Differential Geometry III
Gauge Theory

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1 Covering theory

Here are some good references for covering theory: Jänich [Jän05, Kapitel 9], Fulton [Ful95, Parts VI and VII], May [May99, §1-§4], and Hatcher [Hat02, §1].

1.1 Covering maps

Definition 1.1. Let X, B be topological spaces. A **covering map** is a continuous map $p\colon X \to B$ such that: for every $b \in B$ there are an open neighborhood U of $b \in B$, a discrete space D, and a homeomorphism $\tau\colon p^{-1}(U) \to U \times D$ such that

$$\mathrm{pr}_1 \circ \tau = p|_{p^{-1}(U)}.$$

X is a **covering space** of B if there is a covering map $p\colon X \to B$. •

Example 1.2. Let X be a topological space. Let D be a discrete space. The map $\mathrm{pr}_1\colon X \times D \to X$ is a covering map. ♠

Example 1.3. The projection maps $p\colon \mathbf{R} \to S^1 := \mathbf{R}/2\pi Z$ and $q\colon S^n \to \mathbf{R}P^n := S^n/\{\pm 1\}$ are covering maps. ♠

Example 1.4. For $k \in \mathbf{N}$ the map $p_k\colon S^1 \to S^1$ defined by $p_k([x]) = [kx]$ is a covering map. ♠

Example 1.5. `some covering spaces of the figure eight` ♠

Example 1.6. The map $\exp\colon \mathbf{C} \to \mathbf{C}^\times$ is a covering map. ♠

Example 1.7. Let $d \in \mathbf{N}_0$. Let $P_d \subset \mathbf{C}[x]$ be the subset of polynomials with distinct roots. Set $R_d := \{(x,p) \in \mathbf{C} \times P_d : p(x) = 0\}$. The map $\mathrm{pr}_2\colon R_d \to P_d$ is a covering map. ♠

Example 1.8. Let $n \in \mathbf{N}$. Let $p \in \mathbf{N}$ and $q_1, \ldots, q_n \in \mathbf{Z}$ such that

$$\gcd(p, q_i) = 1.$$

Identify $\mathbf{R}^{2n} = \mathbf{C}^n$ and define $\phi \in \mathrm{SO}(2n)$ by

$$\phi(z_1, \ldots, z_n) := (e^{2\pi i q_1/p} z_1, \cdots, e^{2\pi i q_n/p} z_1).$$

By construction, the subgroup $\langle\phi\rangle \subset \mathrm{SO}(2n)$ is cyclic of order p and acts freely on S^{2n-1}. The **lens space** $L(p; q_1, \ldots, q_n)$ is the quotient

$$L(p; q_1, \ldots, q_n) := S^{2n-1}/\langle\phi\rangle.$$

The projection map $p\colon S^{2n-1} \to L(p; q_1, \ldots, q_n)$ is a covering map. ♠

Example 1.9. Denote by $\mathbf{H}$ the $\mathbf{R}$–algebra of the quaternions. Set $\mathrm{Sp}(1) \coloneqq \{q \in \mathrm{Sp}(1) : |q| = 1\}$. The map $\mathrm{Ad}\colon \mathrm{Sp}(1) \to \mathrm{SO}(\mathrm{Im}\,\mathbf{H})$

$$\mathrm{Ad}(q)x \coloneqq qxq^*$$

is a covering map. ♠

Proposition 1.10. *If $p\colon X \to B$ is a proper local homeomorphism, then it is a covering map.* ■

Exercise 1.11. Prove Proposition 1.10.

Proposition 1.12. *Let $p\colon X \to B$ be a covering space with B connected. For every $b_0, b_1 \in B$ there is a bijection $p^{-1}(b_0) \to p^{-1}(b_1)$.*

Proof. For $b_0 \in B$ set

$$S_{b_0} \coloneqq \{b \in B : \text{there is a bijection } p^{-1}(b_0) \to p^{-1}(b)\}.$$

Trivially, $b_0 \in S_{b_0}$. If $b \in S_{b_0}$ and U is an open neighborhood of $b \in B$ as in Definition 1.1, then $U \subset S_{b_0}$. Therefore, S_{b_0} is open. Finally,

$$B \backslash S_{b_0} = \bigcup_{b \notin S_{b_0}} S_b.$$

Therefore, S_{b_0} is closed. Since B is connected, $S_{b_0} = B$. ■

Definition 1.13. Let $p\colon X \to B$ be a proper covering map. The **degree of p** is the map $\deg_\cdot(p)\colon B \to \mathbf{N}_0$ defined by

$$\deg_b(p) \coloneqq \# p^{-1}(b).$$

•

1.2 Quotients and covering maps

Definition 1.14. Let G be a topological group. Let X be a topological space.

(1) A **left action** of G on X is a continuous map $L\colon G \times X \to X$ satisfying

$$L(\mathbf{1}, \cdot) = \mathrm{id}_X \quad \text{and} \quad L(g, L(h, x)) = L(gh, x).$$

Define $L_g \in \mathrm{Homeo}(X)$ by $L_g \coloneqq L(g, \cdot)$ and abbreviate

$$g \cdot x \coloneqq L(g, x).$$

(2) The **quotient of X by L** is the topological space

$$G \backslash X \coloneqq X/{\sim_L}$$

with $x \sim_L y$ if and only if $x = g \cdot y$ for some $g \in G$.

(3) A left action L of G on X is **properly discontinuous** if every $x \in X$ has a neighborhood U such that

$$U \cap (g \cdot U) \neq \varnothing \quad \text{if and only if} \quad g = \mathbf{1} \in G.$$

•

Proposition 1.15. *Let G be a group. Let X be a topological space. Let L be a properly discontinuous left action of G on X. Set $B \coloneqq G \backslash X$.*

1. *The canonical projection map $p\colon X \to B$ is a covering map.*
2. *Let S be a discrete space. Let $\lambda\colon G \to \mathrm{Homeo}(S) = \mathrm{Bij}(S)$ be a group homomorphism. Set*

$$Y \coloneqq X \times_G S \coloneqq G \backslash (X \times S)$$

with $g \cdot (x, s) \coloneqq (g \cdot x, \lambda(g)s)$. The canonical projection map $q\colon Y \to B$ is a covering map. ■

Exercise 1.16. Prove Proposition 1.15.

If B is not too pathological (e.g., if B is a manifold or CW complex), then (up to isomorphism) every covering map $p\colon X \to B$ arises from the construction in Proposition 1.15.

1.3 Lifting along covering maps, I

Definition 1.17. Let $p\colon X \to B$ and $f\colon A \to B$ be a continuous maps. A **lift of f along p** is a continuous map $\tilde{f}\colon A \to X$ such that

$$p \circ \tilde{f} = f.$$ •

The following diagram illustrates the situation in Definition 1.17:

$$\begin{tikzcd} & X \arrow[d, "p"] \\ A \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B. \end{tikzcd}$$

The key to the theory of covering spaces is to solve the lifting problem: which maps f admit a lift along p?

Proposition 1.18. *Let $p\colon X \to B$ be a covering map. Let $f\colon A \to B$ be a continuous map. Let $a_0 \in A$ and $x_0 \in p^{-1}(f(a_0))$. If A is connected, then there is at most one lift $\tilde{f}\colon A \to X$ of f along p with $\tilde{f}(a_0) = x_0$.*

The proof is an immediate consequence of the following.

Lemma 1.19. *Let $p\colon X \to B$ be a covering map. Set*

$$X \times_B X \coloneqq \{(x, y) \in X \times X : p(x) = p(y)\} \quad \text{and} \quad \Delta \coloneqq \{(x, x) \in X \times_B X\}.$$

The subset $\Delta \subset X \times_B X$ is open and closed.

Proof. For $x \in X$ denote by V_x an open neighborhood of $x \in X$ such that $p(V_x)$ is open and $p|_{V_x}\colon V_x \to p(V_x)$ is a homeomorphism. $(V \times V) \cap (X \times_B X)$ is an open neighborhood of $(x, x) \in X \times_B X$ and contained in Δ. Therefore, Δ is open.

Let $(x, y) \in (X \times_B X) \backslash \Delta$. Choose V_x, V_y as above with $V_x \cap V_y = \varnothing$. $(V_x \times V_y) \cap X \times_B X$ is an open neighborhood of $(x, y) \in X \times_B X$ and does not intersect Δ. Therefore, Δ is closed. ■

Proof of Proposition 1.18. Suppose $\tilde{f}_1, \tilde{f}_2\colon A \to X$ are lifts of f along p with $\tilde{f}_i(a_0) = x_0$. By Lemma 1.19, $S := (\tilde{f}_1, \tilde{f}_2)^{-1}(\Delta)$ is open and closed. Since $a_0 \in S$ and A is connected, $S = A$; hence: $\tilde{f}_1 = \tilde{f}_2$. ∎

Example 1.20. The map $p_k\colon S^1 \to S^1$ does not admit a lift along the projection $p\colon \mathbf{R} \to S^1$. ♠

This illustrates that not every map $f\colon A \to B$ can be lifted along p. However, it is quite easy to see that this posssible if $A = [0,1]$. It is quite easy to see that paths can be lifted along covering maps. In fact, this can be done in families.

Definition 1.21. Let A be a topological space. A continuous map $p\colon X \to B$ has the **homotopy lifting property (HLP) with respect to** A if for every homotopy $h\colon [0,1] \times A \to B$ and lift $\tilde{h}_0\colon A \to X$ of $h(0,\cdot)$ there is a homotopy $\tilde{h}\colon [0,1] \times A \to X$ which is a lift of h with $\tilde{h}(0,\cdot) = \tilde{h}_0$. •

The following diagram illustrates the situation in Definition 1.21:

$$\begin{tikzcd}
A \arrow[r, "\tilde{h}_0"] \arrow[d, hook] & X \arrow[d, "p"] \\
{[0,1] \times A} \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B.
\end{tikzcd}$$

Definition 1.22. A continuous map $p\colon X \to B$ is a **Hurewicz fibration** if it has the HLP with respect to every topological space. •

Lemma 1.23. *If $p\colon X \to B$ is a covering map, then is a Hurewicz fibration.*

Proof. If D is a discrete space, then $\mathrm{pr}_1\colon B \times D \to B$ is a Hurewicz fibration. Consequently, every $b \in B$ has a neighborhood U such that $p|_{p^{-1}(U)}$ is a Hurewicz fibration.

Let A be a topological space. Let $h\colon [0,1] \times A \to B$ be a homotopy. For every $(t,a) \in [0,1] \times A$ choose a neighborhood $U_{t,a}$ of $h(t,a)$ as above. Since $[0,1]$ is compact, there are $0 = t_0 < t_1 < \cdots < t_n = 1$ and an open neighborhood V_a of $a \in A$ with $[t_i, t_{i+1}] \times V_a \subset h^{-1}(U_{t,a})$ for some $t \in [0,1]$. Let $\tilde{h}_0$ be a lift of $h(0,\cdot)$. Since $p|_{p^{-1}(h([t_i,t_{i+1}]\times V_a))}$ is a Hurewicz fibration, a finite induction argument constructs a lift $\tilde{h}_{V_a}$ of $h|_{[0,1]\times V_a}$ with $\tilde{h}_{V_a}(0,\cdot) = \tilde{h}_0|_{V_a}$.

By Proposition 1.18 and because $[0,1]$ is connected, $\tilde{h}_{V_a}$ and $\tilde{h}_{V_b}$ agree on $[0,1] \times (V_a \cap V_b)$. Therefore, they assemble into a lift $\tilde{h}$ of h with $\tilde{h}(0,\cdot) = \tilde{h}_0$. ∎

1.4 The monodromy representation

The theory of covering maps is intricately intertwined with concept of fundamental group(oid).

Definition 1.24. Let X be a topological space. Let $\gamma, \delta\colon [0,1] \to X$ be paths.

(1) If $\gamma(1) = \delta(0)$, then the **concatenation** of γ and δ is the path $\gamma * \delta\colon [0,1] \to X$ defined by

$$(\gamma * \delta)(t) := \begin{cases} \gamma(2t) & \text{if } t \in [0,1/2], \\ \delta(2t-1) & \text{if } t \in [1/2,1]. \end{cases}$$

(2) The **reverse** of γ is the path $\bar{\gamma}\colon [0,1] \to X$ defined by

$$\bar{\gamma}(t) \coloneqq \gamma(1-t).$$

(3) Let (X, x) be a pointed topological space. Its **fundamental group** is the set $\pi_1(X, x)$ of homotopy classes rel $\{0,1\}$ of paths $\gamma\colon [0,1] \to X$ with $\gamma(0) = \gamma(1) = x$ and the multiplication defined by

$$[\gamma] \cdot [\delta] \coloneqq [\delta * \gamma].$$

The unit $\mathbf{1} \in \pi_1(X, x)$ is the homotopy class of the constant path $[x]$ and $[\gamma]^{-1} = [\bar{\gamma}]$. •

The above definition of the group structure on $\pi_1(X, x)$ might appear backwards. This convention is justified by Proposition 1.25 (2). It is also the group structure inherited from the composition in a fundamental groupoid $\Pi_1(X)$.

Proposition 1.25. *Let $p\colon X \to B$ be a covering map.*

(1) *For every path $\gamma\colon [0,1] \to B$ there is a map $\mathrm{tra}_\gamma\colon p^{-1}(\gamma(0)) \to p^{-1}(\gamma(1))$ such that if $\tilde{\gamma}$ is a lift of γ, then*

$$\mathrm{tra}_\gamma(\tilde{\gamma}(0)) = \tilde{\gamma}(1).$$

(2) *If $\gamma, \delta\colon [0,1] \to X$ are paths with $\gamma(1) = \delta(0)$, then*

$$\mathrm{tra}_{\gamma*\delta} = \mathrm{tra}_\delta \circ \mathrm{tra}_\gamma .$$

(3) *For every path $\gamma\colon [0,1] \to B$ the map tra_γ is a bijection; indeed:*

$$\mathrm{tra}_{\bar{\gamma}} = \mathrm{tra}_\gamma^{-1} .$$

(4) *The map tra_γ depends only on the homotopy class of γ rel $\{0,1\}$.*

Proof. By Proposition 1.18, tra_γ is well-defined. This proves (1).

If $\tilde{\gamma}$ is a lift of γ and $\tilde{\delta}$ is a lift of δ with $\tilde{\delta}(0) = \tilde{\gamma}(0)$, then $\tilde{\gamma} * \tilde{\delta}$ is a lift of $\gamma * \delta$ with $(\tilde{\gamma} * \tilde{\delta})(0) = \tilde{\gamma}(0)$. This proves (2).

If $\tilde{\gamma}$ is a lift of γ, then $\bar{\tilde{\gamma}}$ is a lift of $\bar{\gamma}$. This proves (3).

Let $\Gamma\colon [0,1] \times [0,1] \to B$ be a homotopy rel $\{0,1\}$. If $\tilde{\Gamma}$ is a lift of Γ along p, then $\tilde{\Gamma}(\cdot, 1)$ is constant it maps to $p^{-1}(\Gamma(0,1))$ and the latter is discrete. This proves (4). ■

Example 1.26. Consider Example 1.7 for $d = 2$. By the quadratic formula the roots of $p = x^2+ax+b$ are

$$-\frac{1}{2}\Big(a \pm \sqrt{a^2 - 4b}\Big).$$

Therefore, $P_2 = \{x^2 + ax + b \in \mathbf{C}[x] : a^2 \neq 4b\}$. Consider the path $p\colon [0,1] \to P_2$ defined by

$$p(t) \coloneqq x^2 - e^{2\pi i t} = (x + e^{\pi i t})(x - e^{\pi i t}).$$

The lift $\tilde{p}\colon [0,1] \to R_2$ of p along $\mathrm{pr}_2\colon R_2 \to P_2$ with $\tilde{p}(0) = (1, p(t))$ is

$$\tilde{p}(t) = (e^{\pi i t}, p(t)).$$

Therefore, $\mathrm{tra}_p(1) = -1$. ♠

Definition 1.27. Let $p\colon X \to B$ be a covering map. Let $b \in B$. The **monodromy representation** of p is the homomorphism $\mathrm{tra}\colon \pi_1(B,b) \to \mathrm{Bij}(p^{-1}(b))$ defined by

$$\mathrm{tra}(\gamma) \coloneqq \mathrm{tra}_\gamma. \qquad \bullet$$

Definition 1.28. Let $p\colon (X,x_0) \to (B,b_0)$ be a pointed covering map. The **characteristic subgroup** of p and x_0 is the subgroup

$$C(p,x_0) \coloneqq \mathrm{im}(p_*\colon \pi_1(X,x_0) \to \pi_1(B,b_0)) < \pi_1(B,b_0). \qquad \bullet$$

Definition 1.29. Let G be a group. Let $H < G$ be a subgroup. The **normal core of** $H < G$ is the normal subgroup

$$\mathrm{Core}_G(H) \coloneqq \bigcap_{g \in G} gHg^{-1} \triangleleft G. \qquad \bullet$$

Proposition 1.30. *Let $p\colon (X,x_0) \to (B,b_0)$ be a pointed covering map.*

(1) *The homomorphism $p_*\colon \pi_1(X,x_0) \to C(p,x_0) < \pi_1(B,b_0)$ is an isomorphism.*

(2) *For every $[\gamma] \in \pi_1(B,b_0)$ and $y \coloneqq \mathrm{tra}([\gamma])(x)$*

$$C(p,y) = [\gamma]C(p,x_0)[\gamma]^{-1}.$$

(3) *For every $[\gamma] \in \pi_1(B,b_0)$*

$$\mathrm{tra}([\gamma])(x) = x \quad \text{if and only if} \quad [\gamma] \in C(p,x_0).$$

(4) *If X is path-connected, then*

$$\mathrm{tra}(\cdot)(x)\colon \pi_1(B,b_0)/C(p,x_0) \to p^{-1}(b)$$

is a bijection; in particular,

$$\deg(p) = |C(p,x_0) : \pi_1(B,b_0)|.$$

(5) *If X is path-connected, then*

$$\ker(\mathrm{tra}\colon \pi_1(B,b_0) \to \mathrm{Bij}(p^{-1}(b_0)) = \mathrm{Core}_{\pi_1(B,b_0)}(C(p,x_0)).$$

Proof. If $[\tilde\gamma] \in \ker p_*$, then there is a homotopy $\Gamma\colon [0,1]\times[0,1] \to B$ rel $\{0,1\}$ with $\Gamma(0,\cdot) = \gamma \coloneqq p \circ \tilde\gamma$ and $\Gamma(1,\cdot) = b$. Denote by $\tilde\Gamma\colon [0,1]\times[0,1] \to X$ the lift of Γ with $\tilde\Gamma(0,\cdot) = \tilde\gamma$. $\tilde\Gamma(\cdot,0)$, $\tilde\Gamma(\cdot,1)$, and $\tilde\Gamma(1,\cdot)$ are lifts of the constant path b; hence: equal to the constant path x. Therefore, $\tilde\Gamma$ is a homotopy rel $\{0,1\}$ and $H(1,\cdot) = x$ is the constant path x. Consequently, $[\tilde\gamma] = \mathbf{1} \in \pi_1(X,x_0)$. This proves (1).

If $\tilde\gamma\colon [0,1] \to X$ is a path with $\tilde\gamma(0) = x$ and $\tilde\gamma(1) = y$, then the map $\pi_1(X,x_0) \to \pi(X,y)$ defined by

$$[\delta] \mapsto [\bar{\tilde\gamma} * \delta * \tilde\gamma]$$

is an isomorphism. This implies (2).

If $[\gamma] \in C(p, x_0)$, then $\mathrm{tra}([\gamma])(x_0) = x_0$. If $\mathrm{tra}([\gamma])(x_0) = x_0$, then the lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = x_0$ satisfies $\tilde{\gamma}(1) = x_0$; hence: it defines an element $[\tilde{\gamma}] \in \pi_1(X, x_0)$. Evidently, $p_*[\tilde{\gamma}] = [\gamma]$. This proves (3).

By (3), $\mathrm{tra}(\cdot)(x_0)$ is injective. To prove that $\mathrm{tra}(\cdot)(x_0)$ is surjective, let $x \in p^{-1}(b_0)$. Since X is path-connected, there is path $\tilde{\gamma}\colon [0,1] \to X$ with $\tilde{\gamma}(0) = x_0$ and $\tilde{\gamma}(1) = x$. By construction, $\gamma := p \circ \tilde{\gamma}$ satisfies $\gamma(0) = \gamma(1) = b_0$, and $\mathrm{tra}([\gamma])(x_0) = x$. This proves (4)

By (3),

$$\ker \mathrm{tra} = \bigcap_{x \in p^{-1}(b_0)} C(p, x).$$

This, (2), and (4) imply (5). ∎

1.5 Lifting along covering maps, II

Definition 1.31. A topological space X is **locally path connected** if for every $x \in X$ and every neighborhood U of $x \in X$ there is a path connected, open neighborhood $V \subset U$ of $x \in X$. •

Theorem 1.32 (Lifting criterion). *Let $p\colon X \to B$ be a covering map. Let $f\colon A \to B$ be a continuous map. Suppose that A is connected and locally path connected. Let $a \in A$ and $x \in p^{-1}(f(a))$. Set $b := p(x)$. There is a lift $\tilde{f}\colon A \to X$ of f along p with $\tilde{f}(a) = x$ if and only if*

$$\mathrm{im}(f_*\colon \pi_1(A, a) \to \pi_1(B, b)) < C(p, x).$$

Proof. If f admits as lift $\tilde{f}$ along p with $\tilde{f}(a) = x$, then

$$f_*\pi_1(A, a_0) = p_*\tilde{f}_*\pi_1(A, x_0) < p_*\pi_1(X, x_0) = C(p, x_0).$$

For $a \in A$ choose a path $\gamma\colon [0,1] \to A$ with $\gamma(0) = a_0$ and $\gamma(1) = a$ and set

$$\tilde{f}(a) := \mathrm{tra}_{f\circ\gamma}(x_0).$$

This does not depend on the choice of γ. Indeed, if $\delta\colon [0,1] \to A$ is path $\delta(0) = a_0$ and $\delta(1) = a$, then $f_*([\gamma]^{-1}[\delta]) \in \mathrm{im}\, p_*$. Therefore, by Proposition 1.30 (3), $\mathrm{tra}_{f\circ\gamma}(x_0) = \mathrm{tra}_{f\circ\delta}(x_0)$.

It remains to prove that $\tilde{f}$ is continuous. If $p = \mathrm{pr}_1\colon B \times D \to B$ with D discrete and $x_0 = (b_0, d_0)$, then the above construction yields the continuous map $\tilde{f} = (f, d_0)$. A moment's thought shows that repeating the above construction with a_0 and x_0 replaced with a and $\tilde{f}(a)$ respectively produces the same map $\tilde{f}$. Since A is locally path connected, every $b \in B$ has a path connected neighborhood V with $f(V) \subset B$ contained in an open subset U as in Definition 1.1. Therefore, $\tilde{f}$ is continuous. ∎

1.6 The classification of covering maps

If one wants to stick to the fundamental group (instead of the fundamental groupoid), then it is convenient to introduce base-points throughout and in the category of pointed topological spaces and pointed continuous map.

Definition 1.33. Two pointed covering maps $p\colon (X,x_0) \to (B,b_0)$ and $q\colon (Y,y_0) \to (B,b_0)$ are **isomorphic** if there is a pointed homeomorphism $\phi\colon (X,x_0) \to (Y,y_0)$ such that $q \circ \phi = p$. •

Theorem 1.34. *Let (B,b_0) be a connected, locally path-connected, pointed topological space. Two pointed covering maps $p\colon (X,x_0) \to (B,b_0)$ and $q\colon (Y,y_0) \to (B,b_0)$ with X, Y connected are isomorphic if and only if*

$$C(p,x_0) = C(q,y_0).$$

Proof. This is a straight-forward consequence of Theorem 1.32. ■

Definition 1.35. A topological space X is **semi-locally simply-connected** if every $x \in X$ has a neighborhood such that every loop $\gamma\colon [0,1] \to U$ with $\gamma(0) = \gamma(1) = x$ is homotopic rel $\{0,1\}$ to a constant loop in X. •

Theorem 1.36. *Let (B,b_0) be a connected, locally path-connected, semi-locally simply-connecteed, pointed topological space. For every $C < \pi_1(B,b_0)$ there is a pointed covering map $p\colon (X,x_0) \to (B,b_0)$ with X connected and*

$$C(p,x_0) = C.$$

Proof.

Step 1. *Construction of the set X and $p\colon X \to B$.*

Denote by P_b the set of paths $\gamma\colon [0,1] \to B$ with $\gamma(0) = b$. Define the equivalence relation $\sim$ on P_b by $\gamma \sim \delta$ if and only if $\gamma(1) = \delta(1)$ and $[\gamma * \bar{\delta}] \in C$. Denote the equivalence class of γ with respect to $\sim$ by $\langle\gamma\rangle$ Set

$$X \coloneqq P_b/{\sim}$$

and define $p\colon X \to B$ by

$$p(\langle\gamma\rangle) \coloneqq \gamma(1).$$

Step 2. *Construction of the topology on X.*

For $\gamma\colon [0,1] \to B$ with $\gamma(0) = b$ and $s \in [0,1]$ define $\gamma_s\colon [0,1] \to P_b$ by $\gamma_s(t) \coloneqq \gamma(st)$. By construction, $p(\langle\gamma_s\rangle) = \gamma(s)$. Let $U \subset X$ an path-connected open subset and $\langle\gamma\rangle \in p^{-1}(U)$. Denote by $V(U,[\gamma]) \subset X$ to be subset of elements of the form $\langle\gamma * \delta\rangle$ with $\delta\colon [0,1] \to U$ and $\delta(0) = \gamma(1)$. Evidently, these form the basis of a topology.

Step 3. *Proof that p is continuous and open.*

Since B is locally path-connected, p is continuous. Since U is path-connected, $p(V(U,[\gamma])) = U$. Therefore, p is open.

Step 4. *Proof that p is a covering map.*

Let $c \in B$. Let U be a path-connected neighborhood of $c \in B$ such that every $\varepsilon\colon [0,1] \to U$ with $\varepsilon(0) = \varepsilon(1) = b$ is homotopic rel $\{0,1\}$ to the constant loop c in B. Let $\langle\gamma\rangle, \langle\delta\rangle \in p^{-1}(b)$. If $V(U,[\gamma]) \cap V(U,[\delta]) \neq \varnothing$, then there are paths $\varepsilon, \phi\colon [0,1] \to U$ with $\varepsilon(0) = \gamma(1)$, $\phi(0) = \delta(1)$, and

$$[\gamma * \varepsilon * \bar{\phi} * \bar{\delta}] \in C.$$

By construction, $\varepsilon * \bar{\phi}$ is homotopic rel $\{0, 1\}$ to the constant path c. Therefore, $[\gamma * \bar{\delta}]$; hence: $\langle\gamma\rangle = \langle\delta\rangle$. This proves that $p^{-1}(c)$ is discrete.

If $p(\langle\gamma\rangle) \in U$, then there is a path $\delta\colon [0, 1] \to U$ with $\delta(0) = \gamma(1)$ and $\delta(1) = c$. Evidently $[\gamma] \in V(U, \langle\gamma * \delta\rangle)$. Therefore,

$$p^{-1}(U) = \coprod_{\langle\gamma\rangle \in p^{-1}(b)} V(U, \langle\gamma\rangle).$$

It remains to prove that $p\colon V(U, \langle\gamma\rangle) \to U$ is a homeomorphism. It already is continuous and open. Since U is path-connected, this map is surjective. Since every loop in U based at $\gamma(1)$ can be contracted, the map is injective.

Step 5. *Proof that X is path connected.*

X is path-connected since the maps $s \mapsto \langle\gamma_s\rangle$ are continuous.

Step 6. *Proof that $C(p, x_0) = C$.*

Finally, it remains to verify that $C(p, \langle b\rangle) = C$. A loop $\gamma\colon [0, 1] \to B$ with $\gamma(0) = \gamma(1) = b$ represents an element of $C(p, \langle b\rangle)$ if and only if its lift $s \mapsto \langle\gamma_s\rangle$ to X satisfies $b = \langle\gamma_0\rangle = \langle\gamma_1\rangle = \langle\gamma\rangle$. By construction, the latter is equivalent to $[\gamma] \in C$. ■

1.7 Deck transformations

Definition 1.37. Let $p\colon X \to B$ be a covering map. A **deck transformation** of p is a homeomorphism $\phi\colon X \to X$ such that

$$p \circ \phi = p.$$

These form the **deck transformation group** denoted by

$$\mathrm{Deck}(p).$$ •

Proposition 1.38. *Let $p\colon X \to B$ be a covering map. If X is connected, then action of $\mathrm{Deck}(p)$ on X is properly discontinuous.*

Definition 1.39. Let G be a group. Let $H < G$ be a subgroup. The **normaliser** of $H < G$ is

$$N_G(H) \coloneqq \{g \in G : gHg^{-1} = H\}.$$ •

Proposition 1.40. *Let $p\colon X \to B$ be a covering map. Let $b \in B$ and $x, y \in p^{-1}(b)$.*

(1) *Let $\phi \in \mathrm{Deck}(p)$. If $\phi(x) = y$, then $C(p, x) = C(p, y)$.*

(2) *If X is connected and locally path-connected, and $C(p, x) = C(p, y)$, then there is a unique $\phi \in \mathrm{Deck}(p)$ with $\phi(x) = y$.*

(3) *If X is connected and locally path-connected, then there is a unique anti-isomorphism*

$$\tau\colon \operatorname{Deck}(p) \to N_{\pi_1(B,b)}(C(p,x))/C(p,x)$$

with the following property: if $\phi \in \operatorname{Deck}(p)$ and $\gamma\colon [0,1] \to X$ is a path with $\gamma(0) = x$ and $\gamma(1) = \phi(x)$, then

$$\tau(\phi) = [p \circ \gamma]. \tag{1.41}$$

(4) *If X is connected and locally path-connected, then* $\operatorname{Deck}(p)$ *acts transitively on $p^{-1}(b)$ if and and only if $C(p,x) < \pi_1(B,b)$ is normal.*

Proof. (1) and (2) are consequences of Theorem 1.32 and Proposition 1.18.

(1) and (2) the map $\mathrm{ev}_x\colon \operatorname{Deck}(p) \to \{y \in p^{-1}(x) : C(p,x_0) = C(p,y_0)\}$ defined by

$$\mathrm{ev}_x(\phi) = \phi(x)$$

is bijective. By Proposition 1.30 (2) and (4), the map

$$\mathrm{tra}(\cdot)(x)\colon N_{\pi_1(B,b_0)}(C(p,x_0))/C(p,x_0) \to \{y \in p^{-1}(x) : C(p,x_0) = C(p,y_0)\}$$

is bijective. The map

$$\tau \coloneqq (\mathrm{tra}(\cdot)(x))^{-1} \circ \mathrm{ev}_x$$

satisfies (1.41) and is bijective.

To verify that τ is an anti-homomorphism, let $\phi, \psi \in \operatorname{Deck}(p)$. Let $\gamma, \delta\colon [0,1] \to X$ be a path with $\gamma(0) = \delta(0) = x$, $\gamma(1) = \phi(x)$, and $\delta(1) = \psi(x)$. Since $\varepsilon \coloneqq \delta * (\psi \circ \gamma)$ satisfies $\varepsilon(0) = x$ and $\varepsilon(1) = \psi(\phi(x))$,

$$\tau(\psi\phi) = [p \circ (\delta * (\psi \circ \gamma))] = \tau(\phi)\tau(\psi).$$

Therefore, τ is an anti-homomorphism. This proves (3).

(4) is a direct consequence of Proposition 1.30 (4). ■

Exercise 1.42. Compute $\operatorname{Deck}(p)$ for Example 1.5. (This assumes that you already have some tools to understand the fundamental group in this case.)

Definition 1.43. A covering map $p\colon X \to B$ is **normal** if $C(p,x_0) < \pi_1(B,b_0)$ is normal. •

If $p\colon X \to B$ is a normal covering map with X connected and locally path-connected, then it induces a homeomorphism $\operatorname{Deck}(p)\backslash X \cong B$.

1.8 Universal covering maps

Definition 1.44. A covering map $p\colon X \to B$ is **universal** if X is path-connected and simply-connected. •

If G is a group with group operation $(g,h) \mapsto gh$, then G^{op} denotes the opposite group operation $(g,h) \mapsto hg$.

Proposition 1.45. *If $p\colon X \to B$ is a universal covering map, then it is normal and for every $x \in X$ and $b := p(x)$ there is an isomorphism* $\mathrm{Deck}(p) \cong \pi_1(B, b_0)^{\mathrm{op}}$. ■

Exercise 1.46. Compute $\pi_1(S^1, [0])$.

Proposition 1.47. *Let (B, b_0) be a connected, locally path-connected, pointed topological space. Let $p\colon (X, x_0) \to (B, b_0)$ and $q\colon (Y, y_0) \to (B, b_0)$ be pointed covering maps with X, Y connected. Suppose that $C(p, x_0) < C(q, y_0)$. Denote by $f\colon (X, x_0) \to (Y, y_0)$ the unique lift of p along q.*

(1) *The map f is a covering map and* $\mathrm{Deck}(f) < \mathrm{Deck}(p)$.

(2) *The anti-isomorphism $\tau\colon \mathrm{Deck}(p) \to N_{\pi_1(B,b_0)}(C(p, x_0))/C(p, x_0)$ maps $\mathrm{Deck}(f) < \mathrm{Deck}(p)$ to $N_{C(q,y_0)}(C(p, x_0))/C(p, x_0)$.*

(3) *If p is normal, then f is normal. In particular,*

$$Y \cong \mathrm{Deck}(f)\backslash X \cong X \times_{\mathrm{Deck}(f)} S \quad \textit{with} \quad S := \mathrm{Deck}(p)/\mathrm{Deck}(f).$$

Proof. Denote by $r\colon (Z, z) \to (Y, y_0)$ a pointed covering map with Z connected and $C(r, z) = C(p, x_0) < C(q, y_0) \cong \pi_1(Y, y_0)$. Denote by ϕ the lift of f along r. Denote by ψ the lift of $q \circ r$ along p. The following diagram summarises this situation:

$$\begin{tikzcd}
& (Z, z_0) \arrow[d, "r"] \arrow[dl, dashed, bend right, "\psi"] \\
(X, x_0) \arrow[ur, dashed, bend left, "\phi"] \arrow[r, dashed, "f"] \arrow[dr, "p"'] & (Y, y_0) \arrow[d, "q"] \\
& (B, b_0).
\end{tikzcd}$$

Since

$$p \circ \psi \circ \phi = q \circ r \circ \phi = q \circ f = p,$$

$\psi \circ \phi$ is the lift of p along p and thus agrees with id_X. Since

$$q \circ f \circ \psi = q \circ r,$$

$f \circ \psi$ is the lift of $q \circ r$ along q; but so is r; hence: $f \circ \psi = r$. Therefore,

$$r \circ \phi \circ \psi = f \circ \psi = r;$$

that is: $\phi \circ \psi$ is the lift of r along r and thus agrees with id_Z. This proves (1).

(2) and (3) are obvious. ■

1.9 The classification of G–principal covering maps

Warning: here we use right instead of left actions.

Definition 1.48. Let G be a group. A G**–principal covering map** $p\colon X \to B$ is a normal covering map with a free right action $\rho\colon G \to \mathrm{Deck}(p)^{\mathrm{op}}$ such that p induces a homeomorphism $X/G \cong B$. •

Definition 1.49. Two pointed G–principal covering maps $(p\colon (X, x_0) \to (B, b_0), \rho)$ and $(q\colon (Y, y_0) \to (B, b_0), \sigma)$ are **isomorphic** if there is a homeomorphism $\phi\colon (X, x_0) \to (Y, y_0)$ with

$$q \circ \phi = p \quad \text{and} \quad \phi\rho\phi^{-1} = \sigma. \qquad \bullet$$

Definition 1.50. Let $(p\colon (X, x_0) \to (B, b_0), \rho)$ be a pointed principal G–covering map. The **monodromy representation** of (p, λ) is the homomorphism $\mu\colon \pi_1(B, b_0) \to G$ characterised by

$$\mathrm{tra}([\gamma])(x) = \lambda(\mu([\gamma]))(x). \qquad \bullet$$

Example 1.51. Let (B, b_0) be a connected, locally path-connected, pointed topological space. Let $p\colon (X, x_0) \to (B, b_0)$ be a universal covering map. Proposition 1.40 (3) gives an isomorphism $\tau\colon \pi_1(B, b_0) \cong \mathrm{Deck}(p)^{\mathrm{op}}$. This exhibits p is a $\pi_1(B, b_0)$–principal covering map. Let $\mu \in \mathrm{Hom}(\pi_1(B, b_0), G)$ be a homomorphism. Let $\pi_1(B, b_0)$ act on $X \times G$ via

$$(x, g)[\gamma] := (\tau([\gamma])(x), g\mu([\gamma])).$$

Set

$$Y := \pi_1(B, b_0)\backslash(X \times G) \quad \text{and} \quad y := [x, 1].$$

The projection map $q\colon (Y, y_0) \to (B, b_0)$ is a pointed covering map. The map $\sigma\colon G \to \mathrm{Deck}(q)^{\mathrm{op}}$ is induced by right-multiplication on G makes q into a principal G–covering map. ♠

Theorem 1.52. *Let (B, b_0) be a connected, locally path-connected, semi-locally simply-connected, pointed topological space*

1. *Two pointed G–principal covering maps are isomorphic if and only if their monodromy representations agree.*
2. *Every $\mu \in \mathrm{Hom}(\pi_1(B, b_0), G)$ is the monodromy representation of a pointed G–principal covering map.*

Proof sketch. Fix a pointed universal covering map $r\colon (Z, z_0) \to (B,_0 b)$. Let $(p\colon (X, x_0) \to (B, b_0), \rho)$ be a pointed G–principal covering map. Let $(q\colon (Y, y_0) \to (B, b_0), \sigma)$ be the G–principal covering constructed in the previous example from r and the monodromy representation μ of p.

Denote by $f\colon (Z, z_0) \to (X, x_0)$ the lift of r along p. A moments thought shows that

$$f(\tau([\gamma])x) = \rho(\mu([\gamma]))^{-1} f(x).$$

Define $\Phi\colon Z \times G \to X$ by

$$\Phi(\zeta, g) = \rho(g)(f_{x_0}(\zeta)).$$

Let $[\gamma] \in \pi_1(B, b_0)$. Denote by $\tau\colon \pi_1(B, b_0) \cong \mathrm{Deck}(r)^{\mathrm{op}}$ the isomorphism from Proposition 1.40 (3). The map Φ is $\pi_1(B, b_0)$–invariant; indeed:

$$\begin{aligned}
\Phi((x,g)[\gamma]) &= \Phi(\tau([\gamma])(x), g\mu([\gamma])) \\
&= \rho(g \cdot \mu([\gamma]))f_x(\tau([\gamma])x) \\
&= \rho(g)f_x(x) \\
&= \Phi(x, g).
\end{aligned}$$

Therefore, Φ induces a continuous map $\phi\colon (Y, y_0) \to (X, x_0)$. Evidently,

$$q \circ \phi = p.$$

A moment's thought shows that ϕ is an isomorphism of pointed G–principal covering maps.

To complete prove it remains to verify that the construction in the preceding example indeed gives a G–principal covering map with monodromy given by μ. This is an exercise. ■

1.10 The Seifert–van Kampen theorem

Theorem 1.53 (Seifert–van Kampen). *Let X be a topological space. Let $\{U_1, U_2\}$ be an open cover of X. Suppose that $X, U_1, U_2, U_1 \cap U_2$ are connected, locally path-connected, semi-locally simply-connected. Let $x \in U_1 \cap U_2$. Denote by*

$$\iota_i\colon \pi_1(U_1 \cap U_2, x_0) \to \pi_1(U_i, x_0) \quad \textit{and} \quad \jmath_i\colon \pi_1(U_i, x_0) \to \pi_1(X, x_0)$$

the maps induced by inclusion. The fundamental group $\pi_1(X, x_0)$ has the following universal property: if $\phi_i\colon \pi_1(U_i, x_0) \to G$ are homomorphisms with $\phi_1 \circ \iota_1 = \phi_2 \circ \iota_2$, then there is a unique $\phi\colon \pi_1(X, x_0) \to G$ such that

$$\phi \circ \jmath_i = \phi_i.$$

The following diagram illustrates Theorem 1.53:

$$\begin{tikzcd}
\pi_1(U_1 \cap U_2, x_0) \arrow[r, "i_1"] \arrow[d, "i_2"] & \pi_1(U_1, x_0) \arrow[d, "j_1"] \arrow[ddr, bend left, "\phi_1"] & \\
\pi_1(U_2, x_0) \arrow[r, "j_2"] \arrow[drr, bend right, "\phi_2"'] & \pi_1(X, x_0) \arrow[dr, dashed, "\exists!", "\phi"'] & \\
& & G.
\end{tikzcd}$$

Exercise 1.54. Proof this from the classification of pointed G–principal covering maps.

1.11 The topological proof of the Nielsen–Schreier Theorem

Definition 1.55. Let S be a set. The **free group on** S is the group $F(S)$ generated by S. A group G is **free** if it is isomorphic to $F(S)$ for some S. The **rank** of G is $\mathrm{rk}(G) := \#S$. •

Theorem 1.56 (Nielsen–Schreier Theorem). *If G is a free group, then every subgroup $H < G$ is free. If $\mathrm{rk}(G) = r \in \mathbf{N}_0$ and $|H : G| = i \in \mathbf{N}$, then $\mathrm{rk}(H) = i(r-1) + 1$.*

The proof relies on realising G as a fundamental group and covering theory.

Definition 1.57.

(1) A **graph** is a triple $\Gamma = (V, E, \alpha)$ with V a set, E a set of unordered pairs, and a map $\alpha\colon \bigcup E \to V$. The **vertices** and **edges** of Γ are the elements of V and E respectively. An edge e **connects** $x, y \in V$ if $\alpha(e) = \{x, y\}$.

(2) For every unordered pair $e = \{x, y\}$ set

$$I_e := (e \times [0,1])/\sim$$

with $\sim$ denoting the equivalence relation generated by $(x, t) \sim (y, 1-t)$.

(3) The **topological realisation** of Γ is

$$X(\Gamma) := \left(V \amalg \coprod_{e \in E} I_e\right) \Big/ \sim$$

with $\sim$ denoting the equivalence realation generated by $[x, 0] \sim \alpha(x)$. •

Example 1.58. Let S be a set. Set $V := \{*\}$ and $E := \{0,1\} \times S$. There is a unique map $\alpha\colon E \to V$. The graph $\Gamma = (V, E, \alpha)$ has a unique vertices $\star$ and an edge connecting $*$ to itself for every $s \in S$. The topological realisation $X(\Gamma)$ of Γ is homeomorphic to a **bouquet of circles** indexed by S:

$$X(\Gamma) \cong \bigvee_{s \in S} \{s\} \times S^1 := \left(\coprod_{s \in S} \{s\} \times S^1\right) \Big/ \sim$$

with $\sim$ denoting the equivalence relation generated by $(s, [0]) \sim (t, [0])$. By Theorem 1.53,

$$\pi_1(X(\Gamma), *) \cong F(S).$$

♠

Definition 1.59. Let $\Gamma = (V, E, \alpha)$ be a graph.

(1) A **subgraph** of a graph $\Gamma = (V, E, \alpha)$ is a graph $\Delta = (W, F, \beta)$ with $W \subset V$, $F \subset E$, and $\beta = \alpha|_F$.

(2) A **path** in Γ is a is a sequence of vertices $v_0, \dots, v_n$ together with a sequence of edges $e_0, \dots, e_n$ such that e_i connects v_i and v_{i+1}. A **cycle** in Γ is a path with $n \geqslant 1$, $v_0 = v_n$, and $e_i \neq e_{i+1}$.

(3) Γ is **connected** if for every $v, w \in V$ there is a path with $v_0 = v$ and $v_n = w$.

(4) A **forest** is a graph without cycles. A **tree** is a connected forest. •

Proposition 1.60. *Let Γ be a connected graph. $X(\Gamma)$ is homotopy equivalent to a bouquet of circles.*

Proof sketch. Denote by $\mathscr{T}$ the set of subgraphs of Γ which are trees. There is an obvious order on $\mathscr{T}$. Use Zorn's lemma to construct a maximal $T \in \mathscr{T}$. A moment's thought shows that T has the same vertices as Γ. The subspace $X(T) \subset X(\Gamma)$ is contractible. $X(\Gamma)/X(T)$ is homeomorphic to a bouquet of circles. Finally, the projection $X(\Gamma) \to X(\Gamma)/X(T)$ is a homotopy equivalence. ■

Proposition 1.61. *Let Γ be a graph. If $p\colon Y \to X(\Gamma)$ is a covering map, then Y is homeomorphic to $X(\Delta)$ for some graph Δ.*

Proof. Exercise. ■

Proof of Theorem 1.56. Let G be a free group. Construct a graph Γ with $\pi_1(X(\Gamma)) \cong G$. If $H < G$ is a subgroup, then there is a covering map $p\colon Y \to X(\Gamma)$ with characteristic subgroup isomorphic to H. By the above, $\pi_1(Y)$ is free.

If F has rank r and $|H : G| = i$, then $\deg(p) = i$; hence:

$$1 - \mathrm{rk}(H) = \chi(Y) = i\chi(X(\Gamma)) = i(1 - r).$$

This implies $\mathrm{rk}(H) = i(r - 1) + 1$. ■

2 Fibre bundles

The purpose of this section is to develop the theory of Ehresmann connections on fibre bundles. [Ste51] is the classical reference of the topological theory of fibre bundles. [Hus94] is a more modern reference. The theory of connections of fibre bundles is due to Ehresmann [Ehr51]. Kolář, Michor, and Slovák [KMS93]

2.1 Definition and examples

Definition 2.1. Let X, B be smooth manifolds. A **fibre bundle** is a smooth map $p\colon X \to B$ such that for every $b \in B$ there are an open subset $b \in U \subset B$, a smooth manifold F, and a diffeomorphism $\tau\colon p^{-1}(U) \to U \times F$ such that

$$\mathrm{pr}_1 \circ \tau = p;$$

that is: the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\tau"] \arrow[dr, "p"'] & U \times F \arrow[d, "\mathrm{pr}_1"] \\ & U \end{tikzcd}$$

commutes. The **total space** of p is X. The **base space** of p is B. For $b \in B$ the **fibre of p over b** is

$$X_b := p^{-1}(b).$$ •

Definition 2.2. Let $p\colon X \to B$ and $q\colon Y \to C$ fibre bundles. A **morphism of fibre bundles** $(\phi, f)\colon p \to q$ is a pair of smooth maps $\phi\colon X \to Y$ and $f\colon B \to C$ such that

$$q \circ \phi = f \circ p;$$

that is: the diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "p"] & Y \arrow[d, "q"] \\ B \arrow[r, "f"] & C \end{tikzcd}$$

commutes. If $B = C$, then a **morphism of fibre bundles over** B $\phi\colon p \to q$ is a smooth map $\phi\colon X \to Y$ such that (ϕ, id_B) is a morphism of fibre bundles. •

Example 2.3. Let B, F be smooth manifolds. The **trivial fibre bundle** over B with fibre F is the projection map $\mathrm{pr}_1\colon B \times F \to B$. ♠

Example 2.4. The **Hopf bundle** is the projection $p\colon S^{2n+1} \subset \mathbf{C}^{n+1}\backslash\{0\} \to \mathbf{C}P^n$. ♠

Example 2.5. Let X, B be smooth manifolds. If $p\colon X \to B$ is a covering map, then it is a fiber bundle. ♠

Example 2.6. Let B be a smooth manifold. If $p\colon E \to B$ is a vector bundle, then it is a fiber bundle. ♠

Example 2.7. Let B be a manifold. Let V be an Euclidean vector bundle over B. The **sphere bundle**

$$p\colon S(V) \to B \quad \text{with} \quad S(V) := \{v \in V : |v| = 1\}.$$

is a fibre bundle. ♠

Example 2.8. Let B be a smooth manifold and $p\colon V \to B$ be a vector bundle. For $r \in \mathbf{N}_0$ denote by

$$\mathrm{Gr}_r(V) := \{(b, \Pi) : b \in B, \Pi \subset V_b \text{ with } \dim \Pi = r\}$$

the **Grassmannian of** r**–planes in** V. $\mathrm{Gr}_r(V)$ admits the structure of a smooth manifold such that the map $q\colon \mathrm{Gr}_r(V) \to B$ obtained by restriction of pr_1 is a fibre bundle. ♠

Example 2.9. Let B be a manifold. Let $p\colon V \to B$ be a vector bundle. Denote by

$$\mathrm{Fr}(V) := \{(b, \phi) : b \in B, \phi\colon \mathbf{R}^{\mathrm{rk}_b V} \to V_b \text{ isomomorphism}\}$$

the **frame bundle of** V. $\mathrm{Fr}(V)$ admits the structure of a smooth manifold such that the map $q\colon \mathrm{Fr}(V) \to B$ obtained by restriction of pr_1 is a fibre bundle. ♠

Example 2.10. Let $k \in \mathbf{N}_0$. Let X, Y be smooth manifolds. Let U, V be open neighborhoods of $x \in X$. Two maps $f \in C^\infty(U, Y)$ and $g \in C^\infty(V, Y)$ have the same k**–jet at** x if $f(x) = g(x) =: y$ and for every chart ϕ on X with $\phi(x) = 0$ and ψ on Y with $\psi(y) = 0$ the maps $\tilde{f} := \psi \circ f \circ \phi^{-1}$ and $\tilde{g} := \psi \circ g \circ \phi^{-1}$ satisfy

$$\partial^\alpha \tilde{f}(0) = \partial^\alpha \tilde{g}(0)$$

for every $\alpha \in \mathbf{N}_0^{\dim_x X}$ with $|\alpha| \leqslant k$ and every chart ϕ on X and ψ on Y. Having the same k–jet at x is an equivalence relation. Denote the set of equivalence classes by $J_x^k(X, Y)$. An element of $J_x^k(X, Y)$ is a k**–jet at** x. The k**–jet space space** of maps $X \to Y$ is

$$J^k(X, Y) := \coprod_{x \in X} J_x^k(X, Y)$$

$J^k(X, Y)$ admits the structure of a smooth manifold such that the canonical projection maps $p\colon J^k(X, Y) \to X$, $q\colon J^k(X, Y) \to Y$ are fibre bundle. ♠

Exercise 2.11. Construct the above mentioned structures of a smooth manifold on $S(V)$, $\mathrm{Gr}_r(V)$, $\mathrm{Fr}(V)$, and $J^k(X, Y)$.

