Embedded Contact Homology

last time: M connected, closed, oriented 3-mfd + pointed Heegard diagram,
acs on $\mathbb{R}\times M$ $\Rightarrow$ Heegaard-Floer homology $\widehat{HF}(M)$

1. Embedded Contact Homology

M like above, with (global) contact form α, ξ = ker α contact structure

R Reeb vf   idea: use periodic orbits of R to construct homology theory

Def: · A Reeb orbit $\gamma: S^1 = \mathbb{R}/T\mathbb{Z} \to M$ ($\dot\gamma(t) = R(\gamma(t))$) is nondegenerate

if $d\varphi_T^R|_{\xi_{\gamma(0)}}: \xi_{\gamma(0)} \to \xi_{\gamma(0)}$ does not have 1 as an eigenvalue

· α nondegenerate if all periodic Reeb orbits are nondeg.

· α Morse-Bott if all per. orbits are nondeg. ($\Rightarrow$ isolated)

or belong to an S^1-family and are nondeg. in the normal direction.

Assumption: α nondegenerate

$\Rightarrow$ $d\varphi_T^R|_\xi$ linear symplectic map in $(\xi_{\gamma(0)}, d\alpha)$

$\Rightarrow$ has eigenvalues $\{\lambda, \lambda^{-1}\}$

Def: A Reeb orbit is · hyperbolic if $\lambda, \lambda^{-1} \in \mathbb{R}$

· elliptic if $\lambda, \lambda^{-1} \in S^1$

$\mathcal{P} := \{$ simple Reeb orbits $\}$

Def: $ECC(M,\alpha)$ generated over $\mathbb{Z}_2$ by finite orbit sets $\gamma = \{(\gamma_i, m_i)\}$ with

· $\gamma_i \in \mathcal{P}$, $\gamma_i \neq \gamma_j$ for $i \neq j$ · $m_i \in \mathbb{N}$

· $\gamma_i \in$ hyperbolic orbit $\Rightarrow$ $m_i = 1$

notation: $\gamma = \prod_i \gamma_i^{m_i}$ with $\gamma_i^2 = 0$ if γ_i hyperbolic

$\emptyset$ is allowed and will be denoted by 1

Every orbit sets defines a homology class: $[\gamma] = \sum_i m_i [\gamma_i] \in H_1(M)$

symplectisation $\mathbb{R}\times M$ with an admissible acs J, i.e.

· J is $\mathbb{R}$-invariant

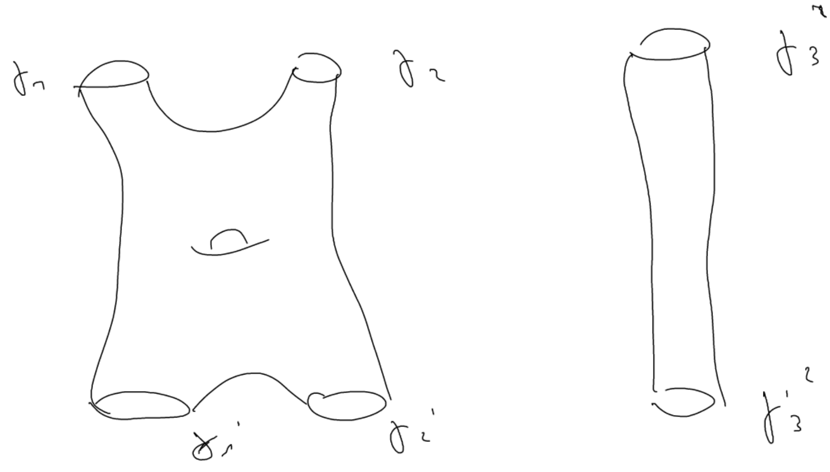
- $J|_\xi$ is $d\alpha$-compatible
- $J(\partial_a) = R$, $J(R) = -\partial_a$

$\gamma = \{(\gamma_i, m_i)\}$, $\gamma' = \{(\gamma_i', m_i')\}$ with $[\gamma] = [\gamma'] \in H_1(M)$

