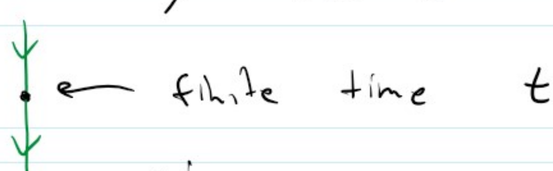
- Differential is given by counting isolated gradient flow lines of $\mathcal{A}_0^F$ with cascades.
- Continuation map $HF(\mathcal{A}_H^F) \to RFH(V, \Sigma)$ counts (unbroken):

$H_- = H_{-1} = H$

$H_+ = H_1 = 0$



Remark: More than one cascade in region $s<0$ is ruled out by "Morseness"



gives one parameter family by changing t