Exercise 2.12. Construct diffeomorphism $J^0(X, Y) \to X \times Y$ and $J^1(X, \mathbf{R}) \to T^*X \times \mathbf{R}$.

Proposition 2.13. *Let $p\colon X \to B$ be a fiber bundle. If B is connected and $b_0, b_1 \in B$, then X_{b_0} and X_{b_1} are diffeomorphic.* ■

Theorem 2.14 (Ehresmann fibration theorem). *Let X, B be smooth manifolds. If $p\colon X \to B$ is a proper submersion, then p is a fibre bundle.*

2.2 Constructions: Product, disjoint union, pullback

Proposition 2.15. *Let $p\colon X \to B$ and $q\colon Y \to B$ be fibre bundles.*

(1) *The subset*

$$X \times_B Y := \{(x, y) \in X \times Y : p(x) = q(y)\}$$

is a submanifold and the projection map $q\colon X \times_B Y \to B$ is a fiber bundle.

(2) *The map $p \amalg q\colon X \amalg Y \to B$ is a fiber bundle.*

■

Proposition 2.16. *Let $p\colon X \to B$ be a fibre bundle. Let $f\colon A \to B$ a smooth map.*

(1) *The subset*

$$f^*X := \{(a, x) \in A \times X : f(a) = p(x)\}$$

is a smooth submanifold and the map $f^*p\colon f^*X \to A$ obtained as the restriction of pr_1 is a fibre bundle.

(2) *Denote by $\phi\colon f^*X \to X$ the restriction of pr_2. If $q\colon Y \to A$ is a fibre bundle and $(\psi, f)\colon q \to p$ is a morphism, then $f^*\psi := (q, \psi)\colon q \to f^*p$ is the unique morphism over A such that*

$$(\phi, f) \circ (f^*\psi, \mathrm{id}_A) = (\psi, f);$$

that is: the diagram

$$
\begin{tikzcd}
Y \arrow[rr, "\psi"] \arrow[dr, dashed, "f^*\psi"] \arrow[ddd, "q"'] & & X \arrow[ddd, "p"] \\
& f^*X \arrow[ur, "\phi"'] \arrow[d, "f^*p"] & \\
& A \arrow[dl, equal] \arrow[dr, "f"] & \\
A \arrow[rr, "f"] & & B
\end{tikzcd}
$$

commutes. ∎

Definition 2.17. Let $p\colon X \to B$ be a fibre bundle. The fibre bundle $f^*p\colon f^*X \to A$ is the **pullback of** p **via** f. •

Definition 2.18. Let $p\colon X \to B$ be a fibre bundle. Let $f\colon A \to B$ be a smooth map. A smooth map $\tilde{f}\colon A \to X$ is a **lift of** f **along** p if $p \circ \tilde{f} = f$. A **section of** p is lift of id_B along p; that is: a smooth map $s\colon B \to X$ satisfying $p \circ s = \mathrm{id}_B$. •

Applying Proposition 2.16 with $q = \mathrm{id}_A$ yields.

Corollary 2.19. *Let $p\colon X \to B$ be a fibre bundle. Let $f\colon A \to B$ a smooth map. Denote by $f^*p\colon f^*X \to A$ the pullback of p via f. Let $\phi\colon f^*X \to X$ be as above. Composition with ϕ induces a bijection*

$$\{\text{sections of } f^*X\} \to \{\text{lifts of } f\}, s \mapsto \phi \circ s.$$ ∎

2.3 Ehresmann connections

It is not terribly difficult to prove the following.

Theorem 2.20. *If $p\colon X \to B$ is a fibre bundle, then it is a Hurewicz fibration.* ∎

However, fibre bundles usually lack the unique path lifting property. This defect is can be overcome by choosing an Ehresmann connections.

Definition 2.21. Let $p\colon X \to B$ be a fibre bundle. The **vertical tangent bundle** of p is the vector bundle

$$V_p := \ker(Tp\colon TX \to p^*TB) \to X.$$ •

Remark 2.22. There is a mild abuse of notation in the above: usually Tp is a map $TX \to TB$. It would be more correct to write p^*Tp for the above map (but that makes the notation quite heavy). ♣

Definition 2.23. Let $p\colon X \to B$ be a fibre bundle. An **Ehresmann connection** on p is a splitting of the exact sequence

$$0 \to V_p \xrightarrow{\iota} TX \xrightarrow{Tp} p^*TB \to 0;$$

that is: an isomorphism

$$A\colon TX \to V_p \oplus p^*TB$$

such that the diagram

$$\begin{tikzcd} V_p \arrow[r, "\iota"] \arrow[d, equal] & TX \arrow[r, "Tp"] \arrow[d, "A"] & p^*TB \arrow[d, equal] \\ V_p \arrow[r] & V_p \oplus p^*TB \arrow[r] & p^*TB \end{tikzcd}$$

commutes. The **horizontal subbundle of** A is the subbundle $H_A \subset TX$ defined by

$$H_A \coloneqq A^{-1}(p^*TB).$$

The **connection** 1**–form of** A is the 1–form $\theta_A \in \Omega^1(X, V_p)$ defined by

$$\theta_A \coloneqq \mathrm{pr}_{V_p} \circ A.$$

The set of Ehresmann connections on p is denoted by $\mathscr{A}(p)$. •

Example 2.24. Let B, F be smooth manifolds. For the trivial fibre bundle $\mathrm{pr}_B\colon B \times F \to B$

$$V_{\mathrm{pr}_B} = \mathrm{pr}_F^* TF.$$

The **product connection** A_0 is the Ehresmann connection

$$A_0 \coloneqq (T\mathrm{pr}_F, T\mathrm{pr}_B)\colon TX \to \mathrm{pr}_F^*TF \oplus \mathrm{pr}_B^*TB.$$

The horizontal subbundle of A_0 is $H_{A_0} = \ker T\mathrm{pr}_F$. The connection 1–form of A_0 is $\theta_{A_0} = T\mathrm{pr}_F$. ♠

Example 2.25. Continue with the above situation. For every $a \in \mathrm{Hom}(\mathrm{pr}_B^*TB, \mathrm{pr}_F^*TF_B)$

$$A \coloneqq \begin{pmatrix} \mathbf{1} & a \\ 0 & \mathbf{1} \end{pmatrix} \circ A_0$$

defines an Ehresmann connection on pr_B. In fact, every Ehresmann connection on pr_B is of this form. ♠

Example 2.26. Let X, B be smooth manifolds. If $p\colon X \to B$ is a covering map, then $V_p = 0$; hence: p admits a unique Ehresmann connection. ♠

Proposition 2.27. *Let $p\colon X \to B$ be a fibre bundle.*

(1) *A subbundle $H \subset TX$ is the horizontal subbundle of an Ehresmann connection if and only if the map $Tp\colon H \to p^*TB$ is an isomorphism.*

(2) *A 1–form $\theta \in \Omega^1(X, V_p)$ is the connection 1–form of an Ehresmann connection if and only if*

$$\theta \circ \iota = \mathrm{id}_{V_p}.$$

Proof. If $Tp\colon H \to p^*TB$ is an isomorphism, then $TX = V_p \oplus H$ and $A \coloneqq \mathrm{id}_{V_p} \oplus Tp\colon TX = V_p \oplus H \to V_p \oplus p^*TB$ is an Ehresmann connection with $H_A = H$.

If $\theta \circ \iota = \mathrm{id}_{V_p}$, then $A = (\theta, Tp)\colon TX \to V_p \oplus p^*TB$ is an Ehresmann connection with $\theta_A = \theta$. ■

Definition 2.28. Let $p\colon X \to B$ be a fibre bundle. Let V be a vector bundle over X. A differential form $\alpha \in \Omega^\bullet(X, V)$ is **horizontal** if for every $v \in V_p$

$$i_v\alpha = 0.$$

The subspace of basic (or horizontal) differential forms is denoted by

$$\Omega^\bullet_{\mathrm{hor}}(X, V).$$ •

Proposition 2.29. *Let $p\colon X \to B$ be a fibre bundle. There is an Ehresmann connection A on p. If A_0, A are Ehresmann connections on p, then there is an $a \in \Omega^1_{\mathrm{basic}}(X, V_p) \subset \mathrm{End}(TX)$ such that*

$$A = A_0 + a;$$

moreover, for every $a \in \Omega^1_{\mathrm{hor}}(X, V_p)$, $A_0 + a$ is an Ehresmann connection.

Proof. Choose an open cover $\mathscr{U} = \{U_\alpha : \alpha \in A\}$ of X such that for every $\alpha \in A$ there are a smooth manifold F_α and a diffeomorphism $\tau_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times F_\alpha$ such that

$$\mathrm{pr}_1 \circ \tau_\alpha = p.$$

For every $\alpha \in A$ the 1–form $\theta_\alpha \in \Omega^1(p^{-1}(U_\alpha), V_p)$ defined by

$$\theta_\alpha \coloneqq \tau_\alpha^*(T\mathrm{pr}_2).$$

Denote by $\{\chi_\alpha : \alpha \in A\}$ a partition of unity subordinate to $\mathscr{U}$. The 1–form θ defined by

$$\theta \coloneqq \sum_{\alpha \in A} \chi_\alpha \cdot \theta_\alpha$$

satisfies $\theta \circ \iota = \mathrm{id}_{V_p}$; hence: it is the connection 1–form of an Ehresmann connection.

Let A_0 be an Ehresmann connection. $A \in \mathrm{Hom}(TX, V_p \oplus p^*TB)$ defines an Ehresmann connection if and only if

$$A \circ A_0^{-1} = \begin{pmatrix} \mathbf{1} & * \\ 0 & \mathbf{1} \end{pmatrix}$$

This proves the remaining assertions. ■

Remark 2.30. Here is a different proof of the existence part: Choose a Riemannian metric on X and let H be the orthogonal complement of V_p. ♣

Proposition 2.31. *Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. Let $f\colon A \to B$ be a smooth map. Denote by $f^*p\colon f^*X \to A$ the pullback of p via f and denote by $(\phi, f)\colon f^*p \to p$ the associated morphism.*

(1) *The map $T\phi\colon Tf^*X \to \phi^*TX$ induces an isomorphism*

$$\Psi\colon V_{f^*p} \to \phi^*V_p.$$

(2) *The* 1*–form*

$$\Psi^{-1} \circ \phi^*\theta_A \in \Omega^1(f^*X, V_{f^*p})$$

is the connection 1*–form of an Ehresmann connection f^*A.*

Proof. Since $p \circ \phi = f \circ (f^*p)$, $T\phi(V_{f^*p}) \subset \phi^*V_p$. Therefore, $T\phi$ induces a morphism $\Psi\colon V_{f^*p} \to V_p$. Since ϕ (tautologically) induces an diffeomorphism $(f^*p)(a) \to p^{-1}(f(a))$, Ψ is an isomorphism. This proves (1).

To prove (2), let $v \in V_{f^*p}$ and compute

$$(\Psi^{-1} \circ \phi^*\theta_A)(v) = \Phi^{-1}\theta_A(T\phi(v)) = \Phi^{-1}\theta_A(\Psi(v)) = \Phi^{-1}(\Psi(v)) = v. \qquad \blacksquare$$

Definition 2.32. In the situation of Proposition 2.31, f^*A is the **pullback of** A **via** f. •

2.4 Parallel transport

Definition 2.33. Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. Let $b_0 \in B$ and $x_0 \in p^{-1}(b_0)$. Let $v \in T_{b_0}B$. The A**–horizontal lift** of v to x_0 is the unique $\tilde{v} \in H_{A,x_0}$ with $T_{x_0}p(\tilde{v}) = v$. •

Definition 2.34. Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. A smooth map $f\colon A \to X$ is a A**–horizontal** if

$$\operatorname{im} T_x f \subset f^*H_A;$$

or, equivalently,

$$f^*\theta_A = 0. \qquad \bullet$$

The correspondence between lifts and sections of the pullback from Corollary 2.19 interacts with being horizontal as follows.

Proposition 2.35. *Let $p\colon X \to B$ be a fibre bundle. Let $f\colon A \to B$ be a smooth map. Denote by $f^*p\colon f^*X \to A$ the pullback of p. Denote by $\phi\colon f^*X \to X$ the canonical map. Let $A \in \mathscr{A}(p)$ and denote its pullback via f by f^*A. Composition with ϕ induces a bijection*

$$\{(f^*A)\text{–horizontal sections of } f^*X\} \to \{A\text{–horizontal lifts of } f\}, s \mapsto \phi \circ s.$$

Proof. A section $s\colon A \to f^*X$ of f^*p is f^*A–horizontal if and only if

$$0 = s^*\theta_{f^*A} = \Psi^{-1} \circ s^*\phi^*\theta_A = \Psi^{-1} \circ (\phi \circ s)^*\theta_A.$$

Since Ψ is an isomorphism, this is equivalent to $(\phi \circ s)^*\theta_A = 0$; that is: $(\phi \circ s)$ being A–horizontal. ■

Proposition 2.36. *Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. Let $\gamma\colon [0,1] \to B$ be a smooth path. For every $x_0 \in p^{-1}(\gamma(0))$ there is at most one horizontal lift $\tilde{\gamma}\colon [0,1] \to X$ with $\tilde{\gamma}(0) = x_0$.*

Proof. By Proposition 2.35, it suffices to consider $B = [0,1]$ and $\gamma = \mathrm{id}_B$. There is a unique A–horizontal vector field $v_A \in \Gamma(H_A) \subset \mathrm{Vect}(X)$ which is p–related to $\partial_t \in \mathrm{Vect}([0,1])$. A section $s\colon [0,1] \to X$ is A–horizontal if and only if it is an integral curve of v_A. The assertion therefore follows from the Picard–Lindelöf Theorem. ∎

The above proof also tell us that constructing horizontal lifts of paths amounts to integrating a vector field.

Definition 2.37. Let $p\colon X \to B$ be a fibre bundle. An Ehresmann connection $A \in \mathscr{A}(p)$ is **complete** if for every smooth path $\gamma\colon [0,1] \to B$ and every $x_0 \in p^{-1}(\gamma(0))$ there is an A–horizontal lift $\tilde{\gamma}\colon [0,1] \to X$ with $\tilde{\gamma}(0) = x_0$. •

Remark 2.38. The theory of Ehresmann connections developed so far does not require p to be a fibre bundle. It would suffice that p is a submersion so that $V_p = \ker Tp$ is a vector bundle of rank complementary to TB. However,if p admits a complete Ehresmann connection, then p is a fibre bundle. One can also prove that if p is a fibre bundle, then it admits a complete Ehresmann connection [dHoy16]. Of course, if p is proper, then every Ehresmann connection on p is complete. ♣

Definition 2.39. Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$ be a complete Ehresmann connection. Let $\gamma\colon [0,1] \to B$ be a smooth path. The **parallel transport along** γ is the diffeomorphism $\mathrm{tra}_\gamma^A\colon p^{-1}(\gamma(0)) \to p^{-1}(\gamma(1))$ characterised by

$$\mathrm{tra}_\gamma^A(\tilde{\gamma}(0)) \coloneqq \tilde{\gamma}(1)$$

for every A–horizontal lift $\tilde{\gamma}$ of γ. •

Example 2.40. For covering maps this reconstructs the transport map. ♠

Example 2.41 (Parallel transport and line integrals). Let B be a smooth manifold. Set $X \coloneqq B \times \mathbf{R}$. Consider the trivial bundle $\mathrm{pr}_B\colon X \to B$. Identifying $V_{\mathrm{pr}_B} = \underline{\mathbf{R}}$ every $\alpha \in \Omega^1(B)$ defines a connection 1–form

$$\theta_A \coloneqq T\mathrm{pr}_{\mathbf{R}} + \alpha.$$

A smooth path $\tilde{\gamma} \eqqcolon (I, \gamma)\colon [0,1] \to X$ is A–horizontal if and only if

$$0 = \theta(\dot{\tilde{\gamma}}) = \dot{I} + \alpha(\dot{\gamma}) = \dot{I} + \gamma^*\alpha$$

or, equivalently,

$$I(t) = I(0) - \int_{[0,t]} \gamma^*\alpha.$$

Consequently,

$$\mathrm{tra}_\gamma^A(x_0, I_0) = \left(x_0, I_0 - \int_{[0,1]} \gamma^*\alpha\right).$$ ♠

Definition 2.42. Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. Let $b_0 \in B$. The **holonomy group of** A **based at** b_0 is the subgroup $\mathrm{Hol}_{b_0}(A) < \mathrm{Diff}(p^{-1}(b_0))$ defined by

$$\mathrm{Hol}_{b_0}(A) \coloneqq \left\{\mathrm{tra}_\gamma^A : \gamma\colon [0,1] \to B \text{ piecewise smooth with } \gamma(0) = \gamma(1) = b_0\right\}.$$ •

2.5 Curvature

Example 2.41 illustrates that the parallel transport tra_γ^A depends on γ and not just its homotopy class rel $\{0,1\}$. It does not factor through the fundamental groupoid $\Pi_1(B)$. Of course, tra_γ^A is invariant under reparametrisation. Piecewise smooth paths up to reparametrisations build up the thin path groupoid $P_1(B)$. Parallel transport factors though $P_1(B)$.

Proposition 2.43. *Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. There is a unique horizontal 2–form $F_A \in \Omega^2_{\mathrm{hor}}(X, V_p)$ such that for every $v, w \in \mathrm{Vect}(X)$*

$$F_A(v,w) = -\theta_A([v - \theta_A(v), w - \theta_A(w)]).$$

*F_A is the **curvature of** A.*

Proof. For $f \in C^\infty(X)$ and $v, w \in \Gamma(H_A)$

$$\theta_A([fv,w]) = f\theta_A([v,w]) - \theta_A(\mathscr{L}_w f \cdot v) = f\theta_A([v,w]),$$

and, similarly, $\theta_A([v,fw]) = f\theta_A([v,w])$. Therefore, $v \wedge w \mapsto \theta_A([v,w])$ is tensorial. ∎

Example 2.44. Consider the Hopf bundle $p\colon\ S^{2n+1} \subset \mathbf{C}^{n+1}\backslash\{0\} \to \mathbf{C}P^n$. The vertical tangent bundle is spanned by the vector field $\partial_\alpha \in \mathrm{Vect}(S^{n+1})$ defined by

$$\partial_\alpha(z) = iz.$$

Define a connection A by

$$H_A \coloneqq \{v \in TS^{2n+1} : v \perp \partial_\alpha\}.$$

Let $v, w \in \mathrm{Vect}(S^{2n+1})$ The curvature of A is

$$\begin{aligned} F_A(v,w) &= -\langle [v,w], iz\rangle \otimes iz \\ &= -\langle \nabla_v w - \nabla_w v, iz\rangle \otimes iz \\ &= -2\langle v, iw\rangle \otimes iz \\ &= -2\pi \cdot p^*\omega_{\mathrm{FS}} \otimes \partial_\alpha. \end{aligned}$$

Here $\omega_{\mathrm{FS}} \in \Omega^2(\mathbf{C}P^n)$ is the Fubini–Study form on $\mathbf{C}P^n$. ♠

Exercise 2.45. Let $U_a \coloneqq \{[z_0, \dots, z_n] \in \mathbf{C}P^n : z_a \neq 0\}$ and define $\phi_a\colon\ U_a \to \mathbf{C}^n$ by

$$\phi([z_0,\dots,z_n]) \coloneqq [z_0/z_a : \dots : \widehat{z_a/z_a} : \dots : z_n/z_a].$$

Prove that there is a unique 2–form ω_{FS} on $\mathbf{C}P^n$ satisfying

$$(\phi_a)_*\omega_{\mathrm{FS}} = \frac{i}{2\pi}\left(\sum_{b=1}^n \frac{\mathrm{d}z_b \wedge \mathrm{d}\bar z_b}{1+|z|^2} - \sum_{b,c=1}^n \frac{\bar z_c \mathrm{d}z_c \wedge z_b \mathrm{d}\bar z_b}{(1+|z|^2)^2}\right).$$

Prove that the above formula for F_A indeed holds.

Definition 2.46. Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. $F_A \in \Omega^2_{\mathrm{hor}}(X, V_p)$ **curvature of** A. The Ehresmann connection A is **flat** if $F_A = 0$. •

Proposition 2.47 (Flat connections and covering maps). *Let $p\colon X \to B$ be a fibre bundle. If $A \in \mathscr{A}(p)$ is complete and flat, then there is a covering map $q\colon S \to B$ and a bijective immersion $\iota\colon S \looparrowright X$ such that $q = p \circ \iota$ and $T\iota\colon TS \hookrightarrow \iota^*TX$ induces an isomorphism $TS \cong \iota^*H_A$; in particular:* tra^A_γ *depends only on the homotopy class rel $\{0,1\}$ of γ.*

$$\begin{tikzcd} S \arrow[r, hook, "\iota"] \arrow[dr, "q"'] & X \arrow[d, "p"] \\ & B. \end{tikzcd}$$

Proof of Proposition 2.47. A is flat if and only if the distribution H_A is involutive. Frobenius's theorem guarantees the existence of a bijective immersion $\iota\colon S \looparrowright X$ such that $T\iota\colon TS \hookrightarrow \iota^*TX$ induces an isomorphism $TS \cong \iota^*H_A$.

To prove that $q \coloneqq p \circ \iota$ is a covering map, let $b_0 \in B$ and let U be a connected, simply-connected, open neighborhood of b_0. Let V be a connected component of $S \cap \iota^{-1}(U)$. It remains to prove that $q|_V\colon V \to U$ is a diffeomorphism. By construction, $q|_V$ is a local diffeomorphism. Since U is path-connected, $q|_V$ is surjective. To prove that $q|_V$ is injective, let $s_0, s_1 \in q^{-1}(b_0) \cap V$. Since V is path-connected, there is a smooth path $\tilde{\gamma}\colon [0,1] \to S$ with $\tilde{\gamma}(0) = s_0$ and $\tilde{\gamma}(1) = s_1$. Since U is simply-connected, there is a smooth homotopy $\Gamma\colon [0,1] \times [0,1] \to B$ rel $\{0,1\}$ with $\Gamma(0,\cdot) = q \circ \tilde{\gamma}$ and $\Gamma(1,\cdot) = b_0$. The task at hand is to find an A–horizontal lift $\tilde{\Gamma}\colon [0,1] \times [0,1] \to X$ of Γ along p with $\tilde{\Gamma}(0,0) = x_0 \coloneqq \iota(s_0)$.

By Proposition 2.35, it suffices to consider $B = [0,1] \times [0,1]$ and $\gamma = \mathrm{id}_B$. Denote by v_1, v_2 the A–horizontal lifts of ∂_1, ∂_2. The lift

$$\tilde{\Gamma}(t_1, t_2) \coloneqq \mathrm{flow}^{t_1}_{v_1} \circ \mathrm{flow}^{t_2}_{v_2}(x_0)$$

maps into the maximal integral submanifold through x_0; hence, it is A–parallel. ■

Proposition 2.48. *Let $p\colon X \to B$ be a fibre bundle. Let $f\colon A \to B$ be a smooth map. Denote by $f^*p\colon f^*X \to A$ the pullback of p. Denote by $\phi\colon f^*X \to X$ the canonical map. Let $A \in \mathscr{A}(p)$ and denote its pullback via f by f^*A. The curvature of A and f^*A are related by*

$$F_{f^*A} = \Psi^{-1} \circ \phi^* F_A.$$

Proof sketch. Let $v, w \in H_{f^*A, x_0}$ and set $\tilde{v} \coloneqq T\phi(v)$, $\tilde{w} \coloneqq T\phi(w)$. Extend v, w and $\tilde{v}, \tilde{w}$ to ϕ–related vector field in H_{f^*A} and H_A. (This part of the proof actually is a little fishy. It is not obvious how to construct these. A clean way of doing this is to prove it only after the introduction of the Fröhlicher–Nijenhuis bracket; see Proposition 2.71.) Since $[v,w]$ and $[\tilde{v},\tilde{w}]$ are ϕ–related, $F_{f^*A}([v,w]) = -\theta_{f^*A}([v,w]) = -\Psi^{-1} \circ \theta_A([\tilde{v},\tilde{w}]) = \Psi^{-1} \circ \phi^* F_A(v,w)$. ■

Here is another (more direct) way to see that $F_A = 0$ implies homotopy-independence of parallel transport.

Lemma 2.49. *Let $p\colon X \longrightarrow B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$ be a complete Ehresmann connection. Let $b_0 \in B$ and $x_0 \in p^{-1}(b_0)$ Let $\Gamma\colon [0,1] \times [0,1] \longrightarrow B$ with $\Gamma(0,0) = b_0$. Define $\tilde{\Gamma}\colon [0,1] \times [0,1] \longrightarrow X$ by*

$$\tilde{\Gamma}(s,t) := \mathrm{tra}^A_{\Gamma(\cdot,t)|_{[0,s]}} \circ \mathrm{tra}^A_{\Gamma(0,\cdot)|_{[0,t]}}(x_0).$$

The derivative of $\tilde{\Gamma}(1,\cdot)$ satisfies

$$\partial_t \tilde{\Gamma}(1,\cdot) = \widetilde{\partial_t \Gamma} \circ \tilde{\Gamma}(1,\cdot) + \int_0^1 ((\mathrm{flow}^{\sigma}_{\widetilde{\partial_s \Gamma}})_* F_A(\widetilde{\partial_s \Gamma}, \widetilde{\partial_t \Gamma})) \circ \tilde{\Gamma}(1,\cdot)\, \mathrm{d}\sigma.$$

Remark 2.50. The integrals in Lemma 2.49 measures the deviation of $\tilde{\Gamma}(1,\cdot)$ from being horizontal. ♣

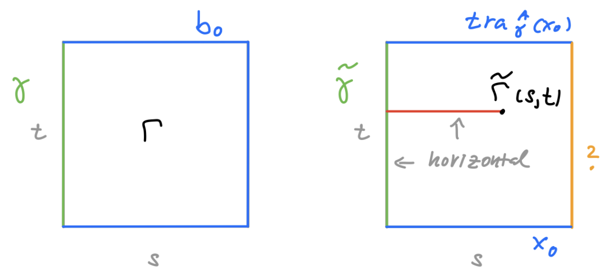


Figure 1: An illustration of Lemma 2.49 for a null-homotopy Γ.

Proof of Lemma 2.49. It suffices to prove this for $B = [0,1]^2$ and $\Gamma = \mathrm{id}_B$. Since $\tilde{\Gamma}(s,t) = \mathrm{flow}^s_{\tilde{\partial}_s} \circ \mathrm{flow}^t_{\tilde{\partial}_t}(x_0)$,

$$\partial_t \tilde{\Gamma}(1,\cdot) = ((\mathrm{flow}^1_{\tilde{\partial}_s})_* \tilde{\partial}_t) \circ \tilde{\Gamma}(1,\cdot).$$

Since $[\partial_s, \partial_t] = 0$,

$$\begin{aligned}
(\mathrm{flow}^1_{\tilde{\partial}_s})_* \tilde{\partial}_t &= \widetilde{(\mathrm{flow}^1_{\partial_s})_* \partial_t} + \theta_A((\mathrm{flow}^1_{\tilde{\partial}_s})_* \tilde{\partial}_t) \\
&= \tilde{\partial}_t + \int_0^1 \partial_\sigma \theta_A((\mathrm{flow}^{\sigma}_{\tilde{\partial}_s})_* \tilde{\partial}_t)\, \mathrm{d}\sigma \\
&= \tilde{\partial}_t - \int_0^1 \theta_A((\mathrm{flow}^{\sigma}_{\tilde{\partial}_s})_* [\tilde{\partial}_s, \tilde{\partial}_t])\, \mathrm{d}\sigma \\
&= \tilde{\partial}_t - \int_0^1 (\mathrm{flow}^{\sigma}_{\tilde{\partial}_s})_* \theta_A([\tilde{\partial}_s, \tilde{\partial}_t])\, \mathrm{d}\sigma \\
&= \tilde{\partial}_t + \int_0^1 (\mathrm{flow}^{\sigma}_{\tilde{\partial}_s})_* F_A(\tilde{\partial}_s, \tilde{\partial}_t)\, \mathrm{d}\sigma.
\end{aligned}$$
■

Corollary 2.51. *Let $p\colon X \longrightarrow B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$. The parallel transport tra^A_γ depends only on the homotopy class rel $\{0,1\}$ of γ if and only if A is flat.* ■

2.6 Digression: Ehresmann connections and Riemannian metrics

Question 2.52. Let $p\colon X \to B$ be a fibre bundle. Let g be a Riemannian metric on X. Denote by A the Ehresmann connection with $H_A = V_p^\perp$. What is F_A?

Let us take a slightly broader perspective. Let X be a smooth manifold. Let $p\colon E \to X$ be a vector bundle equipped with a covariant derivative $\nabla\colon \Gamma(E) \to \Omega^1(X, E)$. Suppose a direct sum decomposition $E = E' \oplus E''$ given. Denote by $P' \in \mathrm{End}(E)$ the projection onto E' respectively. ∇ induces a covariant derivative

$$\nabla' := P' \circ \nabla\colon \Gamma(E') \to \Omega^1(X, E')$$

on E' respectively. If $s \in \Gamma(E')$, then

$$\nabla s = \nabla' s + \mathrm{II}' s \quad \text{with} \quad \mathrm{II}' := \nabla P' \in \Omega^1(X, \mathrm{Hom}(E', E'')).$$

To see this, observe that $(\mathbf{1} - P')\nabla s = \nabla((1 - P')s) - (\nabla(\mathbf{1} - P'))s$.

If $E = TX$ and ∇ is torsion-free, then for $v, w \in \Gamma(E') \subset \mathrm{Vect}(X)$

$$[v, w] = \nabla_v w - \nabla_w v = \nabla'_v w - \nabla'_w v + \mathrm{II}'(v)w - \mathrm{II}'(w)v.$$

To answer the initial question set $E' = H_A$, $E'' = V_p$, and choose ∇ to be the Levi-Civita connection of g. The above formula implies

$$F_A(v, w) = -\mathrm{II}'(v)w + \mathrm{II}'(w)v.$$

Therefore, A is flat if and only if the section of $\mathrm{Hom}(H_A \otimes H_A, V_p)$ induced by II' is symmetric. In fact, if A is flat, then the latter is the second fundamental form of the integral submanifold of H_A.

I recommend to read [Kar99] for more on the Riemannian geometry computations on fibre bundles. Karcher also gives a neat proof of Frobenius' theorem.

2.7 The Gauß–Manin connection

Let X be a manifold. The **de Rham complex** is $(\Omega^\bullet(X), \mathrm{d})$. (The $\bullet$ here just indicates that $\Omega^\bullet(X)$ is graded and we take all of the graded pieces.) Its cohomology $\mathrm{H}^\bullet(\Omega^\bullet(X), \mathrm{d})$ is the **de Rham cohomology** of X:

$$\mathrm{H}^k_{\mathrm{dR}}(X) := \frac{\ker(\mathrm{d}\colon \Omega^k(X) \to \Omega^{k+1}(X))}{\mathrm{im}(\mathrm{d}\colon \Omega^{k-1}(X) \to \Omega^k(X))}.$$

If $f\colon X \to Y$ is a smooth map, then the pullback $f^*\colon \Omega^\bullet(Y) \to \Omega^\bullet(X)$ descends to

$$f^*\colon \mathrm{H}^k_{\mathrm{dR}}(Y) \to \mathrm{H}^k_{\mathrm{dR}}(X).$$

The latter depends only on the homotopy class of f. If f is a diffeomorphism, then f^* is an isomorphism.

Let $p\colon X \to B$ be a fibre bundle. Set

$$\mathscr{H}^k_{\mathrm{dR}}(p) := \coprod_{b\in B} \mathrm{H}^k_{\mathrm{dR}}(p^{-1}(b)).$$

Denote by $q\colon \mathscr{H}^k_{\mathrm{dR}}(p) \to B$ the canonical projection. This can be given the structure of a flat vector bundle.

Let $\{U_\alpha : \alpha \in A\}$ be an open cover of B, and for every $\alpha \in A$ let F_α be a smooth manifold and let $\tau_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times F_\alpha$ be a diffeomorphism such that $\mathrm{pr}_{U_\alpha} \circ \tau_\alpha = p$. Define

$$\phi_\alpha\colon q^{-1}(U_\alpha) = \coprod_{b\in U_\alpha} \mathrm{H}^k_{\mathrm{dR}}(p^{-1}(b)) \to U_\alpha \times \mathrm{H}^k_{\mathrm{dR}}(F_\alpha).$$

by

$$\phi_\alpha(b,[\alpha]) := (b, (\tau_\alpha^{-1}(b,\cdot))^*[\alpha]).$$

These maps are bijections and the transition maps $\phi_\beta \circ \phi_\alpha^{-1}\colon (U_\alpha \cap U_\beta) \times F_\alpha \to (U_\alpha \cap U_\beta) \times F_\beta$ satisfy

$$\phi_\beta \circ \phi_\alpha^{-1}(b,[\alpha]) = \Big(b, \big(\tau_\alpha(b,\cdot) \circ \tau_\beta^{-1}(b,\cdot)\big)^*[\alpha]\Big).$$

The map $\psi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{Hom}(\mathrm{H}^k_{\mathrm{dR}}(F_\alpha), \mathrm{H}^k_{\mathrm{dR}}(F_\alpha))$, $b \mapsto \big(\tau_\alpha(b,\cdot)\circ\tau_\beta^{-1}(b,\cdot)\big)^*$ is locally constant by the homotopy invariance of de Rham cohomology. Therefore, the ϕ_α define a smooth structure on $\mathscr{H}^k_{\mathrm{dR}}(p)$ which makes q into a flat vector bundle. Explicitly, the flat covariant derivative $\nabla\colon \Gamma(\mathscr{H}^k_{\mathrm{dR}}(p)) \to \Omega^1(B, \mathscr{H}^k_{\mathrm{dR}}(p))$ is defined as follows. Let $s \in \Gamma(\mathscr{H}^k_{\mathrm{dR}}(p))$. For $x \in U_\alpha$

$$\phi_\alpha \circ s(x) = (x, \tilde{s}_\alpha(x))$$

for some map $\tilde{s}_\alpha\colon U_\alpha \to \mathrm{H}^k_{\mathrm{dR}}(F_\alpha)$. These are related by

$$\tilde{s}_\beta(x) = \psi_{\beta\alpha}\tilde{s}_\alpha(x).$$

Since $\mathrm{d}\psi_{\beta,\alpha} = 0$,

$$\mathrm{d}\tilde{s}_\beta(x) = \psi_{\beta\alpha}(x)\mathrm{d}\tilde{s}_\alpha(x).$$

Therefore, there is a $\nabla s \in \Omega^1(X, \mathscr{H}^k_{\mathrm{dR}}(p))$ such that for $x \in U_\alpha$

$$\phi_\alpha \circ \nabla s(x) = (x, \mathrm{d}\tilde{s}_\alpha(x)).$$

The covariant derivative defined in this way is the **Gauß–Manin connection**.

Example 2.53. Let F be smooth manifold. Let $f \in \mathrm{Diff}(F)$. Denote X_f the **mapping torus** of f; that is:

$$X_f := ([0,1] \times F)/\sim$$

with denoting the equivalence relation generated by $(0,x) \sim (1, f(x))$. X_f is a smooth manifold and the projection map $p\colon X \to S^1 = \mathbf{R}/\mathbf{Z}$ is a fibre bundle. The monodromy of the Gauß–Manin connection on $\mathscr{H}^\bullet_{\mathrm{dR}}(p)$ is precisely the action of $\mathbf{Z}$ on $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$ generated by f^*. ♠

Let $p\colon X \to B$ be a fibre bundle. Let $b_0 \in B$ and $\gamma\colon [0,1] \to B$ be a smooth loop based at b_0. The pullback $\gamma^* p\colon \gamma^* X \to [0,1]$ is isomorphic to a trivial bundle $\mathrm{pr}_{[0,1]}\colon [0,1] \times F \to [0,1]$. The chain of diffeomorphisms

$$p^{-1}(b_0) \cong (\gamma^* p)^{-1}(0) \cong F \cong (\gamma^* p)^{-1}(1) \cong p^{-1}(b_0)$$

defines a diffeomorphism $f \in \mathrm{Diff}(p^{-1}(b_0))$. The monodromy of the Gauß–Manin connection around γ is precisely $f^* \in \mathrm{End}(\mathrm{H}^\bullet_{\mathrm{dR}}(p^{-1}(b_0)))$.

Remark 2.54. A **local system** on X is a vector bundle E equipped with a flat connection ∇. For every local system $(\Omega^\bullet(X,E), \mathrm{d}_\nabla)$ is the **twisted de Rham complex**. Its cohomology is

$$\mathrm{H}^k_{\mathrm{dR}}(X,E) := \frac{\ker(\mathrm{d}_\nabla\colon \Omega^k(X,E) \to \Omega^{k+1}(X,E))}{\mathrm{im}(\mathrm{d}_\nabla\colon \Omega^{k-1}(X,E) \to \Omega^k(X,E))}.$$

The above shows that $\mathscr{H}^\bullet_{\mathrm{dR}}(p)$ is a local system on B. Its tempting to ask: what is the relation between $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$ and $\mathrm{H}^\bullet_{\mathrm{dR}}(B, \mathscr{H}^\bullet_{\mathrm{dR}}(p))$? The answer to this question is (an instance of) the Leray–Serre spectral sequence. It will lead us to do far astray to discuss this in detail. An excellent reference is [BT82, §14]. ♣

2.8 Graded derivations of the exterior algebra

Our next goal is to understand how the de Rham complex $(\Omega^\bullet(X), \mathrm{d})$ refines if X is the total space of a fibre bundle equipped with an Ehresmann connection. To prepare this task it is helpful to first ponder graded derivations of $\Omega^\bullet(X)$. [KMS93, §8] contains a more detailed treatment of the material discussed below.

Definition 2.55. Let $k \in \mathbf{Z}$. A **graded derivation of degree** k on $\Omega^\bullet(X)$ is an $\mathbf{R}$–linear map $\delta\colon \Omega^\bullet(X) \to \Omega^{\bullet+k}(X)$ satisfying the **graded Leibniz rule**

$$\delta(\alpha \wedge \beta) = (\delta\alpha) \wedge \beta + (-1)^{k\cdot\ell} \alpha \wedge (\delta\beta)$$

for every $\alpha \in \Omega^\ell(X), \beta \in \Omega^\bullet(X)$. The graded derivations of $\Omega^\bullet(X)$ form a graded Lie algebra $\mathrm{Der}_\bullet(\Omega^\bullet(X))$. •

Exercise 2.56. Verify the last sentence in the above definition.

Example 2.57. The exterior derivative d is a graded derivation of degree 1 of $\Omega^\bullet(X)$. If $v \in \mathrm{Vect}(X)$, then i_v is a graded derivation of degree -1 of $\Omega^\bullet$. By Cartan's formula, their graded commutator is the Lie derivative

$$\mathscr{L}_v = \mathrm{d}i_v + i_v\mathrm{d} = [i_v, \mathrm{d}];$$

itself a graded derivation of degree 0. ♠

The derivations of $C^\infty(X)$ are precisely the vector fields on X: $\mathrm{Der}(C^\infty(X)) \cong \mathrm{Vect}(X)$. Is there an analogous result for $\mathrm{Der}_\bullet(\Omega^\bullet(X))$?

Definition 2.58. Let $k \in \mathbf{N}_0$. Denote by $i_\cdot\colon \Omega^{k+1}(X,TX) \to \mathrm{Der}_k(\Omega^\bullet(X))$ the unique linear map satisfying

$$i_{\xi\otimes v}\alpha = \xi \wedge i_v\alpha \quad \text{for} \quad \xi \in \Omega^{k+1}(X) \quad \text{and} \quad v \in \mathrm{Vect}(X).$$

Define $\mathscr{L}_\cdot\colon \Omega^k(X,TX) \to \mathrm{Der}_k(\Omega^\bullet(X))$ by

$$\mathscr{L}_\Xi \coloneqq [i_\Xi, \mathrm{d}]. \qquad \bullet$$

Exercise 2.59. Prove that ι and $\mathscr{L}$ are injective.

Exercise 2.60. For $\Xi = \sum_i \xi_i \otimes v^i \in \Omega^k(X,TX)$ prove that

$$\mathscr{L}_\Xi\alpha = \sum_i \xi_i \wedge \mathscr{L}_{v^i}\alpha + (-1)^k(\mathrm{d}\xi_i) \wedge i_{v^i}\alpha.$$

Proposition 2.61. *Let X be a smooth manifold. Let $k \in \mathbf{Z}$. The map $\mathscr{L} + \iota\colon \Omega^k(X,TX) \oplus \Omega^{k+1}(X,TX) \to \mathrm{Der}_k(\Omega^\bullet(X))$ is an isomorphism. Moreover, $\delta \in \operatorname{im}\mathscr{L}$ if and only if $[\delta,\mathrm{d}] = 0$; and $\varepsilon \in \operatorname{im}\mathscr{L}$ if and only if $\varepsilon(\Omega^0(X)) = 0$.*

Proof. Every $\delta \in \mathrm{Der}_k(\Omega^\bullet(X))$ is determined by its restriction to $\Omega^0(X) \oplus \Omega^1(X)$. If $v_1,\ldots,v_k$, then the map

$$f \mapsto (\delta f)(v_1,\ldots,v_k)$$

is a derivation of $\Omega^0(X) = C^\infty(X)$. Hence, there is a unique vector field $\Xi(v_1,\ldots,v_k)$ such that

$$(\delta f)(v_1,\ldots,v_k) = \mathscr{L}_{\Xi(v_1,\ldots,v_k)}f.$$

A moment's thought shows that $(v_1,\ldots,v_k) \mapsto \Xi(v_1,\ldots,v_k)$ is tensorial. Therefore, it defines a $\Xi \in \Omega^k(X,TX)$. The derivation $\varepsilon \coloneqq \delta - \mathscr{L}_\Xi$ vanishes on $\Omega^0(X)$.

If $f \in C^\infty(X)$ and $\alpha \in \Omega^1(X)$, then

$$\varepsilon(f\alpha) = f \cdot \varepsilon\alpha;$$

that is: $\varepsilon\colon \Omega^1(X) \to \Omega^k(X)$ is tensorial. Therefore, there is a $\Theta \in \Omega^k(X,TX)$ with $\varepsilon = i_\Theta$.

By construction, $\delta = i_\Theta + \mathscr{L}_\Xi$ on $\Omega^0(X) \oplus \Omega^1(X)$. This proves the first assertion. The vanishing criterion for Θ is obvious. A brief computation shows that $[\mathscr{L}_\Xi, \mathrm{d}] = 0$. Since

$$[i_\Theta, \mathrm{d}] = \mathscr{L}_\Theta,$$

the final assertion follows. ∎

Exercise 2.62. What are Θ and Ξ for $\delta = \mathrm{d}$?

Exercise 2.63. Use Proposition 2.61 to prove Cartan's formula $\mathscr{L}_v = \mathrm{d}i_v + i_v\mathrm{d}$.

The identification can be used to define the Lie bracket $[\cdot,\cdot]$ on $\mathrm{Vect}(X)$. Since

$$[[\mathscr{L}_\Theta, \mathscr{L}_\Xi], \mathrm{d}] = 0,$$

one obtains a graded Lie bracket on $\Omega^\bullet(X,TX)$.

Definition 2.64. The **Fröhlicher–Nijenhuis bracket** is the map

$$[\cdot,\cdot]\colon\ \Omega^\bullet(X,TX)\otimes\Omega^\bullet(X,TX)\to\Omega^\bullet(X,TX)$$

characterised by

$$[\mathscr{L}_\Theta,\mathscr{L}_\Xi]=\mathscr{L}_{[\Theta,\Xi]}.$$ •

It turns out (somewhat miraculously in my opinion) that Fröhlicher–Nijenhuis bracket consistently shows up as an obstruction to integrability.

Proposition 2.65. *If $\theta\in\Omega^1(X,TX)$, then the **Nijenhuis tensor***

$$N_\theta := -\frac{1}{2}[\theta,\theta]\in\Omega^2(X,TX)$$

satisfies

$$N_\theta(v,w)=-\theta(\theta([v,w]))-[\theta(v),\theta(w)]+\theta([\theta(v),w]+[v,\theta(w)]).$$

The proof relies in the following

Proposition 2.66. *For $\theta,\alpha\in\Omega^1(X,TX)$ and $v,w\in\mathrm{Vect}(X)$*

$$(\mathscr{L}_\theta\alpha)(v,w)=\mathscr{L}_{\theta(v)}(\alpha(w))-\mathscr{L}_{\theta(w)}(\alpha(v))-\alpha([\theta(v),w])-\alpha([v,\theta(w)])+\alpha(\theta([v,w])).$$

Proof. Let $\theta,\alpha\in\Omega^1(X,TX)$ and $v,w\in\mathrm{Vect}(X)$. Since

$$(\mathrm{d}\alpha)(v,w)=\mathscr{L}_v(\alpha(w))-\mathscr{L}_w(\alpha(v))-\alpha([v,w]),$$

by definition of $\mathscr{L}_\theta$,

$$\begin{aligned}(\mathscr{L}_\theta\alpha)(v,w)&=(i_\theta\mathrm{d}\alpha-\mathrm{d}i_\theta a)(v,w)\\&=\mathscr{L}_{\theta(v)}(\alpha(w))-\mathscr{L}_w(\alpha(\theta(v)))-\alpha([\theta(v),w])\\&\quad+\mathscr{L}_v(\alpha(\theta(w)))-\mathscr{L}_{\theta(w)}(\alpha(v))-\alpha([v,\theta(w)])\\&\quad-\mathscr{L}_v(\alpha(\theta(w)))+\mathscr{L}_w(\alpha(\theta(v)))+\alpha(\theta([v,w]))\\&=\mathscr{L}_{\theta(v)}(\alpha(w))-\alpha([\theta(v),w])\\&\quad-\mathscr{L}_{\theta(w)}(\alpha(v))-\alpha([v,\theta(w)])\\&\quad+\alpha(\theta([v,w])).\end{aligned}$$ ■

Proof of Proposition 2.65. $N_\theta\in\Omega^2(X,TX)$ is determined by the action on $\Omega^0(X)=C^\infty(X)$. For $\theta\in\Omega^k(TX,X)$

$$(\mathscr{L}_\theta f)(v_1,\dots,v_k)=\mathscr{L}_{\theta(v_1,\dots,v_k)}f;$$

in particular, $(\mathscr{L}_\theta f)(v)=\mathscr{L}_{\theta(v)}f$. Therefore, using Proposition 2.66,

$$\begin{aligned}(\mathscr{L}_\theta\mathscr{L}_\theta f)(v,w)&=\mathscr{L}_{\theta(v)}\mathscr{L}_{\theta(w)}f-\mathscr{L}_{\theta([\theta(v),w])}f\\&\quad-\mathscr{L}_{\theta(w)}\mathscr{L}_{\theta(v)}f-\mathscr{L}_{\theta([v,\theta(w)])}f\\&\quad+\mathscr{L}_{\theta(\theta([v,w]))}f\\&=\mathscr{L}_{[\theta(v),\theta(w)]}f-\mathscr{L}_{\theta([\theta(v),w])}f\\&\quad-\mathscr{L}_{\theta([v,\theta(w)])}f+\mathscr{L}_{\theta(\theta([v,w]))}f.\end{aligned}$$

This implies the assertion. ■

Remark 2.67. If $J \in \mathrm{End}(TX)$ is an almost complex structure (that is: $J^2 = -\mathbf{1}$), then the vanishing of N_J characterises the integrability of J. Indeed, the Newlander–Nirenberg theorem asserts that $N_J = 0$ if and only if X admits a holomorphic structure which induces the almost complex structure J. ♣

Exercise 2.68. Let $p\colon X \to B$ be fibre bundle. Let $A \in \mathscr{A}(p)$. Regard the connection 1–form $\theta_A\colon \Omega^1(X, V_p)$ as TX–valued 1–form. Prove that

$$F_A = N_{\theta_A}$$

Remark 2.69. The graded Jacobi identity implies the **Bianchi identity**

$$[\theta_A, F_A] = 0.$$ ♣

Definition 2.70. Let X, Y be smooth manifolds Let $f\colon X \to Y$ be a smooth map. $\Theta \in \Omega^k(X, TX)$ and $\Xi \in \Omega^k(Y, TY)$ are f**–related** if for every $x \in X, v_1, \ldots, v_k \in T_xX$

$$T_xf(\Theta(v_1, \ldots, v_k)) = \Xi(T_xf(v_1), \ldots, T_xf(v_k)).$$ •

Proposition 2.71. *Let X, Y be smooth manifolds Let $f\colon X \to Y$ be a smooth map. Let $\Theta_1, \Theta_2 \in \Omega^\bullet(X, TX)$ and $\Xi_1, \Xi_2 \in \Omega^\bullet(Y, TY)$. If Θ_i and Ξ_i are f–related, then $[\Theta_1, \Theta_2]$ and $[\Xi_1, \Xi_2]$ are f–related.*

Proof. Exercise. ■

2.9 Differential forms on fibre bundles

Definition 2.72. Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$.