Def: $\mathcal{M}_J(\gamma, \gamma')$ contains J-hol. maps

$$u : (\dot\Sigma, j) \longrightarrow (\mathbb{R}\times M, J) \quad \text{mod rep. s.t.}$$

1. (Σ, j) closed Riemann surface, $\dot\Sigma := \Sigma \setminus \{x_1, \dots, x_n\}$
2. nbhds of the punctures are mapped asymptotically to cylinders over Reeb orbits.
3. at positive end of $\mathbb{R}\times M$: u asymptotic to $\mathbb{R}\times\gamma_i$ with multiplicity m_i
4. analogously for (γ_i', m_i') at negative end



analysis: $J_{reg} \subset J$ is of 2nd Baire category.

ECH-index $I(u)$

Lemma: $J \in J_{reg}$, $u \in \mathcal{M}_J(\gamma, \gamma')$

1. If $I(u) = 0$ then u is a disjoint union of standard covers of cylinders over simple Reeb orbits. Such u's are called connectors
2. If $I(u) = 1$ resp. $I(u) = 2$ $\Rightarrow$ u is a disjoint union of a connector and an embedding u' with $I(u') = \operatorname{ind}(u') = 1$

resp. $I(u') = \mathrm{ind}(u') = 2$

$\hat{\mathcal{M}}_J(\gamma, \gamma')$: we identify curves that differ by connectors and mod out the $\mathbb{R}$-action.

Def: $J \in J_{reg}$ $\quad \partial : ECC(M, \alpha, J) \longrightarrow ECC(M, \alpha, J)$

$$\gamma \longmapsto \sum_{\gamma'} |\hat{\mathcal{M}}_J^{I=1}(\gamma, \gamma')| \, \gamma'$$

Theorem [Taubes, Hutchings]: $\partial^2 = 0$

Def: $ECH(M, \alpha, J) := H_*(ECC(M, \alpha, J), \partial)$ <u>embedded contact homology</u>

Theorem: [Taubes] $ECH(M, \alpha, J)$ does not depend on α, J, ξ and only on M $\quad \rightarrow$ notation: $ECH(M)$

Let $z \in \mathbb{R} \times M$ be a generic point and $\hat{\mathcal{M}}_J(\gamma, \gamma', z)$ pointed moduli space of curves passing through z

Def: U-map: $\quad U_z : ECC(M, \alpha) \longrightarrow ECC(M, \alpha)$

$$U_z \gamma = \sum_{\gamma'} |\hat{\mathcal{M}}_J^{I=2}(\gamma, \gamma', z)| \, \gamma'$$

U is a chain map $\rightsquigarrow$ $\widehat{ECC}(M, \alpha)$ cone of U_z

$\rightarrow \widehat{ECH}(M)$

$C(U)$: $ECC^* \oplus ECC^{*-1} \longleftrightarrow ECC^{*+1} \oplus ECC^{*} \ldots$

$$d_c = \begin{pmatrix} \partial & 0 \\ U & \partial \end{pmatrix}$$

goal: $\widehat{ECH}(M) \cong \widehat{HF}(M)$

2. ECH for mflds with torus boundary

N 3-dim mfld with $\partial N \cong T^2$, α contact form on N s.t. ∂N is a negative Morse-Bott torus.

α Morse-Bott, $\{v_1, v_2\}$ oriented basis for ξ at $p \in \partial N$ s.t. $v_1 \pitchfork T\partial N$ and $v_2 \in T(\partial N)$ $\rightarrow$ $d\varphi_{\xi}^{R} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ $\quad a < 0$

can perturb α by a Morse fctn $\rightarrow$ get a maximum (a hyperbolic orbit h) and minimum (an elliptic orbit e).