(1) For $p, q \in \mathbf{N}_0$ set

$$\Lambda_A^{p,q}T^*X := \Lambda^p H_A^* \otimes \Lambda^q V_p^* \quad \text{and} \quad \Omega_A^{p,q}(X) := \Gamma(\Lambda_A^{p,q}T^*X).$$

These define bi-gradings on $\Lambda^\bullet T^*X$ and $\Omega^\bullet(X)$. (A bigrading is a grading by $\mathbf{N}_0 \times \mathbf{N}_0$; instead of $\mathbf{N}_0$.)

(2) For $p, q \in \mathbf{Z}$ denote by $\mathrm{d}_A^{p,q}$ the component of d of bidegree (p, q).

(3) For every $b \in B$ denote by $\mathrm{res}_b\colon \Omega_A^{p,q}(X) \to \Lambda^pT_b^*B \otimes \Omega^q(p^{-1}(b))$ the composition of the maps

$$\Omega_A^{p,q}(X) \to \Gamma(p^{-1}(b), \Lambda^pH_A^* \otimes \Lambda^qV_p^*) \cong \Gamma(\Lambda^pT_b^*B \otimes \Lambda^qV_p^*) \cong \Lambda^pT_b^*B \otimes \Omega^q(p^{-1}(b)).$$ •

Remark 2.73. Dualizing the exact sequence defining V_p yields

$$0 \to p^*T^*B \to T^*X \to V_p^* \to 0.$$

The image of p^*T^*B in T^*X is V_p^0, the annihilator of V_p. Consequently, $\Omega_A^{p,0}(X)$ is independent of A. Indeed: $\Omega_A^{p,0}(X) = \Omega_{\mathrm{hor}}^p(X)$. ♣

Proposition 2.74. *Let $p\colon X \to B$ be a fibre bundle. Let $A \in \mathscr{A}(p)$.*

(1) *The exterior derivative decomposes into three components of bidegree $(1,0)$, $(0,1)$, and $(2,-1)$:*
$$\mathrm{d} = \mathrm{d}_A^{1,0} + \mathrm{d}_A^{0,1} + \mathrm{d}_A^{2,-1}.$$

(2) *These components satisfy*
$$\mathrm{d}_A^{1,0} = \mathscr{L}_{\mathrm{id}_{TX}-\theta_A} - 2i_{F_A}, \quad \mathrm{d}_A^{0,1} = \mathscr{L}_{\theta_A} + i_{F_A}, \quad \textit{and} \quad \mathrm{d}_A^{2,-1} = i_{F_A}.$$

(3) *For every $b \in B$*
$$\begin{tikzcd} \Omega_A^{p,q}(X) \arrow[rr, "\mathrm{d}_A^{0,1}"] \arrow[d, "\mathrm{res}_b"] & & \Omega_A^{p,q+1}(X) \arrow[d, "\mathrm{res}_b"] \\ \Lambda^p T_b^*B \otimes \Omega^q(p^{-1}(b)) \arrow[rr, "\mathrm{id}\otimes\mathrm{d}"] & & \Lambda^p T_b^*B \otimes \Omega^{q+1}(p^{-1}(b)). \end{tikzcd}$$

(4) *The operators $\mathrm{d}_A^{1,0}$, $\mathrm{d}_A^{0,1}$, $\mathrm{d}_A^{2,-1}$ satisfy*
$$\begin{aligned} \left(\mathrm{d}_A^{1,0}\right)^2 + \mathrm{d}_A^{0,1}\mathrm{d}_A^{2,-1} + \mathrm{d}_A^{2,-1}\mathrm{d}_A^{0,1} &= 0, \\ \left(\mathrm{d}_A^{0,1}\right)^2 &= 0, \\ \left(\mathrm{d}_A^{2,-1}\right)^2 &= 0, \\ \mathrm{d}_A^{1,0}\mathrm{d}_A^{0,1} + \mathrm{d}_A^{0,1}\mathrm{d}_A^{1,0} &= 0, \quad \textit{and} \\ \mathrm{d}_A^{1,0}\mathrm{d}_A^{2,-1} + \mathrm{d}_A^{2,-1}\mathrm{d}_A^{1,0} &= 0. \end{aligned}$$

Proof. The proof has several steps.

Step 7. $\mathrm{d}_A^{p,q} \in \mathrm{Der}_{p+q}(\Omega^\bullet(X))$ *and vanishes unless* $(p,q) \in \{(2,-1),(0,1),(1,0),(2,-1)\}$.

Since d is a graded derivation of $\Omega^\bullet(X)$, so are its components $\mathrm{d}_A^{p,q}$. Evidently, $\mathrm{d}_A^{p,q} = 0$ vanishes unless $p + q = 1$. A graded derivation of $\Omega^\bullet(X)$ is determined by its restriction to $\Omega^0(X) \oplus \Omega^1(X)$. Therefore, $\mathrm{d}_A^{p,q} = 0$ unless $p, q \geqslant -1$.

Step 8. $\mathrm{d}_A^{-1,2} = 0$ *and* $\mathrm{d}_A^{2,-1} = i_{F_A}$.

It suffices to verify these identities on $C^\infty(X)$ and $\Omega^1(X)$. For $f \in C^\infty(X)$
$$\mathrm{d}_A^{-1,2} f = 0 = \mathrm{d}_A^{2,-1} f$$
for degree reasons. Similarly, $\mathrm{d}_A^{-1,2}\alpha = 0$ for $\alpha \in \Omega_A^{1,0}(X)$ and $\mathrm{d}_A^{2,-1}\alpha = 0 = i_{F_A}\alpha$ for $\alpha \in \Omega_A^{0,1}(X)$. For $\alpha \in \Omega^{1,0}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta_A = 0$,
$$\begin{aligned} (\mathrm{d}_A^{-1,2}\alpha)(v,w) &= (\mathrm{d}\alpha)(\theta_A(v), \theta_A(w)) \\ &= -\alpha([\theta_A(v), \theta_A(w)]). \end{aligned}$$

The latter vanishes since $[\theta_A(v), \theta_A(w)]$ is a vertical vector field. For $\alpha \in \Omega^{0,1}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta_A = \alpha$,

$$\begin{aligned}(\mathrm{d}_A^{-1,2}\alpha)(v,w) &= (\mathrm{d}\alpha)(v - \theta_A(v), v - \theta_A(w)) \\ &= -\alpha(\theta_A[v - \theta_A(v), w - \theta_A(w)]) \\ &= (i_{F_A}\alpha)(v,w).\end{aligned}$$

The next two steps determine explicit formulae for $\mathrm{d}_A^{1,0}$ and $\mathrm{d}_A^{0,1}$. The computations are longish and not particularly illuminating. `Skip the proofs in class.`

Step 9. $\mathrm{d}_A^{1,0} = \mathscr{L}_{\mathrm{id}_{TX} - \theta_A} - 2i_{F_A}$.

It suffices to verify the identity on $C^\infty(X)$ and $\Omega^1(X)$.

For $f \in C^\infty(X)$

$$\begin{aligned}\mathrm{d}_A^{1,0} f &= \mathrm{d}f \circ (\mathrm{id}_{TX} - \theta_A) \\ &= (\mathscr{L}_{\mathrm{id}_{TX} - \theta_A} - 2i_{F_A})f.\end{aligned}$$

For $\alpha \in \Omega^1(X)$ and $v, w \in \mathrm{Vect}(X)$

$$\begin{aligned}(\mathscr{L}_{\mathrm{id}_{TX} - \theta_A}\alpha)(v,w) &= \mathscr{L}_{v-\theta_A(v)}(\alpha(w)) - \mathscr{L}_{w - \theta_A(w)}(\alpha(v)) \\ &\quad - \alpha([v - \theta_A(v), w]) - \alpha([v, w - \theta_A(w)]) + \alpha([v,w] - \theta_A([v,w])).\end{aligned}$$

For $\alpha \in \Omega_A^{1,0}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = 0$,

$$\begin{aligned}(\mathrm{d}_A^{1,0}\alpha)(v,w) &= \mathrm{d}\alpha(v - \theta_A(v), w - \theta_A(w)) \\ &= \mathscr{L}_{v-\theta_A(v)}(\alpha(w)) - \mathscr{L}_{w-\theta(w)}(\alpha(v)) - \alpha([v - \theta_a(v), w - \theta_a(w)]) \\ &= (\mathscr{L}_{\mathrm{id}_{TX} - \theta_A}\alpha - 2i_{F_A}\alpha)(v,w) \\ &\quad + \alpha([v - \theta_A(v), w]) + \alpha([v, w - \theta_A(w)]) \\ &\quad - \alpha([v,w]) - \alpha([v - \theta_a(v), w - \theta_a(w)]).\end{aligned}$$

The sum of the last four term vanishes because

$$[v - \theta_A(v), w] + [v, w - \theta_A(w)] - [v,w] - [v - \theta_a(v), w - \theta_a(w)] = -[\theta_A(v), \theta_A(w)]$$

is a vertical vector field.

For $\alpha \in \Omega_A^{0,1}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = \alpha$,

$$\begin{aligned}\mathrm{d}_A^{1,0}\alpha(v,w) &= \mathrm{d}\alpha(\theta_A(v), w - \theta_A(w)) + \mathrm{d}\alpha(v - \theta_A(v), \theta_A(w)) \\ &= \mathscr{L}_{v-\theta_A(v)}(\alpha(w)) - \mathscr{L}_{w-\theta_A(w)}(\alpha(v)) \\ &\quad - \alpha([\theta_a(v), w - \theta_a(w)]) - \alpha([v - \theta_a(v), \theta_a(w)]) \\ &= \big(\mathscr{L}_{\mathrm{id}_{TX} - \theta_A}\alpha - 2i_{F_A}\alpha\big)(v,w) \\ &\quad + \alpha([v - \theta_A(v), w]) + \alpha([v, w - \theta_A(w)]) \\ &\quad - 2\alpha([v - \theta_A(v), w - \theta(w)] \\ &\quad - \alpha([\theta_a(v), w - \theta_a(w)]) - \alpha([v - \theta_a(v), \theta_a(w)]).\end{aligned}$$

The sum of the last five terms vanishes.

Step 10. $\mathrm{d}_A^{0,1} = \mathscr{L}_{\theta_A} + i_{F_A}$.

For $f \in C^\infty(X)$

$$\begin{aligned}\mathrm{d}_A^{0,1} f &= \mathrm{d}f \circ \theta_A \\ &= (\mathscr{L}_{\theta_A} + i_{F_A}) f.\end{aligned}$$

For $\alpha \in \Omega^1(X)$ and $v, w \in \mathrm{Vect}(X)$

$$\begin{aligned}(\mathscr{L}_{\theta_A}\alpha)(v,w) = {}&\mathscr{L}_{\theta_A(v)}(\alpha(w)) - \mathscr{L}_{\theta_A(w)}(\alpha(v)) \\ &- \alpha([\theta_A(v), w]) - \alpha([v, \theta_A(w)]) + \alpha(\theta_A([v,w])).\end{aligned}$$

For $\alpha \in \Omega_A^{1,0}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = 0$,

$$\begin{aligned}(\mathrm{d}_A^{0,1}\alpha)(v,w) &= \mathrm{d}\alpha(\theta_A(v), w - \theta_A(w)) + \mathrm{d}\alpha(v - \theta_A(v), \theta_A(w)) \\ &= \mathscr{L}_{\theta_A(v)}(\alpha(w)) - \mathscr{L}_{\theta_A(w)}(\alpha(v)) \\ &\quad - \alpha([\theta_A(v), w - \theta_A(w)]) - \alpha([v - \theta_A(v), \theta_A(w)]) \\ &= (\mathscr{L}_{\theta_A}\alpha + i_{F_A}\alpha)(v,w) \\ &\quad + \alpha([\theta_A(v), w]) + \alpha([v, \theta_A(w)]) \\ &\quad - \alpha([\theta_A(v), w - \theta_A(w)]) - \alpha([v - \theta_A(v), \theta_A(w)]).\end{aligned}$$

The sum of the last four term vanishes because

$$[\theta_A(v), w] + [v, \theta_A(w)] - [\theta_A(v), w - \theta_A(w)] - [v - \theta_A(v), \theta_A(w)] = 2[\theta_A(v), \theta_A(w)]$$

is a vertical vector field.

For $\alpha \in \Omega_A^{0,1}(X)$ and $v, w \in \mathrm{Vect}(X)$, since $\alpha \circ \theta = \alpha$,

$$\begin{aligned}(\mathrm{d}_A^{0,1}\alpha)(v,w) &= \mathrm{d}\alpha(\theta_A(v), \theta_A(w)) \\ &= \mathscr{L}_{\theta_A(v)}(\alpha(w)) - \mathscr{L}_{\theta(w)}(\alpha(v)) - \alpha([\theta_A(v), \theta_A(w)]) \\ &= (\mathscr{L}_{\theta_A}\alpha + i_{F_A}\alpha)(v,w) \\ &\quad + \alpha([\theta_A(v), w]) + \alpha([v, \theta_A(w)]) - \alpha([v,w]) \\ &\quad + \alpha([v - \theta_A(v), w - \theta_A(w)]) - \alpha([\theta_A(v), \theta_A(w)]).\end{aligned}$$

The sum of the last five term vanishes.

At this point (1) and (2) are established. (3) is obvious and (4) is a consequence of $\mathrm{d}^2 = 0$ and degree considerations. ∎

It is evident from Proposition 2.74 that

$$E_1^{p,q} := \frac{\ker(\mathrm{d}_A^{0,1} \colon \Omega_A^{p,q}(X) \to \Omega_A^{p,q+1}(X))}{\mathrm{im}(\mathrm{d}_A^{0,1} \colon \Omega_A^{p,q-1}(X) \to \Omega_A^{p,q}(X))} \cong \Omega^p(B, \mathscr{H}_{\mathrm{dR}}^q(p)).$$

Since
$$\mathrm{d}_A^{1,0}\mathrm{d}_A^{0,1} + \mathrm{d}_A^{0,1}\mathrm{d}_A^{1,0} = 0 \quad \text{and} \quad (\mathrm{d}_A^{1,0})^2 = -(\mathrm{d}_A^{0,1}\mathrm{d}_A^{2,-1} + \mathrm{d}_A^{2,-1}\mathrm{d}_A^{0,1}),$$
$\mathrm{d}_A^{1,0}$ descends to an operator $\mathrm{d}_1^{0,1}$ on $E_1 := \bigoplus_{p,q} E_1^{p,q}$ and satisfies $\left(\mathrm{d}_1^{0,1}\right)^2 = 0$. Indeed, $\mathrm{d}_1^{1,0}$ corresponds to d_∇ on $\Omega^p(B, \mathscr{H}_{\mathrm{dR}}^q(p))$ arising from the Gauß–Manin connection. The cohomology
$$E_2 := \mathrm{H}(E_1, \mathrm{d}_1^{1,0}) \cong \mathrm{H}_{\mathrm{dR}}^\bullet(B, \mathscr{H}_{\mathrm{dR}}^\bullet(p))$$
does not typically compute $\mathrm{H}_{\mathrm{dR}}^\bullet(X)$. There is a differential $\mathrm{d}_2^{2,-1}$ on E_2 (whose computation requires a bunch of work). $E_3 := \mathrm{H}(E_2, \mathrm{d}_2^{2,-1})$ does not typically compute $\mathrm{H}_{\mathrm{dR}}^\bullet(X)$ either. This procedure can be repeated indefinitely, after finitely many steps $\mathrm{d}_k^{k,k-1} = 0$, and $E_k = E_{k+1} = \cdots$ does compute $\mathrm{H}_{\mathrm{dR}}^\bullet(X)$. This is called the Leray–Serre spectral sequence. An excellent treatment for the Leray–Serre spectral sequence can be found in [BT82, §14]. The lecture notes [**Bunke**] are also outstanding.

It is tempting to guess that the differential $\mathrm{d}_2^{2,-1}$ on E_2 corresponds to i_{F_A} and, therefore, if A is flat, then the Leray–Serre spectral sequence terminates at E_2. **This is false!** If $F_A = 0$, then $\mathrm{d} = \mathrm{d}_A^{0,1} + \mathrm{d}_A^{0,1}$ and $\Omega^\bullet(X)$ is a double complex. The cohomology of a double complex typically does not agree with the iterated cohomology.

Exercise 2.75. Define a double complex by
$$C^{p,q} := \begin{cases} \mathbf{R} & \text{if } (p,q) \in \{(0,1),(0,2),(1,0),(1,1)\} \\ 0 & \text{otherwise} \end{cases},$$
with $\mathrm{d}^{1,0}\colon C^{0,1} \to C^{1,1}$, $\mathrm{d}^{0,1}\colon C^{0,1} \to C^{0,2}$, $\mathrm{d}^{0,1}\colon C^{1,0} \to C^{1,1}$ agreeing with $\mathrm{id}_{\mathbf{R}}$ and with zero otherwise. Set $\mathrm{d} := \mathrm{d}^{1,0} + \mathrm{d}^{0,1}$. Compute
$$H(C, \mathrm{d}), \quad H(H(C, \mathrm{d}^{1,0}), \mathrm{d}^{0,1}), \quad \text{and} \quad H(H(C, \mathrm{d}^{0,1}), \mathrm{d}^{1,0}).$$

Example 2.76 ([Ban13, §6], [MS74, Appendix C]). $\mathrm{PSL}_2(\mathbf{R})$ acts on $H := \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$ by Möbius transformations.
$$\lambda_g(z) := \frac{az+b}{cz+d} \quad \text{for} \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
Define $P\colon STH \to H \times (\mathbf{R} \cup \{\infty\})$ by
$$P(z,v) := \lim_{t\to\infty} \exp_x(tv).$$
Here $\exp_x$ is computed with respect to the hyperbolic metric g_{-1} on H. `Draw a picture to illustrate this.` A brief computation reveals that
$$P(z,v) = \begin{cases} \infty & \text{if } v = i, \\ 0 & \text{if } v = -i, \\ z + \frac{\operatorname{Im} z}{\operatorname{Re} v}(1 - iv) & \text{otherwise.} \end{cases}$$
But that is not very helpful. $\mathrm{PSL}_2(\mathbf{R})$ acts on STH by
$$\Lambda_g(z,v) := (\lambda_g(z), T_z\lambda_g(v))$$

and on $\mathbf{R} \cup \{\infty\}$ by Möbius transformations:

$$\lambda_g^\infty(x) \coloneqq \frac{ax+b}{cx+d} \quad \text{for} \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

A moments thought shows that P is $\mathbf{PSL}_2(\mathbf{R})$–equivariant; that is:

$$P \circ \Lambda_g = (\lambda_g \times \lambda_g^\infty) \circ P.$$

Let Σ be a Riemann surface of genus $g \geqslant 2$. By the uniformization theorem there is a $\Gamma < \mathbf{PSL}(2, \mathbf{R})$ such that $\Sigma = \Gamma \backslash H$. Evidently, $ST\Sigma = \Gamma \backslash STH$. By the above, P induces an fibre bundle isomorphism

$$\Gamma \backslash STH \cong H \times_\Gamma (\mathbf{R} \cup \{\infty\})$$

The latter inherits flat connection.

Using, for example, the Gysin sequence it can be proved that $H^1(ST\Sigma) \cong H^1(\Sigma)$. Therefore, the Leray–Serre spectral sequence can collapse at E_2. ♠

Despite all of the above, it is true that the Leray–Serre spectral sequences terminates at E_2 if X is compact and for some Riemannian metric on $F \coloneqq p^{-1}(b_0)$ the monodromy of A acts by Riemannian isometries [Ban13, Theorem 5.2]. (If A has finite monodromy group G, then this can be derived from Künneth's formula and using G–invariance.)

Exercise 2.77. Prove that the Hopf bundle $p\colon S^{2n+1} \to \mathbf{C}P^n$ does not admit a flat connection.

2.10 Fibre integration

Definition 2.78. Let $p\colon X \to B$ be a fibre bundle. A **fibre orientation** on p is a an orientation on the line bundle $\det(V_p) \coloneqq^r V_p \to X$ with $r \coloneqq \operatorname{rk} V_p$. If B is oriented, then a fibre orientation on p induces an orientation on X via the above isomorphism. •

Proposition 2.79. *Let $d \in \mathbf{N}_0$. Let $p\colon X \to B$ be a fibre bundle with compact fibres of dimension d.*

(1) *There is a unique linear map $p_*\colon \Omega^\bullet(X) \to \Omega^{\bullet-d}(B)$ such that for every $\alpha \in \Omega^{d+k}(X)$, $b \in B$, and $\tilde{v}_1, \ldots, \tilde{v}_k \in \Gamma(TX|_{p^{-1}(b)})$ lifts of $v_1, \ldots, v_k \in T_bB$*

$$(p_*\alpha)(v_1, \ldots, v_k) = \int_{p^{-1}(b)} i_{\tilde{v}_k} \cdots i_{\tilde{v}_1} \alpha|_{p^{-1}(b)}. \tag{2.80}$$

(2) *For every $\alpha \in \Omega^\bullet(X)$ and $\beta \in \Omega^\bullet(B)$*

$$p_*(\alpha \wedge p^*\beta) = p_*\alpha \wedge \beta.$$

(3) *Suppose that B is oriented. For every $\alpha \in \Omega^\bullet(X)$*

$$\int_X \alpha = \int_B p_*\alpha$$

(4) *Suppose that $\partial B = \varnothing$. Set $\partial p \coloneqq p|_{\partial X}\colon\ \partial X \to B$. For every $\alpha \in \Omega^\bullet(X)$*

$$p_* \mathrm{d}\alpha - (-1)^d \mathrm{d} p_* \alpha = \partial p_* \alpha.$$

Proof. The right-hand side of (2.80) is independent of the lifts $\tilde{v}_1, \ldots, \tilde{v}_k$. To verify that (2.80) does define the map p_* it suffices to it suffices the require smoothness. It is enough to verify this for $\mathrm{pr}_B\colon\ B \times F \to B$. This proves (1).

(2) is evident from the construction. (3) follows from Fubini's theorem.

(4) is a consequence of Stokes' theorem; indeed: for every $\alpha \in \Omega^{k+d}(X)$ and $\beta \in \Omega^\bullet(B)$

$$\begin{aligned}
\int_B (p_* \mathrm{d}\alpha) \wedge \beta &= \int_B p_*(\mathrm{d}\alpha \wedge p^*\beta) \\
&= \int_X \mathrm{d}\alpha \wedge p^*\beta \\
&= \int_{\partial X} \alpha \wedge p^*\beta + (-1)^{k+d} \int_X \alpha \wedge p^* \mathrm{d}\beta \\
&= \int_{\partial X} \alpha \wedge p^*\beta + (-1)^{k+d} \int_B (p_*\alpha) \wedge \mathrm{d}\beta \\
&= \int_{\partial X} \alpha \wedge p^*\beta + (-1)^{d} \int_B (\mathrm{d}p_*\alpha) \wedge \beta.
\end{aligned}$$

■

Definition 2.81. In the situation of Proposition 2.79, the map p_* is the **fibre integration**. •

If $\partial X = \partial B = \varnothing$, then p_* descends to de Rham homology $p_*\colon\ \mathrm{H}^\bullet_{\mathrm{dR}}(X) \to \mathrm{H}^{\bullet-d}_{\mathrm{dR}}(B)$. Set $K^\bullet \coloneqq \ker p_*\colon\ \Omega^\bullet(X) \to \Omega^{\bullet-d}(B)$. The short exact sequence

$$0 \to K^\bullet \to \Omega^\bullet(X) \to \Omega^{\bullet-d}(B) \to 0$$

induces a long exact sequence

$$\cdots \to \mathrm{H}^k(K^\bullet) \to \mathrm{H}^k_{\mathrm{dR}}(X) \to \mathrm{H}^{k-d}_{\mathrm{dR}}(B) \xrightarrow{\delta} \mathrm{H}^{k+1}(K^\bullet) \cdots .$$

If p is an oriented sphere bundle, then the map $p^*\colon\ \mathrm{H}^k(B) \to \mathrm{H}^k(K^\bullet)$ is an isomorphism. The Bockstein homomorphism δ arises from taking the wedge product with the **Euler class** of p (more about that later). This instance of the above elong exact sequence is the **Gysin sequence**.

3 Lie groups

In this section I will introduce (review?) the concept of a Lie group, that is, a group in the category of manifolds. For the purpose of this course Lie groups will be a tool and a source of examples of manifolds. The theory of Lie groups is a vast subject and we will not even scrape the surface. A good reference is Bump [Bum13].

3.1 Definition

Definition 3.1. A **Lie group** is a smooth manifold G together with a group structure on G such that the maps $m\colon G \times G \to G$ defined by $m(g, h) := g \cdot h$, and $i\colon G \to G$ defined by $i(g) := g^{-1}$ are smooth. Let G and H be Lie groups. A **Lie group homomorphism** from G to H is a smooth group homomorphism $\rho\colon G \to H$. •

Example 3.2. $S^1 = \mathbf{R}/\mathbf{Z}$, $\mathrm{GL}_n(\mathbf{R})$, $\mathrm{GL}_n(\mathbf{C})$, $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ are Lie groups. ♠

Example 3.3. Let V be a vector space. If $\omega \in \Lambda^2 V^*$ is a non-degenerate 2–form on V, then $H = H(V, \omega)$, the **Heisenberg group** of (V, ω), is defined by $H := \mathrm{U}(1) \times V$ with the group operation

$$(e^{i\alpha}, v) \cdot (e^{i\beta}, w) := (e^{i\alpha + i\beta + 2\pi i \omega(v,w)}, v + w).$$

H is a Lie group. ♠

Theorem 3.4 (`reference?`). *Let G be a Lie group. Let $H < G$ be a subgroup. If H is closed, then it is a submanifold; hence: H is a Lie group.*

Theorem 3.5 (Yamabe [Yam50]; see also Goto [Got69]). *Let G be a Lie group. Let $H < G$ be a subgroup. If H is path-connected, then H is an immersed submanifold; hence: H is a Lie group.*

3.2 Lie group actions

Definition 3.6. Let X be a smooth manifold. Let G be a Lie group.

(1) A **(left) action** of G on X is a smooth map $L\colon G \times X \to X$ satisfying

$$L(\mathbf{1}, \cdot) = \mathrm{id}_X \quad \text{and} \quad L(g, L(h, \cdot) = L(gh, \cdot).$$

Define $L_g \in \mathrm{Diff}(X)$ by $L_g := L(g, \cdot)$ and abbreviate $g \cdot x = L(g, x)$.

(2) The **orbit** of $x \in X$ is

$$G \cdot x := \{g \cdot x : g \in G\}.$$

(3) The **stabiliser** of $x \in X$ is

$$G_x := \{g \in G : g \cdot x = x\}.$$

(4) The action of G on X is **free** if $G_x = \mathbf{1}$ for every $x \in X$.

(5) The action of G on X is **proper** if the map $(L, \mathrm{pr}_X)\colon G \times X \to X \times X$ is proper. •

(6) A **right action** of G on X is a smooth map $R\colon X \times G \to G$ satisfying

$$R(\cdot, \mathbf{1}) = \mathrm{id}_X \quad \text{and} \quad R(R(\cdot, g), h) = R(\cdot, gh).$$

Set $R_g(\cdot) := R(\cdot, g)$ and abbreviate $x \cdot g := R(x, g)$. If R is a right action, then $L(g, x) := R(x, g^{-1})$ defines a left action. The notions orbit, stabiliser, free, proper carry over to right actions in the obvious way.

In this section, actions are assumed to be left actions unless explicitly stated otherwise.

Example 3.7. If G is a Lie group, then G acts on itself on the left by left multiplication $L\colon\ G\times G \to G$,

$$L(g,h) \coloneqq g\cdot h.$$

The same formula also defines the action of G on itself on the right by right multiplication $R\colon\ G\times G$,

$$R(h,g) = h\cdot g.$$

These actions commute and G acts on itself on the left by conjugation $C\colon\ G\to G\to G$,

$$C(g,h) \coloneqq ghg^{-1}. \qquad \spadesuit$$

Exercise 3.8. Let G be a Lie group. Let $H < G$ be a closed subgroup. Prove that the action of H on G is free and proper.

Example 3.9. $\mathrm{U}(1)$ acts on S^{2n+1} via $e^{i\alpha}\cdot z = e^{i\alpha}z$. ♠

Example 3.10. Let $\theta\in\mathbf{R}$. $\mathbf{R}$ acts on $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ via $L(t,[x,y]) \coloneqq [x+t, y+\theta t]$. ♠

3.3 The slice theorem

Definition 3.11. Let X be a manifold. Let G be a Lie group acting on X. A **quotient** of X by G is a smooth manifold X/G together with a smooth G–invariant map $p\colon\ X\to X/G$ such that every G–invariant map $f\colon\ X\to Y$ uniquely factors though p. •

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"] & Y \\ X/G. \arrow[ur, dashed] & \end{tikzcd}$$

$$C^\infty(X/G,\cdot) \cong C^\infty(X,\cdot)^G$$

Which actions admit quotients?

Proposition 3.12. *Let X be a manifold. Let G be a Lie group. If G acts freely and properly on X, then it admits a quotient.*

Proof assuming that G is compact. Denote by X/G the topological quotient space and by $p\colon\ X\to X/G$ the projection map. X/G is paracompact and Hausdorff, and p is open. (Exercise!)

Let $x\in X$. The map $G\to X, g\mapsto gx$ is a proper injective immersion. Therefore, the orbit $G\cdot x\subset X$ is a submanifold. Choose a G–invariant metric g on X. (This is a red herring. The proof requires no Riemannian geometry, but it psychologically helpful.) Identify

$$N_x(G\cdot x) \cong T_x(G\cdot x)^\perp \subset T_xX$$

For $\varepsilon > 0$ set $V_x := B_\varepsilon(0) \subset N_x(G \cdot x)$ and define $\jmath_x \colon G \times V_x \to X$ by

$$\jmath_x(g, v) := g \exp_x(v).$$

Provided $\varepsilon \ll 1$, $\jmath$ is a G–equivariant embedding. Set $S_x := \jmath_x(\{\mathbf{1}\} \times V_x)$ and $U_x := p(\tilde{U}_x)$. The map $p|_{S_x} \colon S_x \to U_x$ is a homeomorphism. Define $\phi_x \colon U_x \to V_x$ by

$$\phi_x := \mathrm{pr}_{V_x} \circ \jmath_x^{-1} \circ \left(p|_{S_x}\right)^{-1}.$$

The task at hand is to prove that the maps ϕ_x form a smooth atlas. Let $x, y \in X$. $U_x \cap U_y \neq \varnothing$ if and only if $(G \cdot S_x) \cap S_y \neq \varnothing$. By construction, $(g \cdot S_x) \cap S_x = \varnothing$ unless $g = 1$. Therefore, there is a unique map $\gamma_x^y \colon (G \cdot S_x) \cap S_y \to G$ satisfying $\gamma_x^y(z) \cdot z \in S_x$ or, equivalently, $\mathrm{pr}_G \circ \jmath_y^{-1}(\gamma_x^y(z) \cdot z) = \mathbf{1}$. By the implicit function theorem, γ_x^y is smooth. A moment's thought shows that the transition map $\phi_x \circ \phi_y^{-1}$ satisfies

$$\phi_x \circ \phi_y^{-1}(z) = \mathrm{pr}_{V_x} \circ \jmath_x\left(\gamma_x^y(\jmath_y^{-1}(\mathbf{1}, z)) \cdot \jmath_y^{-1}(\mathbf{1}, z)\right).$$

Therefore, it is smooth. This finishes the construction of the smooth atlas on X/G.

The universal property is evident from the construction. ■

Remark 3.13. For non-compact G one first proves that $G \cdot x$ is a submanifold and then produces an S_x in some (quite arbitrary way). ♣

Definition 3.14. A **homogeneous space** is a smooth manifold X together with a transitive G action. •

Proposition 3.15. *If X is a homogeneous space, then the map $G/G_{x_0} \to X$ induced by $g \mapsto g \cdot x_0$ is a diffeomorphism.* ■

Example 3.16. $\mathbf{C}P^n \cong S^{2n+1}/\mathrm{U}(1)$. ♠

Example 3.17. $\mathrm{Gr}_r(\mathbf{R}^n) \cong O(n)/(\mathrm{O}(r) \times \mathrm{O}(n-r))$. ♠

`For future versions: One could do a lengthy discussion of the slice theorem and its consequences here.`

3.4 Lie algebra

Proposition 3.18. *Let G be a Lie group. Denote by*

$$\mathrm{Lie}(G) := \mathrm{Vect}(G)^L := \{\xi \in \mathrm{Vect}(G) : L_g^*\xi = \xi \text{ for every } g \in G\}.$$

the space of left-invariant vector fields on G.

(1) $\mathrm{Lie}(G) \subset \mathrm{Vect}(G)$ *is a Lie subalgebra.*

(2) *For $g \in G$ and $\xi \in \mathrm{Lie}(G)$, $R_g^*\xi \in \mathrm{Lie}(G)$.*

Proof. (1) is obvious. (2) holds because R_g and L_g commute. ■

Definition 3.19. Let G be a Lie group. The **Lie algebra of** G is the Lie algebra of left-invariant vector fields:

$$\mathfrak{g} = \mathrm{Lie}(G) := \mathrm{Vect}(G)^L.$$

The **adjoint representation** $\mathrm{Ad}\colon G \to \mathrm{End}(\mathrm{Lie}(G))$ is defined by

$$\mathrm{Ad}(g)\xi := R_g^*\xi.$$

The **adjoint representation** $\mathrm{ad}\colon \mathrm{Lie}(G) \to \mathrm{End}(\mathrm{Lie}(G))$ is defined by

$$\mathrm{ad}(\xi)\eta := [\xi, \eta].$$

•

Proposition 3.20. *Let G be a Lie group.*

(1) *The evaluation map* $\mathrm{ev}_{\mathbf{1}}\colon \mathrm{Vect}(G)^L \to T_{\mathbf{1}}G$ *is an isomorphism.*

(2) *For $g \in G$ and $\xi \in \mathrm{Vect}(G)^L$*

$$\mathrm{Ad}(g)\xi = \mathrm{ev}_{\mathbf{1}}^{-1} \circ T_{\mathbf{1}}C_g \circ \mathrm{ev}_{\mathbf{1}}(\xi).$$

(3) *For $\xi, \eta \in \mathrm{Vect}(G)^L$*

$$T_{\mathbf{1}}\,\mathrm{Ad}(\mathrm{ev}_{\mathbf{1}}(\xi))\eta = [\xi, \eta].$$

(4) *If $\rho\colon G \to H$ is a Lie group homomorphism, then* $\mathrm{Lie}(\rho)\colon \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ *defined by*

$$\mathrm{Lie}(\rho) = \mathrm{ev}_{\mathbf{1}}^{-1} \circ T_{\mathbf{1}}\rho \circ \mathrm{ev}_{\mathbf{1}}$$

is a Lie algebra homomorphism.

Proof. A left-invariant vector field v satisfies

$$v_g = T_{\mathbf{1}}L_g(v_{\mathbf{1}}).$$

Therefore, it is determined by $v_{\mathbf{1}}$. Conversely, the above formula defines a left-invariant vector field. This proves (1).

To prove (2), compute

$$\begin{aligned}
\mathrm{ev}_{\mathbf{1}}(R_g^*\xi) &= T_g R_{g^{-1}}(\xi_g) \\
&= T_g R_{g^{-1}} T_{\mathbf{1}} L_g(\xi_{\mathbf{1}}) \\
&= T_{\mathbf{1}} C_g(\xi_{\mathbf{1}}).
\end{aligned}$$

To prove (3), observe that

$$\begin{aligned}
\mathrm{flow}_\xi^t(g) &= \mathrm{flow}_\xi^t(L_g(\mathbf{1})) \\
&= L_g(\mathrm{flow}_\xi^t(\mathbf{1})) \\
&= R_{\mathrm{flow}_\xi^t(\mathbf{1})} g.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
T_1 \operatorname{Ad}(\mathrm{ev}_1(\xi))\eta &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} R^*_{\mathrm{flow}^t_\xi(1)}(\eta) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\mathrm{flow}^t_\xi)^*(\eta) \\
&= [\xi, \eta].
\end{aligned}
$$

To prove (4), observe that by (2)

$$\operatorname{Ad}(\rho(g)) \circ \operatorname{Lie}(\rho)(\xi) = \operatorname{Lie}(\rho) \circ \operatorname{Ad}(g)(\xi).$$

By (3), this implies that $\operatorname{Lie}(\rho)$ is a Lie algebra homomorphism. ■

The following gadget turns out to be important for us later.

Definition 3.21. Let G be a Lie group. The **Maurer–Cartan form** $\mu \in \Omega^1(G, \operatorname{Lie}(G))$ is defined by

$$\mu_g(\xi) \coloneqq \mathrm{ev}_1^{-1} \circ T_g L_{g^{-1}}(\xi).$$ •

Proposition 3.22. *Let G be a Lie group.*

(1) *The Maurer–Cartan form μ satisfies $\mu(\xi) = \xi$ for every $\xi \in \operatorname{Lie}(G)$.*

(2) *For every $g \in G$*

$$R_g^* \mu = \operatorname{Ad}(g^{-1}) \circ \mu.$$

(3) *The Maurer–Cartan form μ satisfies the* ***Maurer–Cartan equation***

$$\mathrm{d}\mu + \tfrac{1}{2}[\mu \wedge \mu] = 0$$

Proof. (1) is obvious.

To prove (2), for $g \in G$ and $\xi \in \operatorname{Lie}(G)$ compute

$$(R_g^*\mu)(\xi) = \mu((R_g)_*\xi) = (R_{g^{-1}})^*\xi = \operatorname{Ad}(g^{-1})\xi.$$

To prove (3), compute

$$
\begin{aligned}
\big(\mathrm{d}\mu + \tfrac{1}{2}[\mu \wedge \mu]\big)(\xi, \eta) &= \mathscr{L}_\xi(\mu(\eta)) - \mathscr{L}_\eta(\mu(\xi)) - \mu([\xi, \eta]) \\
&\quad + \frac{1}{2}([\mu(\xi), \mu(\eta)] - [\mu(\xi), \mu(\eta)]) \\
&= 0.
\end{aligned}
$$ ■

Exercise 3.23. Let $\rho\colon G \to H$ be a Lie group homomorphism. Prove that

$$\rho^* \mu_H = \operatorname{Lie}(\rho) \circ \mu_G.$$

3.5 Exponential map

Definition 3.24. Let G be a Lie group. The **exponential map** $\exp\colon \mathrm{Lie}(G) \to G$ is defined by

$$\exp(\xi) := \mathrm{flow}^1_\xi(\mathbf{1}). \qquad \bullet$$

Exercise 3.25. (1) Prove that exp is well-defined.

(2) Let $\rho\colon G \to H$ be a Lie group homomorphism. Prove that

$$\rho \circ \exp(\xi) = \exp \circ \mathrm{Lie}(\rho)(\xi).$$

(3) Prove that

$$C_g \circ \exp = \exp \circ \mathrm{Ad}_g .$$

Definition 3.26. Let X be smooth manifold. Let G be a Lie group. Let $L\colon G\times X \to X$ be a smooth left action. The **infinitesimal action** of $\mathrm{Lie}(G)$ on X is the map $v = v^L\colon \mathrm{Lie}(G) \to \mathrm{Vect}(X)$ defined by

$$v_\xi(x) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L_{\exp(t\xi)}(x).$$

$\bullet$

Proposition 3.27. *Let X be smooth manifold. Let G be a Lie group. Let $L\colon G\times X \to X$ be a smooth left action. Denote by $v\colon \mathrm{Lie}(G) \to \mathrm{Vect}(X)$ the corresponding infinitesimal action.*

(1) *For every $\xi \in \mathrm{Lie}(G)$*

$$L_{\exp(t\xi)} = \mathrm{flow}^t_{v_\xi} .$$

(2) *For every $g \in G$ and $\xi \in \mathrm{Lie}(G)$*

$$v_{\mathrm{Ad}(g)\xi} = L^*_{g^{-1}} v_\xi$$

(3) *The infinitesimal action v is an Lie algebra anti-isomorphism; that is: for every $\xi, \eta \in \mathrm{Lie}(G)$*

$$v_{[\xi,\eta]} = -[v_\xi, v_\eta].$$

Remark 3.28. If R is a right action and L is the corresponding left-action, then $v^R = -v^L$. In particular, v^R is a Lie algebra homomorphism. ♣

Proof. (1) is obvious.

To prove (2), compute

$$\begin{aligned} v_{\mathrm{Ad}(g)\xi}(x) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L_{g\exp(t\xi)g^{-1}}(x) \\ &= T_{L_g(x)}L_g\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L_{\exp(t\xi)}L_{g^{-1}}(x)\right) \\ &= T_{L_g(x)}L_g\Big(v_\xi(L_{g^{-1}}(x))\Big) \\ &= (L^*_{g^{-1}}v_\xi)(x). \end{aligned}$$

To prove (3), differentiate

$$v_{\mathrm{Ad}(\exp(t\xi))\eta} = L^*_{\exp(-t\xi)} v_\eta = \left(\mathrm{flow}^t_{v_\xi}\right)^* v_\eta. \qquad \blacksquare$$

3.6 Haar volume form

Proposition 3.29. *Let G be a Lie group. Set $d := \dim G$. There is a unique left-invariant volume form up to multiplication by a non-zero constant:*

$$\dim \Omega^d(G)^L := \{v \in \Omega^d(G) : L_g^* v = v\} = 1.$$

Definition 3.30. Let G be a Lie group. A **Haar volume form** on G is a left-invariant volume form v on G. v is **normalised** if $\int_G v = 1$. •

Proof of Proposition 3.29. If $v \in \Omega^d(G)$ is left-invariant, then

$$v_g = v_1 \circ \Lambda^d T_g L_{g^{-1}}.$$

Therefore, v is uniquely determined by $v_1 \in \Lambda^d T_1^* G$. Conversely, every $v_1 \in \Lambda^d T_1^G$ determines a left-invariant $v \in \Omega^d(G)$. ■

Exercise 3.31. Let G be a Lie group. Let v be a Haar volume form on G. For every $g \in G$, $R_g^* v$ is a Haar volume form. The **modular function** of G is the function $\Delta \in C^\infty(G, \mathbf{R}^\times)$ defined by

$$\Delta(g) := \frac{R_g^* v}{v}.$$

(1) Prove that $\Delta = 1$ if and only if G admits a right-invariant Haar measure. These groups are **unimodular**.

(2) Prove that $\Delta\colon\ G \to \mathbf{R}^\times$ is a Lie group homomorphism.

(3) Prove that $\Delta = 1$ if G is compact.

(4) Define $i\colon\ G \to G$ by $i(g) := g^{-1}$. Prove that $i^* v = \Delta v$.

(5) Consider the Lie group

$$G := \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0, y \in \mathbf{R} \right\}.$$

Compute modular function of G.

3.7 The Killing form

Definition 3.32. Let $\mathfrak{g}$ be a Lie algebra. The **Killing form** $B \in S^2\mathfrak{g}^*$ is defined by

$$B(\xi, \eta) := \operatorname{tr}(\operatorname{ad}(\xi) \circ \operatorname{ad}(\eta)).$$ •

Exercise 3.33. Prove that

$$B([\xi, \eta], \zeta) = B(\eta, [\xi, \zeta]).$$

Definition 3.34. A Lie algebra is called **semisimple** if B is negative definite. G **semisimple** if $\mathrm{Lie}(G)$ is semisimple. •

Exercise 3.35. Prove that if $\mathfrak{g} = \mathfrak{gl}(n)$, then

$$B(\xi, \eta) = 2n \operatorname{tr}(\xi\eta) - 2 \operatorname{tr}(\xi) \operatorname{tr}(\eta).$$

Exercise 3.36. Prove that if $\mathfrak{g} = \mathfrak{su}(n)$, then

$$B(\xi, \eta) = 2n \operatorname{tr}(\xi\eta).$$

3.8 de Rham cohomology of manifolds with G–actions

Let X be a manifold. Let G be a Lie group. Let $L\colon G\times X \to X$ be an action. Such an action can tremendously simplify the computation of $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$. To see this define

$$\Omega^\bullet(X)^L := \{\alpha \in \Omega^\bullet(X) : L_g^*\alpha = \alpha \text{ for every } g \in G\}.$$

Exercise 3.37. Prove that $\mathrm{d}\Omega^\bullet(X)^L \subset \Omega^\bullet(X)^L$.

We have an inclusion of cochain complexes $i\colon \Omega^\bullet(X)^L \to \Omega^\bullet(X)$.

Proposition 3.38. *If G is connected and compact, then i induces an isomorphism $H^\bullet(i)\colon \mathrm{H}^\bullet(\Omega^\bullet(X)^L) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(X)$.*

Proof. Let ν be a normalized Haar volume form on G. Define $\mathrm{av}\colon \Omega^\bullet(X) \to \Omega^\bullet(X)^L$ by

$$\mathrm{av}(\alpha) := \int_G L_g^*\alpha\, \nu(g)$$

This is a cochain map and

$$\mathrm{av}\circ i = \mathrm{id}_{\Omega^\bullet(X)^L};$$

hence, $H^\bullet(\mathrm{av})$ is a left inverse of $H^\bullet(i)$.

To show that $H^\bullet(\mathrm{av})$ also is a right inverse we proceed as follows. Denote by

$$\int_G \colon \Omega^\bullet(G\times X) \to \Omega^{\bullet-\dim G}(X)$$

the fibre integration map defined by the property that

$$\left(\int_G \alpha\right)_x (v_1,\dots,v_k) = \int_{g\in G} \alpha_{(g,x)}(v_1,\dots,v_k,\dots)$$

for $(v_1,\dots,v_k) \in T_xM$.

Exercise 3.39. Verify that this is a chain map.

We can now write $i\circ\mathrm{av}$ as the composition

$$i\circ\mathrm{av} = \mathrm{av}_\nu := \int_G \circ\, (\nu\wedge\cdot)\circ L^*.$$

$H^\bullet(\nu\wedge\cdot)$ depends only on $[\nu]$; hence, if $\eta \in \Omega^{\dim G}(G)$ with $\int_G \eta = 1$, i.e., $[\mu] = [\eta]$, then $H^\bullet(i)\circ H^\bullet(\mathrm{av}) = H^\bullet(\mathrm{av}_\eta)$.

Heuristically, if we could take η to be a δ volume form at $\mathbf{1}\in G$, then $\mathrm{av}_\delta = \mathrm{id}_{\Omega^\bullet(X)}$; and as the induced map on cohomology does not change when as η becomes closer and closer to δ the proof is complete. It is not terribly difficult to make the above heuristic rigorous, but we will follow the standard route and chose η supported in neighbourhood U of $\mathbf{1}\in G$ which is smoothly contractible.

If $j\colon U \times X \to G \times X$, then

$$\mathrm{av}_\eta = \int_U \circ (\eta \wedge \cdot) \circ j^* \circ L^*.$$

Since U is smoothly contractible, on cohomology $j^* \circ \rho^* = (\rho \circ j)^*$ is the same as pulling back via the projection $\mathrm{pr}_X\colon U \times X \to X$. However,

$$\int_U \circ (\eta \wedge \cdot) \circ \mathrm{pr}_X^* = \mathrm{id}_{\Omega^\bullet(X)}. \qquad \blacksquare$$

Remark 3.40. The advantage of not following the heuristic argument, is that one can (at least in principle) write down a chain homotopy h such that

$$i \circ \mathrm{av} - \mathrm{id} = \mathrm{d} \circ h + h \circ \mathrm{d}. \qquad \clubsuit$$

Let us now use Proposition 3.38 to compute the de Rham cohomology in a few simple cases.

Example 3.41. $G = \mathrm{SO}(n+1)$ acts transitively on S^n. The stabilizer of any $x \in S^n$ is $\mathrm{SO}(T_x S^n) \cong \mathrm{SO}(n)$. A moments thought shows that

$$\begin{aligned} \Omega^\bullet(S^n)^G &= (\Lambda^* T_x S^n)^{\mathrm{SO}(T_x S^n)} \\ &= (\Lambda^*(\mathbf{R}^n)^*)^{\mathrm{SO}(n)} \\ &= \mathbf{R} \cdot 1 \oplus \mathbf{R} \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \\ &= \mathbf{R}[0] \oplus \mathbf{R}[n]. \end{aligned}$$

The differential necessarily vanishes (for dimension reasons if $n > 1$); hence, this already is $\mathrm{H}^\bullet_{\mathrm{dR}}(S^n)$. The last step in the above computation is a fact from the representation theory of $\mathrm{SO}(n)$. $\spadesuit$

Example 3.42. $G = \mathrm{U}(n+1)$ acts transitively on $\mathbf{C}P^n$ with stabiliser of any $\mathbf{C}\cdot z \in \mathbf{C}P^n$ isomorphic to $\mathrm{U}(z^\perp) = \mathrm{U}(T_z \mathbf{C}P^n) \cong \mathrm{U}(n)$. We compute

$$\Omega^\bullet(\mathbf{C}P^n)^{U(n+1)} \otimes \mathbf{C} = (\Lambda^*(\mathbf{C}^n)^*)^{\mathrm{U}(n)}.$$

The latter is generated as a $\mathbf{C}$*–algebra* by the standard symplectic form

$$\omega := \sum_{i=1}^n \frac{i\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i}{2};$$

that is,

$$\begin{aligned} (\Lambda^*(\mathbf{C}^n)^*)^{\mathrm{U}(n)} &= \mathbf{C} \cdot 1 \oplus \mathbf{C} \cdot \omega \oplus \cdots \oplus \mathbf{C} \cdot \omega^n \\ &= \mathbf{C}[\omega]/(\omega^{n+1}). \end{aligned}$$

Since this complex is supported in even degrees, the differential vanishes and this already is $H^\bullet_{\mathrm{dR}}(\mathbf{C}P^n) \otimes \mathbf{C}$. $\spadesuit$

Example 3.43. Let G be a Lie group. Let $\mathfrak{g} := \mathrm{Lie}(G)$. If we consider the L action of G on itself, then

$$\Omega^\bullet(G)^L = \Lambda^*\mathfrak{g}^* = \mathrm{Hom}(\Lambda^*\mathfrak{g}, \mathbf{R}).$$

The differential, which is usually denoted by δ, does not vanish. It can be computed to be

$$\begin{aligned}(\delta\alpha)(\xi_1, \dots, x_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}\xi_i \cdot \alpha(\xi_1, \dots, \widehat{\xi_i}, \dots, \xi_{k+1}) \\ &+ \sum_{i<j=1}^{k+1}(-1)^{i+j}\alpha([\xi_i, \xi_j], \xi_1, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_{k+1});\end{aligned}$$

in fact, since the Lie algebra acts trivially on $\mathbf{R}$ the first term vanishes. $(\mathrm{Hom}(\Lambda^*\mathfrak{g}, \mathbf{R}), \delta)$ is the **Chevalley–Eilenberg cochain complex** (although it was discovered decades before Chevalley–Eilenberg by Cartan). It is defined for every Lie algebra $\mathfrak{g}$. Its cohomology

$$\mathrm{H}^\bullet(\mathfrak{g}) := \mathrm{H}^\bullet(C^\bullet(\mathfrak{g}), \delta).$$

is the **Lie algebra cohomology** of $\mathfrak{g}$.

If V is any representation of $\mathfrak{g}$, then δ as defined above makes $\mathrm{Hom}(\Lambda^*\mathfrak{g}, M)$ into a cochain complex. $\mathrm{H}^\bullet(\mathfrak{g}; V) := \mathrm{H}^\bullet(\mathrm{Hom}(\Lambda^*\mathfrak{g}, V))$ is called the **Lie algebra cohomology** of $\mathfrak{g}$ with coefficients in V. Proposition 3.38 shows that $\mathrm{H}^\bullet_{\mathrm{dR}}(G) = \mathrm{H}^\bullet(\mathfrak{g}; \mathbf{R})$. The notion of Lie algebra cohomology goes back to Chevalley and Eilenberg [CE48]. ♠

Remark 3.44. Let $\rho\colon G \to \mathrm{GL}(V)$ be a Lie group representation. Consider the trivial vector bundle $\mathrm{pr}_G\colon \underline{V} = G \times V \to G$. G acts on $\underline{V}$ by $L \times \rho$. This turns $\underline{V}$ into a G–equivariant vector bundle. The formula $\mathrm{d}_\nabla s := \mathrm{d}s + (\mathrm{Lie}(\rho) \circ \mu)s$ defines a G–equivariant flat connection on $\underline{V}$. A moment's thought shows that $\mathrm{H}^\bullet_{\mathrm{dR}}(G, \underline{V}) = \mathrm{H}^\bullet(\mathfrak{g}, V)$. ♣

Example 3.45. Let G be a connected compact Lie group. Let $H < G$ be a connected closed Lie subgroup. Set $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{h} := \mathrm{Lie}(H)$. Set $C^\bullet(\mathfrak{g}) := \mathrm{Hom}(\Lambda^\bullet\mathfrak{g}, \mathbf{R})$ and define δ as above. Denote by $C^\bullet(\mathfrak{g}, \mathfrak{h})$ the subcomplex of those $\alpha \in C^\bullet(\mathfrak{g})$ with

$$i_\xi\alpha = 0 \quad \text{and} \quad i_\xi\delta\alpha = 0 \quad \text{for every} \quad \xi \in \mathfrak{h}.$$

The **relative Lie algebra cohomology** of $\mathfrak{g} \supset \mathfrak{h}$ is

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{h}) := \mathrm{H}^\bullet(C^\bullet(\mathfrak{g}, \mathfrak{h}), \delta).$$

The adjoint action of the Lie algebra $\mathfrak{h}$ on $\mathfrak{g}$ descends to $\mathfrak{g}/\mathfrak{h}$. Denote by $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h}, \mathbf{R})^\mathfrak{h}$ the corresponding invariant subspace of $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h}, \mathbf{R})$. $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h}, \mathbf{R})^\mathfrak{h}$ can be regarded as a subspace $C^\bullet(\mathfrak{g})$. A moment's thought identifies it as $C^\bullet(\mathfrak{g}, \mathfrak{h})$. Moreover, $\mathrm{Hom}(\Lambda^\bullet\mathfrak{g}/\mathfrak{h}, \mathbf{R})^\mathfrak{h} \cong \Omega^\bullet(G/H)^H$ and the differentials δ and d agree. Therefore,

$$\mathrm{H}^\bullet_{\mathrm{dR}}(G/H) \cong \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{h}).$$ ♠

Exercise 3.46. Show that $\mathrm{H}^1(\mathfrak{g}, \mathbf{R}) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$.

Example 3.47. Set $\tilde{R}(g,h) := hg^{-1}$. If we consider the action $L \times \tilde{R}$ of $G \times G$ on G, then

$$\Omega^\bullet(G)^{L\times\tilde{R}} = (\Lambda^\bullet \mathfrak{g}^*)^{\mathrm{Ad}}.$$

Here Ad denotes the coadjoint action. Suppose $\alpha \in \Omega^k(G)^{L\times\tilde{R}}$, that is, α is left invariant and right invariant. Since derivative of the map $i\colon G \to G, g \mapsto g^{-1}$ is

$$\mathrm{d}_g i = -\mathrm{d}L_{g^{-1}} \circ \mathrm{d}R_{g^{-1}},$$

we have

$$i^*\alpha = (-1)^k \alpha.$$

It follows that

$$\mathrm{d}\alpha = (-1)^k \mathrm{d}i^*\alpha = (-1)^k i^*\mathrm{d}\alpha = -\mathrm{d}\alpha;$$

hence, the differential vanishes on $\Omega^\bullet(G)^{L\times\tilde{R}}$ and

$$\mathrm{H}^\bullet_{\mathrm{dR}}(G) = (\Lambda^* \mathfrak{g}^*)^{\mathrm{Ad}}.$$

The formula

$$\gamma(\xi,\eta,\zeta) := B([\xi,\eta],\zeta)$$

defines an element $\gamma \in (\Lambda^3\mathfrak{g}^*)^{\mathrm{Ad}}$. If G is semisimple, then $\gamma \neq 0$; hence $b_3(G) \geqslant 1$. ♠

Example 3.48. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation. $\mathrm{Lie}(\rho)\colon \mathfrak{g} \to \mathfrak{gl}(V)$ can be regarded as an element of $\theta_\rho \in \mathfrak{g}^* \otimes \mathfrak{gl}(V)$. Evidently, θ_ρ is invariant under the action induces by Ad and ρ and so is $\theta_\rho^{\wedge k} \in \Lambda^k\mathfrak{g}^* \otimes \mathfrak{gl}(V)$. Therefore, $\mathrm{tr}(\theta_\rho^{\wedge k}) \in \Lambda^k\mathfrak{g}^*$. ♠

Remark 3.49. The multiplication map $m\colon G \times G \to G$ induces a map $\Delta\colon \mathrm{H}^\bullet(G) \to \mathrm{H}^\bullet(G) \otimes \mathrm{H}^\bullet(G)$. This turns $\mathrm{H}^\bullet(G)$ into a **Hopf algebra**. ♣

4 Principal bundles

4.1 Definition and examples

Definition 4.1. Let G be a Lie group. A G–**principal fibre bundle** is a smooth map $p\colon P \to B$ together with a right action $R\colon P \times G \to P$ such that for every $b \in B$ there are an open subset $b \in U \subset B$, and a G–equivariant diffeomorphism $\tau\colon p^{-1}(U) \to U \times G$ such that

$$\mathrm{pr}_U \circ \tau = p;$$

that is: the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\tau"] \arrow[dr, "p"'] & U \times G \arrow[d, "\mathrm{pr}_U"] \\ & U \end{tikzcd}$$

commutes. G is the **structure group** of (p,R). •

Remark 4.2. In the situation of Definition 4.1, $p\colon P \to B$ is a quotient $P \to P/G$, R is free, fibre-preserving, and its restriction to any fibre $p^{-1}(b)$ is transitive. ♣

Example 4.3. The **trivial** G**–principal bundle** over B is $\mathrm{pr}_B\colon B \times G \to B$ with $R((b,g),h) := (b, gh)$. ♠

Example 4.4. Let B be a smooth manifold. Let $V \to B$ be a vector bundle of rank r. Denote by

$$\mathrm{Fr}(V) := \{(b,\phi) : b \in B, \phi\colon \mathbf{R}^r \to V_b \text{ isomorphism}\}$$

the **frame bundle of** V. Denote by $p\colon \mathrm{Fr}(V) \to B$ the projection map. $\mathrm{GL}_r(\mathbf{R})$ acts on the right of $\mathrm{Fr}(V)$ via

$$R((b,\phi),\tau) := (b, \phi \circ \tau).$$

There is a unique smooth structure on $\mathrm{Fr}(V)$ such that (p,R) is a $\mathrm{GL}_r(\mathbf{R})$–principal bundle. ♠

Example 4.5. Let $k, r \in \mathbf{N}$ with $k < r$. The **Stiefel manifold** $\mathrm{St}^*_k(\mathbf{R}^r)$ is the submanifold

$$\mathrm{St}^*_k(\mathbf{R}^r) := \{(v_1,\ldots,v_k) \in (\mathbf{R}^r)^k : v_1,\ldots,v_k \text{ linearly independent}\}$$

or, equivalently,

$$\mathrm{St}^*_k(\mathbf{R}^r) := \{A \in \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^r) : A \text{ is injective}\}.$$

$\mathrm{GL}_k(\mathbf{R})$ acts on the right of $\mathrm{St}^*_k(\mathbf{R}^r)$ via $R(A,\tau) := A \circ \tau$. The map $p\colon \mathrm{St}^*_k(\mathbf{R}^r) \to \mathrm{Gr}_k(\mathbf{R}^r)$ defined by

$$p(A) := \operatorname{im} A$$

together with R is a $\mathrm{GL}_k(\mathbf{R})$–principal bundle. Of course, $\mathrm{St}^*_k(\mathbf{R}^r)$ is the frame bundle of the tautological bundle over $\mathrm{Gr}_k(\mathbf{R}^r)$. ♠

Proposition 4.6. *Let G be a compact Lie group. Let P be smooth manifold. If $R\colon P \times G \to P$ is a free right action, then $p\colon P \to B := P/G$ together with R is a G–principal fibre bundle.*

Proof. Exercise. ■

Exercise 4.7. The Hopf bundle $p\colon S^{2n+1} \to \mathbf{C}P^n$ together with the right action $R\colon S^{2n+1} \times \mathrm{U}(1) \to S^{2n+1}$ defined by $R(z, e^{i\alpha}) := ze^{i\alpha}$ is a $\mathrm{U}(1)$–principal bundle.

Exercise 4.8. Let $n \in \mathbf{N}$. $\mathrm{Sp}(1) := \{q \in \mathbf{H} : |q| = 1\}$ acts on $S^{4n+3} \subset \mathbf{H}^{n+1}$ by $R\colon S^{4n+3} \times \mathrm{Sp}(1) \to S^{4n+3}$ with

$$R_+(x,q) := q^{-1}x.$$

The quotient of R is the $\mathbf{H}P^n$, the space of $\mathbf{H}$–left modules $L \subset \mathbf{H}^{n+1}$ of dimension 1. The projection map $q\colon S^{4n+3} \to \mathbf{H}P^n$ together with R is an $\mathrm{Sp}(1)$–principal bundle—the **quaternionic Hopf bundle**.

∼

Definition 4.9. Let $(p\colon P \to B, R)$ and $(q\colon Q \to B, S)$ be G–principal fibre bundles. A **morphism** $(p, R) \to (q, S)$ is a G–equivariant smooth map $\phi\colon P \to Q$ satisfying $q \circ \phi = p$. A **gauge transformation** of (p, R) is an isomorphism $(p, R) \to (p, R)$. The **gauge group** of (p, R) is the group of all gauge transformations of (p, R) and denoted by

$$\mathscr{G}(p, R).$$ •

Exercise 4.10. Prove that every morphism of G–principal bundles is an isomorphism.

Exercise 4.11. Let (p, R) be a G–principal bundle. Prove that (p, R) is isomorphic to the trivial G–principal bundle if and only if p admits a section.

The gauge group plays a very important role. The following concrete description of the gauge group is quite useful.

Proposition 4.12. *Let $(p\colon P \to B, R)$ be G–principal fibre bundles. Denote by $C^\infty(P, G)^C$ the subspace of $u \in C^\infty(P, G)$ satisfying*

$$R_g^* u = C_{g^{-1}} u, \quad \textit{i.e.,} \quad u(xg) = g^{-1} u(x) g$$

for every $g \in G$

(1) *For every $u \in C^\infty(P, G)^C$ the map $\tilde{u} \in C^\infty(P, P)$ defined by*

$$\tilde{u}(x) = x \cdot u(x)$$

is a gauge transformation.

(2) *The map $\tilde{\cdot}\colon C^\infty(P, G)^C \to \mathscr{G}(p, R)$ defined by the above is a bijection; in fact, a group isomorphism.*

Proof. To prove (1), it suffices to verify that $\tilde{u}$ is G–equivariant:

$$\tilde{u}(xg) = xg \cdot u(xg) = xg \cdot g^{-1} u(x) g = x u(x) g = \tilde{u}(x) g.$$

Evidently, the map $\tilde{\cdot}\colon C^\infty(P, G)^C \to \mathscr{G}(p, R)$ is injective. To prove that it is surjective, observe that if $\tilde{u} \in \mathscr{G}(p, R)$ then for every $x \in P$ there is a unique $u(x) \in G$ such that $\tilde{u}(x) = x \cdot u(x)$. The map $u \in \mathrm{Map}(P, G)$ thus defined is smooth. The G–equivariance of $\tilde{u}$ follows from the G–equivariance of u.

It remains to prove that $\tilde{\cdot}$ is a group homomorphism. To see this observe that

$$\tilde{v}(\tilde{u}(x)) = x \cdot u(x) \cdot v(x \cdot u(x)) = x \cdot v(x) \cdot u(x).$$ ■

Definition 4.13. Denote by $\gamma_\cdot\colon \mathscr{G}(p, R) \to C^\infty(P, G)^C$, $u \mapsto \gamma_u$ the inverse of $\tilde{\cdot}\colon C^\infty(P, G)^C \to \mathscr{G}(p, R)$. `(This notation is not standard and probably a bad choice. I'm not aware of any standard notation for this map.)` •

Example 4.14. If G is an abelian group, then $C^\infty(P,G)^C$ consists precisely of the G–invariant maps $C^\infty(P,G)^G \cong C^\infty(B,G)$. Therefore, $\mathscr{G}(p,R) \cong C^\infty(B,G)$.

Let G be an arbitrary Lie group. Denote by $Z(G) := \{g \in G : gh = hg \text{ for every } h \in G\}$ the **center of** G. Evidently,

$$C^\infty(B, Z(G)) \cong C^\infty(P, Z(G))^C \hookrightarrow \mathscr{G}(p,R). \qquad \spadesuit$$

Example 4.15. Let $V \to B$ be a vector bundle of rank r. Consider the frame bundle $(p\colon \mathrm{Fr}(V) \to B, R)$. For every $\lambda \in \mathbf{R}^*$ the map $\varepsilon\colon \mathrm{Fr}(V) \to \mathrm{Fr}(V)$ defined by

$$\varepsilon(b,(v_1,\dots,v_r)) := \varepsilon(b,(\lambda v_1,\dots,\lambda v_r))$$

is an example of a gauge transformation. ♠

Example 4.16. For the trivial G–principal bundle $(p\colon B\times G \to B, R)$ the map $C^\infty(B,G) \to \mathscr{G}(p,R)$ defined by $C^\infty(B,G) \ni \gamma \mapsto u_\gamma$ defined by $u_\gamma(b,g) := (b,\gamma(b)g)$ is a group isomorphism. (Observe that for general G–principal bundle the left-multiplication is not available.) ♠

Proposition 4.17. *Let $p\colon P \to B$ with R be a G–principal bundle. Let $f\colon A \to B$ be a smooth map. Denote by $f^*p\colon f^*P \to A$ the pullback of $p\colon P \to B$. Define $f^*R\colon f^*P \times G \to f^*P$ by*

$$f^*R((a,p),g) = (a, R(p,g)).$$

(f^*p, f^*R) is a G–principal fibre bundle.

Definition 4.18. The G–principal bundle (f^*p, f^*R) constructed above is the **pullback** of (p,R) via f. •

∼

Example 4.19. Let G be a Lie group. Here is how to construct a fiber bundle $p\colon X \to B$ with fibres diffeomorphic to G but which cannot be turned into a G–principal fibre bundle. Let $\phi \in \mathrm{Diff}(G)$. Denote by $X_\phi := ([0,1]\times G)/\!\sim$ with $\sim$ generated by $(1,x) \sim (0,\phi(x))$ the mapping torus of ϕ. The projection $p\colon X_\phi \to S^1$ is a fibre bundle whose fibres are diffeomorphic to G.

The right action of G on $[0,1]\times G$ descends to X_ϕ if and only if $\phi(g) = hg$ for some $h \in G$. Therefore, usually, p cannot be turned into a G–principal fibre in the obvious way. In fact, often, p cannot be turned into a G–principal at all. To see this, observe that if there is a $g \in G$ such that g and $\phi(g)$ lie in the same connected component of G, then p admits a section. Therefore, p is isomorphic to $\mathrm{pr}_{S^1}\colon S^1 \times G \to S^1$. However, this implies that ϕ is isotopic to id_G.

To make this concrete consider the orientation reversing diffeomorphism $\phi \in \mathrm{Diff}(\mathrm{U}(1))$ defined by $\phi(e^{i\alpha}) := e^{-i\alpha}$. The mapping torus T_ϕ is the Klein bottle; hence, not diffeomorphic to $S^1 \times \mathrm{U}(1)$. However, the projection $p\colon T_\phi \to S^1$ admits a section $s(b) := [b, \mathbf{1}]$. ♠

4.2 G–principal connections

Proposition 4.20. *Let $(p\colon P \to B, R\colon P\times G \to P)$ be a G–principal fibre bundle. The map $\kappa\colon P\times \mathfrak{g} \to V_p$ defined by*

$$\kappa(p,\xi) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R(p,\exp(t\xi)) = T_1R_p \circ \mathrm{ev}_1(\xi).$$

is an isomorphism. ■

The isomorphism κ simplifies the theory of connections (or at least it makes it more concrete).

Definition 4.21. Let $(p\colon P \to B, R\colon P\times G \to P)$ be a G–principal fibre bundle. For $\xi \in \mathfrak{g}$ define $v_\xi \in \Gamma(V_p)$ by

$$v_\xi(p) := \kappa(p,\xi).$$ •

Exercise 4.22. Prove that $\mathfrak{g} \to \mathrm{Vect}(P)$, $\xi \mapsto v_\xi$ is a Lie algebra homomorphism.

Exercise 4.23. Construct a fibre bundle $p\colon X \to B$ whose fibres are diffeomorphic to S^1 but which cannot be equipped with a S^1 action R making (p,R) into an S^1–principal bundle.

Definition 4.24. Let $(p\colon P \to B, R\colon P\times G \to P)$ be a G–principal fibre bundle. A **G–principal connection** is an Ehresmann connection $A\colon TP \to V_p \oplus p^*TB$ satisfying

$$A \circ TR_g = (TR_g \oplus \mathrm{id}_{p^*TB}) \circ A.$$

The **G–principal connection 1–form** of A is the 1–form $\theta_A \in \Omega^1(P,\mathfrak{g})$ defined by

$$\theta_A := \mathrm{pr}_{\mathfrak{g}} \circ \kappa^{-1} \circ \mathrm{pr}_{V_p} \circ A.$$

The space of connections on (p,R) is denoted by $\mathscr{A}(p,R)$. •

Proposition 4.25. *Let $(p\colon P \to B, R\colon P\times G \to P)$ be a G–principal fibre bundle.*

(1) *An Ehresmann connection A on p is a G–principal connection if and only if for every $g \in R$*

$$TR_g(H_A) = H_A.$$

(2) *An Ehresmann connection A on p is a G–principal connection if and only if the 1–form*

$$\theta_A := \mathrm{pr}_{\mathfrak{g}} \circ \kappa^{-1} \circ \mathrm{pr}_{V_p} \circ A.$$

satisfies

$$R_g^*\theta_A = \mathrm{Ad}(g^{-1}) \circ \theta_A$$

for every $g \in G$. ■

Proposition 4.26. *Every G–principal connection is complete.*

Proof. Exercise. ■

Definition 4.27. Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let V be a finite-dimensional vector space. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G. G acts on $\Omega^\bullet(P, V)$ via

$$g \cdot \alpha \coloneqq \rho(g) \circ R_g^* \alpha.$$

Set

$$\Omega^\bullet_{\mathrm{hor}}(P, V)^G = \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho \coloneqq \{\alpha \in \Omega^\bullet_{\mathrm{hor}}(P, V) : g \cdot \alpha = \alpha \text{ for every } g \in G\}. \qquad \bullet$$

Remark 4.28. The above construction is particularly important for the adjoint representation $\rho\colon G \to \mathrm{GL}(\mathrm{Lie}(G))$. ♣

Proposition 4.29. *Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. $\mathscr{A}(p, R)$ is an affine space modelled on $\Omega^1_{\mathrm{hor}}(P, \mathrm{Lie}(G))^{\mathrm{Ad}}$.*

Proof. Exercise; cf. Proposition 2.29. ■

Proposition 4.30. *Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let $A \in \mathscr{A}(p, R)$. Denote by $\sigma_A\colon TP \to H_A$ the projection onto H_A. Let V be a finite-dimensional vector space. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G. Define $\mathrm{d}_A\colon \Omega^\bullet_{\mathrm{hor}}(P, V) \to \Omega^{\bullet+1}_{\mathrm{hor}}(P, V)$ by*

$$(\mathrm{d}_A\alpha)(v_1, \ldots, v_k) \coloneqq (\mathrm{d}\alpha)(\sigma_A(v_1), \ldots, \sigma_A(v_k)).$$

(This is essentially the map $\mathrm{d}_A^{1,0}$ from Proposition 2.74.)

1. *Let $\alpha \in \Omega^\bullet_{\mathrm{hor}}(P, V)$. If α is G–invariant, then so is $\mathrm{d}_A\alpha$. Therefore, d_A induces a map $\mathrm{d}_A\colon \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho \to \Omega^{\bullet+1}_{\mathrm{hor}}(P, V)^\rho$.*
2. *The map $\mathrm{d}_A\colon \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho \to \Omega^{\bullet+1}_{\mathrm{hor}}(P, V)^\rho$ satisfies*

$$\mathrm{d}_A\alpha = \mathrm{d}\alpha + (\mathrm{Lie}(\rho)\theta_A) \wedge \alpha.$$

Proof. To prove (1), observe that $\sigma_A \circ TR_g = TR_g \circ \sigma_A$ and compute

$$\begin{aligned}
\rho(g)R_g^*\mathrm{d}_A\alpha(v_1, \ldots, v_k) &= \rho(g)\mathrm{d}\alpha(\sigma_A \circ TR_g(v_1), \ldots, \sigma_A \circ TR_g(v_k)) \\
&= \rho(g)\mathrm{d}\alpha(TR_g \circ \sigma_A(v_1), \ldots, TR_g \circ \sigma_A(v_k)) \\
&= (\rho(g)R_g^*\mathrm{d}\alpha)(\sigma_A(v_1), \ldots, TR_g \circ \sigma_A(v_k)).
\end{aligned}$$

The assertion follows since

$$R_g^*\mathrm{d}\alpha = \mathrm{d}R_g^*\alpha = \rho(g^{-1})\mathrm{d}\alpha.$$

One can deduce (2) more or less directly from Proposition 2.74 (2) (but we didn't prove that in class). Instead let us verify this directly. It suffices to verify the formula on $v_0, \ldots, v_k$ which are each either vertical or horizontal. If they are all horizontal, then second term on the RHS vanishes and the formula hold. If at least two are vertical, then both sides vanish. Therefore, without loss of generality, v_0 is vertical and $v_1, \dots, v_k$ are horizontal. Extend the v_j to local G–invariant vector fields with $v_0 = v_\xi$ and $v_2, \dots, v_{k+1}$ horizontal. In this case,

$$\begin{aligned}
\mathrm{d}\alpha(v_\xi, v_1 \dots, v_k) &= \mathscr{L}_{v_\xi}(\alpha(v_1, \dots, v_k)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (R^*_{\exp(t\xi)}\alpha)(v_1, \dots, v_k)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho(\exp(-t\xi)) \circ \alpha(v_1, \dots, v_k)) \\
&= -\operatorname{Lie}(\rho)(\xi) \circ \alpha(v_1, \dots, v_k) \\
&= -(\operatorname{Lie}(\rho)(\theta_A) \wedge \alpha)(v_\xi, v_1, \dots, v_k).
\end{aligned}$$

■

Remark 4.31. The maps $\mathrm{d}_A \colon \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho \to \Omega^{\bullet+1}_{\mathrm{hor}}(P, V)^\rho$ are compatible with the usual operations on representations, in particular, $\otimes$ and $\oplus$. One consequence of this is that $\Omega^\bullet_{\mathrm{hor}}(P, V)$ is a (left-)module over $\Omega^\bullet_{\mathrm{hor}}(P)^G \cong \Omega^\bullet(B)$ and d_A satisfies the corresponding Leibniz rule. ♣

Proposition 4.32. *Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let $A \in \mathscr{A}(p, R)$.*

(1) *There is a unique G–invariant horizontal 2–form $F_A \in \Omega^2_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$ such that for every $v, w \in \Gamma(H_A)$*

$$F_A(v, w) = -\theta_A([v, w]).$$

*F_A is the **curvature** of A.*

(2) *F_A can be computed as*

$$F_A = \mathrm{d}\theta_A + \frac{1}{2}[\theta_A \wedge \theta_A]$$

(3) *F_A satisfies the **Bianchi identity***

$$\mathrm{d}_A F_A = 0.$$

(4) *If $\rho\colon G \to \mathrm{GL}(V)$ is a finite-dimensional representation of G, then $\mathrm{d}_A \colon \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho \to \Omega^{\bullet+1}_{\mathrm{hor}}(P, V)^\rho$ satisfies*

$$\mathrm{d}_A \circ \mathrm{d}_A = (\operatorname{Lie}(\rho) \circ F_A) \wedge \cdot.$$

Remark 4.33. One sometimes sees the formula $F_A = \mathrm{d}_A\theta_A$. This is correct, but it easily leads to confusion. The issue is that one is tempted to forget the original definition of d_A and use Proposition 4.30 (2) instead; however: θ_A is (not at all) horizontal and this formula obviously does not hold for θ_A. ♣

Proof of Proposition 4.32. Define $F_A \coloneqq \mathrm{d}\theta_A + \frac{1}{2}[\theta_A \wedge \theta_A]$ and compute

$$F_A(v, w) = \mathscr{L}_v\theta_A(w) - \mathscr{L}_w\theta_A(v) - \theta_A([v, w]) + [\theta_A(v), \theta_A(w)].$$

This matches $-\theta_A(v, w)$ for $v, w \in \Gamma(H_A)$.

If $\xi \in \mathfrak{g}$ and $w \in \Gamma(H_A)$ is G–invariant, then

$$F_A(v_\xi, w) = \mathscr{L}_w \xi - \theta_A([v_\xi, w]) = 0.$$

The first term vanishes because ξ is constant. The second term vanishes because w is G–invariant.

For $\xi, \eta \in \mathfrak{g}$

$$F_A(v, w) = \mathscr{L}_{v_\xi}\eta - \mathscr{L}_{v_\eta}\xi - \theta_A([v_\xi, v_\eta]) + [\xi, \eta].$$

The first two term vanish because ξ,η are constant. The last two terms cancel because $\xi \mapsto v_\xi$ is a Lie algebra homomorphism. (This is, of course, essentially the proof of the Maurer–Cartan equation.)

The G–invariance of F_A follows from the G–invariance of θ_A. Thus (1) and (2) are proved.

$$\begin{aligned}
\mathrm{d}_A F_A &= (\mathrm{d} + [\theta_A \wedge \cdot])(\mathrm{d}\theta_A + \frac{1}{2}[\theta_A \wedge \theta_A]) \\
&= \frac{1}{2}([\mathrm{d}\theta_A \wedge \theta_A] - [\theta_A \wedge \mathrm{d}\theta_A]) + [\theta_A \wedge \mathrm{d}\theta_A] + \frac{1}{2}[\theta_A \wedge [\theta_A \wedge \theta_A]] \\
&= 0
\end{aligned}$$

because the first three term cancel and the last term vanishes by the Jacobi identity.

(4) follows by direct computation. ■

Proposition 4.34. *Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let $A \in \mathscr{A}(p, R)$. Let $\gamma\colon [0,1] \to B$ be a piecewise smooth path. The parallel transport tra_γ^A is G–equivariant; that is: for every $x \in p^{-1}(\gamma(0))$ and $g \in G$*

$$\mathrm{tra}_\gamma^A(x) \cdot g = \mathrm{tra}_\gamma^A(x \cdot g).$$

Proof. This is a consequence of A being G–invariant. ■

Remark 4.35. Let $x_0 \in p^{-1}(b_0)$. There is a $g \in G$ such that $\mathrm{tra}_\gamma^A(x_0) = x_0 \cdot g$. Every element of $p^{-1}(b_0)$ is of the form $x_0 \cdot h$ for some $h \in G$. Since tra_A^γ is G–invariant,

$$\mathrm{tra}_\gamma^A(x_0 h) = \mathrm{tra}_\gamma^A(x_0)h = x_0 \cdot gh. \quad ♣$$

Definition 4.36. Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let $A \in \mathscr{A}(p, R)$. Let $b_0 \in B$ and $x_0 \in p^{-1}(b_0)$. The **holonomy group** of A based at x_0 is the subgroup $\mathrm{Hol}_{x_0}(A) < G$ defined by

$$\mathrm{Hol}_{x_0}(A) \coloneqq \{g \in G : (\star)\}$$

with $(\star)$ meaning that there is a piecewise smooth loop $\gamma\colon [0,1] \to B$ based at b_0 with $\mathrm{tra}_\gamma^A(x_0) = x_0 \cdot g$. The **restricted holonomy group** of A based at x_0 is the subgroup $\mathrm{Hol}_{x_0}^0(A) < G$ defined by

$$\mathrm{Hol}_{x_0}^0(A) \coloneqq \{g \in G : (\dagger)\}$$

with $(\dagger)$ meaning that there is a null-homotopic piecewise smooth loop $\gamma\colon [0,1] \to B$ based at b_0 with $\mathrm{tra}_\gamma^A(x_0) = x_0 \cdot g$. •

Proposition 4.37. *The holonomy group and the restricted holonomy group are Lie subgroups of G.*

Proof sketch. $\mathrm{Hol}^0_{x_0}(A)$ is path-connected and therefore a Lie subgroup of G. Parallel transport defines a group homomorphism $\pi_1(B, b_0) \to \mathrm{Hol}_{x_0}(A)/\mathrm{Hol}^0_{x_0}(A)$. With Γ denoting its image

$$\mathrm{Hol}_{x_0}(A) = \coprod_{\gamma \in \Gamma} g \cdot \mathrm{Hol}^0_{x_0}(A).$$

Use this to construct a smooth structure on $\mathrm{Hol}_{x_0}(A)$. ■

Remark 4.38. The above underlines that G–principal connections are really much simpler than general Ehresmann connections. The holonomy group of an Ehresmann connection is a subgroup of $\mathrm{Diff}(p^{-1}(b_0))$, a possibly wild infinite-dimensional beast; while that of a G–principal connection sits inside a fixed finite dimensional Lie group. ♣

`One could discuss the relation between` F_A `and Hol here, but it is probably better to do this later in a section on Ambrose–Singer..`

∼

Proposition 4.39. *Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let $A \in \mathscr{A}(p, R)$. Let $u \in \mathscr{G}(p, R)$. There is a unique $u^*A \in \mathscr{A}(p, R)$ such that*

$$\theta_{u^*A} = u^*\theta_A.$$

Moreover, the following hold:

(1) *The connection 1–form θ_{u^*A} can be written as*

$$\theta_{u^*A} = \mathrm{Ad}(\gamma_u^{-1}) \circ \theta_A + \gamma_u^*\mu$$

with $\mu \in \Omega^1(G, \mathfrak{g})$ denoting the Maurer–Cartan form.

(2) *The horizontal subspaces of A and u^*A are related by*

$$H_{u^*A} = Tu^{-1}(H_A).$$

(3) *The pullback via the gauge transformation u preserves $\Omega^\bullet(P, V)^\rho$ and*

$$\mathrm{d}_{u^*A} = u^* \circ \mathrm{d}_A \circ (u^{-1})^*.$$

(4) *The curvatures of A and u^*A are related by*

$$F_{u^*A} = \mathrm{Ad}(\gamma_u^{-1})F_A.$$

(5) *The parallel transports of A and u^*A are related by*

$$\mathrm{tra}_\gamma^{u^*A} = u^{-1} \circ \mathrm{tra}_\gamma^A \circ u.$$

The above defines a *right* action of $\mathscr{G}(p,R)$ on $\mathscr{A}(p,R)$.

Proposition 4.39. Since u is G–equivariant, $\theta_{u^*A} := u^*\theta_A$ is the connection 1–form of a G–principal connection u^*A. Evidently, $H_{u^*A} = Tu^{-1}(H_A)$.

Since

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R_{g\exp(t\xi)}x &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R_{\exp(t\xi)}R_g(x) \\ &= v_\xi(R_g(x)),\end{aligned}$$

the derivative of the map $x \mapsto R_{\gamma_u(x)}(y)$ is

$$v_{\gamma_u^*\mu}(R_{\gamma_u(x)}(y)).$$

Since $u(x) = R_{\gamma_u(x)}(x)$,

$$T_x u(\hat{x}) = T_x R_{\gamma_u(x)}(\hat{x}) + v_{(\gamma_u^*\mu)(\hat{x})}(R_{\gamma_u(x)}(x)).$$

Therefore,

$$\begin{aligned}\theta_{u^*A} &= u^*\theta_A \\ &= \mathrm{Ad}(\gamma_u^{-1}) \circ \theta_A + \gamma_u^*\mu.\end{aligned}$$ ∎

4.3 The tangent group

The following is a preparation for the study of associate fibre bundles, but it is also a interesting observation by itself.

Proposition 4.40. *Let G be a Lie group with multiplication $m\colon G \times G \to G$, inversion $i\colon G \to G$, and unit $\mathbf{1}$. Set $\mathfrak{g} := \mathrm{Lie}(G)$.*

(1) *The tangent bundle TG together with $Tm\colon TG \times TG \to TG$, $Ti\colon TG \to TG$, and $(\mathbf{1}, 0) \in TG$ is a Lie group.*

(2) *Denote by $G \ltimes \mathfrak{g} = G \ltimes_{\mathrm{Ad}} \mathfrak{g}$ the Lie group $G \times \mathfrak{g}$ with group multiplication*

$$(g,\xi)(h,\eta) := (gh, \xi + \mathrm{Ad}(g)\eta).$$

The map $G \ltimes \mathfrak{g} \to TG$ defined by

$$(g,\xi) \mapsto (g, TR_g(\mathrm{ev}_{\mathbf{1}}(\xi))$$

is a Lie group isomorphism.

Proof. To prove (1) formulate the group conditions as commutative diagrams and apply the tangent functor. (2) follows from a computation. ∎

Definition 4.41. TG is the **tangent group** associated with G. •

Proposition 4.42. *Let G be a Lie group. Let X be a smooth manifold.*

(1) *If $R\colon X \times G \to X$ is a right action, then $TR\colon TX \times TG \to TG$ is a right action.*

(2) *With respect to the isomorphism $G \ltimes \mathfrak{g} \cong TG$, TR is given by*

$$(x, v) \cdot (g, \xi) = (R_g x, TR_g(v + v_\xi(x))).$$

Proof. To prove (1), Write the condition for R to be a right action as a commutative diagram and apply the tangent functor.

To prove (2) compute. ■

Proposition 4.43. *Let G be a Lie group. If $(p\colon P \to B, R\colon P \times G \to P)$ is a G–principal fibre bundle, then $(Tp\colon TP \to TB, TR\colon TP \times TG \to TP)$ is a TG–principal fibre bundle.*

Proof. Exercise. *Hint:* apply the tangent functor to the local trivialisations. ■

Remark 4.44. The above exhibits $Tp\colon TP \to TB$ as a quotient $TP \to TP/TG$. For every $b \in B$ and $x \in p^{-1}(b)$ the linear map

$$T_x P/\mathfrak{g} \to Tp^{-1}(b)/TG \to T_b B$$

is an isomorphism. ♣

4.4 Associated fibre bundles

The following construction gives a way to "replace" the fibres of a principal bundle with a manifold F equipped with a G–action.

Proposition 4.45. *Let G be a Lie group. Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let F be a smooth manifold. Let $L\colon G \times F \to F$ be a left action. Define a right action $S\colon (P \times F) \times G \to P \times F$ by*

$$S((x, f), g) \coloneqq (xg, g^{-1} f).$$

(1) *The action S admits a quotient*

$$q\colon P \times F \to (P \times F)/G \eqqcolon P \times_L F = P \times_G F.$$

(2) *The pair $(q\colon P \times F \to P \times_L F, S)$ is a G–principal fibre bundle.*

(3) *The map $r\colon P \times_G F \to B$ induced by $p \circ \mathrm{pr}_P\colon P \times F \to B$ is a fibre bundle. This is the* ***fibre bundle associated with*** *(p, R)* ***and*** *L.*

(4) *The map $\tau^{-1}\colon P \times F \to p^*(P \times_L F)$ defined by*

$$\tau^{-1}(x, f) \coloneqq (x, [x, f]).$$

is an isomorphism $\phi\colon \mathrm{pr}_P \to p^* r$ from the trivial bundle pr_P to $p^* r$. Denote the inverse of τ^{-1} by τ.

(5) *Consider the G–equivariant maps*

$$C^\infty(P,F)^L := \{\hat{s} \in C^\infty(P,F) : \hat{s}(xg) = g^{-1}\hat{s}(x)\}$$

and

$$\Gamma(r) := \{s \in C^\infty(B, P \times_L F) : r \circ s = \mathrm{id}_B\}.$$

If $s \in \Gamma(r)$, then $\hat{s} := \mathrm{pr}_F \circ p^* s \in C^\infty(P,F)^L$. This defines a bijection

$$\hat{\cdot}\colon \Gamma(r) \to C^\infty(P,F)^L.$$

Here is a diagram summarising the above

$$\begin{tikzcd} P \times F \arrow[r, "q"] \arrow[d, "\mathrm{pr}_P"] & P \times_L F \arrow[d, "r"] \\ P \arrow[r, "p"] & B. \end{tikzcd}$$

Proof. For the trivial G–principal bundle $(\mathrm{pr}_B\colon P := B \times G, R)$ the quotient is $r = (\mathrm{pr}_B, L)\colon B \times G \times F \to B \times F$. (Verify this!)

Denote by $r\colon P \times F \to (P \times F)/G =: P \times_G F$ the topological quotient. The map $p \circ \mathrm{pr}_P$ is G–invariant and thus descends to a continuous map $q\colon P \times_G G \to B$. For $b \in B$ let U_b and $\tau_b\colon p^{-1}(U_b) \to U_b \times G$ as in Definition 4.1. For $b_1, b_2 \in B$ there is a smooth map $\phi_{b_2}^{b_1}\colon U_{b_1} \cap U_{b_2} \to G$ such that

$$\tau_{b_2} \circ \tau_{b_1}^{-1}(b,g) = (b, \phi_{b_2}^{b_1}(b) \cdot g).$$

By the trivial case $q^{-1}(U_{b_i})$ is canonically homeomorphic to $U_{b_i} \times X$ and the transition between these homemorphisms is given by

$$(b,f) \mapsto (b, \phi_{b_2}^{b_1}(b) \cdot f).$$

This endows $P \times_G X$ with the structure of a smooth manifold and exhibits $q\colon P \times_G X \to B$ as a fibre bundle. This proves (1), (2), and (3).

(4) and (5) are exercises. ■

Example 4.46. Let $V \to B$ be a real (complex) vector bundle of rank r. Denote by $(p\colon \mathrm{Fr}(V) \to B, R)$ the frame bundle of $V \to B$. $\mathrm{GL}_r(\mathbf{R})$ $(\mathrm{GL}_r(\mathbf{C}))$ acts on $\mathbf{R}P^n$ $(\mathbf{C}P^n)$ The associated fibre bundle $\mathrm{Fr}(V) \times_{\mathrm{GL}_r(\mathbf{R})} \mathbf{R}P^n$ $(\mathrm{Fr}(V) \times_{\mathrm{GL}_r(\mathbf{R})} \mathbf{C}P^n)$ is the **projectivisation** of V and denoted by $\mathbf{P}(V) \to B$. ♠

Example 4.47. Let $n \in \mathbf{Z}$. The **Hirzebruch surface** Σ_n is

$$\Sigma_n := \mathbf{P}(\mathscr{O}_{\mathbf{P}^1} \oplus \mathscr{O}_{\mathbf{P}^1}(n)). \quad ♠$$

Proposition 4.48. *Assume the situation of Proposition 4.45. Let $A \in \mathscr{A}(p,R)$.*

(1) *The V_r fits into the following diagram of exact sequences:*

$$\begin{tikzcd}
(P \times F) \times \mathfrak{g} \arrow[r, equal] \arrow[d, hook] & (P \times F) \times \mathfrak{g} \arrow[d, hook] & \\
V_p \times TF \arrow[r, hook] \arrow[d, two heads, "Tq"] & TP \times TF \arrow[r, two heads, "T(p \circ \mathrm{pr}_P)"] \arrow[d, two heads, "Tq"] & p^*TB \arrow[d, equal] \\
q^*V_r \arrow[r, hook] & q^*T(P \times_L F) \arrow[r, two heads, "q^*Tr"] & q^*r^*TB.
\end{tikzcd}$$

Here $(P \times F) \times \mathfrak{g} \to TP \times TF$ is defined by

$$((x,f),\xi) \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} S((x,f),\exp(t\xi)) = (v_\xi^R(x), -v_\xi^L(f)).$$

(2) *There is a unique Ehresmann connection $\tilde{A}$ on $r\colon P \times_L F \to B$ such that*

$$H_{\tilde{A}} = Tq(\mathrm{pr}_P^* H_A) \quad \text{and} \quad q^*\theta_{\tilde{A}} = \mathrm{pr}_P^*(\kappa \circ \theta_A) \oplus \mathrm{id}_{TF}.$$

*This is the **Ehresmann connection induced by** A.*

(3) *Let $s \in C^\infty(P,F)^L$. The corresponding section $\hat{s} \in \Gamma(r)$ is $\tilde{A}$–horizontal if and only if $Ts|_{V_p \times TF} = 0$.*

(4) *The parallel transports of A and $\tilde{A}$ are related by*

$$\mathrm{tra}_\gamma^{\tilde{A}}(q(x,f)) = q(\mathrm{tra}_\gamma^A(x), f).$$

(5) *The curvatures of A and $\tilde{A}$ are related by*

$$F_{\tilde{A}}(T_x q(v), T_x q(w)) = Tq(\kappa \circ F_A(v,w) \oplus 0) = Tq(0 \oplus v^L_{F_A(v,w)}).$$

Proof. (1) is an exercise.

Since H_A is R–invariant, $\mathrm{pr}_P^* H_A$ is S–invariant. Therefore, it descends to a distribution $H_{\tilde{A}} \subset T(P \times_L F)$. Evidently, $Tr\colon H_{\tilde{A}} \to r^*TB$ is an isomorphism. Therefore, $H_{\tilde{A}}$ defines an Ehresmann connection $\tilde{A}$ on $q\colon P \times_L F \to B$. Similarly, $\mathrm{pr}_p^*(\kappa \circ \theta_A) \oplus \mathrm{id}_{TF}$ is S–invariant; hence it descends to $\theta_{\tilde{A}}$. By construction, $\ker \theta_{\tilde{A}} = H_{\tilde{A}}$. This proves (2).

(3) holds by construction. This in turn implies (4).

To prove the fist formula in (5) choose lifts and compute on $P \times F$. The second formula follows from (1). ■

Proposition 4.49. *Let G be a Lie group. Let $(p\colon P \to B, R\colon P \times G \to P)$ be a G–principal fibre bundle. Let $\rho\colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation. $r\colon P \times_\rho V \to B$.*

(1) *There are unique vector space structure on the fibres of r such that r becomes a vector bundle and the map $\tau\colon p^*(P \times_\rho V) \to P \times V$ is an isomorphism of vector bundles. This is the **vector bundle associated with** (p,R) **and** ρ.*

(2) *If* $\alpha \in \Omega^\bullet(B, P \times_\rho V)$, *then* $\hat{\alpha} := \mathrm{pr}_V p^*\alpha \in \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho$. *This defines a bijection*

$$\hat{\cdot}\colon\ \Omega^\bullet(B, P \times_\rho V) \to \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho.$$

(3) *There is a unique covariant derivative corresponding* d_A *on* $P \times_\rho V$ *such that*

$$\begin{tikzcd} \Omega^\bullet(B, P \times_\rho V) \arrow[r, "\mathrm{d}_A"] \arrow[d, "\hat{\cdot}"] & \Omega^{\bullet+1}(B, P \times_\rho V) \arrow[d, "\hat{\cdot}"] \\ \Omega^\bullet_{\mathrm{hor}}(P, V)^\rho \arrow[r, "\mathrm{d}_A"] & \Omega^{\bullet+1}_{\mathrm{hor}}(P, V)^\rho. \end{tikzcd}$$

Moreover, if $G = \mathrm{GL}(V)$ *and* $\rho = \mathrm{id}$, *then every covariant derivative on* $P \times_\rho V$ *arises from a unique* $\mathrm{GL}(V)$*–principal connection* A*.*

Proof. It suffices to prove (1) for the trivial bundle $\mathrm{pr}_B\colon\ B \times G \to B$. In this case the map $q\colon\ (B \times G) \times V \to B \times V$ defined by $((b, g), v) \mapsto (b, \rho(g)v)$ is G–invariant and every G–invariant map from $(B \times G) \times V$ factors through q. Therefore, $(B \times G) \times_\rho V$ is (isomorphic) to $B \times V$.

(2) is an exercise.

The first part of (3) follows from the fact that d_A is compatible with tensor products. It suffices to verify the second part for the trivial bundle. In this case $P = B \times \mathrm{GL}(V)$ and the quotient map $q\colon\ P \times V \to P \times_\rho V = B \times V$ is $q(b, g, v) := (b, \rho(g)v)$. The map $r\colon\ P \times_\rho V = B \times V \to B$ is $r = \mathrm{pr}_B$. The product connection gives rise to the covariant derivative $\mathrm{d}\colon\ C^\infty(B, V) \to \Omega^1(B, V)$. Any other covariant derivative is of the form $\mathrm{d} + a$ with $a \in \Omega^1(B, \mathfrak{gl}(V))$. Denote by θ_0 the connection 1–form of the trivial connection. Then $\theta_0 + \mathrm{Ad}(\mathrm{pr}_G)^{-1} \cdot r^*a$ is a connection 1–form and induces $\mathrm{d} + a$. ■

Example 4.50. Let X be a smooth manifold of dimension n. Denote by $p\colon\ \mathrm{Fr}(TX) \to X$ the frame bundle of TX. The orientation double cover of X is the fibre bundle associated with (p, R) and the action of $\mathrm{GL}_n(R)$ on $\{\pm 1\}$ induced by the group homomorphism $\mathrm{GL}_n(\mathbf{R}) \to \{\pm 1\}$ given by

$$\phi \mapsto \frac{\det \phi}{|\det \phi|}.$$ ♠

Example 4.51. Let $s \geqslant 0$ and $n \in \mathbf{N}_0$. A s**–density** on $\mathbf{R}^n$ is a map $\mu\colon\ (\mathbf{R}^n)^{\times n} \to \mathbf{R}$ such that for every $A \in \mathrm{GL}_n(\mathbf{R})$ and $v_1, \ldots, v_n \in \mathbf{R}^n$

$$\mu(\phi(v_1), \ldots, \phi(v_n)) = |\det \phi| \mu(v_1, \ldots, v_n).$$

The set of s–densities is a 1–dimensional vector space: $D^s(\mathbf{R}^n)$.

Let X be a smooth manifold of dimension n. Denote by $p\colon\ \mathrm{Fr}(TX) \to X$ the frame bundle of TX. The bundle of s–densities on X is

$$D^s(TX) := \mathrm{Fr}(TX) \times_{\mathrm{GL}_n(R)} D^s(\mathbf{R}^n).$$

A **density** is a 1–density and $D(TX) := D^1(TX)$. ♠

Exercise 4.52. Let X be a closed smooth manifold (possibly not oriented). Construct a linear map

$$\int_X \colon\ D(TX) \to \mathbf{R}$$

(worthy of its notation).