$ECC(N, \alpha)$ gen. over $\mathbb{F}_2$ by orbit sets in $\mathrm{int}(N)$, h, e

the differential counts "very nice Morse-Bott buildings"

$ECC^e(N,\alpha)$ gen. by closed Reeb orbits in int(N) and e
is subcomplex of $ECC(N,\alpha)$

recall notation $\prod_i \gamma_i^{m_i}$. $\langle e-1\rangle$ ideal generated by $e-1$

Def: $ECC(N,\partial N,\alpha) := ECC^e(N,\alpha)/_{\langle e-1\rangle}$

$\widehat{ECC}(N,\partial N,\alpha) := ECC(N,\alpha)/_{\langle e-1\rangle}$

$A \in H_1(N) \rightsquigarrow ECC_A(N,\alpha)$ gen. by orbits sets gen. A

$$ECC_A(N,\alpha) \longrightarrow ECC_{A+[e]}(N,\alpha)$$
$$\gamma \longmapsto e\gamma$$

relative version: $\bar{A} \in H_1(N)/_{\langle [e]\rangle}$ $ECC_{\bar{A}}(N,\partial N,\alpha)$ subcomplex of $ECC(N,\partial N,\alpha)$

Lemma: Suppose $[e] \neq 0 \in H_1(N)$

(*)
$$ECH_{\bar{A}}(N,\partial N,\alpha) = \varinjlim \{ECH^e_{A+k[e]}(N,\alpha)\}_{k\in\mathbb{N}}$$
$$\widehat{ECH}_{\bar{A}}(N,\partial N,\alpha) = \varinjlim \{ECH^e_{A+k[e]}(N,\alpha)\}_{k\in\mathbb{N}}$$

Theorem [Colin, Ghiggini, Honda]

N^3 mfd with torus boundary and ∂N negative Morse-Bott torus

M Dehn filling of N along e and $\exists\ \varphi \in H^1(N)$ s.t.
$\varphi([\gamma]) \geq 0\ \forall$ closed Reeb orbits γ and $\varphi([e]) > 0$. $\Rightarrow$
$\exists$ isomorphisms

$$\sigma_* : ECH(N,\partial N,\alpha) \longrightarrow ECH(M)$$
$$\hat{\sigma}_* : \widehat{ECH}(N,\partial N,\alpha) \longrightarrow \widehat{ECH}(M)$$

s.t. the following commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & \widehat{ECH}(N,\partial N) \arrow[r] \arrow[d, "\hat{\sigma}_*"] & ECH(N,\partial N) \arrow[r] \arrow[d, "\sigma_*"] & ECH(N,\partial N) \arrow[r] \arrow[d, "\sigma_*"] & \cdots \\
\cdots \arrow[r] & \widehat{ECH}(M) \arrow[r] & ECH(M) \arrow[r, "U"] & ECH(M) \arrow[r] & \cdots
\end{tikzcd}$$

Connection to OBD

recall: An abstract open book is a pair (S,ϕ). S oriented compact
surface with boundary, $\phi: S \to S$ monodromy, i.e. $\phi: S \to S$

diffeom. with $\phi = id$ on $Op(\partial S)$

assume ∂S is connected

$$M \cong N \cup_\phi S^1 \times D^2$$

mapping torus $(S \times [0,1])/_\sim$ for ϕ

the core of $S^1 \times D^2$ is the binding B of an open book decomposition.

$\rightsquigarrow$ $N \cong M \setminus \{\text{tubular nbhd of } B\}$

$\rightarrow$ N is a 3-mfd with torus boundary

one can construct a contact form α on M which is supported by the OBD (i.e. $\alpha > 0$ on B, $d\alpha$ pos. area form on the pages).

$e_1^R = \emptyset$ in N and ∂N is a negative Morse-Bott torus.

$$ECH_e(N,\alpha) := \bigoplus_{A \cdot [S] = e} ECH_A(N,\alpha)$$

$A \in H_1(N)$

$[S] \in H_2(N, \partial N)$ rel. homology class of a page

Apply Lemma (*):

$$\widehat{ECH}(N, \partial N, \alpha) = \varinjlim ECH_e(N,\alpha)$$

$$\widehat{ECH}(M) \quad (=)$$

$\uparrow$ def. by an OBD