Example 4.53. Let $V \to B$ be a vector bundle rank r. Denote the corresponding frame bundle by $\mathrm{Fr}(V)$ of isomorphisms $\phi\colon\ \mathbf{R}^r \to V_x$. The map $\mathrm{ev}\colon\ \mathrm{Fr}(V) \times \mathbf{R}^r \to V$ defined by

$$((x,\phi),v) \coloneqq (x,\phi(v))$$

is G–invariant and exhibits V as (isomorphic to) $\mathrm{Fr}(V) \times_\rho \mathbf{R}^r$ with $\rho \coloneqq \mathrm{id}\colon\ \mathrm{GL}_r(\mathbf{R}) \to \mathrm{GL}_r(\mathbf{R})$. ♠

Remark 4.54. The frame bundle formalism is a convenient way to carry linear algebra constructions over to vector bundles:

(1) Denote by $\rho^*\colon\ \mathrm{GL}_r(\mathbf{R}) \to \mathrm{GL}((\mathbf{R}^r)^*)$ the contragredient representation defined by

$$\rho^*(g)\lambda \coloneqq \lambda \circ \rho(g^{-1}).$$

$\mathrm{Fr}(V) \times_{\rho^*} (\mathbf{R}^r)^* \cong V^*$.

(2) Denote by $\Lambda^k\rho\colon\ \mathrm{GL}_r\, r(\mathbf{R}) \to \mathrm{GL}(\Lambda^k(\mathbf{R}^r))$ the representation defined by

$$(\Lambda^k\rho)(g)\alpha \coloneqq \Lambda^k(\rho(g))\alpha.$$

$\mathrm{Fr}(V) \times_{\Lambda^k\rho} \Lambda^k\mathbf{R}^r \cong \Lambda^k V$. ♣

4.5 Extension and reduction of the structure group

Proposition 4.55. *Let G, H be Lie groups. Let $\rho\colon\ G \to H$ be a Lie group homomorphism. If $(p\colon\ P \to B, R)$ is a G–principal bundle, then $(q\colon\ P \times_\rho H, S)$ with $S([x,h],g) \coloneqq [x, h\rho(g)]$ is a H–principal bundle.* ■

Definition 4.56. $P \times_\rho H$ is called an **extension** of P. •

This raises the question of when one can undo this construction.

Definition 4.57. Let G be a Lie group. Let $(p\colon\ P \to B, R)$ be G–principal bundle, Let $H < G$ be an immersed Lie subgroup. An H–principal bundle $(q\colon\ Q \to B, R)$ together with a H–equivariant fibre-preserving smooth map $i\colon\ Q \to P$ is called a **reduction** of (p, R). We say that the structure group of (p, R) is reducible to H. •

Remark 4.58. In the situation of Definition 4.57, the map

$$Q \times_H G \to P,\ [q,g] \mapsto i(q)g$$

is an isomorphism of G–principal bundles. ♣

Example 4.59. A trivialization is a reduction to the trivial group. ♠

Example 4.60. Let (V, g) be a Euclidean vector bundle over B. Denote by $p\colon \operatorname{Fr}(V) \to B$ and $q\colon \operatorname{Fr}_{\mathrm{O}}(V) \to B$ the frame bundle and the orthogonal frame bundle. The inclusion $\operatorname{Fr}_{\mathrm{O}}(V) \to \operatorname{Fr}(V)$ is a reduction. ♠

Proposition 4.61. *Let G be a Lie group. Let $(p\colon P \to B, R)$ be G–principal bundle. Let $H < G$ be a closed Lie subgroup. The set of isomorphism classes of reductions of (p, R) to H is bijective to the set of sections of $P \times_G G/H$.*

Proof. A section s of $P \times_G G/H$ is equivalent to a G–equivariant map $\hat{s}\colon P \to G/H$. Define

$$Q := \{p \in P : s(p) = \mathbf{1}H\}.$$

Since p is G–equivariant, if $p \in Q$ and $h \in H$, then

$$s(ph) = h^{-1}s(p) = h^{-1}\mathbf{1}H = \mathbf{1}H;$$

that is, $ph \in Q$; hence, H acts on Q. Suppose $p, pg \in Q$. Then

$$\mathbf{1}H = s(pg) = g^{-1}\mathbf{1}H.$$

To see that Q is a principal H–bundle, what is left is to provide local trivialisations. For this we can assume that P itself is trivial, i.e., $P = B \times G$. Then clearly $Q = Bx \times H$.

Now suppose Q is a reduction of P. Then $P \cong Q \times_H G$ and $s\colon P \to G/H$ defined by $(q, g) \mapsto g^{-1}H$ is the desired section. ■

Remark 4.62. The proposition also holds (morally) if H is just immersed, even-though $G/might$ H not be a smooth manifold. In this case one has to interpret $\Gamma(P \times_G G/H) = C^\infty(P, G/H)^G$ as the set of maps $\operatorname{Map}(P, G/H)^G$ which locally can be lifted to smooth maps $P \to G$. Below we will use the result in this form.

♣

Example 4.63. Let V be a real vector bundle over B of rank n. Denote by $\operatorname{Fr}(V) \to B$ its frame bundle.

(1) A reduction of $\operatorname{Fr}(V)$ to $\mathrm{GL}(\mathbf{C}^{n/2})$ corresponds to a section of $\operatorname{Fr}(V) \times_{\mathrm{GL}(\mathbf{R}^n)} \mathrm{GL}(\mathbf{R}^n)/\mathrm{GL}(\mathbf{C}^{n/2})$. The latter correspond precisely to the almost complex structures on V.

(2) A reduction of $\operatorname{Fr}(V)$ to $\mathrm{GL}^+(n)$ corresponds to a section of $\operatorname{Fr}(V) \times_{\mathrm{GL}(\mathbf{R}^n)} \mathrm{GL}(\mathbf{R}^n)/\mathrm{GL}^+(\mathbf{R}^n)$. The latter correspond precisely to the orientations on V.

(3) A reduction of $\operatorname{Fr}(V)$ to $\mathrm{O}(n)$ corresponds to a section of $\operatorname{Fr}(V) \times_{\mathrm{GL}(\mathbf{R}^n)} \mathrm{GL}(\mathbf{R}^n)/\mathrm{O}(n)$. The latter correspond precisely to Euclidean inner products on V. ♠

It is a natural question to ask: given a G–principal bundle, what is the smallest possible reduction? This question is quite difficult. But if the reduction is required to be compatible with a connection, it becomes easy.

Theorem 4.64 (Reduction to the holonomy group). *Let $(p\colon P \to B, R)$ be a G–principal bundle. Let $A \in \mathscr{A}(p, R)$. Suppose B is connected. Let $x \in P$. Define*

$$P_{A,x} := \{y \in P : \text{ there is a } A\text{–horizontal path from } x \text{ to } y\}.$$

The following hold:

1. *$P_{A,x} \subset P$ is an immersed submanifold and $(q\colon P_{A,x} \to B, S)$ with $q := p|_{P_{A,x}}$ and $S := R|_{P_{A,x}\times \mathrm{Hol}_x(A)}$ a $\mathrm{Hol}_p(A)$–principal bundle.*
2. *There is a unique connection $\tilde{A} \in \mathscr{A}(q, S)$ with $H_{\tilde{A},y} = H_{A,y}$ for every $y \in P_{A,x}$*
3. *$P_{A,x} \xrightarrow{\cdot g} P_{A,xg}$ is a isomorphism of principal bundles.*

Proof. Set $H := \mathrm{Hol}_p(A)$. Define a map $s\colon P \to G/H$ as follows. For every $y \in P$, pick a path $\gamma\colon [0, 1] \to B$ from $p(x)$ to $p(y)$. Let $\tilde{\gamma}\colon [0, 1] \to P$ be a horizontal lift starting at y. Then there is a unique $g \in G$ such that $y = \tilde{\gamma}(1) \cdot g$. Define $s(y) = gH$. This is a smooth G–equivariant map. By Proposition 4.61 there is an associated structure reduction to H and inspection of the proof of Proposition 4.61 this reduction is exactly $P_{A,x}$. This proves (1).

Suppose $y \in P_{A,x}$ and $v \in H_{A,y}$. Let γ be a horizontal curve starting in q with $\dot{\gamma}(0) = v$. We can extend γ to a horizontal curve passing though p for some $t < 0$. This shows that $H_{A,y} \subset T_y P_{A,x}$. Thus A induces a connection on $P_{A,x}$.

The last assertion is clear. ■

Remark 4.65. $P_{A,x}$ is the minimal reduction of P compatible with A. ♣

Definition 4.66. We say that A is **irreducible** if $P_{A,p} = P$. We say that A is **reducible** if it is not irreducible. •

4.6 Ambrose–Singer Theorem

Definition 4.67. The **holonomy Lie algebra** $\mathfrak{hol}_x(A)$ based at $x \in P$ is the Lie algebra of the Lie group $\mathrm{Hol}_x(A)$. •

Theorem 4.68 (Ambrose and Singer [AS53]). *The holonomy Lie algebra satisfies*

$$\mathfrak{hol}_x(A) = \langle \{F_A(u, v) : y \in P_{A,x}, u, v \in T_y P\}\rangle.$$

Proof. Without loss of generality $\mathrm{Hol}_p(A) = G$ and $P_{A,x} = P$. Set

$$\mathfrak{k} := \langle \{F_A(u, v) : x \in P, u, v \in T_y P\}\rangle$$

and define $E \subset TP$ by

$$E_y := H_{A,y} \oplus \mathfrak{k} \cdot y.$$

This clearly is a distribution on TP. If we can show that $E = TP$, we are done.

Next we show that E is involutive. By definition of curvature $[H_{A,y}, H_{A,y}] \subset \mathfrak{f} \cdot y$. Because the distribution H_A is G–invariant, $[H_{A,y}, \mathfrak{f} \cdot y] \subset H_{A,y}$. Finally if $\xi \in \mathfrak{f}$, then

$$\begin{aligned}[F_A(u,v), \xi] &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{Ad}(\exp(-t\xi))F_p(u,v) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} F_A(\mathrm{d}R_{\exp(t\xi)}u, \mathrm{d}R_{\exp(t\xi)}v) \in \mathfrak{f}.\end{aligned}$$

Let Q be the maximal connected integral submanifold through x. We show that $Q = P$. Since $y \in Q$ if and only if there is a path γ from x to y with $\dot{\gamma} \in E$, clearly $P_{A,x} \subset Q$. But $P_{A,y} = P$.

This proves that $E = TP$ and consequently $\mathfrak{f} = \mathfrak{hol}_x(A)$. ■

Exercise 4.69. Give an example of a connection with $\mathfrak{hol}_x(A)$ is not spanned by the curvature at x itself.

4.7 The cocycle perspective

Definition 4.70. Let X be a topological space. Let $\mathscr{U} = \{U_i : i \in I\}$ be an open cover of X. For $k \in \mathbf{N}_0$ set

$$X^k(\mathscr{U}) \coloneqq \coprod_{i_0,\dots,i_k \in I} U_{i_0} \times \cdots \times U_{i_k}.$$

For $k \in \mathbf{N}_0$ and $\ell \in \{0, \dots, k\}$ denote by $\jmath_\ell \colon X^{k+1}(\mathscr{U}) \to X^k(\mathscr{U})$ the map induced by the inclusions $U_{i_0} \times \cdots \times U_{i_k} \subset U_{i_0} \times \cdots \times \widehat{U}_{i_\ell} \times \cdots \times U_{i_k}$. Denote by $\jmath \colon X^0(\mathscr{U}) \to X$ the map induced by the inclusion $U_i \subset X$. •

Let $(p\colon P \to B, R)$ be a G–principal bundle. There is an open cover $\mathscr{U} = \{U_i : i \in I\}$ such that the pullback of (p, R) to $B^0(\mathscr{U})$ via $\jmath$ is trivial. Since $\jmath \circ \jmath_0 = \iota \circ \jmath_1$, there is a canonical identification

$$\jmath_0^* \jmath^*(p, R) = \jmath_1^* \jmath^*(p, R).$$

Suppose that $\tau\colon \iota^*(p,R) \to (\mathrm{pr}_{B^0(\mathscr{U})}, S)$ is a trivialisation. Then $\iota_k^*\tau\colon \iota^*(p,R) \to \jmath_k^*(\mathrm{pr}_{B^1(\mathscr{U})}, S) = (\mathrm{pr}_{B^1(\mathscr{U})}, T)$ are trivialisations. Therefore,

$$\phi \coloneqq (\jmath_0^*\tau) \circ (\jmath_1^*\tau)^{-1} \in \mathscr{G}(\mathrm{pr}_{B^1(\mathscr{U})}, T).$$

A moment's thought shows that the **cocycle condition**

$$\jmath_0^*\phi \circ \jmath_2^*\phi = \jmath_1^*\phi$$

holds.

Given $\phi \in \mathscr{G}(\mathrm{pr}_{B^1(\mathscr{U})}, T)$ satisfying the cocycle condition, define an equivalence relation on $B^0(\mathscr{U}) \times G$ by

$$(i, b, g) \sim (j, b', g') \iff \jmath(b) = \jmath(b'), g' = \phi|_{(i,j)\times(U_i \cap U_j)}(b)g.$$

The quotient $P \coloneqq (B^0(\mathscr{U}) \times G)/\sim$ is a smooth manifold and the total space of a G–principal bundle $(p\colon P \to B, R)$ with p induced by $\jmath \circ \mathrm{pr}_{B^0(\mathscr{U})}\colon B^0(\mathscr{U}) \times G \to B$ and R induced by the right action on $B^0(\mathscr{U}) \times G$. Evidently, $\jmath^*(p,R)$ is has a canonical trivialisation and the above construction recovers ϕ.

The gauge transformation $\phi \in \mathscr{G}(\mathrm{pr}_{B^1(\mathscr{U})}, T)$ determines (p,R) *and the trivialisation* τ up to isomorphism. We don't care about τ. So let's figure out how to forget it. Suppose that $\sigma\colon \iota^*(p,R) \to (\mathrm{pr}_{B^1(\mathscr{U})}, T)$ is a further trivialisation and dives rise to $\psi \in \mathscr{G}(\mathrm{pr}_{B^1(\mathscr{U})}, T)$. Then

$$\delta := \sigma \circ \tau^{-1} \in \mathscr{G}(\mathrm{pr}_{B^0(\mathscr{U})}, S)$$

and, evidently,

$$\psi = (\jmath_0^*\delta) \circ \phi \circ (\jmath_1^*\delta)^{-1}.$$

This formula defines an action of $\mathscr{G}(\mathrm{pr}_{B^0(\mathscr{U})}, S)$ on $\mathscr{G}(\mathrm{pr}_{B^1(\mathscr{U})}, T)$. The above shows that (p,R), up to isomorphism, is determined by $[\phi] \in \mathscr{G}(\mathrm{pr}_{B^1(\mathscr{U})}, T)/\mathscr{G}(\mathrm{pr}_{B^0(\mathscr{U})}, S)$ satisfying the cocycle condition.

Definition 4.71. Let G be a Lie group. Let B be a smooth manifold. Let $\mathscr{U} = \{U_i : i \in I\}$ be an open cover of B.

(1) For $i_1, \ldots, i_k \in I$ set $U_{i_1,\ldots,i_k} := U_{i_1} \cap \cdots \cap U_{i_k}$. A k**–cochain** on $\mathscr{U}$ with values in the sheaf $C^\infty(\cdot, G)$ is an I^k–tuple $\underline{\phi} = (\phi_{i_1\cdots i_k} \in C^\infty(U_{i_1\cdots i_k}, G) : i_1, \ldots, i_k \in I)$. Denote the set of k–cochains by $\check{C}^k(\mathscr{U}, G)$.

(2) Define $\tau\colon \check{C}^0(\mathscr{U}, G) \to \mathrm{Bij}(\check{C}^1(\mathscr{U}, G))$ by

$$\tau(\underline{\psi})(\underline{\phi})_{ij} := \psi_j \phi_{ij} \psi_i^{-1}.$$

(3) Define $\delta\colon \check{C}^1(\mathscr{U}, G) \to \check{C}^2(\mathscr{U}, G)$ by

$$(\delta\underline{\phi})_{ijk}(b) := \phi_{ik}(b)^{-1}\phi_{ij}(b)\phi_{jk}(b)$$

for every $b \in U_{ijk}$. Set

(4) Set

$$\check{Z}^1(\mathscr{U}, G) := \ker \delta.$$

and

$$\check{\mathrm{H}}^1(\mathscr{U}, G) := \check{Z}^1(\mathscr{U}, G)/\tau(\check{C}^0(\mathscr{U}, G)) \quad \text{and} \quad \check{\mathrm{H}}^1(B, G) := \varprojlim_{\mathscr{U}} \check{\mathrm{H}}^1(\mathscr{U}, G).$$

The latter is the **first Čech cohomology of** B **with values in** $C^\infty(\cdot, G)$. •

Remark 4.72. In practice, passing to the limit can be avoided by working with a cover by contractible sets. ♣

The above exhibits the set of equivalence classes of G–principal bundles as $\check{\mathrm{H}}^1(B, G)$.

Going back to the previous discussion, let $u \in \mathscr{G}(p, R)$. The pullback $\tau_*\jmath^*u$ is a gauge transformation of $v \in \mathscr{G}(\mathrm{pr}_{B^0(\mathscr{U})}, S)$ satisfying

$$\phi = (\jmath_0^* v)\phi \circ (\jmath_1^* v)^{-1}$$

Therefore,

$$\mathscr{G}(p,R) \cong \{\underline{v} \in \check{C}^0(\mathscr{U},G) : \tau(\underline{v})\underline{\phi} = \underline{\phi}\}.$$

Finally, let us discuss connections from this point of view. Let $A \in \mathscr{A}(p,R)$. Then $\tilde{A} := \tau_* J^* A \in \mathscr{A}(\mathrm{pr}_{B^0(\mathscr{U})}, S)$ is a connection and satisfies

$$J_0^* \tilde{A} = \phi^*(J_1^* \tilde{A}).$$

The latter is also the condition for a connection $\tilde{A}$ on $(\mathrm{pr}_{B^0(\mathscr{U})}, S)$ to descend to (p,R)

If $\tilde{A}$ is regarded as a connection 1–form

$$(\theta_i) \in \coprod_{i\in I} \Omega^1(U_i, \mathfrak{g})$$

and ϕ as an element

$$(\phi_{ij}) \in \coprod_{i\in I} C^\infty(U_{ij}, G),$$

then the above conditions amounts to

$$\theta_j = \mathrm{Ad}(\phi_{ji}^{-1})\theta_i + \phi_{ji}^* \mu.$$

Exercise 4.73. Work out how associated bundles work in the above context.

4.8 Chern–Weil theory

The following notion is very useful to distinguish G–principal bundles.

Definition 4.74. Let $\mathrm{H}^\bullet$ be a cohomology theory, e.g., $\mathrm{H}^\bullet = \mathrm{H}^\bullet_{\mathrm{dR}}$. A **characteristic class** is an assignment of any G–principal bundle $(p\colon P \to B, R)$ to a cohomology class $c(p,R) \in \mathrm{H}^\bullet(B)$ such that if $f\colon A \to B$, then $c(f^*(p,R)) = f^*c(p,R)$. •

Remark 4.75. An excellent reference for the theory of characteristic classes is Milnor and Stasheff [MS74]. A systematic approach uses the **classifying space** BG. This is a topological space together with a G–principal bundle $(p\colon EG \to BG, R)$ such that up to isomorphism every G–principal bundle arises as $f^*(p,R)$ for some $f\colon B \to BG$. This exhibits a bijection between the set of isomorphism classes of G–princpal bundles over B and

$$[B, BG],$$

the set of homotopy classes of continuous maps $B \to BG$. Knowing this a characteristic class is simply a cohomology class $c \in \mathrm{H}^\bullet(BG)$. For $G = \mathrm{U}(1)$, $B\mathrm{U}(1) = \mathbf{C}P^\infty$, $E\mathrm{U}(1) = S^\infty$, and $p\colon E\mathrm{U}(1) \to B\mathrm{U}(1)$ is a (version of the) Hopf fibration. Unfortunately, BG is not a (finite dimensional) smooth manifold. BG can be approximated by smooth manifolds (or be regarded as a smooth stack $BG = [*/G]$). ♣

The above approach still doesn't give us concrete characteristic classes.

$\sim$

Let V be a vector space. A **polynomial** of degree k is a linear map $p \in \mathrm{Hom}(S^k V, \mathbf{R})$. Here $S^k V$ denotes the k–th symmetric product. If $V = \mathbf{R}^n$, then

$$e_1^{k_1} \odot \cdots \odot e_n^{k_n}$$

with $\sum k_i = k$ form a basis of $S^k V$. This gives us a polynomial $P \in \mathbf{R}[x_1, \ldots, x_k]$ of degree k by

$$P := \sum_{k_1+\ldots+k_n=k} p(e_1^{k_1} \odot \cdots \odot e_n^{k_n}) \cdot x_1^{k_1} \cdots x_n^{k_n}.$$

A **formal power series** on V is an element of

$$\mathrm{Hom}(S^\bullet V, \mathbf{R}) = \mathrm{Hom}(\bigoplus_{k=0}^{\infty} S^k V, \mathbf{R}) = \prod_{k=0}^{\infty} \mathrm{Hom}(S^k V, \mathbf{R}).$$

If $V = \mathfrak{g} := \mathrm{Lie}(G)$, then G acts on $\mathrm{Hom}(S^\bullet \mathfrak{g}, \mathbf{R})$ via $g \mapsto S^\bullet \mathrm{Ad}_g$. Denote the Ad–**invariant** polynomials/formal power series by

$$\mathrm{Hom}(S^\bullet \mathfrak{g}, \mathbf{R})^{\mathrm{Ad}}.$$

(This space is very computable using Lie theory, and as we will see concretely later.)

Example 4.76. If $\mathfrak{g} = \mathfrak{u}(n)$, then $p(A) = \mathrm{tr}(A)$, $q(A, B) = \mathrm{tr}(AB)$, $r(A, B, C) = \mathrm{tr}(ABC + BAC + ACB)$ are Ad–invariant polynomials of degree 1, 2, 3. ♠

Let $s_k \in \mathrm{Hom}(S^k \mathfrak{g}, \mathbf{R})^{\mathrm{Ad}}$. Let $(p\colon P \to B, R)$ be a G–principal bundle. Let $A \in \mathscr{A}(p, R)$. Since $F_A \in \Omega^2_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$,

$$F_A^{\wedge k} := \underbrace{F_A \wedge \ldots \wedge F_A}_{k \text{ times}} \in \Omega^{2k}_{\mathrm{hor}}(P, S^k \mathfrak{g})^{\mathrm{Ad}}$$

and

$$s_k(F_A^{\wedge k}) \in \Omega^{2k}_{\mathrm{hor}}(P)^G.$$

Since the latter is horizontal and G–invariant, there is a unique

$$\widetilde{s_k(F_A^{\wedge k})} \in \Omega^{2k}(B) \quad \text{with} \quad p^*[\widetilde{s_k(F_A^{\wedge k})}] = s_k(F_A^{\wedge k}).$$

$\check{-}$ is the inverse of the map $\hat{-}$ from Proposition 4.49 corresponding to the trivial representation of G on $\mathbf{R}$. Observe that, on $\Omega^\bullet_{\mathrm{hor}}(P)^G$, $\mathrm{d}_A = \mathrm{d}$.

Proposition 4.77. *Let $(p\colon P \to B, R)$ be a G–principal bundle. Let $s \in \mathrm{Hom}(S^\bullet \mathfrak{g}, \mathbf{R})$. Let $A \in \mathscr{A}(p, R)$.*

(1) *The differential form*

$$\gamma_{p,R;A}(s) := \widetilde{s(F_A^{\wedge \bullet})} \in \Omega^\bullet(B)$$

is closed.

(2) *The de Rham cohomology class*

$$\gamma_{p,R}(s) \coloneqq [\gamma_{p,R;A}(s)] \in H^\bullet_{\mathrm{dR}}(B)$$

is independent of A. In fact, for every $A_0, A_1 \in \mathscr{A}(p,R)$, there is a ***transgression form*** *$\tau = \tau(A_0, A_1) \in \Omega^{\bullet-1}(B)$ satisfying*

$$\gamma_{p,R;A_1}(s) = \gamma_{p,R;A_0}(s) + \mathrm{d}\tau.$$

(3) *If $f\colon A \to B$ is smooth, then*

$$\gamma_{f^*(p,R)}(s) = f^*\gamma_{p,R}(s).$$

(4) *If $s, t \in \mathrm{Hom}(S^\bullet\mathfrak{g}, \mathbf{R})$, then*

$$\gamma_{p,R}(s \cdot t) = \gamma_{p,R}(s) \cup \gamma_{p,R}(t).$$

Definition 4.78. The homomorphism of graded algebras $\gamma_{p,R}\colon \mathrm{Hom}(S^\bullet\mathfrak{g}, \mathbf{R}) \to \mathrm{H}^\bullet_{\mathrm{dR}}(B)$ defined by Proposition 4.77 is called the **Chern–Weil homomorphism**. •

Proof of Proposition 4.77. Suppose that $s \in \mathrm{Hom}(S^k\mathfrak{g}, \mathbf{R})$. To prove (1), use the Bianchi identity and to compute

$$\mathrm{d}_A s(F_A^{\wedge k}) = s(\mathrm{d}_A(F_A^k)) = 0.$$

To prove (2), set

$$a \coloneqq A_1 - A_0 \in \Omega^1_{\mathrm{hor}}(P, \mathfrak{g}) \quad \text{and} \quad A_t \coloneqq A_0 + ta.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t} F_{A_t} = \mathrm{d}_{A_t} a$$

and using the

$$\begin{aligned} s(F_{A_1}^{\wedge k}) - s(F_{A_0}^{\wedge k}) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} s(F_{A_t}^{\wedge k})\,\mathrm{d}t \\ &= k\int_0^1 s(\mathrm{d}_{A_t} a \wedge F_{A_t}^{\wedge k-1})\,\mathrm{d}t \\ &= \mathrm{d}\tau(A_0, A_1) \end{aligned}$$

with

$$\tau(A_0, A_1) \coloneqq k\int_0^1 s(a \wedge F_{A_t}^{\wedge k-1})\,\mathrm{d}t.$$

Assertions (3) and (4) are obvious. ■

The following theorem asserts that the Chern–Weil homomorphism constructs all characteristic classes (up to torsion).

Theorem 4.79. $\mathrm{Hom}(S^k\mathfrak{g}, \mathbf{R})^{\mathrm{Ad}} = \mathrm{H}^k(BG, \mathbf{R})$.

Sadly, the proof is outside of the scope of this course.

∼

Let us consider the case $G = \mathrm{GL}_n(\mathbf{C})$. For convenience of notation we use the obvious adaptation of the Chern–Weil homomorphism to $\mathbf{C}$ (instead of $\mathbf{R}$).

Proposition 4.80. *The restriction*

$$\mathrm{res}_\Delta\colon\ \mathrm{Hom}(S^\bullet\mathfrak{gl}_n(\mathbf{C}), \mathbf{C})^{\mathrm{Ad}} \to \mathbf{C}[[x_1, \dots, x_n]]^{S_n}$$

to diagonal matrices in $\mathfrak{u}(n)$*, is an isomorphism.*

Proof. Diagonalizable matrices are dense in $\mathfrak{gl}(\mathbf{C}^n)$, Therefore, any $s \in \mathrm{Hom}(S^\bullet\mathfrak{gl}_n(\mathbf{C}), \mathbf{C})^{\mathrm{GL}_n(\mathbf{R})}$ is determined by its values on diagonal matrices. The space of diagonal matrices is $\mathbf{C}^n$ and the stabiliser of this subspace is $S_n \subset \mathrm{GL}(\mathbf{C}^n)$. ∎

Remark 4.81. This is a special case of Chevalley's restriction theorem: If G is a complex connected semi-simple Lie group, $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra and W is the Weyl group, then

$$\mathrm{res}\colon\ \mathrm{Hom}(S^\bullet\mathfrak{g}, \mathbf{C})^{\mathrm{Ad}} \to \mathrm{Hom}(S^\bullet\mathfrak{t}, \mathbf{C})^W$$

is an isomorphism. ♣

For $X \in \mathfrak{gl}_n(\mathbf{C})$ consider the characteristic polynomial

$$\det\left(\mathbf{1} + \lambda\frac{iX}{2\pi}\right) = \sum_{k=0}^{n} p_k(X)\lambda^k.$$

Clearly, $s_k \in \mathrm{Hom}(S^k\mathfrak{gl}_n(\mathbf{C}), \mathbf{R})^{\mathrm{Ad}}$. In terms of Proposition 4.80 we have

$$\mathrm{res}_\Delta(s_k) = \left(\frac{i}{2\pi}\right)^k \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} x_{i_1} \cdots x_{i_k}.$$

Up to the prefactor these are the **elementary symmetric polynomials** of degree k in n–variables. It is not too difficult to see that these generate $\mathbf{C}[[x_1, \dots, x_n]]^{S_n}$.

Definition 4.82. The k–th **Chern class** of P is the characteristic class defined by

$$c_k(p, R) \coloneqq \gamma_{p,R}(s_k) \in H^{2k}_{\mathrm{dR}}(B; \mathbf{C})$$

The **total Chern class** is

$$c(p, R) \coloneqq \sum_{k=0}^{\infty} c_k(p, R).$$

If E is a complex vector bundle of rank n, we also call the Chern classes of $\mathrm{Fr}_{\mathrm{GL}_n(\mathbf{C})}(E)$ the Chern classes of E. •

Exercise 4.83. Write a explicit formulae for $c_0(p,R)$, $c_1(p,R)$ and $c_2(p,R)$ in terms of a connection on (p,R).

Exercise 4.84. Compute $c(TCP^1)$.

Remark 4.85. The normalization $\frac{i}{2\pi}$ might seem strange at this point. It ensures that $c_k(P)$ is **integral**, i.e.,

$$c_k(p,R) \in \operatorname{im}(\mathrm{H}^{2k}(B;\mathbf{Z}) \to \mathrm{H}^{2k}(B;\mathbf{C}))$$

and also is needed to make c_k agree with other definitions of the Chern class. ♣

Exercise 4.86. Prove that if E is a complex rank r vector bundles, then $c_k(E) = 0$ for $k > r$.

Exercise 4.87. Prove that E_1 and E_2 are complex vector bundles, then

$$c(E_1 \oplus E_2) = c(E_1) \cup c(E_2).$$

For $X \in \mathfrak{gl}_n(\mathbf{C})$ consider

$$\operatorname{tr}\exp\left(\lambda \frac{iX}{2\pi}\right) = \sum_{\lambda=1}^{\infty} t_k(X)\lambda^k.$$

In terms of Proposition 4.80 we have

$$\operatorname{res}_\Delta(t_k) = \left(\frac{i}{2\pi}\right)^k \sum_{i=1}^{n} \frac{x_i^k}{k!}.$$

These too, like the s_k, generate all of $\operatorname{Hom}(S^\bullet\mathfrak{gl}_n(\mathbf{C}),\mathbf{C})^{\mathrm{Ad}}$. Expressions of the form $\sum_{i=1}^n x_i^k$ are called **power sums**.

Definition 4.88. The k–th **Chern character** of P is the characteristic class defined by

$$\operatorname{ch}_k(p,R) := \gamma_{p,R}(t_k) \in H^{2k}_{\mathrm{dR}}(M;\mathbf{C})$$

The **total Chern character** is

$$\operatorname{ch}(P) := \sum_{k=0}^{\infty} \operatorname{ch}_k(p,R).$$

If E is a complex vector bundle, then we also call the Chern characters of $\operatorname{Fr}_{\mathrm{GL}_n(\mathbf{C})}(E)$ the Chern characters of E. •

Exercise 4.89. Show that

$$\operatorname{ch}(E_1 \oplus E_2) = \operatorname{ch}(E_1) + \operatorname{ch}(E_2)$$

and

$$\operatorname{ch}(E_1 \otimes E_2) = \operatorname{ch}(E_1) \cup \operatorname{ch}(E_2).$$

Since both the p_k and the q_k generate $\mathbf{C}[\mathfrak{g}]^G$, it follows from Proposition 4.80 that the c_k can be expressed as a function of the ch_k. The following formulae are used often.

Proposition 4.90. *If E is a complex vector bundle, then*

$$\begin{aligned}\mathrm{ch}_1(E) &= c_1(E),\\ \mathrm{ch}_2(E) &= \frac{1}{2}(c_1(E)^2 - 2c_2(E)) \quad \textit{and}\\ \mathrm{ch}_3(E) &= \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)).\end{aligned}$$

Proof. The first is obvious since $p_1 = q_1$. For the second note that

$$\sum_{i=1}^n x_i^2 = \left(\sum_{i=1}^n x_i\right)^2 - 2\sum_{1\leqslant i<j\leqslant n} x_i x_j.$$

I leave the last identity as an exercise. ∎

Remark 4.91. Underlying the above proposition are certain combinatorial identities know as the **Newton identities**. ♣

∼

Complexification induces an inclusion

$$\iota\colon\ \mathrm{GL}_n(\mathbf{R}) \hookrightarrow \mathrm{GL}_n(\mathbf{C})$$

Therefore, the characteristic classes for $\mathrm{GL}_n(\mathbf{C})$ induce characteristic classes for $\mathrm{GL}_n(\mathbf{R})$:

$$\mathrm{Hom}(S^\bullet\mathfrak{gl}_n(\mathbf{C}), \mathbf{C})^{\mathrm{Ad}} \xrightarrow{\iota^*} \mathrm{Hom}(S^\bullet\mathfrak{gl}_n(\mathbf{R}), \mathbf{C})^{\mathrm{Ad}}.$$

Definition 4.92. The k–th **Pontryagin class** of a real vector bundle $E \to B$ is

$$p_k(E) = (-1)^k c_{2k}(E \otimes_{\mathbf{R}} \mathbf{C}) \in \mathrm{H}^{4k}_{\mathrm{dR}}(B; \mathbf{C}).$$ •

Exercise 4.93. Show that $c_{2k+1}(E \otimes_{\mathbf{R}} \mathbf{C}) = 0$.

Exercise 4.94. Suppose B is a Riemannian closed 4–manifold. Let G be a semi-simple Lie group and P a principal G–bundle. Then minus the Killing form is a metric on $\mathfrak{g}_P := P \times_{\mathrm{Ad}} \mathfrak{g}$. Show that there are constants $c_1 > 0$ and $c_2 \in \mathbf{R}$ such that for any $A \in \mathscr{A}(P)$

$$\mathrm{YM}(A) = c_1 \int_X |F_A + *F_A|^2 + c_2 \int_X p_1(\mathfrak{g}_P).$$

Since the second term on the right-hand side depends only on P, this shows, in particular, that anti-self-dual instantons are absolute minima of YM (and not just critical points).

Finally, let me introduce the Euler class. This requires some linear algebra.

Proposition 4.95. *If $X \in \mathfrak{o}(n)$, then there exists a $g \in \mathrm{SO}(n)$ such that gXg^{-1} is block diagonal with blocks of the form*

$$\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$$

The stabilizer of the space of block diagonal matrices is S_m.

Remark 4.96. If $X \in \mathfrak{o}(2m+1)$, an analogous result holds but one needs to allow for one block of the form (0). ♣

Definition 4.97. The **Pfaffian** is a $\mathrm{SO}(2m)$–invariant degree m polynomial on $\mathfrak{o}(2m)$ defined by

$$\mathrm{Pf}(A) = \prod_{i=1}^{n} \lambda_i. \qquad \bullet$$

Remark 4.98. It is clear from the definition that $\mathrm{Pf}(A)^2 = \det(A)$; ♣

Exercise 4.99. If $X \in \mathfrak{o}(2m)$ and $\omega := \sum_{i<j} X_{ij} e_i \wedge e_j$, then

$$\mathrm{Pf}(X) \cdot e_1 \wedge \cdots \wedge e_{2n} = \frac{\omega^n}{n!}.$$

Definition 4.100. If (p, R) is a principal $\mathrm{SO}(2m)$–bundle, then its **Euler class** is

$$e(p, R) := \gamma_{p,R}\left(\frac{\mathrm{Pf}}{(2\pi)^n}\right).$$

Define the **Euler class** of an oriented (Euclidean) vector bundle of rank $2m$ by as the Euler class of its $\mathrm{SO}(2m)$ frame bundle. •

Remark 4.101. Here is a **warning**: the Pfaffian really is attached to $\mathrm{SO}(2m)$ and *not* $\mathrm{GL}^+(2m)$. Therefore, a $\mathrm{GL}^+(2m)$–principal bundle might very well admit a flat connection, but a reduction of structure group to $\mathrm{SO}(2m)$ does not have a trivial Euler class! ♣

Exercise 4.102. If $(p\colon P \to B, R)$ is a $\mathrm{U}(n)$–principal bundle and $(q\colon Q := P \times_{\mathrm{U}(n)} \mathrm{SO}(2n), S)$ is its associated $\mathrm{SO}(2n)$–principal bundle, then

$$e(q, S) = c_n(p, R).$$

Example 4.103. Let (Σ, g) be an Riemann surface. If (e_1, e_2) is a local orthonormal frame and (e^1, e^2) is the dual coframe, then R_g is of the form

$$R_g = K_g \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes e^1 \wedge e^2$$

for some function K_g. A moments thought shows that K_g does not depend on the choice of local frame and, hence, defines a function $K_g \in C^\infty(M)$. K_g is called the **Gauss curvature** of g. In an *arbitrary* basis (e_1, e_2) we have

$$K_g := -\frac{\langle R_g(e_1, e_2)e_1, e_2\rangle}{|e_1|^2|e_2|^2 - \langle e_1, e_2\rangle^2}.$$

If Σ is oriented, then we can define $e(T\Sigma)$ and the above shows that

$$e(T\Sigma) = \frac{1}{2\pi}[K_g \cdot \mathrm{vol}_g]. \qquad \spadesuit$$

See my Riemannian geometry notes from last semester for a discussion of the Chern–Gauß–Bonnet theorem.

4.9 G–structures on smooth manifolds

Definition 4.104. Let $\rho\colon G \to \mathrm{GL}_n(\mathbf{R})$ be a Lie group homomorphism Let X be a smooth manifold of dimension n. A G–**structure** on X is a ρ reduction of $\mathrm{Fr}(TX)$; that is: a G–principal bundle $(p\colon P \to X, R)$ together with an isomorphism

$$P \times_\rho \mathrm{GL}_n(\mathbf{R}) \cong \mathrm{Fr}(TX).$$

•

Example 4.105. A $\mathrm{GL}_n^+(\mathbf{R})$–structure on X is equivalent to an orientation of X. ♠

Example 4.106. An $\mathrm{O}(n)$–structure on X is equivalent to a Riemannian metric on X. ♠

Example 4.107. An $\mathrm{GL}_{n/2}(\mathbf{C})$–structure on X is equivalent to an almost complex structure. ♠

Example 4.108. An $\mathrm{U}(n/2)$–structure on X is equivalent to an almost Hermitian structure. ♠

Example 4.109. For $n \geqslant 3$, $\pi_1(\mathrm{SO}(n), \mathbf{1}) \cong \mathbf{Z}/2\mathbf{Z}$. The universal cover of $\mathrm{SO}(n)$ is a Lie group, $\mathrm{Spin}(n)$, and comes with the covering map is a Lie group homomorphism: $\rho\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$. A **spin structure** on a *Riemannian manifold* (X, g) is a ρ–reduction of $\mathrm{Fr}_{\mathrm{SO}}(TX, g)$. The above description of $\mathrm{Spin}(n)$, unfortunately, is not that useful. $\mathrm{Spin}(n)$ has more representations than $\mathrm{SO}(n)$ and to understand those one needs (either) a more concrete description of $\mathrm{Spin}(n)$ (or some knowledge of Lie theory). We will learn more about spin geometry in the context of Seiberg–Witten theory next semester. ♠

For an affine connection, that is: a covariant derivative ∇ on TX, there is a notion of torsion $T_\nabla \in \Omega^2(X, TX)$ defined by

$$T_\nabla(v, w) \coloneqq \nabla_v w - \nabla_w v - [v, w]$$

for $v, w \in \mathrm{Vect}(X)$. In the context of G–structures this can be formulated using the solder form.

Definition 4.110. Let $(p\colon P \to X, R)$ be a G–principal bundle. Let $\rho\colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation. A **(ρ–)solder form** σ on (p, R) is a horizontal G–equivariant 1–form

$$\sigma \in \Omega^1_{\mathrm{hor}}(P, V)^\rho$$

such that the map $TX \to P \times_\rho V$, defined by

$$(p(x), v) \mapsto [x, \sigma(\tilde{v})]$$

with $\tilde{v}$ denoting any lift of v, is an isomorphism. •

Example 4.111. Let $(p\colon \mathrm{Fr}(TX) \to X, R)$ be the frame bundle of TX. The canonical solder form σ on (p, R) is defined as follows. For $x \in X$ and $\phi\colon \mathbf{R}^n \to T_xX$ a frame define

$$\sigma_{(x,\phi)}(v) \coloneqq \phi^{-1} \circ T_{(x,\phi)}p(v).$$

By restriction this induces a solder form for every G–structure on X. ♠

Remark 4.112. If (p, R) admits a solder form σ, then that induces an isomorphism $P \times_G \mathrm{GL}(V) \cong \mathrm{Fr}(TX)$ such that σ is the restriction of the canonical solder form. Therefore, it is convenient to regard a G–structure as a G–principal bundle (p, R) together with a solder form. ♣

Proposition 4.113. *Let A be a $\mathrm{GL}_n(\mathbf{R})$–principal connection on $(p\colon \mathrm{Fr}(TX) \to X, R)$. Denote by ∇ the corresponding affine connection. Denote by σ the canonical solder form. The isomorphism $\Omega^2(X, TX) \cong \Omega^2_{\mathrm{hor}}(\mathrm{Fr}(TX), \mathbf{R}^n)^{\mathrm{GL}_n(\mathbf{R})}$ maps the torsion T_∇ of ∇ to*

$$\mathrm{d}_A\sigma = \mathrm{d}\sigma + [\theta_A \wedge \sigma].$$

Proof. Let $v, w \in \mathrm{Vect}(X)$. Let

$$\hat{v}, \hat{w} \in C^\infty(\mathrm{Fr}(TX), \mathbf{R}^n)^{\mathrm{GL}_n(\mathbf{R})}$$

be the lifts to maps. Denote by $\tilde{v}, \tilde{w} \in \mathrm{Vect}(P)$ (arbitrary) lifts to vector fields. The solder form relates these by

$$\sigma(\tilde{v}) = \hat{v} \quad \text{and} \quad \sigma(\tilde{w}) = \hat{w}.$$

According to Proposition 4.49, $\nabla v, \nabla w \in \Omega^1(X, TX)$ lift to

$$\mathrm{d}_A\hat{v}, \mathrm{d}_A\hat{w} \in \Omega^1_{\mathrm{hor}}(\mathrm{Fr}(TX), \mathbf{R}^n)^{\mathrm{GL}_n(\mathbf{R})}.$$

Therefore, $T_\nabla(v, w)$ lifts to

$$\mathrm{d}_A\hat{v}(\tilde{w}) - \mathrm{d}_A\hat{w}(\tilde{v}) - \sigma([\tilde{v}, \tilde{w}]) \in C^\infty(\mathrm{Fr}(TX), \mathbf{R}^n)^{\mathrm{GL}_n(\mathbf{R})}.$$

This is precisely $(\mathrm{d}_A\sigma)(\tilde{v}, \tilde{w})$. ■

We know that X always admits a torsion-free affine connection, even one compatible with a choice of Riemannian metric, i.e. a $\mathrm{O}(n)$–structure. What about other G–structures?

Definition 4.114. Let (p, R) be G–principal bundle on X together with a solder form σ. Let $A \in \mathscr{A}(p, R)$. The **torsion** of A is $\mathrm{d}_A\sigma$. If A' is another G–principal connection, then $A' = A + a$ with $a \in \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})$ and

$$\mathrm{d}_{A'}\sigma = \mathrm{d}_A\sigma + \rho(a) \wedge \sigma.$$

The **intrinsic torsion** of $(p, R; \sigma)$ is defined by

$$T(p, R; \sigma) \coloneqq [\mathrm{d}_A\sigma] \in \mathrm{coker}\big[\rho(-) \wedge \sigma\colon\ \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}} \to \Omega^2_{\mathrm{hor}}(P, V)^\rho\big].$$

If $T(p, R; \sigma) = 0$, then $(p, R; \sigma)$ is **torsion-free**. •

Obviously:

Proposition 4.115. *(p, R) admits a torsion-free connection if and only if the intrinsic torsion vanishes. Moreover, in this case, the space of torsion-free connection is an affine space modelled on*

$$\ker\big[\rho(-) \wedge \sigma\colon\ \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}} \to \Omega^2_{\mathrm{hor}}(P, V)^{\rho}\big].$$

■

Exercise 4.116. Prove that the map

$$\rho(-) \wedge \sigma\colon\ \Omega^1_{\mathrm{hor}}(P, \mathfrak{o}(n))^{\mathrm{Ad}} \to \Omega^2_{\mathrm{hor}}(P, \mathbf{R}^n)^{\mathrm{O}(n)}$$

is an isomorphism. (This algebraic fact implies the fundamental theorem of Riemannian geometry: the existence and uniqueness of the Levi–Civita connection.)

Exercise 4.117. Let X be a smooth manifold of dimension $2n$. Let I be an almost complex structure on TX. Prove that the corresponding $\mathrm{GL}_n(\mathbf{C})$–structure has vanishing intrinsic torsion if and only if the Nijenhuis tensor N_I vanishes.

5 Aspects of Yang–Mills theory

5.1 The Yang–Mills functional

Let (X, g) be an oriented pseudo-Riemannian manifold. Let G be a Lie group. Set $\mathfrak{g} \coloneqq \mathrm{Lie}(G)$

Definition 5.1. Let $(p\colon\ P \to X, R)$ be a G–principal bundle. The **adjoint bundle** associated with (p, R) is the vector bundle

$$\mathrm{Ad}(P) \coloneqq P \times_{\mathrm{Ad}} \mathfrak{g} \to X.$$

•

$\mathrm{Ad}(P)$ plays an important role because

$$\Omega^k_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}} \cong \Omega^k(X, \mathrm{Ad}(P)).$$

Therefore, $\mathscr{A}(p, R)$ can be regarded as an affine spaces modelled on $\Omega^1(X, \mathrm{Ad}(P))$, and the curvature F_A of a connection A can and (mostly) will be regarded as an $\mathrm{Ad}(P)$–valued 2–form.

Every Ad–invariant bilinear form $B \in \mathrm{Hom}(S^2\mathfrak{g}, \mathbf{R})^{\mathrm{Ad}}$ induces a bilinear form on $\mathrm{Ad}(P)$. If G is a matrix group; that is: $G < \mathrm{GL}_n(\mathbf{R})$, then a natural choice is

$$B(\xi, \eta) \coloneqq \mathrm{tr}(\xi\eta).$$

In fact, there always is a canonical choice.

Definition 5.2. The **Killing form** is the Ad–invariant bilinear form $B \in \mathrm{Hom}(S^2\mathfrak{g}, \mathbf{R})^{\mathrm{Ad}}$ defined by

$$B(\xi, \eta) \coloneqq \mathrm{tr}(\mathrm{ad}(\xi) \circ \mathrm{ad}(\eta)).$$

•

Choose a $B \in \mathrm{Hom}(S^2\mathfrak{g}, \mathbf{R})^{\mathrm{Ad}}$. Define

$$|F_A|^2 \in C^\infty(X, \mathbf{R})$$

using B and the pseudo-Riemannian metric g.

Definition 5.3. The **Yang–Mills functional** $\mathrm{YM}\colon \mathscr{A}(p, R) \to \mathbf{R}$ is defined by

$$\mathrm{YM}(A) := \frac{1}{2}\int_X |F_A|^2 \,\mathrm{vol}_g. \qquad \bullet$$

If $u \in \mathscr{G}(p, R)$, then

$$\mathrm{YM}(u^*A) = \mathrm{YM}(A).$$

(Prove this!) Therefore, YM descends to a map

$$\mathrm{YM}\colon \mathscr{A}(p, R)/\mathscr{G}(p, R) \to \mathbf{R}.$$

The Yang–Mills functional should be regarded as an energy functional. (At this stage, despite the notation, $|F_A|^2$ need not be non-negative. This should not deter us.)

Suppose that B is non-degenerate. The covariant derivative

$$\mathrm{d}_A\colon \Omega^\bullet(X, \mathrm{Ad}(P)) \to \Omega^{\bullet+1}(X, \mathrm{Ad}(P))$$

has a formal adjoint

$$\mathrm{d}_A^*\colon \Omega^\bullet(X, \mathrm{Ad}(P)) \to \Omega^{\bullet-1}(X, \mathrm{Ad}(P))$$

with respect to B and g.

Proposition 5.4. *For $A \in \mathscr{A}(p, R)$ and $a \in \Omega^1(X, \mathrm{Ad}(P))$*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{YM}(A + ta) = \int_X \langle \mathrm{d}_A^* F_A, a\rangle \,\mathrm{vol}_g$$

Proof. This is a consequence of

$$F_{A+ta} = F_A + t\mathrm{d}_A a + \frac{1}{2}t^2[a \wedge a].$$

Indeed,

$$\begin{aligned}
\mathrm{YM}(A + ta) &= \frac{1}{2}\int_X |F_A + t\mathrm{d}_A a|^2 \,\mathrm{vol}_g + O(t^2) \\
&= \mathrm{YM}(A) + t\int_X \langle F_A, \mathrm{d}_A a\rangle \,\mathrm{vol}_g + O(t^2) \\
&= \mathrm{YM}(A) + t\int_X \langle \mathrm{d}_A^* F_A, a\rangle \,\mathrm{vol}_g + O(t^2). \qquad \blacksquare
\end{aligned}$$

Definition 5.5. The **Yang–Mills equation** is

$$\mathrm{d}_A^* F_A = 0. \qquad \bullet$$

Remark 5.6. The Yang–Mills equation should be understood as a second order equation on the connection A. ♣

The Yang–Mills equation stands at the interface of physics and geometry, similar to (the vacuum) Einstein equations. Mysteriously, its study has had remarkable impact on mathematics: ranging across representation theory, algebraic geometry, partial differential equations, and topology.

5.2 Maxwell's equations

Maxwell's equations (without charges and currents) governing electro-magnetism are an instance of the Yang–Mills equation. In Maxwell's theory, there are two fields: the electric field $E = (E_1, E_2, E_3)$ and the magnetic field $B = (B_1, B_2, B_3)$ satisfying

$$\nabla \cdot B = 0, \quad \nabla \times E + \partial_t B = 0, \quad \nabla \cdot E = 0, \quad \text{and} \quad \nabla \times B - \partial_t E = 0.$$

In the presence of charges and currents, the last two equations are modified. The first two equations imply that there are ϕ, the electric potential, and $\mathbf{A} = (A_1, A_2, A_3)$, the vector potential, such that

$$E = \nabla\phi - \partial_t \mathbf{A} \quad \text{and} \quad B = \nabla \times \mathbf{A}.$$

Of course, $\mathbf{A}$ and ϕ are not uniquely determined by E and B.

It is convenient to package $\mathbf{A}$ and ϕ together as

$$A := i\sum_{a=1}^{3} A_a \mathrm{d}x_a + \phi \mathrm{d}t \in \Omega^1(\mathbf{R}^4, \mathfrak{u}(1)).$$

This can be regarded as a U(1)–principal connection on the trivial U(1)–principal bundle over $\mathbf{R}^4$. Its curvature is

$$F_A = \mathrm{d}A = \frac{i}{2}\sum_{a,b=1}^{3} \underbrace{(\partial_a A_b - \partial_b A_a)}_{B_c} \mathrm{d}x_a \wedge \mathrm{d}x_b + i\sum_{a=1}^{3} \underbrace{(\partial_a \phi - \partial_t A_a)}_{E_a} \mathrm{d}x_a \wedge \mathrm{d}t.$$

The Bianchi equation $\mathrm{d}_A F_A = 0$ encodes precisely the first two of Maxwell's equations. The ambiguity in choosing $\mathbf{A}$ and ϕ corresponds the possibly gauge transformations u^*A of A which (because U(1) is abelian) have the same curvature $F_{u^*A} = F_A$.

To obtain the last two of Maxwell's equations, equip $\mathbf{R}^4$ with the Minkowski metric

$$g = \mathrm{d}x_1 \odot \mathrm{d}x_1 + \mathrm{d}x_2 \odot \mathrm{d}x_2 + \mathrm{d}x_3 \odot \mathrm{d}x_3 - \mathrm{d}t \odot \mathrm{d}t.$$

Denote by ε_{abc} the Levi–Civita symbol. A brief computation reveals that

$$\begin{aligned} \mathrm{d}_A^* F_A &= i(\partial_2 B_3 - \partial_3 B_2 - \partial_t E_1)\mathrm{d}x_1 + i(\partial_3 B_1 - \partial_1 B_3 - \partial_t E_2)\mathrm{d}x_2 \\ &+ i(\partial_1 B_2 - \partial_2 B_1 - \partial_t E_3)\mathrm{d}x_3 - i\sum_{a=1}^{3} \partial_a E_a \,\mathrm{d}t. \end{aligned}$$

Therefore, $\mathrm{d}_A^* F_A = 0$ is equivalent to the last two of Maxwell's equations.

5.3 Anti-self-duality

Let (X, g) be an oriented Riemannian manifold. Let B be an Euclidean inner product on $\mathfrak{g}$.

Flat connections trivially satisfy the Yang–Mills equation. Indeed, they are absolute minima of the Yang–Mills functional. Moreover, the condition to be flat is a first order equation on the metric, while the Yang–Mills equation is a second order equation. Chern–Weil theory gives rise to numerous obstructions for G–principal bundles to admit flat connections.

Suppose that

$$\dim X = 4.$$

The miracle of anti-self-duality appears:

$$*\colon\ \Omega^2(X) \to \Omega^2(X) \quad \text{and} \quad ** = \mathrm{id}.$$

Therefore, $*$ has two eigenvalues $+1$ and -1. By the Bianchi identity,

$$*F_A = \pm F_A \implies \mathrm{d}_A^* F_A = 0.$$

Definition 5.7. A connection $A \in \mathscr{A}(p, R)$ is **anti-self-dual (ASD)** if

$$*F_A = -F_A. \qquad \bullet$$

Remark 5.8. Initially, whether one studies the anti-self-duality equation $*F_A = -F_A$ or the (possibly more natural seeming) self-duality equation $*F_A = F_A$ seems to not matter. After all, one is free to flip the orientation on X and that exchanges these notions. It turns out, however, that for Kähler 4–manifolds, the complex structure selects a preferred orientation and for that orientation the anti-self-duality equation interacts well with the theory of holomorphic vector bundles. ♣

Remark 5.9. There are versions of anti-self-duality in higher dimension, but these all require X to have special geometry; e.g., it must be Kähler manifold, a G_2–manifold, or a $\mathrm{Spin}(7)$–manifold, etc. ♣

Example 5.10. Flat connections are ASD. ♠

Example 5.11. If $G = \mathrm{U}(1)$, then $F_A \in \Omega^2(X, i\mathbf{R})$. By the Bianchi identity, $\mathrm{d}F_A = \mathrm{d}_A F_A = 0$. Therefore $[F_A] \in \mathrm{H}^2_{\mathrm{dR}}(X, i\mathbf{R})$. If $[\Omega] \in \mathrm{im}(\mathrm{H}^2(X, \mathbf{Z}) \to \mathrm{H}^2_{\mathrm{dR}}(X, \mathbf{R}))$, then there always is a $\mathrm{U}(1)$–principal bundle with a connection A such that $F_A = -2\pi i \Omega$. Therefore, $\mathrm{U}(1)$ ASD connections (up to gauge transformations) are essentially classified by the anti-self-dual harmonic 2–forms in $\mathrm{im}(\mathrm{H}^2(X, \mathbf{Z}) \to \mathrm{H}^2_{\mathrm{dR}}(X, \mathbf{R}) \cong \mathscr{H}^2(X, g))$ ♠

Proposition 5.12. *Suppose that G is semi-simple and $-B$ is the Killing form. For every $A \in \mathscr{A}(p, R)$*

$$\mathrm{YM}(A) = \mp 4\pi^2 \int_X p_1(\mathrm{Ad}(P)) + \frac{1}{4}\int_X |F_A \pm *F_A|^2 \,\mathrm{vol}_g$$

In particular, (anti-)self-dual connections are absolute minima of the Yang–Mills functional.

Proof. By definition,

$$p_1(\mathrm{Ad}(P)) = -c_2(\mathrm{Ad}(P)\otimes \mathbf{C}).$$

Since

$$c_2 = -\mathrm{ch}_2 + \frac{1}{2}\mathrm{ch}_1^2, \quad \mathrm{ch}_k(\mathrm{Ad}(P)\otimes \mathbf{C}) = \left[\mathrm{tr}\,\frac{1}{k!}\big(\tfrac{i}{2\pi}\,\mathrm{ad}\circ F_A\big)^{\wedge k}\right],$$

and tr vanishes on $\mathfrak{o}(n)$,

$$\begin{aligned} p_1(\mathrm{Ad}(P)) &= -\frac{1}{8\pi^2}[\mathrm{tr}(\mathrm{ad}\circ F_A \wedge \mathrm{ad}\circ F_A)] \\ &= \frac{1}{8\pi^2}\langle F_A \wedge F_A\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{4}\int_X |F_A \pm *F_A|^2 \,\mathrm{vol}_g &= \frac{1}{4}\int_X \langle (F_A \pm *F_A)\wedge(*F_A \pm F_A)\rangle \\ &= \frac{1}{2}\int_X |F_A|^2 \pm \frac{1}{2}\langle F_A \wedge F_A\rangle \\ &= \mathrm{YM}(A) \pm 4\pi^2 p_1(\mathrm{Ad}(P)). \end{aligned}$$

■

Remark 5.13. In dimension 4, YM depends on the conformal class of g only; that is: YM computed with respect to g is identical to YM computed with respect to $e^{2f}g$. Moreover, $*\colon \Lambda^2T^*X \to \Lambda^2T^*X$ also depends only on the conformal class of g. Even more is true: $*$ determines the conformal structure. More precisely: the wedge product defines an symmetric bilinear form of signature $(3,3)$ on $\Lambda^2(\mathbf{R}^4)^*$. If $\Lambda^+ \subset \Lambda^2(\mathbf{R}^4)^*$ is a maximal positive definite subspace, then there is a unique conformal class $[g]$ such that Λ^+ is the $+1$–eigenspace of $*\colon \Lambda^2(\mathbf{R}^4)^* \to \Lambda^2(\mathbf{R}^4)^*$. ♣

5.4 The BPST instanton

The BPST instanton is an important example of an anti-self-dual connection on the trivial $\mathrm{SU}(2)$–principal bundle over $\mathbf{R}^4$ discovered by Belavin, Polyakov, Schwartz, and Tyupkin [BPST75]. Much of the following discussion stems from Atiyah's wonderful book [Ati79].

To understand Belavin, Polyakov, Schwartz, and Tyupkin [BPST75]'s construction it is useful to use the quaternions $\mathbf{H} = \mathbf{R}\langle 1, i, j, k\rangle$. Denote by $q \in C^\infty(\mathbf{H},\mathbf{H})$ the identity map and define $q_0, q_1, q_2, q_3 \in C^\infty(\mathbf{H})$ by

$$q =: q_0 + q_1 i + q_2 j + q_3 k.$$

Denote by $\bar{q}$ the conjugate; that is:

$$\bar{q} = q_0 - q_1 i - q_2 j - q_3 k.$$

Define

$$- \wedge -\colon\ \Omega^\bullet(X,\mathbf{H}) \otimes \Omega^\bullet(X,\mathbf{H}) \to \Omega^\bullet(X,\mathbf{H})$$

as the composition of the $- \wedge -\colon\ \Omega^\bullet(X,\mathbf{H}) \otimes \Omega^\bullet(X,\mathbf{H}) \to \Omega^\bullet(X,\mathbf{H}) \otimes \Omega^\bullet(X,\mathbf{H}\otimes\mathbf{H})$ and multiplication $\mathbf{H}\otimes\mathbf{H} \to \mathbf{H}$. Since $\mathbf{H}$ is not commutative, $- \wedge -$ is not graded commutative. Indeed,

$$\begin{aligned} \mathrm{d}q \wedge \mathrm{d}\bar{q} = &-2(\mathrm{d}q_0 \wedge \mathrm{d}q_1 + \mathrm{d}q_2 \wedge \mathrm{d}q_3) \otimes i \\ &- 2(\mathrm{d}q_0 \wedge \mathrm{d}q_2 + \mathrm{d}q_3 \wedge \mathrm{d}q_1) \otimes j \\ &- 2(\mathrm{d}q_0 \wedge \mathrm{d}q_3 + \mathrm{d}q_1 \wedge \mathrm{d}q_2) \otimes k \end{aligned}$$

but

$$\begin{aligned} \mathrm{d}\bar{q} \wedge \mathrm{d}q = &\,2(\mathrm{d}q_0 \wedge \mathrm{d}q_1 - \mathrm{d}q_2 \wedge \mathrm{d}q_3) \otimes i \\ &+ 2(\mathrm{d}q_0 \wedge \mathrm{d}q_2 - \mathrm{d}q_3 \wedge \mathrm{d}q_1) \otimes j \\ &+ 2(\mathrm{d}q_0 \wedge \mathrm{d}q_3 - \mathrm{d}q_1 \wedge \mathrm{d}q_2) \otimes k. \end{aligned}$$

Observe that the coefficients of $\mathrm{d}q \wedge \mathrm{d}\bar{q}$ are a basis of $\Lambda^+\mathbf{H}^*$ and the coefficients of $\mathrm{d}\bar{q} \wedge \mathrm{d}q$ span $\Lambda^-\mathbf{H}^*$.

The Lie group $\mathrm{Sp}(1) \coloneqq \{q \in \mathbf{H} : q\bar{q} = 1\}$ is isomorphic to $\mathrm{SU}(2)$ ($\mathbf{H} = \mathbf{C} \oplus \mathbf{C}j$). Observe that

$$\mathfrak{sp}(1) \coloneqq \mathrm{Lie}(\mathrm{Sp}(1)) = \mathrm{Im}\,\mathbf{H}.$$

Therefore, a $\mathrm{Sp}(1)$–connection on the trivial $\mathrm{Sp}(1)$–bundle over $\mathbf{H}$ can be regarded as a 1–form $A \in \Omega^1(\mathbf{H}, \mathrm{Im}\,\mathbf{H})$. $\mathrm{Sp}(1)$ acts on $\mathbf{H}$ by

$$R(g)x \coloneqq xg^*$$

and on $\mathrm{Im}\,\mathbf{H}$ by

$$\mathrm{Ad}(g)\xi = g\xi g^*.$$

If A is required to satisfy the invariance condition

$$[R(q)]^*A = \mathrm{Ad}(q)A,$$

then it must be of the form

$$A = 2f(|q|^2)\,\mathrm{Im}(\bar{q}\mathrm{d}q) = f(|q|^2)(\bar{q}\mathrm{d}q - \mathrm{d}\bar{q}q)$$

To facilitate the computation of F_A, observe that

$$2\,\mathrm{Im}(\bar{q}\mathrm{d}q) = 2\bar{q}\mathrm{d}q - \mathrm{d}|q|^2 = -2\mathrm{d}\bar{q}q + \mathrm{d}|q|^2.$$

Hence,

$$\begin{aligned} 4\,\mathrm{Im}(\bar{q}\mathrm{d}q) \wedge \mathrm{Im}(\bar{q}\mathrm{d}q) &= (-2\mathrm{d}\bar{q}q + \mathrm{d}|q|^2) \wedge (2\bar{q}\mathrm{d}q - \mathrm{d}|q|^2) \\ &= -4|q|^2\mathrm{d}\bar{q} \wedge \mathrm{d}q + 2\mathrm{d}|q|^2 \wedge (\bar{q}\mathrm{d}q - \mathrm{d}\bar{q}q) \\ &= -4|q|^2\mathrm{d}\bar{q} \wedge \mathrm{d}q + \mathrm{d}|q|^2 \wedge 4\,\mathrm{Im}(\bar{q}\mathrm{d}q). \end{aligned}$$

Therefore, the curvature of A can be computed to be

$$F_A = \left[2f(|q|^2) - 4|q|^2 f(|q|^2)^2\right] \mathrm{d}\bar{q} \wedge \mathrm{d}q + \left[f'(|q|^2) + 2f(|q|^2)^2\right] \mathrm{d}|q|^2 \wedge 2\,\mathrm{Im}(\bar{q}\mathrm{d}q).$$

The first term is anti-self-dual. To make the second term vanish, one needs to solve the ODE $f' + 2f^2 = 0$:

$$f(|q|^2) = \frac{1}{2}\frac{\mu^2}{\mu^2|q|^2 + 1}$$

with $\mu > 0$. Therefore, we arrive at

$$A_\mu = \frac{\mathrm{Im}(\mu^2\bar{q}\mathrm{d}q)}{\mu^2|q|^2 + 1}.$$

The above computation singles out the origin in $\mathbf{H}$. The **BPST instanton** of scale μ and center b is

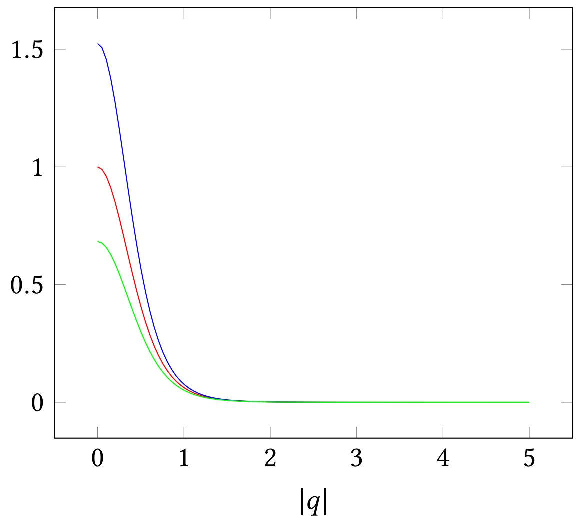
$$A_{\mu,b} := \frac{\mathrm{Im}(\mu^2(\bar{q} - \bar{b})\mathrm{d}q)}{\mu^2|q - b|^2 + 1}.$$

A without any indices shall always refer to $A_{1,0}$.

Observe that

$$F_{A_{\mu,b}} = \frac{\mu^2 \mathrm{d}\bar{q} \wedge \mathrm{d}q}{(\mu^2|q - b|^2 + 1)^2}.$$

Here are plots of $1/(|q|^2 + 1/\mu^2)^2$ for $\mu^2 \in \{0.9, 1, 1.1\}$.



Let us compute $\mathrm{YM}(A)$.

$$F_A = \frac{\mathrm{d}\bar{q} \wedge \mathrm{d}q}{(|q|^2 + 1)^2}.$$

To compute the norm of i, j, k with respect to the the negative of the Killing form observe that $\mathrm{ad}(\xi)$ vanishes on ξ and acts as 2ξ on $\xi^\perp$. Therefore,

$$-B(i,i) = -B(j,j) = -B(k,k) = 8.$$

Consequently,

$$|\mathrm{d}\bar{q} \wedge \mathrm{d}q|^2 = 3 \cdot 8 \cdot 8 = 192.$$

Therefore,

$$\mathrm{YM}(A) = 96 \cdot \mathrm{vol}(S^3) \int_0^\infty \frac{r^3}{(r^2+1)^4} \mathrm{d}r = 16\pi^2.$$

This uses $\mathrm{vol}(S^3) = 2\pi^2$ and evaluates the integral to $1/12$.

Exercise 5.14. Compute $\mathrm{YM}(A)$!

Exercise 5.15. Prove that the parameters μ, b are determined by $|F_{A_{\mu,b}}|$ and, hence, the gauge equivalence class of $A_{\mu,b}$.

Remark 5.16. $\mathrm{Sp}(1)$ also acts on $\mathbf{H}$ via $L(g)x := gx$. This leads to a similar expression with q and $\bar{q}$ exchanged. The corresponding connection has self-dual curvature. ♣

5.4.1 The BPST instanton on S^4

Here is another perspective on the BPST instanton. $\mathrm{Sp}(1)$ acts on the right of

$$S^7 := \{(q_1, q_2) \in \mathbf{H}^2 : |q_1|^2 + |q_2|^2 = 1\}$$

via

$$R((q_1, q_2), q) := (q_1 q, q_2 q).$$

The quotient

$$\mathbf{H}P^1 := S^7/\mathrm{Sp}(1)$$

parametrizes rank 1 right $\mathbf{H}$–submodules $\ell \subset \mathbf{H}^2$. Denote by $p\colon S^7 \to \mathbf{H}P^1$ the canonical projection. Define $\theta_A \in \Omega^1(S^7, \mathfrak{sp}(1))$ by

$$\theta_A := \mathrm{Im}(\bar{q}_1 \mathrm{d}q_1 + \bar{q}_2 \mathrm{d}q_2).$$

A moment's thought shows that θ_A is a $\mathrm{Sp}(1)$–principal connection 1–form. Denote the corresponding $\mathrm{Sp}(1)$–principal connection by A.

Define $\iota_\pm\colon \mathbf{H} \to \mathbf{H}P^1$ by $\iota_+(q) := (q, 1)$ and $\iota_-(q) := (1, q)$. The map $s_\pm\colon \mathbf{H} \to S^7$ defined by

$$s_+(q) := \frac{(q,1)}{\sqrt{|q|^2+1}} \quad \text{and} \quad s_-(q) := \frac{(1,q)}{\sqrt{|q|^2+1}}$$

defines a trivialisation of $\iota_\pm^*(p, R)$. Moreover, a short computation reveals that

$$s_\pm^* \theta_A = \frac{\mathrm{Im}(\bar{q}\mathrm{d}q)}{|q|^2+1}.$$

Remark 5.17. The above discussion shows the BPST instanton $A := A_{1,0}$ on $\mathbf{H}$ can be extended to the conformal compactifications $\mathbf{H}P^1$. Uhlenbeck's removable singularities theorem [Uhl82b] says that this can always be done provided $\mathrm{YM}(A) < \infty$. ♣

Remark 5.18. Since $\mathrm{YM}(A) = 16\pi^2$, $p_1(\mathrm{Ad}(P)) = -4$. This is consistent with the fact that the underlying rank 2 complex bundle E has $c_1(E) = 0$ and $c_2(E) = 1$. Let E be a Hermitian vector bundle of rank n. Denote by P the corresponding $\mathbf{PU}(n)$–principal bundle. The complexification of $\mathrm{Ad}(P)$ is $\mathrm{End}_0(E)$ with the subscript meaning trace-free. A simple computation using Exercise 4.89 and Proposition 4.90 shows that

$$p_1(\mathrm{Ad}(P)) = r(c_1(E)^2 - 2c_2(E)) = -2r\left(c_2(E) - \frac{1}{2}c_1(E)^2\right). \qquad \clubsuit$$

Remark 5.19. The orientation preserving conformal inversion $\jmath$ of $\mathbf{H}^\times$ defined by $f(q) := q^{-1}$ lifts to S^7 ($\mathbf{H}P^1$) as $(q_1, q_2) \mapsto (q_2, q_1)$ ($[q_1 : q_2] \mapsto [q_2 : q_1]$). Obviously, θ_A is invariant under this inversion. This shows that $\jmath^*A$ is gauge equivalent to A. Indeed, the gauge transformation is $q \mapsto u(q) = q/|q|$. This can also be verified by direct computation. ♣

Set

$$S^4 := \{(q, t) \in \mathbf{H} \oplus \mathbf{R} : |q|^2 + t^2 = 1\}.$$

Define the stereographic projections $\sigma_\pm \colon U_\pm := S^4 \backslash \{(0, \mp 1)\} \to \mathbf{H}$ by

$$\sigma_\pm(q, t) := \frac{q}{1 \mp t}.$$

The map $\phi \colon S^4 \to \mathbf{H}P^1$ defined by

$$\phi(q, t) := \begin{cases} \iota_+(\sigma_+(q, t)) & \text{if } (q, t) \in U^+, \\ \iota_-(\overline{\sigma_-(q, t)}) & \text{if } (q, t) \in U^- \end{cases}$$

is a diffeomorphism. Since $\sigma_\pm$ are conformal, ϕ^*A is an anti-self-dual connection on $\phi^*(p, R)$ defined over S^4.

Remark 5.20. $\mathbf{H}P^1$ carries a natural metric g_{FS}, the Fubini–Study metric. The standard metric on $S^7 \subset \mathbf{H}^2$ descends along p because it is $\mathrm{Sp}(1)$–invariant. To obtain a formula proceed as follows. Denote by $\hat{p} \colon \mathbf{H}^2 \backslash \{0\} \to \mathbf{H}P^1$ the canonical projection. The Riemannian metric g_{FS} is characterised by the condition that

$$(\hat{p}^* g_{\mathrm{FS}})_x(v, w) = |x|^{-2}\langle v, w\rangle = |x|^{-2}\,\mathrm{Re}(w^*v)$$

whenever $v, w \perp x \cdot \mathbf{H}$, that is: $x^*v = x^*w = 0$. Since $v \mapsto v - xx^*v/|x|^2$ is the projection to $(x \cdot \mathbf{H})^\perp$,

$$\begin{aligned}(\hat{p}^* g_{\mathrm{FS}})_x(v, w) &= |x|^{-2}\,\mathrm{Re}[(w - xx^*w/|x|^2)^*(v - xx^*v/|x|^2)] \\ &= |x|^{-2}\,\mathrm{Re}[(w^* - w^*xx^*/|x|^2)^*(v - xx^*v/|x|^2)] \\ &= \frac{\mathrm{Re}(w^*v)}{|x|^2} - \frac{\mathrm{Re}((w^*x)(x^*v))}{|x|^4}.\end{aligned}$$

Therefore,

$$(\iota_+^* g_{\mathrm{FS}})_q(v, w) = \frac{\mathrm{Re}(w^*v)}{1 + |q|^2} - \frac{|q|^2\,\mathrm{Re}(w^*v)}{(1 + |q|^2)^2} = \frac{\mathrm{Re}(w^*v)}{(1 + |q|^2)^2}.$$

This reveals that: $4g_{\mathrm{FS}} = g_{S^4}$. Moreover, it shows that with respect to this metric $|F_{A_{1,0}}|$ is constant! ♣

Set

$$\begin{aligned}
\mathrm{SL}_2(\mathbf{H}) &:= \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in M_2(\mathbf{H}) : ad - bc = 1\right\}, \\
\mathrm{Sp}(2) &:= \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in M_2(\mathbf{H}) : |a|^2 + |c|^2 = |b|^2 + |d|^2 = 1, \bar{a}b + \bar{c}d = 0.\right\}, \\
\mathbf{P}\mathrm{SL}_2(\mathbf{H}) &:= \mathrm{SL}_2(\mathbf{H})\{\pm 1\}, \quad \text{and} \\
\mathbf{P}\mathrm{Sp}(2) &:= \mathrm{Sp}(2)/\{\pm 1\}.
\end{aligned}$$

$\mathbf{P}\mathrm{SL}_2(\mathbf{H})$ acts on $\mathbf{H}P^1$ via

$$\begin{bmatrix} a & b \\ c & d\end{bmatrix}[q_1 : q_2] = [aq_1 + bq_2, cq_1 + dq_2].$$

In fact, $\mathbf{P}\mathrm{SL}_2(\mathbf{H})$ is the orientation-preserving conformal group of $\mathbf{H}P^1$. This action does not lift to S^7, but the underlying action of $\mathrm{SL}_2(\mathbf{H})$ does. The best way to see this is to observe that $\mathrm{SL}_2(\mathbf{H})$ acts on $\mathbf{H}^2\backslash\{0\}$ and to identify $S^7 = (\mathbf{H}^2\backslash\{0\})/\mathbf{R}^+$.

Remark 5.21. $\mathbf{P}\mathrm{Sp}(2)$ acts on

$$V := \left\{\begin{pmatrix} t & \bar{q} \\ q & -t\end{pmatrix} : t \in \mathbf{R}, q \in \mathbf{H}\right\} \cong \mathbf{H} \oplus \mathbf{R},$$

the quaternionic self-adjoint matrices, via conjugation. The latter have an natural inner product and, obviously, this action orthogonal (and orientation preserving). This exhibits an isomorphism $\mathbf{P}\mathrm{Sp}(2) \cong \mathrm{SO}(V) = \mathrm{SO}(5)$ and $\mathrm{Sp}(2) = \mathrm{Spin}(5)$. However, the diffeomorphism $\phi\colon S^4 \to \mathbf{H}P^1$ is *not* $\mathrm{SO}(5)$–equivariant. There is another inclusion $\mathrm{SO}(5) = \mathrm{Isom}(S^4) = \mathrm{Isom}(\mathbf{H}P^1) \hookrightarrow \mathbf{P}\mathrm{SL}_2(\mathbf{H})$. ♣

An simple computation shows that $\mathrm{Sp}(2) < \mathrm{SL}_2(\mathbf{H})$ preserves θ_A. Via $\iota_+\colon \mathbf{H} \to \mathbf{H}P^1$ this gives the partially defined action

$$\begin{bmatrix} a & b \\ c & d\end{bmatrix} q = (aq + b)(cq + d)^{-1}$$

by Möbius transformations. The map $q \mapsto \mu(q + b)$ lifts to

$$\begin{bmatrix} \sqrt{\mu} & \sqrt{\mu}b \\ 0 & 1/\sqrt{\mu}\end{bmatrix} = \begin{bmatrix} \sqrt{\mu} & 0 \\ 0 & 1/\sqrt{\mu}\end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1\end{bmatrix}.$$

Every element of $g \in \mathbf{P}\mathrm{SL}_2(\mathbf{H})$ can be written uniquely as $g = nak$ with

$$n = \begin{bmatrix} 1 & b \\ 0 & 1\end{bmatrix}, \quad a = \begin{bmatrix} \sqrt{\mu} & 0 \\ 0 & 1/\sqrt{\mu}\end{bmatrix}$$

and $g \in \mathbf{P}\mathrm{Sp}(2)$. (This is an Iwasawa decomposition of $\mathbf{P}\mathrm{SL}_2(\mathbf{H})$.) Let us compute

$$\begin{pmatrix} \sqrt{\mu} & \sqrt{\mu}b \\ 0 & 1/\sqrt{\mu}\end{pmatrix}^* \theta_A = \frac{\mathrm{Im}[\mu^2(\bar{q}_1 - \bar{q}_2\bar{b})\mathrm{d}q_1 + (\mu^2\bar{q}_1 + \bar{q}_2)\mathrm{d}q_2]}{\mu^2|q_1 + bq_2|^2 + |q_2|^2}.$$

Obviously this yields $A_{\mu,b}$ (by construction). This shows that the BPST instantons all arise from the actions of $\mathrm{SL}_2(\mathbf{H})$ on S^7 and $\mathbf{H}P^1$. Therefore, the space of BPST instantons is

$$\mathrm{SL}_2(\mathbf{H})/\mathrm{Sp}(2) = \mathbf{P}\mathrm{SL}_2(\mathbf{H})/\mathbf{P}\mathrm{Sp}(2) \cong \mathbf{H} \times \mathbf{R}^+.$$

5.4.2 The BPST instanton on $\mathbf{R} \times S^3$

Consider the conformal diffeomorphism $\phi\colon \mathbf{R} \times \mathrm{Sp}(1) \to \mathbf{H}^\times$ defined by $\phi(t, g) := e^t g$. A brief computation shows that

$$A^{\mathrm{cyl}} := \phi^* A = \frac{g^{-1}\mathrm{d}g}{1 + e^{-2t}} = \frac{\mu_{\mathrm{Sp}(1)}}{1 + e^{-2t}}$$

with μ denoting the Maurer–Cartan form on $\mathrm{Sp}(1)$. As $t \to -\infty$, $\phi^* A$ tends to the trivial connection θ_0. As $t \to +\infty$, $\phi^* A$ tends to $u^*\theta_a$ with $u = \mathrm{id}_{\mathrm{Sp}(1)}\colon \mathrm{Sp}(1) \to \mathrm{Sp}(1)$.

Observe that

$$\partial_t A^{\mathrm{cyl}}(t, -) = \frac{2e^{-2t}\mu_{\mathrm{Sp}(1)}}{(1 + e^{-2t})^2}$$

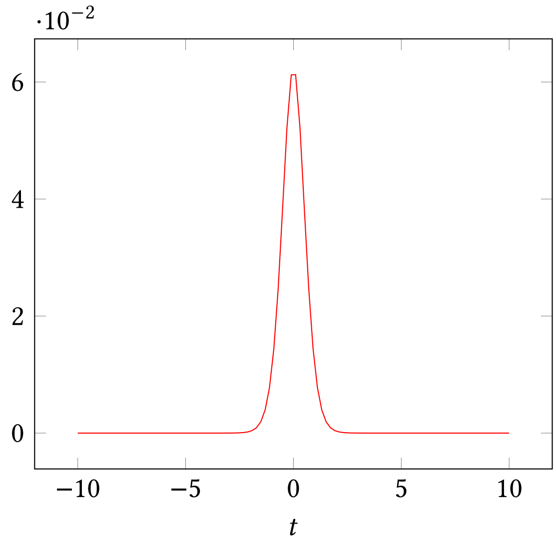
and

$$\begin{aligned} F_{A^{\mathrm{cyl}}(t,-)} &= \frac{\mathrm{d}\mu}{1 + e^{-2t}} + \frac{[\mu \wedge \mu]}{2(1 + e^{-2t})^2} \\ &= \left(-\frac{1}{2(1 + e^{-2t})} + \frac{1}{2(1 + e^{-2t})^2} \right) [\mu \wedge \mu] \\ &= -\frac{e^{-2t}}{2(1 + e^{-2t})^2} [\mu \wedge \mu]. \end{aligned}$$

Since $[\mu \wedge \mu] = 4 * \mu$, this shows that

$$\partial_t A^{\mathrm{cyl}}(t, -) = -F_{A^{\mathrm{cyl}}(t,-)}.$$

Here is a plot of $\frac{e^{-4t}}{2(1+e^{-2t})^4}$.



Exercise 5.22. What is the effect of the scale parameter μ in this perspective?

5.5 Hyperkähler manifolds and hyperkähler reduction

Atiyah, Drinfeld, Hitchin, and Manin [ADHM78] managed to construct every ASD instanton over $\mathbf{H}$ with finite Yang–Mills energy and $G = \mathrm{SU}(r)$ using "linear algebra". This was one of the early major achievements in mathematical gauge theory. The proper context for their construction are hyperkähler manifolds and hyperkähler reduction.

Definition 5.23. Let X be a smooth manifold of dimension $4n$. A **hyperkähler structure** on X consists of a Riemannian metric g and a triple of almost complex structures I_a ($a = 1, 2, 3$) such that

$$g(I_a -, I_a -) = g, \quad I_1 I_2 = I_3, \quad \text{and} \quad \nabla I_a = 0.$$

(This is equivalent to a torsion-free $\mathrm{Sp}(n)$–structure.) A **hyperkähler manifold** is a manifold with a hyperkähler structure.

Observe that $\omega_a := g(I_a -, -) \in \Omega^2(X)$ defines a symplectic (or Kähler) form on X. It is often convenient to encode I_a and ω_a ($a = 1, 2, 3$) as a **hypercomplex structure** and a **hyperkähler form**

$$\begin{aligned} \mathbf{I} &:= i^* \otimes I_1 + j^* \otimes I_2 + k^* \otimes I_3 \in (\operatorname{Im} \mathbf{H})^* \otimes \Gamma(\operatorname{End}(X)) \quad \text{and} \\ \boldsymbol{\omega} &:= i^* \otimes \omega_1 + j^* \otimes \omega_2 + k^* \otimes \omega_3 \in (\operatorname{Im} \mathbf{H})^* \otimes \Omega^2(X). \end{aligned}$$ •

It is a mildly non-trivial fact, that $\boldsymbol{\omega}$ determines $\mathbf{I}$ and g.

Remark 5.24. The hypercomplex structure $\mathbf{I}$ equips every tangent space $T_x X$ with the structure of an $\mathbf{H}$ left-module: for $q = t + \xi$ with $t \in \mathbf{R}$ and $\xi \in \operatorname{Im} \mathbf{H}$

$$q \cdot v := tv + I_\xi v \quad \text{with} \quad I_\xi := \langle \mathbf{I}, \xi \rangle.$$ ♣

Example 5.25. For every $n \in \mathbf{N}$, $\mathbf{H}^n$ is a hyperkähler manifold with g denoting the standard inner product, $I_1 = i$, $I_2 = j$, $I_3 = k$. In this case,

$$\boldsymbol{\omega} = \sum_{a=1}^{n} \mathrm{d}q_a \wedge \mathrm{d}\bar{q}_a.$$

If $\Lambda \subset \mathbf{H}^n$ is a lattice, then $\mathbf{H}^n / \Lambda$ is a hyperkähler manifold. ♠

Non-flat compact hyperkähler manifolds are notoriously difficult to construct. The first example is the $K3$ surface and the construction of the hyperkähler structure requires the use of Yau's solution of the Calabi conjecture. For mysterious(?) reasons, non-compact hyperkähler manifolds habitually emerge as moduli spaces in gauge theory. (More about that later.)

If S is a hyperkähler manifold and G acts on S, then S/G (if the quotient exists) is typically not hyperkähler.

Definition 5.26. Let S by a hyperkähler manifold with hyperkähler form $\boldsymbol{\omega}$. Let G be a compact, connected Lie group. Set $\mathfrak{g} := \mathrm{Lie}(G)$ A **hypersymplectic action** of G on X is a smooth action $\lambda \colon G \to \mathrm{Diff}(S)$ such that for every $g \in G$

$$\lambda(g)^* \boldsymbol{\omega} = \boldsymbol{\omega}.$$

For $\xi \in \mathfrak{g}$ set

$$v_\xi := \mathrm{Lie}(\lambda)(\xi) \in \mathrm{Vect}(S).$$

A **hyperkähler moment map** for λ is a G–equivariant smooth map $\mu\colon S \to (\mathrm{Im}\,\mathbf{H})^* \otimes \mathfrak{g}^*$ such that for every $x \in S$ and $\xi \in \mathfrak{g}$

$$\langle T_x\mu, \xi\rangle = i_{v_\xi(x)}\boldsymbol{\omega}. \tag{5.27}$$

•

Definition 5.28. (1) A **quaternionic Hermitian vector space** is a left $\mathbf{H}$–module S together with an inner product $\langle\cdot,\cdot\rangle$ such that i, j, k acts by isometries.

(2) The **unitary symplectic group** $\mathrm{Sp}(S)$ is the subgroup of $\mathrm{GL}_{\mathbf{H}}(S)$ preserving $\langle\cdot,\cdot\rangle$. A **quaternionic representation** of a Lie group G is a Lie group homomorphism $\lambda\colon G \to \mathrm{Sp}(S)$.

(3) The **distinguished hyperkähler moment map** of λ is the map $\mu\colon S \to (\mathrm{Im}\,\mathbf{H})^* \otimes \mathfrak{g}^*$ defined by

$$\langle \mu(x), q \otimes \xi\rangle := \frac{1}{2}\langle q \cdot \mathrm{Lie}(\rho)(\xi)x, x\rangle.$$

•

Exercise 5.29. $\mathbf{H}^2$ is a quaternionic Hermitian vector space. Consider the quaternionic representation $\lambda\colon \mathrm{U}(1) \to \mathrm{Sp}(\mathbf{H}^2)$ defined by

$$\lambda(e^{i\alpha})(q_1, q_2) := (q_1 e^{i\alpha}, q_2 e^{i\alpha}).$$

Compute μ.

Example 5.30. Let G be a compact, connected Lie group. $\mathbf{H} \otimes \mathfrak{g}$ is a quaternionic Hermitian vector space. Consider the quaternionic representation $\lambda\colon G \to \mathrm{Sp}(\mathbf{H} \otimes \mathfrak{g})$ defined by

$$\lambda(g) := \mathbf{1} \otimes \mathrm{Ad}_g .$$

Identifying $(\mathrm{Im}\,\mathbf{H})^* = \mathrm{Im}\,\mathbf{H}$ and $\mathfrak{g}^* = \mathfrak{g}$,

$$\begin{aligned}\mu(\xi) &= \frac{1}{2}[\xi, \xi] \\ &= ([\xi_2, \xi_3] + [\xi_0, \xi_1]) \otimes i + ([\xi_3, \xi_1] + [\xi_0, \xi_2]) \otimes j + ([\xi_1, \xi_2] + [\xi_0, \xi_3]) \otimes k\end{aligned}$$

for $\xi = \xi_0 \otimes 1 + \xi_1 \otimes i + \xi_2 \otimes j + \xi_3 \otimes k \in \mathbf{H} \otimes \mathfrak{g}$. A computation shows that

$$|\mu|^2 = \frac{1}{2}\sum_{a,b=0}^{4} |[\xi_a, \xi_b]|^2.$$

Therefore, $\mu(\xi) = 0$ if and only if the components of ξ are in an abelian subalgebra of $\mathfrak{g}$. ♠

The following is a direct consequence of (5.27):

Proposition 5.31. *In the above situation, $x \in S$ is a regular point of μ if and only if the stabilizer G_x is discrete; indeed:*

$$T_1 G_x = \{\xi \in \mathfrak{g} : v_\xi(x) = 0\} = \{\xi \in \mathfrak{g} : \langle T_x\mu, \xi\rangle = 0\}.$$

■

Proposition 5.32. *Let $x \in S$. Set*

$$V_x := \{v_\xi(x) : \xi \in \mathfrak{g}\} \quad \textit{and} \quad H_x := \ker T_x\mu \cap V_x^\perp.$$

H_x and $V_x \oplus (\ker T_x\mu)^\perp$ are perpendicular $\mathbf{H}$–submodules of T_xS. Moreover,

$$V_x \oplus (\ker T_x\mu)^\perp = \mathbf{H} \cdot V_x,$$

Proof. By (5.27), $\ker T_x\mu = (\operatorname{Im}\mathbf{H} \cdot V_x)^\perp$. Therefore, $H_x = (\mathbf{H} \cdot V_x)^\perp$; hence: it is an $\mathbf{H}$–submodule. Since $\mathbf{H} \cdot V_x = H_x^\perp = V_x \oplus (\ker T_x\mu)^\perp$, the latter is an $\mathbf{H}$–submodule. ■

Since G is compact,

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z} \quad \text{with} \quad \mathfrak{z} := \ker(\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})).$$

Define

$$\mathfrak{z}^* := [\mathfrak{g}, \mathfrak{g}]^0 \subset \mathfrak{g}^*,$$

the annihilator of $[\mathfrak{g}, \mathfrak{g}]$. One can identify $\mathfrak{z}^*$ with the dual of $\mathfrak{z}$. (This justifies the notation.) The importance of $\mathfrak{z}^*$ is that its elements are G–invariant. Since μ is G–invariant, for $\zeta \in (\operatorname{Im}\mathbf{H})^* \otimes \mathfrak{z}^*$, G acts on $\mu^{-1}(\zeta)$. Denote by

$$\Delta$$

the interior of the set of regular value of μ in $(\operatorname{Im}\mathbf{H})^* \otimes \mathfrak{z}^*$.

Proposition 5.33. *For $\zeta \in \Delta$ set*

$$P_\zeta := \mu^{-1}(\zeta) \quad \textit{and} \quad X_\zeta := \mu^{-1}(\zeta)/G.$$

(1) *For $x \in P_\zeta$, G_x is finite. Therefore, X_ζ is an orbifold and*

$$p_\zeta\colon P_\zeta \to X_\zeta$$

is an orbifold principal G–bundle. (If you don't know what an orbifold is, then just assume that G_x is trivial.)

(2) *For $x \in P_\zeta$, $V_x = \ker T_x p_\zeta$.*

(3) *$H_\zeta := \coprod_{x \in P_\zeta} H_x \subset TP_\zeta$ defines a principal G–connection A_ζ on P_ζ.*

(4) *The 2–form $\boldsymbol{\omega}|_{P_\zeta}$ is G–invariant and H_ζ–horizontal. It descends to a hyperkähler structure $\check{\boldsymbol{\omega}}_\zeta$ on X_ζ.*

(5) *The curvature* $F_{A_\zeta} \in \Omega^2(P_\zeta, \mathfrak{g})$ *satisfies*

$$\langle F_{A_\zeta}, \boldsymbol{\omega}|_{X_\zeta} \rangle = 0.$$

Proof. (2) is obvious.

(3) and (4) follow from Proposition 5.32.

(5) is [GN, Theorem 1.3]. It can be proved directly as follows. Let $v, w \in \Gamma(H_\zeta)^G$ and $\xi \in \mathfrak{g}$. Since $\langle v_\xi, v \rangle = \langle v_\xi, w \rangle = 0$, $\mathscr{L}_{v_\xi}\langle v, w \rangle = 0$, and $[v_\xi, v] = [v_\xi, w] = 0$,

$$\begin{aligned}
\langle F_{H_\zeta}(v, w), \xi \rangle &= \langle [v, w], v_\xi \rangle \\
&= \langle \nabla_v w - \nabla_w v, v_\xi \rangle \\
&= \langle v, \nabla_w v_\xi \rangle - \langle w, \nabla_v v_\xi \rangle \\
&= \langle v, \nabla_{v_\xi} w \rangle - \langle w, \nabla_{v_\xi} v \rangle = 2\langle v, \nabla_{v_\xi} w \rangle.
\end{aligned}$$

Therefore and since $\mathbf{I}$ is parallel,

$$F_{H_\zeta}(I_\xi -, I_\xi -) = F_{H_\zeta}. \qquad \blacksquare$$

Definition 5.34. The hyperkähler manifold $(X_\zeta, \check{\boldsymbol{\omega}}_\zeta)$ is the **hyperkähler quotient of** $(S, \boldsymbol{\omega})$ **by** G **at level** ζ. This is often denoted as

$$X_\zeta \coloneqq S /\!\!/\!\!/_\zeta G.$$

If ζ is omitted, then $\zeta = 0$ is assumed. •

Remark 5.35. *Here is an important observation.* If $\dim X_\zeta = 4$, then A_ζ is an ASD instanton on $p_\zeta \colon P_\zeta \to X_\zeta$. ♣

Example 5.36. Consider the adjoint representation $G \to \mathrm{Sp}(\mathbf{H} \otimes \mathfrak{g})$. Let $T < G$ be a maximal torus. Set $\mathfrak{t} = \mathrm{Lie}(T) \subset \mathfrak{g}$. If $\mu(\xi) = 0$ then there is a $g \in G$ such that $\mathrm{Ad}_g(\xi) \in \mathbf{H} \otimes \mathfrak{t}$. If $\xi \in \mathbf{H} \otimes \mathfrak{t}$ and $\mathrm{Ad}_g(\xi) \in \mathbf{H} \otimes \mathfrak{t}$, then g is in the normaliser $N(T)$ and T acts trivially. Therefore,

$$(\mathbf{H} \otimes \mathfrak{g}) /\!\!/\!\!/ G = (\mathbf{H} \otimes \mathfrak{t})/W$$

with $W \coloneqq N(T)/T$ denoting the Weyl group. For $G = \mathrm{U}(n)$, $\mathfrak{t} = i\mathbf{R}^n$ and $W = S_n$. Therefore,

$$(\mathbf{H} \otimes \mathfrak{u}(n)) /\!\!/\!\!/ \mathrm{U}(n) = \mathbf{H}^n / S_n \eqqcolon \mathrm{Sym}^n \mathbf{H},$$

the n**–fold symmetric product** of $\mathbf{H}$. This is, of course, an orbifold, but varying ζ generically gives smooth hyperkähler manifolds. ♠

As ζ varies in a connected component of Δ, the diffeomorphism type X_ζ persists but $\check{\boldsymbol{\omega}}_\zeta$ varies. The question of how to determine this variation (in cohomology) has been considered by Duistermaat and Heckman [DH82, §2]. [XXX: skip this in class.]

Proposition 5.37. *Set*

$$P := \{(x,\zeta) \in S \times \Delta : \mu(x) = \zeta\} \quad \textit{and} \quad X := P/G.$$

The tangent bundle TP decomposes as

$$T_{(x,\zeta)}P = H \oplus K \oplus V$$

with

$$V := \ker T\pi, \quad H := \ker T\mu \cap V^\perp, \quad \textit{and} \quad K := (\ker T\mu)^\perp = (H \oplus V)^\perp.$$

(1) *The projection $\pi\colon P \to X$ is a principal G–bundle. The projection $\rho\colon X \to \Delta$ is a fiber bundle.*

(2) *$H \oplus K$ is the horizontal distribution of a G–principal connection on π.*

(3) *K is a G–invariant Ehresmann connection on $\rho \circ \pi$. Hence, it descends to an Ehresmann connection on $\check{K}$ on ρ. The vertical tangent bundle of ρ lifts to H.*

(4) *The form $(\mathrm{pr}_S^*\boldsymbol{\omega})^{2,0}$ is G–equivariant and $(H \oplus K)$–horizontal. Therefore, it descends to $\check{\boldsymbol{\omega}}$ on X. It is of bi-degree $(0,2)$ with respect to $\check{K}$ and satisfies*

$$\check{\boldsymbol{\omega}}|_{X_\zeta} = \check{\boldsymbol{\omega}}_\zeta.$$

(5) *Denote by $\tau \in (\operatorname{Im}\mathbf{H})^* \otimes \mathfrak{z}^* \otimes \Omega^1(\Delta)$ the tautological 1–form on Δ. Denote by $\pi_\mathfrak{z} F_H \in \Omega^{0,2}(X,\mathfrak{z})$ the $(0,2)$–form defined by*

$$\pi_\mathfrak{z} F_H|_{X_\zeta} := \pi_\mathfrak{z} F_{H_\zeta}.$$

With respect to $\check{K}$,

$$\mathrm{d}\check{\boldsymbol{\omega}} = \mathrm{d}^{1,0}\check{\boldsymbol{\omega}} = -\langle \rho^*\tau \wedge \pi_\mathfrak{z} F_H\rangle.$$

(6) *Suppose that $\check{K}$ is a complete Ehresmann connection. Let $\boldsymbol{\zeta} \in C^\infty([a,b],\Delta)$ smooth path. With $\mathrm{tra}_{\boldsymbol{\zeta}}\colon X_{\boldsymbol{\zeta}(a)} \to X_{\boldsymbol{\zeta}(b)}$ denoting parallel transport along $\boldsymbol{\zeta}$,*

$$\check{\boldsymbol{\omega}}_{\boldsymbol{\zeta}(a)} = (\mathrm{tra}_{\boldsymbol{\zeta}})^*\check{\boldsymbol{\omega}}_{\boldsymbol{\zeta}(b)} - \int_a^b \langle \dot{\boldsymbol{\zeta}}(t), (\mathrm{tra}_{\boldsymbol{\zeta}|_{[a,t]}})^*\pi_\mathfrak{z} F_{H_{\boldsymbol{\zeta}(t)}}\rangle \,\mathrm{d}t.$$

Proof. (1), (2), and (3) hold by construction.

The triple of 2–forms $\mathrm{pr}_S^*\boldsymbol{\omega} \in (\operatorname{Im}\mathbf{H})^* \otimes \Omega^2(P)$ is closed and G–invariant. However, it fails to be $(H \oplus K)$–horizontal: it has components of bi-digree $(2,0)$ and a $(1,1)$. In fact, by Proposition 5.32 $\mathrm{pr}_S^*\boldsymbol{\omega} \in (\operatorname{Im}\mathbf{H})^* \otimes \Gamma(\Lambda^2 H^* \oplus K^* \otimes V^*)$. This implies (4).

Since $\mathrm{d}\mathrm{pr}_S^*\boldsymbol{\omega} = 0$,

$$\pi^*\mathrm{d}\check{\boldsymbol{\omega}} = \mathrm{d}^{1,0}(\mathrm{pr}_S^*\boldsymbol{\omega})^{2,0} = -\mathrm{d}^{2,-1}(\mathrm{pr}_S^*\boldsymbol{\omega})^{1,1} = -(\mathrm{pr}_S^*\boldsymbol{\omega}_S)^{1,1}(F_{H\oplus K}(\cdot,\cdot),\cdot).$$

Since $\omega|_V = 0$, the above is a section of $(\operatorname{Im}\mathbf{H})^* \otimes \Lambda^2(H \oplus K)^* \otimes K^*$ with the $\Lambda^2(H \oplus K)^*$ factor arising from $F_{H\oplus K}$. By (5.27), $\mathrm{pr}_S^*\omega(v_\xi, \cdot) = \langle T_x\mu \circ \mathrm{pr}_S, \xi\rangle$. Moreover, $T_x\mu \circ \mathrm{pr}_S = (\rho \circ \pi)^*\tau$. Therefore,

$$\mathrm{d}^{1,0}(\mathrm{pr}_S^*\omega)^{2,0} = -\langle \pi^*\rho^*\tau, F_{H\oplus K}\rangle = -\langle \pi^*\rho^*\tau, \pi_{\mathfrak{z}} F_{H\oplus K}\rangle.$$

It remains to show that $\pi_{\mathfrak{z}} F_{H\oplus K} = \pi_{\mathfrak{z}} F_H$. Let $v, w \in \Gamma(H \oplus K)^G$ and $\xi \in \mathfrak{z}$. As in the proof of Proposition 5.33 (5),

$$\begin{aligned}
\langle F_{H_\zeta}(v, w), \xi\rangle &= \langle [v, w], v_\xi\rangle \\
&= \langle \nabla_v w - \nabla_w v, v_\xi\rangle \\
&= \langle v, \nabla_w v_\xi\rangle - \langle w, \nabla_v v_\xi\rangle \\
&= \langle v, \nabla_{v_\xi} w\rangle - \langle w, \nabla_{v_\xi} v\rangle = 2\langle v, \nabla_{v_\xi} w\rangle.
\end{aligned}$$

Therefore, for $w = \gamma(p)v_\eta \in \Gamma(K)$

$$\langle F_{H_\zeta}(v, w), \xi\rangle = 2\langle v, \gamma(p)\nabla_{v_\xi} v_\eta\rangle = 2\langle v, \gamma(p)v_{[\xi,\eta]}\rangle = 0.$$

Consequently, $\pi_{\mathfrak{z}} F_{H\oplus K} = \pi_{\mathfrak{z}} F_H$. Therefore, $\mathrm{d}\check{\omega} = -\langle \rho^*\tau \wedge \pi_\zeta F_H\rangle$. Since the latter is of bi-degree $(1, 2)$, this finishes the proof of (5).

By (5),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{tra}_{\zeta|_{[a,t]}})^*\check{\omega}_{\zeta(t)} &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} (\mathrm{tra}_{\zeta|_{[a,t]}})^*(\mathrm{tra}_{\zeta|_{[t,t+\tau]}})^*\check{\omega}_{\zeta(t+\tau)} \\
&= -\langle \dot{\zeta}(t), (\mathrm{tra}_{\zeta|_{[a,t]}})^*\pi_{\mathfrak{z}} F_{H_{\zeta(t)}}\rangle.
\end{aligned}$$

Integration proves (6). ■

5.6 Aside: The Gibbon–Hawking ansatz

[XXX: The following is a little of an aside and will be discussed in the problem session. It at least shows that there are a lot of hyperkähler 4–manifolds.]

Let U be an open subset of $\mathbf{R}^3$. Denote by $g_{\mathbf{R}^3}$ the restriction of the standard metric on $\mathbf{R}^3$ to U. Let $\pi\colon X \to U$ be a principal $\mathrm{U}(1)$–bundle. Denote by $\partial_\alpha \in \mathrm{Vect}(X)$ the generator of the $\mathrm{U}(1)$–action. Let $i\theta \in \Omega^1(X, i\mathbf{R})$ be a $\mathrm{U}(1)$–connection 1–form and let $f \in C^\infty(U, (0, \infty))$ be a positive smooth function such that

$$\mathrm{d}\theta = - *_3 \mathrm{d}f. \tag{5.38}$$

Set

$$g := f\pi^*g_{\mathbf{R}^3} + \frac{1}{f}\theta \otimes \theta$$

and define complex structures I_1, I_2, I_3 by

$$I_i\partial_\alpha = f^{-1}\partial_{x_i} \quad \text{and} \quad I_i\partial_{x_j} = \sum_{k=1}^3 \varepsilon_{ijk}\partial_{x_k}.$$

The corresponding Hermitian forms are

$$\omega_i := \theta \wedge \mathrm{d}x_i + \frac{1}{2}\sum_{j,k=1}^{3} \varepsilon_{ijk} f \mathrm{d}x_j \wedge \mathrm{d}x_k.$$

Writing (5.54) as

$$\mathrm{d}\theta = -\frac{1}{2}\sum_{\ell,j,k=1}^{3} \varepsilon_{\ell jk}\partial_{x_\ell} f \mathrm{d}x_j \wedge \mathrm{d}x_k,$$

we see that

$$\mathrm{d}\omega_i = \mathrm{d}\theta \wedge \mathrm{d}x_i + \frac{1}{2}\sum_{j,k=1}^{3} \varepsilon_{ijk} \mathrm{d}f \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k = 0.$$

Therefore, we have proved the following.

Proposition 5.39. *(X, g, I_1, I_2, I_3) is hyperkähler manifold.*

This construction is called the **Gibbons–Hawking ansatz**.

Remark 5.40. By construction, the length of the U(1)–orbit over $x \in U$ is $f(x)^{-1/2}$. ♣

Remark 5.41. The fact that

$$i(\partial_\alpha)\omega_i = -\mathrm{d}x_i$$

means that the map $\pi\colon X \to U \subset \mathbf{R}^3$ is a hyperkähler moment map for the action of U(1) on X (with $\mathbf{R}^3$ and $(\mathfrak{u}(1) \otimes \operatorname{Im}\mathbf{H})^*$ identified suitably. ♣

Remark 5.42. By (5.54),

$$\Delta f = 0.$$

Conversely, suppose that $f\colon U \to \mathbf{R}$ is harmonic and the cohomology class of $*_3\mathrm{d}f$ lies in $\operatorname{im}(H^2(U, 2\pi\mathbf{Z}) \to H^2(U,\mathbf{R}))$, then there is a U(1)–bundle X over U together a connection $i\theta$ satisfying

$$\mathrm{d}\theta = - *_3 \mathrm{d}f.$$

♣

Example 5.43 ($\mathbf{R}^4$). Let $U = \mathbf{R}^3\backslash\{0\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) = \frac{1}{2|x|}.$$

This function is harmonic and satisfies

$$- *_3 \mathrm{d}f = \frac{1}{2}\mathrm{vol}_{S^2}.$$

Since $\mathrm{vol}(S^2) = 4\pi$, there is a U(1)–bundle X over U together with a connection $i\theta$ such that (5.54). Therefore, the Gibbins–Hawking ansatz yields a hyperkähler metric on X.

By Chern–Weil theory the first Chern number of the restriction of X to S^2 is

$$\int_{S^2} i\frac{i}{4\pi}\mathrm{vol}_{S^2} = -1.$$

Up to is isomorphism, there is only one principal $\mathrm{U}(1)$–bundle over S^2: the Hopf bundle $\pi\colon S^3 \longrightarrow S^2$ and the $\mathrm{U}(1)$–action given by $e^{i\alpha}\cdot(z_0, z_1) = (e^{-i\alpha}z_0, e^{-i\alpha}z_1)$. If g_{S^3} denotes the standard metric on S^3, then

$$\theta = g_{S^3}(-\partial_\alpha, \cdot)$$

satisfies

$$\mathrm{d}\theta = \pi^*\mathrm{vol}_{S^2}.$$

It follows that

$$X = S^3 \times (0,\infty) = \mathbf{R}^4\backslash\{0\}$$

and the Gibbons–Hawking ansatz gives the metric

$$g = 2r\,\theta\otimes\theta + \frac{1}{2r}(\mathrm{d}r\otimes\mathrm{d}r + r^2 g_{S^2}).$$

The change of coordinates $\rho = \sqrt{2r}$ rewrites this metric as

$$g = \mathrm{d}\rho\otimes\mathrm{d}\rho + \rho^2(\theta\otimes\theta + \frac{1}{4}g_{S^2}) = \mathrm{d}\rho\otimes\mathrm{d}\rho + \rho^2 g_{S^3}.$$

This means that the Gibbons–Hawking ansatz yield the standard metric on $\mathbf{R}^4$. ♠

Example 5.44 (Taub–NUT). Let $U = \mathbf{R}^3\backslash\{0\}$, let $c > 0$, and define $f_c\colon U \longrightarrow \mathbf{R}$ by

$$f_c(x) = \frac{1}{2|x|} + c.$$

This function is harmonic and we have

$$\mathrm{d}f_c = \mathrm{d}f.$$

By the preceding discussion, $X = S^3\times(0,\infty)$ and the Gibbons–Hawking ansatz gives the metric

$$g = \left(\frac{1}{2r}+c\right)^{-1}\theta\otimes\theta + \left(\frac{1}{2r}+c\right)(\mathrm{d}r\otimes\mathrm{d}r + r^2 g_{S^2}).$$

As r tends to zero this metric is asymptotic to

$$c^{-1}\theta\otimes\theta + g_{\mathbf{R}^3}.$$

Although, the metric appears singular at $r = 0$, the coordinate change $\rho = \sqrt{2r}$ rewrites it as

$$(1+c\rho^2)\mathrm{d}\rho\otimes\mathrm{d}\rho + \rho^2\left((1+c\rho^2)^{-1}\theta\otimes\theta + (1+c\rho^2)\frac{1}{4}g_{S^2}\right)$$

which is smooth.

This metric is called the **Taub–NUT metric**. It is non-flat hyperkähler metric on $\mathbf{R}^4$. It was first discovered by Taub [Tau51] and Newman, Tamburino, and Unti [NTU63]. The Taub–NUT space is the archetype of an **ALF space**. ♠

Remark 5.45. It was observed by LeBrun [LeB91] that the Taub–NUT metric is in fact Kähler for the standard complex structure on $\mathbf{C}^2$. Thus it yields a non-flat Ricci-flat Kähler metric on $\mathbf{C}^2$. ♣

Example 5.46 ($(\mathbf{R}^4\backslash\{0\})/\mathbf{Z}_k$). Let $k \in \{1, 2, 3, \dots\}$ Let $U = \mathbf{R}^3\backslash\{0\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) := \frac{k}{2|x|}.$$

This function is harmonic and it satisfies

$$- *_3 \, \mathrm{d}f = k\mathrm{vol}_{S^2}.$$

Thus, the Gibbons–Hawking ansatz applies. Denote by (X_k, g_k) the Riemannian manifold obtained in this way. If $k = 1$, then this $\mathbf{R}^4$ with its standard metric. Let us understand the cases $k \geqslant 2$.

The restriction of X_k to S^2 has Chern number $-k$. This $\mathrm{U}(1)$–bundle is $S^3/\mathbf{Z}_k \to S^2$. Consequently,

$$X_k = S^3/\mathbf{Z}_k \times (0, \infty) = \mathbf{R}^4/\mathbf{Z}_k.$$

We can choose the connection 1-form $i\theta_k$ on X_k such that its pullback to X_1 is $ik\theta_1$. It follows that the pullback of g_k to X_1 can be written as

$$2kr\, \theta \otimes \theta + \frac{k}{2r}(\mathrm{d}r \otimes \mathrm{d}r + r^2 g_{S^2}).$$

Up to a coordinate change $r \mapsto kr$ this is the standard metric on $\mathbf{R}^4$. It follows that g_k is the metric induced by the standard metric on $\mathbf{R}^4$. ♠

Example 5.47 (Eguchi–Hanson and multi-center Gibbons–Hawking). Let $x_1, \dots, x_k$ be k distinct points in $\mathbf{R}^3$. Set $U := \mathbf{R}^3\backslash\{x_1, \dots, x_k\}$ and define $f\colon U \to \mathbf{R}$ by

$$f(x) = \sum_{i=1}^{k} \frac{1}{2|x - x_i|}.$$

From the in discussion Example 5.43 it is clear that the Gibbons–Hawking ansatz for f produces a Riemannian manifold whose apparent singularities over $x_1, \dots, x_k$ can be removed. Denote the resulting manifold by (X, g).

Since

$$f(x) = \frac{k}{2|x|} + O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty,$$

(X, g) is asymptotic at infinity to $\mathbf{R}^4/\mathbf{Z}_k$. These spaces are called **ALE spaces of type** A_{k-1}. For $k = 2$, this metric was discovered by Eguchi and Hanson [EH79]. The metrics for $k \geqslant 3$ were discovered by Gibbons and Hawking [GH78].

Let us understand the geometry and topology of these spaces somewhat more. Suppose γ is an arc in $\mathbf{R}^3$ from x_i to x_j avoiding all the other points x_k. The pre-image in X of any interior point of γ is an S^1 while the pre-images of the end points are points. Therefore,

$$\pi^{-1}(\gamma) \subset X$$

is diffeomorphic to S^2. Suppose γ is straight line segment in $\mathbf{R}^3$ with unit tangent vector

$$v = \sum_{i=1}^{3} a_i \partial_{x_i}$$

with $a_1^2 + a_2^2 + a_3^2 = 1$. The tangent spaces to $\pi^{-1}(\gamma)$ are spanned by ∂_α and v. In particular, they are invariant with respect to the complex structure

$$I_v := a_1 I_1 + a_2 I_2 + a_3 I_3.$$

Its volume is given by

$$\int_{\pi^{-1}(\gamma)} a_1\omega_1 + a_2\omega_2 + a_3\omega_3.$$

Therefore,

$$[\pi^{-1}(\gamma)] \neq 0 \in H_2(X, \mathbf{Z}).$$

If necessary we can reorder the points x_i so that for $i = 1, \dots, k-1$, there is a straight-line segment γ_i joining x_i and x_{i+1}. Set

$$\Sigma_i := \pi^{-1}(\gamma_i).$$

It is not difficult to see that $[\Sigma_1], \dots, [\Sigma_{k-1}]$ generate $H_2(M; \mathbf{Z})$. It is an exercise to show that

$$[\Sigma_i] \cdot [\Sigma_j] = \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases}$$

♠

Remark 5.48. Kronheimer [Kro89b] gave an alternative construction of the ALE spaces of type A_{k-1} (in fact, all ALE spaces) as hyperkähler quotients. He also classified these spaces completely [Kro89a]. ♣

Example 5.49. Let $x_1, \dots, x_k$ be k distinct points in $\mathbf{R}^3$ and let $c > 0$. Set $U := \mathbf{R}^3 \backslash \{x_1, \dots, x_k\}$ and define $f \colon U \to \mathbf{R}$ by

$$f(x) = \sum_{i=1}^{k} \frac{1}{2|x - x_i|} + c.$$

The Gibbons–Hawking ansatz for f gives rise to the so-called **multi-center Taub–NUT** metric.

♠

Example 5.50. The following is due to Anderson, Kronheimer, and LeBrun [AKL89]. Let $x_1, x_2, \dots$ be an infinite sequence of distinct points in $\mathbf{R}^3$ and denote by U the complement of these points. If

$$\sum_{j=2}^{\infty} \frac{1}{x_1 - x_j} < \infty,$$

then

$$f(x) := \sum_{j=1}^{\infty} \frac{1}{2|x - x_j|}$$

defines a harmonic function on U. The Gibbons–Hawking ansatz gives rise to a hyperkähler manifold X whose second homology $H_2(X, \mathbf{Z})$ is infinitely generated. Anderson, Kronheimer, and LeBrun prove that the metric g is complete. ♠

This is not a complete list of interesting examples of hyperkähler manifold which can be produced using the Gibbons–Hawking ansatz. The most egregious omission is that of the Ooguri–Vafa metric.

5.7 Anti-self-duality as a moment map

Let G be a compact connecte semi-simple Lie group. Fix $k \in \{2, 3, \ldots\}$. (The choice of k ultimately turns out to be quite insubstantial. For concreteness one could take $k = 2$.) Set

$$\mathscr{A} := W^{k,2}\Omega^1(\mathbf{H}, \mathfrak{g})$$

Here the prefix $W^{k,2}$ denotes taking the completion with respect to the norm $\|-\|_{W^{k,2}}$ defined by

$$\|A\|^2_{W^{k,2}} := \sum_{\ell=0}^{k} \int_{\mathbf{H}} |\nabla^k A|^2.$$

This is a Hilbert manifold (indeed: a Hilbert space). The L^2–inner product defines a Riemannian metric on $\mathscr{A}$ and, obviously, $\operatorname{Im}\mathbf{H}$ acts on $T\mathscr{A}$: for $a \in T_A\mathscr{A}$ and $\xi \in \operatorname{Im}\mathbf{H}$

$$I_\xi a := -a(\xi \cdot -).$$

(The minus sign is neccessary to preserve $I_i I_j = I_k$.)

Therefore, $\mathscr{A}$ is an infinite dimensional hyperkähler manifold.

Set

$$\mathscr{G}_0 := \exp(W^{k+1,2}(\mathbf{H}, \mathfrak{g})) := \{u = \exp(\xi) : \xi \in W^{k+1,2}(\mathbf{H}, \operatorname{End}(\mathfrak{g}))\}.$$

The subscript zero indicates that the gauge transformations $u \in \mathscr{G}_0$. $\mathscr{G}_0$ acts on the right of $\mathscr{A}$ via

$$u^*A := \operatorname{Ad}(u) \circ A + \mu(u) = \text{"}u^{-1}Au + u^{-1}\mathrm{d}u\text{"}.$$

This action preserves the hyperkähler structure on $\mathscr{A}$. The infinitesimal action of $\xi \in \operatorname{Lie}(\mathscr{G}_0) = W^{k+1,2}(\mathbf{H}, \mathfrak{g})$ is

$$v_\xi(A) = \mathrm{d}_A \xi.$$

This is a (Hilbert space) quaternionic representation. Let us compute the distinguished hyperkähler moment map: for $A \in \mathscr{A}$, $q \in \operatorname{Im}\mathbf{H}$, and $\hat{u} \in \operatorname{Lie}(\mathscr{G}_0) = W^{k+1,2}(\mathbf{H}, \mathfrak{g})$

$$\langle \mu(A), q \otimes \xi \rangle = \frac{1}{2}\langle I_q \mathrm{d}_A \xi, A \rangle.$$

To digest this expression observe that

$$\begin{aligned} \mathrm{d}^*(f\omega_q) &= \sum_{a=1}^{4} -(\partial_a f) i_{\partial_{x_a}} \omega_q = \sum_{a=1}^{4} -(\partial_a f)\langle I_q \partial_{x_a}, - \rangle = \sum_{a=1}^{4} (\partial_a f)\langle \partial_{x_a}, I_q - \rangle \\ &= \sum_{a=1}^{4} (\partial_a f) \mathrm{d}x_a \circ I_q = -I_q \mathrm{d}f. \end{aligned}$$

Therefore,

$$\langle \mu(A), q \otimes \xi\rangle = -\frac{1}{2}\langle \mathrm{d}_A^*(\xi\cdot\omega_q), A\rangle = -\frac{1}{2}\langle \xi\cdot\omega_q), \mathrm{d}_A A\rangle = -\frac{1}{2}\langle \xi\cdot\omega_q, F_A\rangle = -\frac{1}{2}\langle \xi\cdot\omega_q, F_A^+\rangle.$$

Identifying, $\mathrm{Lie}(\mathscr{G}_0)$ and its dual and $\mathrm{Im}\,\mathbf{H}^* = \mathrm{Im}\,\mathbf{H} = \Lambda^+\mathbf{H}$ via $q^* \mapsto -\frac{1}{2}\omega_q$ exhibits

$$\mu(A) = F_A^+.$$

Therefore, the anti-self-dual part of the curvature is the hyperkähler moment map! (The phenomenon that an interesting non-linear partial differential equation appears as a moment maps is suprisingly common.)

Now the hyperkähler reduction of the $\mathscr{A}$ by $\mathscr{G}_0$ is

$$\mathscr{M}_G^{\mathrm{fr}} := \{A \in \mathscr{A} : F_A^+ = 0\}/\mathscr{G}_0.$$

This is the **moduli space of framed** G **ASD instantons on H**. The adjective framed has to do with the fact that the quotient is by gauge transformations which decay to the identity at infinity. $\mathscr{M}_G^{\mathrm{fr}}$ has an action by G. The quotient is the actual moduli space (but it does not have the feature of being hyperkähler).

For every $[A] \in \mathscr{M}_G^{\mathrm{fr}}$

$$k := \frac{1}{4\pi^2}\mathrm{YM}(A) \in \mathbf{N}_0.$$

This number k is the **instanton number** or **charge** of A. It is customary to decompose

$$\mathscr{M}_G^{\mathrm{fr}} = \coprod_{k\in\mathbf{N}_0} \mathscr{M}_{G,k}^{\mathrm{fr}}.$$

5.8 Preparation: projections and connections

Let X be a smooth manifold. Denote by $\underline{\mathbf{R}}^n := \mathbf{R}^n \times X$ the trivial vector bundle of rank N over X. Let $P \in \Gamma(\mathrm{End}(\underline{\mathbf{R}}^n))$ be a projection of constant rank; that is:

$$P^2 = P \quad \text{and} \quad \mathrm{rk}\, P = r.$$

Define the complementary projection by

$$Q := \mathbf{1} - P.$$

These define a decomposition

$$\underline{\mathbf{R}}^n = E \oplus F \quad \text{with} \quad E := \mathrm{im}\, P \quad \text{and} \quad F := \mathrm{im}\, Q.$$

With respect to this decomposition

$$\mathrm{d} = \begin{pmatrix} P\mathrm{d} & P\mathrm{d}Q \\ Q\mathrm{d}Q & Q\mathrm{d} \end{pmatrix}.$$

The diagonal components define covariant derivatives $\nabla := P\mathrm{d}$ on E and $\nabla := Q\mathrm{d}$ on F. Since the roles of E and F are interchangable, let us focus on E. If P is an orthogonal projection; that is: $P^* = P$, then ∇ is an orthogonal covariant derivative on E. (Similarly, if P is complex linear, etc.)

To compute the curvature of ∇ it is convenient to define the covariant derivative $\bar{\nabla}$ on $\underline{\mathbf{R}}^n$ by

$$\bar{\nabla}s := (\mathrm{d} + A)s \quad \text{with} \quad A := -Q\mathrm{d}P = Q\mathrm{d}Q.$$

The covariant derivative $\bar{\nabla}$ preserves the subbundle $E \subset \underline{\mathbf{R}}^N$ and induces ∇. The curvature $F_{\bar{\nabla}}$ of $\bar{\nabla}$ is

$$F_{\bar{\nabla}} = \mathrm{d}Q \wedge \mathrm{d}Q = \mathrm{d}P \wedge \mathrm{d}P$$

To see this observe that

$$Q\mathrm{d}Q \wedge Q\mathrm{d}Q = Q(\mathrm{d}Q - Q\mathrm{d}Q) \wedge \mathrm{d}Q = 0$$

Therefore,

$$F_{\nabla} = P(\mathrm{d}P) \wedge (\mathrm{d}P)P.$$

Henceforth, let us assume that P is orthogonal. Suppose that $R\colon \underline{\mathbf{R}}^n \to \underline{\mathbf{R}}^{n-r}$ is a surjective vector bundle morphism and that

$$E = \ker R \quad \text{and} \quad F = \operatorname{im} R^*.$$

A moment's thought shows that

$$P = \mathbf{1} - Q \quad \text{and} \quad Q = R^*(RR^*)^{-1}R$$

(This becomes particularly simple if $RR^* = \mathbf{1}$.) Since $RP = 0$ and $PR^* = 0$, the above considerations yields

$$F_{\nabla} = P(\mathrm{d}R^*) \wedge (RR^*)^{-1}(\mathrm{d}R)P.$$

Remark 5.51. Let $k, r \in \mathbf{N}_0$.

(1) Denote by $\mathrm{Gr}_r(\mathbf{R}^{k+r})$ the Grassmannian of r–planes in $\mathbf{R}^{k+r}$. There is a tautological vector bundle $p\colon V \to \mathrm{Gr}_r(\mathbf{R}^{k+r})$ (indeed, a subbundle of $\underline{\mathbf{R}}^{k+r} \to \mathrm{Gr}_r(\mathbf{R}^{k+r})$ defined by

$$V := \left\{(\Pi, v) \in \mathrm{Gr}_r(\mathbf{R}^{k+r}) \times \mathbf{R}^{k+r} : v \in \Pi\right\}.$$

V inherits an Euclidean inner product from $\underline{\mathbf{R}}^{k+r}$.

(2) The procedure discussed above defines a covariant derivative ∇ on V.

(3) Let X is a smooth manifold. If $f\colon X \to \mathrm{Gr}_r(\mathbf{R}^{k+r})$ is a smooth map, then $f^*p\colon f^*V \to X$ is a Euclidean rank r vector bundle. Upto isomorphism every Euclidean rank r vector bundle over X comes from such a map for some value of k. (This is a basic result in the theory of vector bundles: a baby version of Whitney's embedding theorem.) Of course, ∇ defines a connection on f^*V. It turns out that all ASD instantons on $\mathbf{H}P^1 = S^4$ can be obtained in this way.

(4) Denote by

$$\mathrm{St}_k^*(\mathbf{R}^{k+r}) := \left\{R \in \mathrm{Hom}(\mathbf{R}^{k+r}, \mathbf{R}^k) : R \text{ is surjective}\right\} \quad \text{and}$$
$$\mathrm{St}_k(\mathbf{R}^{k+r}) := \left\{R \in \mathrm{St}_k^*(\mathbf{R}^{k+r}) : RR^* = \mathbf{1}\right\}$$

the Stiefel manifold and the orthogonal Stiefel manifold. (To match this with our earlier definition, observe that R and R^* are equivalent data.) The maps

$$\ker\colon \ \mathrm{St}_k^*(\mathbf{R}^{k+r}) \to \mathrm{Gr}_r(\mathbf{R}^{k+r}) \quad \text{defined by} \quad \ker(R) := \ker R$$

makes $\mathrm{St}_k^*(\mathbf{R}^{k+r})$ into a $\mathrm{GL}_k(\mathbf{R})$–principal bundle and $\mathrm{St}_k(\mathbf{R}^{k+r})$ into a $\mathrm{O}(k)$–principal bundles. Indeed, they are the frame bundle and the orthogonal frame bundle of $V^\perp(= \mathrm{im}\, R^*)$ respectively.

(5) The diffeomorphism $\mathrm{Gr}_k(\mathbf{R}^{k+r}) \cong \mathrm{Gr}_r(\mathbf{R}^{k+r})$ turns $\mathrm{St}_r(\mathbf{R}^{k+r})$ into an $\mathrm{O}(r)$–principal bundle over $\mathrm{Gr}_r(\mathbf{R}^{k+r})$. Indeed, it is the orthogonal frame bundle of V. Therefore, it inherits a $\mathrm{O}(r)$–principal connection A_∇ from the covariant derivative ∇ on V.

(6) $\mathrm{St}_r(\mathbf{R}^{k+r})$ has a canonical $\mathrm{O}(r)$–invariant Riemannian metric g; indeed:

$$g(\hat{R}_1, \hat{R}_2) = \mathrm{tr}(\hat{R}_1^* \hat{R}_2).$$

This equips it with an $\mathrm{O}(r)$–principal connection A_g. Indeed:

$$A_g = A_\nabla!$$

(7) The Gram–Schmidt process defines a map

$$\mathrm{GS}\colon \ \mathrm{St}_k^*(\mathbf{R}^{k+r}) \to \mathrm{St}_k(\mathbf{R}^{k+r}).$$

Indeed,

$$\mathrm{GS}(R) = (RR^*)^{-1/2} R.$$

(8) Finally, the above precedure can be thought of (universally) as computing the the pullback to $\mathrm{St}_k(\mathbf{R}^{k+r})$ of the canonical $\mathrm{O}(r)$–principal connection A on $\mathrm{St}_r(\mathbf{R}^{k+r}) \to \mathrm{Gr}_r(\mathbf{R}^{k+r})$ and determining its curvature F_A. The total space of this the pullback bundle is

$$\left\{(R, S) \in \mathrm{St}_k^*(\mathbf{R}^{k+r}) \times \mathrm{Hom}(\mathbf{R}^{k+r}, \mathbf{R}^r) : S^* R = 0, SS^* = \mathbf{1}\right\}.$$

The formulae for the connection and the curvature from above apply immediately. ♣

5.9 The ADHM construction

The ADHM construction due to Atiyah, Drinfeld, Hitchin, and Manin [ADHM78] is one of the early groundbreaking discoveries in mathematical gauge theory: it gives a concrete description of $\mathscr{M}^{\mathrm{fr}}_{\mathrm{SU}(r),k}$ as a finite-dimensional hyperkähler reduction. This makes use of ideas from many areas of geometry and has ultimately impacted much of mathematics itself. In the following I will only discuss the construction and not give a complete treatment. If you want to learn more about this, read [DK90, §3.3], [Ati79], and/or [ADHM78].

The following perspective on the ADHM construction is taken from [Ati79]. Define a $\mathrm{Sp}(1)$–connection $A \in \Omega^1(\mathbf{H}^k, \mathfrak{sp}(1))$ by

$$A := \sum_{a=1}^{k} \frac{\mathrm{Im}(\bar{q}_a \mathrm{d}q_a)}{1+|q|^2}.$$

This connection satisfies

$$F_A = \frac{\sum_{a=1}^{k} r d\bar{q}_a \mathrm{d}q_a}{(1+|q|^2)^2};$$

in particular,

$$\langle F_A, \boldsymbol{\omega} \rangle = 0 \in C^\infty(\mathbf{H}^k, \mathfrak{sp}(1) \otimes \mathrm{Im}\,\mathbf{H}^*).$$

The latter is an higher dimensional analogue of the ASD condition. The idea is to obtain ASD instantons on $\mathbf{H}$ by pulling back A with a *suitable* map $f\colon \mathbf{H} \to \mathbf{H}^k$. Of course,

$$u^*A = \sum_{a=1}^{k} \frac{\mathrm{Im}(\bar{u}_a \mathrm{d}u_a)}{1+|u|^2}.$$

Atiyah [Ati79] makes the **ansatz**

$$u(q) = \lambda(B-q)^{-1}$$

with $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbf{H}^k$ (a row vector) and B a *symmetric* $k \times k$–matrix of quaternions.

It turns that this gives a ASD instanton if and only if

$$B^*B + \lambda^*\lambda \quad \text{is a a real } k \times k \text{ matrix}$$

and for every $q \in \mathbf{H}$

$$\ker\left(\lambda B - q\right) = 0.$$

The first condition can be seen to correpond to $\mu(B, \lambda) = 0$ for the distinguished hyperkähler moment map of $\mathrm{O}(k)$ acting on

$$S_{1,k} := \mathrm{Sym}(\mathbf{H}^k) \oplus \mathbf{H}^k.$$

For $k = 1$ this gives the BPST instantons (after applying a conformal inversion $q \mapsto q^{-1}$). If $B = \mathrm{diag}(b_1, \dots, b_n)$ with the entries distinct and $\lambda_1, \dots, \lambda_k > 0$, then the above give an type of ASD instanton discovered by 't Hooft.

Theorem 5.52 (Atiyah, Drinfeld, Hitchin, and Manin [ADHM78]). *Every* $\mathrm{Sp}(1)$ *ASD instanton with instanton number k arises from some choice of* (λ, B) *as above. Two such ASD instantons are gauge equivalent if and only if there are* $g \in \mathrm{Sp}(1)$ *and* $T \in \mathrm{O}(k)$ *with*

$$\lambda' = q\lambda T \quad \textit{and} \quad B' = T^{-1}BT.$$

This can also be phrased as

$$\mathscr{M}^{\mathrm{fr}}_{\mathrm{Sp}(1),k} = S^{\circ}_{1,k}/\!\!/\!\!/\mathrm{O}(k).$$

Here the superscript $\circ$ indicates imposing the non-degeneracy condition.

A similar story works for $\mathrm{Sp}(r)$, $\mathrm{SU}(r)$, $\mathrm{O}(r)$. For $\mathrm{Sp}(r)$, λ is replaced by $\Lambda \in \mathbf{H}^{r\times k}$ and u is replaced by

$$U(q) = \Lambda(B - x)^{-1}.$$

The corresponding connection is of the form

$$(\mathbf{1} + U^*U)^{-1/2}(U^*\mathrm{d}U)(\mathbf{1} + U^*U)^{-1/2} + (\mathbf{1} + U^*U)^{1/2}\mathrm{d}(\mathbf{1} + U^*U)^{-1/2}.$$

Ultimately,

$$\mathscr{M}^{\mathrm{fr}}_{\mathrm{Sp}(r),k} = S^{\circ}_{r,k}/\!\!/\!\!/\mathrm{O}(k) \quad \text{with} \quad S_{r,k} \coloneqq \mathrm{Sym}(\mathbf{H}^k) \oplus \mathbf{H}^{r\times k}.$$

I still owe you an explanation for why the above connections indeed are ASD instantons. It is not impossible to do this by a brute-force computation, but here is a nicer explanation. Let

$$C, D \in \mathrm{Hom}_{\mathbf{H}}(\mathbf{H}^{k+r}, \mathbf{H}^k)$$

by $k \times (k + r)$ matrices of quaternions. The subscript $\mathbf{H}$ denotes linearity with respect to the right $\mathbf{H}$–module structure. For $q = (q_1, q_2) \in \mathbf{H}^2\backslash\{0\}$ set

$$R(q) \coloneqq q_1 C + q_2 D.$$

Assume that $R(q)$ is surjective for every $q \in \mathbf{H}^2\backslash\{0\}$. The $\mathbf{H}$ right $\mathbf{H}$–module $\ker R(q)$ depends only on $[q] \in \mathbf{H}P^1$ (which we take to be the left quotient). This defines a quaternionic vector bundle

$$E \coloneqq \ker R \subset \underline{\mathbf{H}}^{k+r}.$$

The curvature of the induced covariant derivative ∇ on E can be computed using the technology from the previous subsection. It suffices to do this over $\mathbf{H} \hookrightarrow \mathbf{H}P^1$. Over $\mathbf{H}$,

$$R(q) = qC + D.$$

Therefore,

$$F_\nabla = PC^*\mathrm{d}\bar{q} \wedge (RR^*)^{-1}\mathrm{d}qCP.$$

If the matrix RR^* is always real for every $q \in \mathbf{H}$, then the above is anti-self-dual. (In fact, this is an if and only if.)

Now R can be brought into the normal form

$$R = \begin{pmatrix} \Lambda^* & (B - q)^* \end{pmatrix}$$

Therefore,

$$RR^* = \Lambda^*\Lambda + B^*B - (B^*q + \bar{q}B) + |q|^2$$

is real if and only if

$$\Lambda^*\Lambda + B^*B \quad \text{and} \quad B^*q + \bar{q}B$$

are real. The latter condition is equivalent to B being symmetric. The final ingredient is to observe that

$$\begin{pmatrix} -\mathbf{1} \\ U \end{pmatrix} (1 + U^*U)^{-1/2}$$

parametrises $\ker R$ and the corresponding connection is also

$$(\mathbf{1} + U^*U)^{-1/2}(U^*\mathrm{d}U)(\mathbf{1} + U^*U)^{-1/2} + (\mathbf{1} + U^*U)^{1/2}\mathrm{d}(\mathbf{1} + U^*U)^{-1/2}.$$

The following perspective is useful. The Grassmannian $\mathrm{Gr}_k^{\mathbf{H}}(\mathbf{H}^{k+r})$ carries a natural $\mathrm{Sp}(r)$–bundle with a connection. The data above specifies a map $\mathbf{H}(\to \mathbf{H}P^1) \to \mathrm{Gr}_k(\mathbf{H}^{k+r})$ which pulls back the standard connection to an ASD instanton.

∼

Here is the ADHM description for $\mathrm{SU}(r)$. Let $r \in \mathbf{N}$, $k \in \mathbf{N}_0$. Set

$$S_{r,k} \coloneqq \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^r, \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k) \oplus \mathbf{H} \otimes_{\mathbf{R}} \mathfrak{u}(k)$$

The tensor product $\otimes_{\mathbf{C}}$ uses the $\mathbf{C}$ right-module structure that arises from right-multiplication by $i \in \mathbf{H}$. $T \in \mathbf{H} \otimes_{\mathbf{R}} \mathfrak{u}(k)$ defines an endomorphism of $\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k$ given by the composition of multiplication in $\mathbf{H}$ and the action of $\mathfrak{u}(k)$ on $\mathbf{C}^k$. For $(\Psi, T) \in S_{r,k}$ and $x \in \mathbf{H}$ define

$$R_x \colon (\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k) \oplus \mathbf{C}^r \to \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k$$

by

$$R_x(\phi, v) \coloneqq (T - x^*)(\phi) + \Psi(v).$$

(Ψ, T) is **non-degenerate** if R_x is surjective for every $x \in \mathbf{H}$. In this case,

$$V \coloneqq \coprod_{x \in \mathbf{H}} \ker R_x \subset \mathbf{H} \times [(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k) \oplus \mathbf{C}^r]$$

is a Hermitian subbundle of rank

$$\mathrm{rk}\, V = r$$

and inherits a covariant dervivative $\nabla = \nabla_{\Psi,T} \colon \Gamma(V) \to \Omega^1(\mathbf{H}, V)$

$$\nabla s \coloneqq (\mathrm{d}s)^\perp$$

with $(-)^\perp$ denoting the orthogonal projection onto V. Set

$$S^\circ_{r,k} \coloneqq \big\{(\Psi, T) \in S_{r,k} : (\Psi, T) \text{ is non-degenerate}\big\}.$$

$G \coloneqq \mathrm{U}(k)$ acts on $S_{r,k}$ and $S^\circ_{r,k}$ by the defining representation on $\mathbf{C}^k$ and the adjoint representation on $\mathfrak{u}(k)$. Evidently, if (Ψ, T) and $g(\Psi, T)$ give rise to gauge equivalent covariant derivatives. The action of G on $S_{r,k}$ is a quaternionic representation. It turns out that the vanishing of the distinguished hyperkähler moment map is precisely the condition for $\nabla_{\Psi,T}$ to have anti-self-dual curvature (in complete analogy with the ∞–dimensional case considered earlier.)

Again, the data (Ψ, T) defines a map to

$$\mathrm{Gr}_r^{\mathbf{C}}(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k \oplus \mathbf{C}^r) = \mathrm{Gr}_r^{\mathbf{C}}(\mathbf{C}^{2k+r}).$$

The factor $\mathbf{C}^r$ has a particular meaning. To see this, observe that R can be extended from $\mathbf{H}$ to $\mathbf{H}^2 \backslash \{0\}$ as

$$R(q_0, q_1) = \begin{pmatrix} q_0^* T - q_1^* & q_0^* \cdot \end{pmatrix}$$

The kernel of R depends only on $[q_0, q_1] \in \mathbf{H}P^1$. The fiber over $\infty = [0, 1]$ is precisely $\mathbf{C}^r$.

5.10 Dimensional reduction

Let G be a Lie group. Let X be an oriented Riemannian manifold. Let $d \in \mathbf{N}$. Let $p\colon P \to X$ be a G–principal bundle. Denote by $\mathrm{pr}_X\colon X \times \mathbf{R}^d \to X$ the canonical projection map. Consider $\mathrm{pr}_Y^* p\colon \mathrm{pr}_Y^* P = P \times \mathbf{R}^d \to \mathbf{R} \times Y$. Let $\mathbf{A} \in \mathscr{A}(\mathrm{pr}_Y^* p) \subset \Omega^1(P \times \mathbf{R}^d, \mathfrak{g})^{\mathrm{Ad}}$. $\mathbf{A}$ can be decomposed as

$$\mathbf{A} = A + \sum_{a=1}^{d} \xi_a \mathrm{d}t_a.$$

with $A \in \Omega^1(P \times \mathbf{R}^d, \mathfrak{g})^{\mathrm{Ad}}$ satisfying $i_{\partial_{t_a}} A = 0$ and $\xi_a \in C^\infty(P \times \mathbf{R}^d, \mathfrak{g})^{\mathrm{Ad}}$ ($a \in \{1, \dots, d\}$). It is useful to think:

$$A \in C^\infty(\mathbf{R}^d, \mathscr{A}(p)) \quad \text{and} \quad \xi_a \in C^\infty(\mathbf{R}^d, \Gamma(\mathrm{Ad}(P)))$$

The curvature of $\mathbf{A}$ is

$$F_{\mathbf{A}} = F_A - \sum_{a=1}^{d} \big(\partial_{t_a} A - \mathrm{d}_A \xi_a\big) \wedge \mathrm{d}t_a + \frac{1}{2} \sum_{a,b=1}^{d} \big(\partial_{t_a} \xi_b - \partial_{t_a} \xi_b + [\xi_a, \xi_b]\big) \mathrm{d}t_a \wedge \mathrm{d}t_b.$$

Here F_A denotes the curvature of A restricted to the slices $\{t\} \times Y$.

Dimensional reduction is to *impose*

$$\partial_{t_a} A = 0 \quad \text{and} \quad \partial_{t_a} \xi_b = 0 \quad (a, b \in \{1, \dots, d\}).$$

In this case, the above expression for $F_{\mathbf{A}}$ simplifies to

$$F_{\mathbf{A}} = F_A + \sum_{a=1}^{d} \mathrm{d}_A \xi_a \wedge \mathrm{d}t_a + \frac{1}{2} \sum_{a,b=1}^{d} [\xi_a, \xi_b] \mathrm{d}t_a \wedge \mathrm{d}t_b.$$

The dimensional reduction of the Yang–Mills functional yields the following **Yang–Mills–Higgs functional**

$$\mathrm{YMH}(A, \xi) \coloneqq \frac{1}{2} \int_X |F_A|^2 + \sum_{a=1}^{d} |\mathrm{d}_A \xi_a|^2 + \frac{1}{4} \sum_{a,b=1}^{d} |[\xi_a, \xi_b]|^2.$$

More generally, for any representation $\rho\colon G \to \mathrm{O}(V)$ and G–invariant function $Q\colon V \to \mathbf{R}$ one can consider the **Yang–Mills–Higgs functional**

$$\mathrm{YMH}(A, \phi) \coloneqq \frac{1}{2} \int_X |F_A|^2 + |\mathrm{d}_A \phi|^2 + Q(\phi)$$

for $A \in \mathscr{A}(p)$ and $\phi \in \Gamma(P \times_\rho V)$.

The dimensional reductions of the anti-self-duality equation to dimensions 3, 2, 1 give rise to the Bogomolny equation (monopoles), the Hitchin equation (Higgs bundles), and Nahm's equation. Let us derive these equations. I'll give you a little survey of these equations afterwards.

Proposition 5.53. *Let (Y, g) be an oriented Riemannian 3–manifold. Let $(p\colon P \to Y, R)$ be a G–principal bundle. Let $A \in \mathscr{A}(p, R)$ and $\xi \in \Gamma(\mathrm{Ad}(P))$. The G–principal connection*

$$\mathbf{A} := A + \xi \mathrm{d}t \in \mathscr{A}(\mathrm{pr}_Y^*(p, R))$$

is anti-self-dual on $Y \times \mathbf{R}$ if and only if the ***Bogomolny equation***

$$F_A = - * \mathrm{d}_A \xi \tag{5.54}$$

holds.

Proof. To prove this, one needs to understand how the Hodge-$*$-operator on $X := Y \times \mathbf{R}$ and Y are related. The orientation on X is so that $\mathrm{vol}_X = \mathrm{vol}_Y \wedge \mathrm{d}t$. As a consequence for $\alpha, \beta \in \Omega^2(Y)$,

$$\alpha \wedge *_X \beta = \langle \alpha, \beta \rangle \mathrm{vol}_X = \langle \alpha, \beta \rangle \mathrm{vol}_Y \wedge \mathrm{d}t = \alpha \wedge *_Y \beta \wedge \mathrm{d}t.$$

Therefore,

$$*_X \beta = *_Y \beta \wedge \mathrm{d}t$$

Similarly, for $\alpha \in \Omega^1(Y)$

$$*_X(\alpha \wedge \mathrm{d}t) = *_Y \alpha.$$

Therefore,

$$*_X F_{\mathbf{A}} = (*_Y F_A) \wedge \mathrm{d}t + *_Y(\mathrm{d}_A \xi).$$

$*_X F_{\mathbf{A}} = -F_{\mathbf{A}}$ thus amounts to the above equation. ■

Proposition 5.55. *Let (Σ, g) be an oriented Riemann surface Let $(p\colon P \to \Sigma, R)$ be a G–principal bundle. Let $A \in \mathscr{A}(p, R)$ and $\xi_1, \xi_2 \in \Gamma(\mathrm{Ad}(P))$. The G–principal connection*

$$\mathbf{A} := A + \xi_1 \mathrm{d}t_1 + \xi_2 \mathrm{d}t_2 \in \mathscr{A}(\mathrm{pr}_Y^*(p, R))$$

is anti-self-dual if and only if ***Hitchin's equation***

$$\begin{aligned} F_A + [\xi_1, \xi_2] &= 0, \\ \mathrm{d}_A \xi_1 + * \mathrm{d}_A \xi_2 &= 0. \end{aligned} \tag{5.56}$$

holds.

Proof. By the above,

$$F_{\mathbf{A}} = F_A + \mathrm{d}_A \xi_1 \wedge \mathrm{d}t_1 + \mathrm{d}_A \xi_2 \wedge \mathrm{d}t_2 + [\xi_1, \xi_2] \mathrm{d}t_1 \wedge \mathrm{d}t_2.$$

The orientation on $X = \mathbf{R}^2 \times Y$ is $\mathrm{vol}_X = \mathrm{d}t_1 \wedge \mathrm{d}t_2 \wedge \mathrm{vol}_\Sigma$. Therefore,

$$*_X(\mathrm{d}t_1 \wedge \mathrm{d}t_2) = \mathrm{vol}_\Sigma$$

and for $\alpha \in \Omega^1(\Sigma)$

$$*_X(\alpha \wedge \mathrm{d}t_1) = -(*_\Sigma \alpha) \wedge \mathrm{d}t_2 \quad \text{and} \quad *_X (\alpha \wedge \mathrm{d}t_2) = (*_\Sigma \alpha) \wedge \mathrm{d}t_1.$$ ■

Proposition 5.57. *Let I be an interval. Let $\xi_0, \xi_1, \xi_2, \xi_3 \in C^\infty(I, \mathfrak{g})$ The G–principal connection*

$$\mathbf{A} \coloneqq \xi_0 \mathrm{d}t_0 + \xi_1 \mathrm{d}t_1 + \xi_2 \mathrm{d}t_2 + \xi_3 \mathrm{d}t_3$$

is anti-self-dual if and only if ***Nahm's equation***

$$\begin{aligned} \dot{\xi}_1 + [\xi_0, \xi_1] + [\xi_2, \xi_3] &= 0, \\ \dot{\xi}_2 + [\xi_0, \xi_2] + [\xi_3, \xi_1] &= 0, \\ \dot{\xi}_3 + [\xi_0, \xi_3] + [\xi_1, \xi_2] &= 0 \end{aligned} \tag{5.58}$$

holds.

Proof. By the above,

$$F_{\mathbf{A}} = \sum_{a=1}^{3} (\dot{\xi}_a + [\xi_0, \xi_a]) \mathrm{d}t_0 \wedge \mathrm{d}t_a + \frac{1}{2} \sum_{a,b=1}^{3} [\xi_a, \xi_b] \mathrm{d}t_a \wedge \mathrm{d}t_b.$$

Of course,

$$*(\mathrm{d}t_0 \wedge \mathrm{d}t_a) = \frac{1}{2} \sum_{b,c=1}^{3} \varepsilon_{abc} \mathrm{d}t_b \wedge \mathrm{d}t_c$$

This implies the assertion directly. ∎

5.11 The Bogomolny equation

Bogomolny [Bog76] Prasad and Sommerfield [PS75]

Proposition 5.59. *Let (Y, g) be an oriented Riemannian 3–manifold. Let $(p \colon P \to Y, R)$ be a G–principal bundle. If $A \in \mathscr{A}(p, R)$ and $\xi \in \Gamma(\mathrm{Ad}(P))$ is a solution of the Bogomolny equation (5.54), then*

$$\mathrm{d}_A * \mathrm{d}_A \xi = 0.$$

If Y is closed, then

$$\mathrm{d}_A \xi = 0 \quad \textit{and} \quad F_A = 0.$$

Proof. The Bianchi identity $\mathrm{d}_A F_A = 0$ immediately implies that $\mathrm{d}_A^* \mathrm{d}_A \xi = 0$. Therefore,

$$\int_Y |\mathrm{d}_A \xi|^2 = \int_Y \langle \xi, \mathrm{d}_A^* \mathrm{d}_A \xi \rangle.$$ ∎

A consequence of the above on typically studies the Bogomolny equation on a non-compact Y or admits A and ξ to have singularities. In fact, the study of the Bogomolny equation has largely focused on $\mathbf{R}^3$.

Examples. Here are two important examples.

Example 5.60. Denote by $(p\colon S^3 \to S^2, R)$ the Hopf bundle. The adjoint bundle $\mathrm{Ad}(S^3)$ is trivial bundle $i\underline{\mathbf{R}}$. The $\mathrm{U}(1)$–principal connection B induced by the standard Riemannian metric on S^3 satisfies

$$F_B = -\frac{i}{2}\mathrm{vol}_{S^2}.$$

For $k \in \mathbf{Z}$ define $\lambda_k\colon \mathrm{U}(1) \to \mathrm{U}(1)$ by $\lambda_k(z) := z^k$ and denote by $p_k\colon P_k := S^3 \times_{\lambda_k} \mathrm{U}(1) \to S^2$ the corresponding $\mathrm{U}(1)$–principal bundle. B induced a $\mathrm{U}(1)$–principal connection B_k on p_k satisfying

$$F_{B_k} = -\frac{ik}{2}\mathrm{vol}_{S^2}.$$

Denote by $\pi\colon \mathbf{R}^3\backslash\{0\} \to S^2$ the projection map. Let $m \in \mathbf{R}$. The **Dirac monopole** of mass m and charge k defined by

$$A_k^{\mathrm{Dirac}} := \pi^* B_k \quad \text{and} \quad \xi_{m,k}^{\mathrm{Dirac}} := \left(m - \frac{k}{r}\right)\frac{i}{2}$$

satisfies (5.54). ♠

Remark 5.61 (Scaling monopoles). Let $\lambda > 0$. Define $s_\lambda\colon \mathbf{R}^3 \to \mathbf{R}^3$ by $s_\lambda(x) := \lambda x$. If (A, ξ) is a solution to (5.54), the so is

$$(s_\lambda^* A, \lambda s_\lambda^* \xi).$$

The mass parameter can be varied by scaling. ♣

Example 5.62. Let $m \in \mathbf{R}$. Let $k \in \mathbf{N}$. Let $x_1, \ldots, x_k \in \mathbf{R}^3$. Set

$$\xi := \left(m - \sum_{a=1}^{k} \frac{1}{|x - x_a|}\right)\frac{i}{2}.$$

There is a $\mathrm{U}(1)$–principal bundle $(p\colon P \to \mathbf{R}^3\backslash\{x_1, \ldots, x_k\}, R)$ and a connection $A \in \mathscr{A}(p, R)$ which together with ξ satisfies (5.54). ♠

Example 5.63. Bogomolny [Bog76] and Prasad and Sommerfield [PS75] discovered the **BPS monopole**, a particular solution of (5.54). Identify $\mathbf{R}^3 = \mathfrak{sp}(1)$. In particular, $S^2 \subset \mathfrak{sp}(1)$. Denote by $\tau \in C^\infty(S^2, \mathfrak{sp}(1))$ the inclusion map. A brief computation shows that

$$\begin{aligned}
\mathrm{d}\tau &= \frac{1}{2}[\tau, *_{S^2}\mathrm{d}\tau], \\
*_{S^2}\mathrm{d}\tau &= -\frac{1}{2}[\tau, \mathrm{d}\tau], \\
[\mathrm{d}\tau \wedge \mathrm{d}\tau] &= 4\tau\mathrm{vol}_{S^2}, \\
[(*_{S^2}\mathrm{d}\tau) \wedge (*_{S^2}\mathrm{d}\tau)] &= 4\tau\mathrm{vol}_{S^2}.
\end{aligned}$$

Therefore, the *ansatz*

$$A = f(r) *_{S^2} \mathrm{d}\tau \quad \text{and} \quad \xi = g(r)\tau$$

leads to

$$\begin{aligned} F_A &= f'\mathrm{d}r \wedge *_{S^2}\mathrm{d}\tau + 2(-f+f^2)\tau\mathrm{vol}_{S^2} \quad \text{and} \\ \mathrm{d}_A\xi &= g'\tau\mathrm{d}r + (g-2fg)\mathrm{d}\tau. \end{aligned}$$

Since $*\mathrm{d}r = r^2\mathrm{vol}_S^2$, the Bogomolny equation therefore amounts to the ODE

$$\begin{aligned} f' &= g - 2fg \quad \text{and} \\ r^2g' &= 2(f-f^2). \end{aligned}$$

Here is a family of solutions ($m > 0$) to this equation

$$f_m(r) = -\frac{1}{2}\Big(\frac{mr}{\sinh mr} - 1\Big) \quad \text{and} \quad g_m(r) = \frac{m}{2}\bigg(\frac{1}{\tanh mr} - \frac{1}{mr}\bigg).$$

Observe that

$$f_m = \frac{1}{2} + O(mre^{-2mr}). \quad \text{and} \quad g_m = \bigg(\frac{m}{2} - \frac{1}{2r}\bigg) + O(e^{-2mr}).$$

This gives the 1–parameter family of solutions

$$(A_m^{\mathrm{BPS}}, \xi_m^{\mathrm{BPS}}) \quad m > 0$$

of (5.54). ♠

Remark 5.64.

(1) Let X be a manifold of dimension at most 3. Let $L \to X$ be a Hermitian line bundle. Set $E \coloneqq L \oplus L^* \to X$. Eventhough, L might not be trivial, E always is. Here is why. Since $3 < 4$, a generic section s of E is nowhere-vanishing. Since E inherits a complex structure i from L, E has at two linearly independent sections: s, is. The Hermitian inner product, defines an i–anti-linear map $j\colon L \to L^*$ that can be extended to a futher almost complex structure j on E satifying $ij = -ji$. Consequently, E is a quaternionic line bundle and s, is, js, ijs are linearly independent.

(2) As a concrete instantance of the above but in setting of $\mathrm{U}(1)$– and $\mathrm{Sp}(1)$–principal bundles. The Hopf bundle $p\colon P \coloneqq S^3 \to S^2$ is non-trivial. Define $\rho\colon \mathrm{U}(1) \to \mathrm{Sp}(1)$ by

$$\rho(e^{i\alpha}) = e^{i\alpha}.$$

The associated $\mathrm{Sp}(1)$–principal bundle $q\colon Q \coloneqq S^3 \times_\rho \mathrm{Sp}(1) \to S^2$ is trivial. Indeed, the section $s\colon S^2 = S^3/\mathrm{U}(1) \to Q$ defined by

$$s([x]) \coloneqq [x, x^*]$$

trivialises Q. Here $S^3 = \mathrm{Sp}(1)$.

(3) The Dirac monopole is built from the standard $\mathrm{U}(1)$–principal connection 1–form $\theta \in \Omega^1(S^3, i\mathbf{R})$ and the constant $\mathrm{U}(1)$–equivariant map $i \in C^\infty(P, i\mathbf{R})^{\mathrm{Ad}}$. The latter induces the $\mathrm{Sp}(1)$–equivariant map $\xi \in C^\infty(Q, \mathfrak{sp}(1))^{\mathrm{Ad}}$ by

$$\xi([x,g]) := \mathrm{Ad}(g)^{-1}(\mathrm{Lie}\,\rho)(i) = g^* i g.$$

The pullback via s satisfies

$$s^*\eta([x]) = xix^*.$$

This is the map $\tau\colon S^2 = S^3/\mathrm{U}(1) \to \mathfrak{sp}(1)$. This shows that the Higgs field of the BPS monopole is (exponentially) asymptotic to the Higgs field of the Dirac monopole.

The same is true for the connection. The connection 1–form θ on P can be written as

$$\theta = i\,\mathrm{Re}(\mathrm{d}x^* x i) = \frac{1}{2}(i\mathrm{d}x^* x i + x^* \mathrm{d}x).$$

The induced connection on Q descends from

$$\frac{1}{2}\,\mathrm{Ad}(g)^{-1}(i\mathrm{d}x^* x i + x^* \mathrm{d}x) + g^{-1}\mathrm{d}g.$$

with $x \in P$ and $g \in \mathrm{Sp}(1)$. The pullback via s is

$$\frac{1}{2}x(i\mathrm{d}x^* x i + x^*\mathrm{d}x)x^* + x\mathrm{d}x^* = \frac{1}{2}(xi\mathrm{d}x^* x i x^* + x\mathrm{d}x^*).$$

Therefore, for $\tilde\tau(x) = xix^*$,

$$\begin{aligned}
-\frac{1}{4}[\tilde\tau, \mathrm{d}\tilde\tau] &= -\frac{1}{4}[xix^*, \mathrm{d}xix^* + xi\mathrm{d}x^*] \\
&= -\frac{1}{4}(xix^*\mathrm{d}xix^* - x\mathrm{d}x^* + \mathrm{d}xx^* - xi\mathrm{d}x^* xix^*) \\
&= \theta.
\end{aligned}$$

This proves the statement about the connection. ♣

Energy and charge. Henceforth, we restrict to $G = \mathrm{SU}(2) = \mathrm{Sp}(1)$ and, in fact, shortly to $Y = \mathbf{R}^3$. Consider the Yang–Mills–Higgs functional

$$\mathrm{YMH}(A, \xi) = \frac{1}{2}\int_Y |F_A|^2 + |\mathrm{d}_A\xi|^2.$$

Obviously,

$$\mathrm{YMH}(A, \xi) = \frac{1}{2}\int_Y |F_A \pm *\mathrm{d}_A\xi|^2 \mp 4\pi N \quad \text{with} \quad N := \frac{1}{4\pi}\int_Y \langle F_A \wedge \mathrm{d}_A\xi\rangle.$$

For $Y = \mathbf{R}^3$, by Stokes' theorem

$$N = \frac{1}{4\pi}\lim_{r\to\infty}\int_{\partial B_r(0)} \langle F_A, \xi\rangle$$

Under suitable boundary conditions this limit exits and indeed agrees with the degree of $\xi\colon S^2_\infty \to S^2 \subset \mathfrak{sp}(1)$. In particular,

$$N \in \mathbf{Z}.$$

This is the anlogue of the energy identity for the (anti-)self-duality equation.

Moduli spaces. It is customary study (5.54) on the trivial $\mathrm{Sp}(2)$–bundle over $\mathbf{R}^3$ with the following boundary condition that, with respect to the isomorphism $\mathrm{Sp}(1) \times (\mathbf{R}^3\backslash\{0\}) \cong Q$ discussed above,

$$(A, \xi) = (A_k^{\mathrm{Dirac}}, \xi_{1,k}^{\mathrm{Dirac}}) + \text{lower order terms at infinity.}$$

(The mass parameter can be adjusted by scaling.) Denote the corresponding space of configurations of (A, ξ) by $\mathscr{C}_k$. The relevant group of gauge transformations $\mathscr{G}_0$ is required to decay to the identity at infinity. The **framed moduli space of charge k monopoles** is then

$$\mathscr{N}_k := \{(A, \xi) \in \mathscr{C}_k : (5.54)\}/\mathscr{G}_0.$$

Theorem 5.65. *$\mathscr{N}_k$ is a hyperkähler manifold of dimension $4k$. $S^1 \times \mathbf{R}^3$ acts freely on $\mathscr{N}_k$.*

The **strongly centred framed monopole moduli space of charge k** is

$$\tilde{\mathscr{N}}_k^0 := \widetilde{\frac{\mathscr{N}_k}{S^1 \times \mathbf{R}^3}}.$$

$\tilde{\mathscr{N}}_2^0$ is **Atiyah–Hitchin manifold**, already a tantalising geometric object.

The fact that $\mathscr{N}_k$ is always non-empty is one of the first major mathematical achievments of Taubes. His idea is to start with a Dirac monopole with k (well-separated) singularities and to repair the singularities by gluing in BPS monopoles. Using cut-off functions this can be done approximately. To make these approximate solutions actual solutions requires a delicate analysis; cf. Jaffe and Taubes [JT80].

Theorem 5.66 (Donaldson [Don84]). *A choice of isometry $\mathbf{R}^3 = \mathbf{R} \times \mathbf{C}$ defines a bijection between $\mathscr{N}_k$ and the space of rational maps $f\colon \mathbf{C}P^1 \to \mathbf{C}P^1$ of degree k satisfying $f([1:0]) = [0:1]$.*

This theorem was the first to give a global uniform understanding of $\mathscr{N}_k$ for all k. This was conjectured in Murray [Mur83, Appendix B]. Donaldson's proof is rather roundabout. Jarvis [Jar00] gave a direct and geometric proof (of an extension of Theorem 5.66.

the cyclic group C_k acts on $\tilde{\mathscr{N}}_k^0$. Denote by μ_k the k–th roots of unity. There are canonical maps $\lambda_\ell\colon C_k \to \mu_k$ which sends the generator of C_k to $e^{2\pi i\ell/k}$. Therefore, it also acts on the cohomology $\mathrm{H}^i(\tilde{\mathscr{N}}_k^0, \mathbf{C})$. Denote by

$$\mathrm{H}^i_\ell(\tilde{\mathscr{N}}_k^0, \mathbf{C})$$

the subspace in which the action of C_k agrees with λ_ℓ. There is an analoge of the above spaces cohomology replaced by L^2 harmonic forms $\mathscr{H}$. **Sen's conjecture** [Sen94] asserts that

(1) If k, ℓ are coprime, then $\mathscr{H}_\ell^{2k-2}(\tilde{\mathscr{N}}_k^0, \mathbf{C}) \cong \mathbf{C}$ and vanishes otherwise.

(2) If k, ℓ are not coprime, then $\mathscr{H}_\ell^i(\tilde{\mathscr{N}}_k^0, \mathbf{C})$ vanishes.

This conjecture was a main driving force in the study of the geometry of moduli spacee $\tilde{\mathscr{N}}_k^0$. The cohomological version of theses statements are have been proved by Segal and Selby [SS96] The first part of these conjectures has now been proved by Fritzsch, Kottke, and Singer [FKS18] (modulo the appearance of part 2 of that paper?).

Monopoles and scattering. Let I be an interval. Let Σ be a Riemann surface. Let $E \to \Sigma$ be a Hermitian vector bundle with $\det E = \mathbf{C}$. Let d_A be a compatible covariant derivative on E and let $\xi \in \Gamma(\mathfrak{su}(E))$. The covariant derivative d_A splits as

$$\mathrm{d}_A = \partial_A + \bar{\partial}_A + \mathrm{d}t \wedge \nabla_{A,\partial_t}.$$

Therefore,

$$F_A = \mathrm{d}_A^2 = \bar{\partial}_A \partial_A + \partial_A \bar{\partial}_A + \mathrm{d}t \wedge ([\nabla_{A,\partial_t}, \bar{\partial}_A] + [\nabla_{A,\partial_t}, \partial_A]).$$

and

$$*\mathrm{d}_A \xi = \nabla_{A,\partial_t} \xi \cdot \mathrm{vol}_\Sigma + i\mathrm{d}t \wedge \partial_A \xi - i\mathrm{d}t \wedge \bar{\partial}_A \xi$$

By direct inspection, (5.54) is equivalent to

$$[\nabla_{A,\partial_t} + i\xi, \bar{\partial}_A] = 0 \quad \text{and}$$
$$\bar{\partial}_A \partial_A + \partial_A \bar{\partial}_A = \nabla_{A,\partial_t} \xi \mathrm{vol}_\Sigma.$$

Suppose now that $I = [0, 1]$. Restriction to t defines a holomorphic structure $\bar{\partial}_{A,t}$ on E for every $t \in [0, 1]$. Consider parallel transport $T_t \colon E \to E$ from $t = 0$ to t associated with the covariant derivative $\nabla_A + i\xi$. The first of the above equations shows that T_t defines an isomorphism of holomorphic vector bundles $\mathscr{E}_0 := (E, \bar{\partial}_{A,0}) \to \mathscr{E}_t := (E, \bar{\partial}_{A,t})$. This map is (sometimes) called the **scattering map**. Under suitable boundary conditions/stability conditions, upto suitable equivalences, etc., given $\bar{\partial}_A$ and $\nabla_{A,\partial_t} + i\xi$ satisfying the first of the above equations, (A, ξ) can be recovered so that the second equation also holds. This is usually studied when (A, ξ) has singularities. In this case the scattering map is not an isomorphism, but a Hecke modification (an isomorphism in the complement of a bunch of points). See Norbury [Nor11] and Charbonneau and Hurtubise [CH11] to learn more about this. Hurtubise [Hur85] gave an explanation of Theorem 5.66 via scattering maps.

5.12 Hitchin's equation

[This will be discussed in the problem session]

[Hit87]

5.13 The moduli space of ASD instantons

Let (X, g) be an closed oriented Riemannian 4–manifold. Let G be a compact semi-simple Lie group. Let $(p \colon P \to X, R)$ be a G–principal bundle. The **moduli space of ASD instantons** on (p, R) is

$$\mathscr{M} := \{A \in \mathscr{A} : F_A^+ = 0\}/\mathscr{G}.$$

At this stage, $\mathscr{M}$ is a topological space. The purpose of this section is to equip $\mathscr{M}$ with more structure and understand its geometry better. (Here and throughout, to ease notation, (p, R) is dropped from the notation.)

It turns out to be benificial to construct the quotient

$$\mathscr{B} := \mathscr{A}/\mathscr{G}$$

and then construct $\mathscr{M}$. A useful framework to proceed in is that of Banach manifolds. This requires us to enlarge $\mathscr{A}$ and $\mathscr{G}$.

5.13.1 Sobolev spaces

Let $U \subset \mathbf{R}^n$ be a bounded open subset (with smooth boundary).

[Review definition of $W^{k,p}(U)$; check what is known: L^p? distributions? weak derivatives?; mention scaling weight $k - n/p$]

Theorem 5.67 (Sobolev inequality/Sobolev embedding theorem). *Let $k, \ell \in \mathbf{N}_0$, $p, q \in [1, \infty)$. If*

$$k > \ell, \quad \text{and} \quad k - \frac{n}{p} \geqslant \ell - \frac{n}{q},$$

then $W^{k,p}(U) \subset W^{\ell,q}(U)$ and the inclusion map is continuous.

Theorem 5.68 (Rellich–Kondrachov). *Let $k, \ell \in \mathbf{N}_0$, $p, q \in [1, \infty)$. If*

$$k > \ell, \quad \text{and} \quad k - \frac{n}{p} > \ell - \frac{n}{q},$$

then the inclusion map $W^{k,p}(U) \subset W^{\ell,q}(U)$ is compact.

Theorem 5.69 (Morrey inequality). *Let $k, r \in \mathbf{N}_0$, $p \in [1, \infty)$, and $\alpha \in (0, 1)$. If*

$$r + \alpha = k - \frac{n}{p},$$

then every $W^{k,p}$ function on U is (representable) by a $C^{r,\alpha}$ function on $\bar{U}$ and the map

$$W^{k,p}(U) \to C^{r,\alpha}(\bar{U})$$

is continuous. ■

Theorem 5.70 (Sobolev multiplication). *Let $k \in \mathbf{N}_0$, $p \in [1, \infty)$. If*

$$k - \frac{n}{p} > 0,$$

then the multiplication map

$$W^{k,p}(U) \times W^{k,p}(U) \to W^{k,p}(U)$$

is continuous. ■

If (X, g) is a Riemannian manifold, V is an Euclidean vector space, $E \to X$ is an Euclidean vector bundle equipped with a covariant derivative ∇, then we define Sobolev spaces $W^{k,p}(X, V)$, $W^{k,p}\Omega^\bullet(X, V)$, $W^{k,p}\Gamma(E)$, $W^{k,p}\Omega^\bullet(X, E)$, etc. with norms

$$\|s\|_{W^{k,p}} := \sum_{\ell=0}^{k} \left(\int_X |\nabla^k s|^p \mathrm{vol}_g \right)^{1/p}.$$

These are Banach spaces. Of course, there are analogous definitions of with $C^{r,\alpha}$ instead of $W^{k,p}$. These are Banach spaces too. The results mentioned above carry over mutatis mutandis.

5.13.2 Sobolev connections and gauge transformations

Let (X, g) be a closed connected oriented Riemannian manifold of dimension n. Let G be a compact semi-simple Lie group. Let $(p\colon P \to X, R)$ be a G–principal bundle. The theory of connections developed in the smooth case largely carries over to the Sobolev setting. Throughout, let $k \in \mathbf{N}_0$, $p \in (1, \infty)$ with

$$k + 2 - \frac{n}{p} > 0.$$

(The significance of this restriction shall be explained shortly.)

Definition 5.71.

(1) A $W^{k+1,p}$ **connection on** (p, R) is a $W^{k+1,p}$ 1–form

$$\theta_A \in W^{k+1,p}\Omega^1(P, \mathfrak{g})$$

such that for almost every $x \in P$ and $\xi \in \mathfrak{g}$

$$\theta_A(v_\xi(x)) = \xi$$

and for every $g \in G$

$$R_g^*\theta_A = \mathrm{Ad}(g)^{-1}\theta_A.$$

Denote the set of $W^{k+1,p}$ connections by $W^{k+1,p}\mathscr{A}(p, R)$.

(2) A $W^{k+2,p}$ **gauge transformation of** (p, R) is a $W^{k+2,p}$ map

$$u\colon W^{k+2,p}(P, G)^C$$

with the super-script C indicating that for every $g \in G$

$$u \circ R_g = C_g^{-1}u.$$

Denote the set of $W^{k+2,p}$ gauge transformations by $W^{k+2,p}\mathscr{G}(p, R)$. •

The theory developed in the smooth case carries over to the Sobolev setting (provided the regularity suffices to write the formulae.) Here are some facts (and consequences of the theory of Sobolev spaces):

(1) $W^{k+1,p}\mathscr{A}(p, R)$ is an affine space modelled on

$$W^{k+1,p}\Omega^1(X, \mathrm{Ad}(P)).$$

(2) Every $A \in W^{k+1,p}\mathscr{A}(p, R) \cap L^{2p}\mathscr{A}(p, R)$ has a curvature

$$F_A \in W^{k,p}\Omega^2(X, \mathrm{Ad}(P)).$$

Indeed, the curvature map

$$F\colon W^{k+1,p}\mathscr{A}(p, R) \cap L^{2p}\mathscr{A}(p, R) \to W^{k,p}\Omega^2(X, \mathrm{Ad}(P))$$

is analytic.

(3) $W^{k+2,p}\mathscr{G}(p,R)$ is a Banach Lie group with Lie algebra

$$W^{k+2,p}\Gamma(\mathrm{Ad}(P)),$$

and it acts smoothly on (the right of) $W^{k+1,p}\mathscr{A}(p,R)$:

$$\theta_{u^*A} = \mathrm{Ad}(u)^{-1}\theta_A + u^*\mu_G.$$

with $\mu_G \in \Omega^1(G,\mathfrak{g})$ denoting the Maurer–Cartan form on G.

Only the last point requires any justification. Ultimately, this fact follows from the Sobolev multiplication theorem. Naively, applything that theorem suggest that we should have imposed the much stronger condition

$$k + 2 - \frac{n + \dim G}{p} > 0.$$

The crucial point is that gauge transformations are G–equivariant and, therefore, the Sobolev multiplication theorem in dimension n can be used.

5.13.3 A slice theorem

Continue with the situation of the previous subsection, but we drop the pre-scripts $W^{k+1,p}$ and $W^{k+2,p}$ of $\mathscr{A}$ and $\mathscr{G}$ (in an attempt to not go insane). Our goal is to understand the quotient

$$\mathscr{B} := \mathscr{A}/\mathscr{G}.$$

This is a task for the slice theorem.

Proposition 5.72. *The action of $\mathscr{G}$ on $\mathscr{A}$ is proper.*

Proof sketch. Let (u_n) be a sequence $\mathscr{G}$ and (A_n) a sequence in $\mathscr{A}$. We have to prove that if (A_n) converges to A in $W^{k+1,p}$ and $u_n^*A_n$ converges to B in $W^{k+1,p}$, then (u_n) has a convergent subsequence in $W^{k+2,p}$. To do this one has to meditate over the identity

$$\theta_{u_n^*A_n} = \mathrm{Ad}(u_n)^{-1}\theta_{A_n} + u_n^*\mu_G = u_n^{-1}\theta_{A_n}u_n + u_n^{-1}\mathrm{d}u_n$$

and use the idea of bootstrapping.

By hypothesis, $\theta_{u_n^*A_n}$ and θ_{A_n} converge in $W^{k+1,p}$. Moreover, *since G is compact*, $\|u_n\|_{L^\infty}$ is bounded.

By the hypothesis $\|\mathrm{Ad}(u_n)^{-1}\theta_{A_n}\|_{L^p}$ is bounded. But then $\|u_n^*\mu_G\|_{L^p}$ is bounded. This implies that $\|\mathrm{d}u_n\|_{L^p}$ is bounded. That is $\|u_n\|_{W^{1,p}}$ is bounded. Using Sobolev embedding, Hölder's inequality etc. this argument can be iterated to obtain that $\|u_n\|_{W^{k+2,p}}$ is bounded. (The details of this are not hard, but a little fiddly.)

Using Rellich–Kondrachov one sees that a subsequence of u_n converges in $W^{k+1,q}$. This is not quite enough. We wanted convergence in $W^{k+2,p}$. The last missing point is to use the above identity to see that in this case $\mathrm{d}u_n$ must also converge in $W^{k+2,p}$. ∎

The discussion from Section 3.3 extends to proper actions and Banach manifolds. The action of $\mathscr{G}$ on $\mathscr{A}$ however is *not free*.

Definition 5.73. Let $A \in \mathscr{A}$. The **isotropy group** of A is defined by

$$\Gamma_A := \{u \in \mathscr{G} : u^*A = A\}. \qquad \bullet$$

Henceforth, fix $x_0 \in P$. Recall the holonomy group $\mathrm{Hol}_{x_0}(A)$ from Definition 4.36.

Proposition 5.74.

(1) *The evaluation map* $\mathrm{ev} = \mathrm{ev}_{x_0} \colon \mathscr{G} \to G$ *defined by* $\mathrm{ev}(u) = u(x_0)$ *defines an injection* $\mathrm{ev} \colon \Gamma_A \hookrightarrow G$.

(2) *The image of* Γ_A *in* G *is precisely the* $C_G(\mathrm{Hol}_{x_0}(A))$ *the centraliser of the holonomy group.*

Proof. By equivariance, $u(x_0)$ determines u on $p^{-1}(p(x_0))$. If u preserves A, then u commutes with the A–parallel transport. Since X is connected, this completely determines u. This proves (1).

The fact that u commutes with A–parallel transport also implies that $\mathrm{ev}(\Gamma_A) \subset C_G(\mathrm{Hol}_{x_0}(A))$. To prove the reverse conclusion observe that if $g \in C_G(\mathrm{Hol}_{x_0}(A))$ then it can be extended a G–equivariant map $p^{-1}(p(x_0)) \to G$ and to a G–equivariant map $u \colon P \to G$ by A–parallel transport. Since u is was constructed to commute with A–parallel transport, it preserves A. ∎

As a consequence of the above, Γ_A *always* contains $Z(G)$, the center of G.

Definition 5.75. A connection A is **irreducible** if $\mathrm{ev}(\Gamma_A) = Z(G)$. Denote the subset of irreducible connections in $\mathscr{A}$ by

$$\mathscr{A}^*.$$

A is **reducible** if it is not irreducible. •

Remark 5.76. The terminology "irreducible" is common but not ideal. $\mathrm{Hol}_{x_A}(A) < G$ might very well be a proper subgroup with centralizer $C(G)$. ♣

The slice theorem now constructs the quotient

$$\mathscr{B}^* := \mathscr{A}^*/(\mathscr{G}/Z(G)) = \mathscr{A}^*/\mathscr{G}$$

as a Banach manifold. If $A_0 \in \mathscr{A}^*$ and $\varepsilon > 0$ is sufficiently small, then a chart of $\mathscr{A}^*/\mathscr{G}$ around $[A_0]$ can be constructed as follows. Consider the *local slice*

$$U_{A_0,\varepsilon} := \{A + a : \mathrm{d}_{A_0}^* a = 0, \|a\|_{W^{k+1,p}} < \varepsilon\}.$$

The map $U_{A_0,\varepsilon} \to \mathscr{A}^*/\mathscr{G}$ is (the inverse of) a chart. Because every $u \in \Gamma_A$ takes values in the center $C(G)$ it acts trivially on $U_{A_0,\varepsilon}$. The slice condition

$$\mathrm{d}_{A_0}^* a = 0$$

is precisely the condition to be L^2 orthogonal to the action of infinitesimal gauge transformations of A_0:

$$\mathrm{d}_{A_0} \colon W^{k+2,p}\Gamma(\mathrm{Ad}(P)) \to W^{k+1,p}\Omega^1(X, \mathrm{Ad}(P)).$$

The above gives us a very useful description within which to understand the moduli space of irreducible ASD instantons. The application we have in mind requires a description of reducible ASD instantons.

Let $\Gamma < \mathscr{G}$ be any subgroup such that $\mathrm{ev}\colon \Gamma \to G$ is injective. Set

$$\begin{aligned} \mathscr{A}_{(\Gamma)} &:= \{A \in \mathscr{A} : \Gamma_A \text{ is conjugate to } \Gamma\}, \\ \mathscr{A}_{\Gamma} &:= \{A \in \mathscr{A} : \Gamma_A = \Gamma\}, \quad \text{and} \\ W_{\mathscr{G}}(\Gamma) &:= N_{\mathscr{G}}(\Gamma)/\Gamma. \end{aligned}$$

A moment's thought shows that the inclusion map induces a homeomorphism

$$\mathscr{A}_{\Gamma}/W_{\mathscr{G}}(\Gamma) \cong \mathscr{A}_{(\Gamma)}/\mathscr{G} =: \mathscr{B}_{(\Gamma)}.$$

This former quotient can again be constructed as a Banach manifold using the slice theorem. Here is how to understand the charts. Let $A_0 \in \mathscr{A}_{\Gamma}$. The tangent space

$$T_{A_0}\mathscr{A} = \Omega^1(X, \mathrm{Ad}(P))$$

decomposes into a Γ−invariant part and its L^2 orthogonal complement. A local slice of the quotient $\mathscr{A}_{\Gamma}/W_{\mathscr{G}}(\Gamma)$ is

$$U^{\Gamma}_{A_0,\varepsilon} := \{A_0 + a : a \text{ is } \Gamma\text{−invariant}, \mathrm{d}^*_{A_0} a = 0, \|a\|_{W^{k+1,p}} < \varepsilon\}.$$

Γ acts trivially on $U^{\Gamma}_{A_0,\varepsilon}$ and the map $U^{\Gamma}_{A_0,\varepsilon} \to \mathscr{A}_{\Gamma}/W_{\mathscr{G}}(\Gamma)$ is (the inverse of) a chart.

Remark 5.77. A gauge tranformation $u \in \mathscr{G} = C^{\infty}(P, G)^C$ acts on $a \in T_A\mathscr{A} = \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$ by

$$u^* a = \mathrm{Ad}(u)^{-1} a.$$

Decompose

$$\mathfrak{g} = \mathfrak{K} \oplus \mathfrak{m} \quad \text{with} \quad \mathfrak{k} = \mathfrak{g}^{\Gamma_A} = \{\xi \in \mathfrak{g} : \mathrm{Ad}(g)\xi = \xi \text{ for every } g \in \Gamma_A\} \quad \text{and} \quad \mathfrak{m} = \mathfrak{k}^{\perp}.$$

$\mathfrak{k} \subset \mathfrak{g}$ is a Lie subalgebra. It contains the holonomy Lie algebra hol but might be larger. Denote by $K < G$ the corresponding Lie subgroup containing Hol. The bundle P admits a reduction Q of structure group to K. $T_A\mathscr{A}_{\Gamma}$ is $\Omega^1(X, \mathrm{Ad}(Q))$. In fact, $\mathscr{A}_{\Gamma}(P) = \mathscr{A}^*(Q)$. ♣

The upshot of the discussion so far is that

$$\mathscr{B} = \mathscr{B}^* \amalg \coprod_{\Gamma \neq Z(G)} \mathscr{B}_{(\Gamma)}$$

with all of the pieces being Banach manifolds.

Here is description of a neigborhood of A_0 in $\mathscr{B}$.

Proposition 5.78. *Let $\varepsilon > 0$ be sufficiently small. Set*

$$U_{A_0,\varepsilon} \coloneqq \{A_0 + a : \mathrm{d}^*_{A_0} a = 0, \|a\|_{W^{k+1,p}} < \varepsilon\}.$$

The map $\phi\colon U_{A_0,\varepsilon}/\Gamma \to \mathscr{B}$ is an open embedding. Moreover, the stabliser of a in Γ is precisely Γ_{A_0+a}.

More globally, there is a vector bundle $\mathscr{V} \to \mathscr{B}_{(\Gamma)}$ obtained as the decend of the vector bundle over $\mathscr{A}_\Gamma$ whose fiber over A_0 is

$$\ker \mathrm{d}_{A_0} \cap \left[W^{k+1,p}\Omega^1(X, \mathrm{Ad}(P))^\Gamma\right]^\perp.$$

Γ acts on $\mathscr{V}$ and the structure of $\mathscr{B}$ normal to $\mathscr{B}_{(\Gamma)}$ is modelled on $\mathscr{V}/\Gamma$.

∼

In our application, we specialise to $G = \mathrm{Sp}(1)$. In this case the only options for Γ are

$$\{\pm\mathbf{1}\} = Z(\mathrm{Sp}(1)), \quad S^1, \quad \text{and} \quad \mathrm{Sp}(1).$$

(This is *not* a completely trivialy fact.) $\Gamma = \mathrm{Sp}(1)$ corresponds to connections A with holonomy in $\{\pm\mathbf{1}\}$. In particular, A must be flat. These, will cannot appear in the spaces we are interested in. If $\Gamma = S^1 = \{\exp t\xi : t \in \mathbf{R}\}$ $\xi \in S^2 \subset \mathrm{Im}\,\mathbf{H}$, then the holonomy is among the following

$$\text{a finite subgroup of } S^1 \quad \text{and} \quad S^1$$

Again, the former correspond to flat connections and cannot appear. In both cases, the rank 2 Hermitian vector bundle E corresponding to P splits as $E = L \oplus L^*$. In this case,

$$\mathrm{Ad}(P) = i\mathbf{R} \oplus L^2$$

and

$$\Omega^1(X, \mathrm{Ad}(P)) = \Omega^1(X) \oplus \Omega^1(X, L^2).$$

The former summand is Γ invariant and Γ acts on the later by multiplication with unit complex numbers (squared). Therefore,

$$\mathscr{B}_{(S^1)} \cong \{A_0 + a \in \Omega^1(X, i\mathbf{R}) : \mathrm{d}^* a = 0\}.$$

(This means: there is a global slice for $\mathscr{B}_{(S^1)}$. This is true in all abelian gauge theories.) The normal structure of $\mathscr{B}_{(S^1)}$ at A_0 is modelled on

$$\{a \in \Omega^1(X, L^2) : \mathrm{d}^*_{A_0} a\}/S^1.$$

Somewhat informally, the latter is

$$\mathbf{C}^\infty/S^1 = \text{the cone on } \mathbf{C}P^\infty.$$

Exercise 5.79. What is the local model around the trivial connection?

5.13.4 Kuranishi models for the moduli space

We are now in an excellent position to understand

$$\mathscr{M} := \{A \in \mathscr{A} : F_A^+ = 0\}/\mathscr{G}.$$

We continue with dropping the Sobolev prescripts. I will explain at the end why this is ultimately justified if one only cares about $\mathscr{M}$.

Over $\mathscr{A}$ there is a trivial Banach space bundle

$$\tilde{\mathscr{E}} := \mathscr{A} \times W^{k,p}\Omega^+(X, \mathrm{Ad}(P)).$$

The anti-self dual part of the curvature defines a section of $\tilde{\mathscr{E}}$:

$$A \mapsto F_A^+.$$

$\tilde{\mathscr{E}}$ descends to a Banach space bundle $\mathscr{E} \to \mathscr{B}$ and s defines as section of $\mathscr{E}$. By definition:

$$\mathscr{M} = s^{-1}(0) \subset \mathscr{B}.$$

Proposition 5.80. *Let A be an ASD instanton. Set*

$$V_{A,\varepsilon} := \left\{A + a : \|a\|_{W^{k+1,p}} < \varepsilon, \mathrm{d}_A^* a = 0, F_{A+a}^+ = \mathrm{d}_A^+ a + \tfrac{1}{2}[a \wedge a]^+ = 0\right\}.$$

The map $V_{A,\varepsilon}/\Gamma_A \to \mathscr{M}$ is an open embedding. Moreover, the stabiliser of a in Γ_A is Γ_{A+a}.

The term $\frac{1}{2}[a \wedge a]^+$ can be treated a small perturbation (as we will see shortly). It is therefore crucial to understand the linearised operator

$$\delta_A := (\mathrm{d}_A^*, \mathrm{d}_A^+)\colon W^{k+1,p}\Omega^1(X, \mathrm{Ad}(P)) \to W^{k,p}\Omega^0(X, \mathrm{Ad}(P)) \oplus W^{k,p}\Omega^+(X, \mathrm{Ad}(P)).$$

Proposition 5.81. *δ_A is an elliptic operator and, therefore,* ***Fredholm****; that is:*

$$\dim\ker\delta_A < \infty \quad \textit{and} \quad \dim\operatorname{coker}\delta_A < \infty.$$

Moreover,

$$\operatorname{index}\delta_A = \dim\ker\delta_A - \dim\operatorname{coker}\delta_A = -2p_1(\mathrm{Ad}(P)) + \dim G \cdot (b^1(X) - 1 - b^+(X)).$$

It is an easy exercise to compute the symbol of δ_A and verify ellipticity. The index can be computed using the Atyiah–Singer index theorem [see Atiyah–Hitchin–Singer].

Proposition 5.82. *Let X, Y be Banach spaces. Let $U \subset X$ be an open neighborhood of $0 \in X$. Let $f\colon U \to Y$ be a smooth map with $f(0) = 0$. Suppose that $T_0 f\colon X \to Y$ is Fredholm. Choose decompositions*

$$X = \ker T_0 f \oplus \operatorname{coim} T_0 f \quad \textit{and} \quad Y = \operatorname{coker} T_0 f \oplus \operatorname{im} T_0 f.$$

There is an open neighborhood V of 0 in X and a diffeomorphism $\phi\colon V \to U$ and a linear isomorphism $I\colon \operatorname{coim} T_0 f \to \operatorname{im} T_0 f$ such that

$$(f \circ \phi)(x, y) = (g(x, y), I(y)).$$

Apply this with

$$U \subset \ker(\mathrm{d}_A^*\colon\ W^{k+1,p}\Omega^1(X,\mathrm{Ad}) \to W^{k,p}\Omega^0(X,\mathrm{Ad})), Y := W^{k+1,p}\Omega^2(X,\mathrm{Ad}(P))$$

and $a := F^+_{A+a}$. It follows that a neighborhood of $[A] \in \mathscr{M}$ is modelled on

$$f^{-1}(0)/\Gamma_A$$

for some Γ_A–equivariant smooth map

$$f\colon\ \ker\delta_A \to \operatorname{coker}\mathrm{d}_A^+.$$

5.13.5 Digression: the deformation complex

[possibly skip this]

If A is an ASD instanton, then

$$0 \to \Omega^0(X,\mathrm{Ad}(P)) \xrightarrow{\mathrm{d}_A} \Omega^1(X,\mathrm{Ad}(P)) \xrightarrow{\mathrm{d}_A^+} \Omega^+(X,\mathrm{Ad}(P)) \to 0$$

is an elliptic complex. Its cohomology groups H^0_A, H^1_A, H^2_A correpond to infinitesimal gauge transformations, infinitesimal deformations, and infinitesimal obstructions. In fact,

$$H^1_A \cong \ker\delta_A \quad \text{and} \quad H^2_A \cong \operatorname{coker}\mathrm{d}_A^+.$$

This encodes the infinitesimal deformation somewhat more naturally. The above map f can be understood as map $f\colon\ H^1_A \to H^2_A$.

5.13.6 Freed–Uhlenbeck transversality theorem

Theorem 5.83 (Freed–Uhlenbeck). *If $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$, then for a generic Riemannian metric g and every irreducible ASD instanton A,* $\operatorname{coker}\mathrm{d}_A^+ = 0$.

As a consequence, in the situation of the Freed–Uhlenbeck theorem $\mathscr{M}^*$, the moduli space of *irreducible* ASD instantons, is a smooth manifold of dimension

$$-2p_1(\mathrm{Ad}(P)) + \dim G \cdot (b^1(X) - 1 - b^+(X)).$$

5.14 Uhlenbeck compactness

The moduli space $\mathscr{M}$ is very rarely compact. Most applications of ASD instantons require either some understanding of how compactness fails or even a suitable compactification $\overline{\mathscr{M}}$. If you want to understand this issue in detail, then [Weh04] is an excellent reference. Let me begin by discussing what can possibly go wrong:

(1) If A is a connection over $\mathbf{R}^n$, $s_\lambda(x) = \lambda^{-1}x$, then $A_\lambda := s_\lambda^* A$ satisfies

$$\begin{aligned}\mathrm{YM}(A_\lambda) &= \frac{1}{2}\int_{\mathbf{R}^n} |F_{A_\lambda}|^2(x)\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbf{R}^n} \lambda^{-4}|F_A|^2(\lambda^{-1}x)\lambda^n \mathrm{d}(\lambda^{-1}x) \\ &= \lambda^{n-4}\mathrm{YM}(A).\end{aligned}$$

That is the scaling-weight is $n-4$. As a conseqeunce: if $n \leqslant 3$, then for a connection to minimise its Yang–Mills energy it is not beneficial to "scale down"; if $n \geqslant 5$, then for a connection to minimise its Yang–Mills energy it is beneficial to "scale down"; and $n = 4$ is the boarder line. Therefore, one should *expect* that: the compactness problem for Yang–Mills connections in dimension $n \leqslant 3$ should be quite easy ("sub-critical"); for $n \geqslant 5$ it should be very hard ("super-critical"), and for $n = 4$ it is something in between ("critical").

(2) In dimension 4, the Yang–Mills functional is not just scaling invariant. It is in fact invariant under conformal changes: $g \mapsto \lambda^2 g$ for $\lambda \in C^\infty(X, (0,\infty))$. This means that the conformal group acts on Yang–Mills solutions. Since the conformal group is non-compact, this might be a source of non-compactness.

(3) We already have examples of the failure of compactness. The curvature of the BPST instanton $A_{\mu,b}$ on $\mathbf{H}$ is given satisfies

$$|F_{A_{\mu,b}}| = \frac{192^{1/2}\mu^2}{(\mu^2|q-b|^2+1)^2}.$$

As μ tends to ∞, this fails to converge at $q = b$. Away from $q = b$, however, this converges to zero and so does the $A_{\mu,b}$. That is: $A_n = A_{\mu_n,b}$ converges on almost all of $\mathbf{R}^4$, but there is one point at which something goes wrong. This point is identifiable by the fact that the Yang–Mills energy in a small (n–independent) ball around x stays quite large:

$$\liminf_{r\to 0}\liminf_{n\to\infty} \mathrm{YM}(A_n|_{B_r(x)}) > 0.$$

The Yang–Mills energy *concentrates* at x.

Another further issue that complicates the compactness analysis in Yang–Mills theory is that because the gauge group $\mathscr{G}$ is severely non-compact one cannot expect any sequence (A_n) of connections to converge without pasing to a gauge transformed seqeunce $(u_n^* A_n)$. (Of course, since $\mathscr{M} \subset \mathscr{A}/\mathscr{G}$ this is not an actuall issue, but it means that one has to be carefull about what to expect.)

The following discussion focuses on dimension $n = 4$ and ASD instantons. In a sense this is ideal, because it gives us a topological energy bound

$$\mathrm{YM}(A) = -\frac{1}{4\pi^2}p_1(\mathrm{Ad}(p))$$

to get started. The theory in dimension $n \leqslant 3$ is much simpler (and can easily be derived from that in dimension $n = 4$). The theory in dimension $n \geqslant 5$ is quite a bit more complicated (and indeed not fully understood). Some extra ideas and observations are needed for $n \geqslant 5$, in particular: the monotonicity formula due to Price [Pri83] and a more delicate ε–regularity theorem.

5.14.1 Uhlenbeck gauge fixing

The upcoming discussion is local. Let $G < \mathrm{O}(N)$ be a Lie group. On $B_1(0) \subset \mathbf{R}^n$ consider the trivial G–principal bundle $(p\colon G \times B_1(0) \to B_1(0), R)$. A **gauge** is a section of p. The trivial gauge is $x \mapsto (\mathbf{1}, x)$. Of course, any other gauge be identified with a map $u\colon B_1(0) \to G$. By comparison with the trivial gauge it can be identified with a gauge transformation. We regard connection on (p, R) as 1–forms on $B_1(0)$ with value in $\mathfrak{g}$.

Theorem 5.84 (Uhlenbeck [Uhl82a, Theorem 2.1]). *Let $n/2 < p$. There are constants $\varepsilon = \varepsilon(p, G)$ and $c = c(p, G)$ such that the following holds. If $A \in \mathscr{A}(p, R)$ satisfies*

$$\|F_A\|_{L^p} \leqslant \varepsilon,$$

then there is a $W^{2,p}$ gauge u such that u^*A satisfies the gauge fixing conditions

$$\begin{aligned} \mathrm{d}^*(u^*A) &= 0, \\ i(\partial_r)(u^*A) &= 0 \quad \text{on } \partial B_1(0) \end{aligned} \tag{5.85}$$

and

$$\|u^*A\|_{W^{1,p}} \leqslant c\|F_A\|_{L^p}. \tag{5.86}$$

Remark 5.87. Uhlenbeck [Uhl82a, Theorem 2.1] is somewhat stronger than Theorem 5.84 (the smallness condition is on $\|F_A\|_{L^{n/2}}$). Uhlenbeck [Uhl82a, Theorem 1.3] proved an even more delicate result at the Sobolev border line $p = n/2$. (In this case u might not be even be continuous.) It turns out that one can get away with the above result, by working a little harder later. ♣

Remark 5.88. The restriction $p < n$ might appear somewhat strange. It has to do with wanting to avoid the borderline Sobolev embedding at $W^{1,n}$. ♣

Sketch of proof of Theorem 5.84. The proof is based on the *continuity method.* The set

$$\mathscr{A}_\varepsilon := \{A \in W^{1,p}\mathscr{A} : \|F_A\|_{L^p} \leqslant \varepsilon\}.$$

is connected. Indeed, $[0, 1] \ni \lambda \mapsto s_\lambda^* A$ with $s_\lambda(x) = \lambda x$ joins any connection in $\mathscr{A}_\varepsilon$ to the connection 0. (The scaling weight of the L^p norm of a 2–form is $\lambda^{(n-2p)/p}$.)xs The strategy is to prove that

$$\mathscr{A}_\varepsilon^\star := \{A \in \mathscr{A}_\varepsilon : \exists u \in W^{2,p}\mathscr{G} : \mathrm{d}(u^*A) = 0, i(\partial_r)(u^*A) = 0 \text{ on } \partial B_1(0), \|u^*A\|_{W^{1,p}} \leqslant c\|F_A\|_{L^p}.\}$$

is open and closed.

Evidently, $\mathscr{A}_\varepsilon^*$ is $W^{2,p}\mathscr{G}$–invariant. The conditions

$$\begin{aligned} \mathrm{d}^*(u^*A) &= 0, \\ i(\partial_r)(u^*A) &= 0, \\ \|u^*A\|_{W^{1,p}} &\leqslant c\|F_A\|_{L^p} \end{aligned}$$

are closed.

To see that $\mathscr{A}_\varepsilon^\star$ is closed, let (A_n) be a sequence in $\mathscr{A}_\varepsilon^\star$ which converges to $A \in \mathscr{A}$. Denote by u_n the gauge transformations such that $u_n^*A_n$ satisfies the gauge fixing condition. The sequence $u_n^*A_n$ has a weak limit in $W^{1,p}$. The argument from the proof that the action of $\mathscr{G}$ on $\mathscr{A}$ is proper shows that u_n converges weakly in $W^{2,p}$ to a limit u. This suffices to obtain the gauge fixing conditions on u^*A. Therefore, $A \in \mathscr{A}_\varepsilon^\star$. This explains why $\mathscr{A}_\varepsilon^\star$ is closed (regardless of the choice of c)

To prove openness of $\mathscr{A}_\varepsilon^\star$ it suffices (by $\mathscr{G}$–invariance) to show that if A satisfies the gauge fixing condition and $\|A\|_{W^{1,p}} \leqslant c\|F_A\|_{L^p} \leqslant c\varepsilon$, then for a with $\|a\|_{W^{1,p}} < \delta \ll 1$ there is a gauge transformation u such that $u^*(A + a)$ satisfies the gauge fixing condition and $\|u^*(A + a)\|_{W^{1,p}} \leqslant c\|F_A\|_{L^p}$. Here is an assertion proving that the last condition will be automatic.

Proposition 5.89. *There are constant $c = c(n, G)$ and $\varepsilon_0 = \varepsilon_0(n, G)$ such that the following holds. If $A \in \mathscr{A}_\varepsilon$ satisfies (5.85) and $\|A\|_{W^{1,p}} < c_s\varepsilon_0$, then*

$$\|A\|_{W^{1,p}} \leqslant c\|F_A\|_{L^p}.$$

Proof. If (5.85) holds, then

$$\begin{aligned} (\mathrm{d} \oplus \mathrm{d}^*)A &= F_A - \frac{1}{2}[A \wedge A] \\ i(\partial_r)A &= 0. \end{aligned}$$

[An integration by parts argument proves that

$$\int_{B_1(0)} |\nabla A|^2 + \int_{\partial B_1(0)} |A|^2 = \int_{B_1(0)} |\mathrm{d}A|^2$$

This shows that the operator on the LHS is well-behaved on $W^{1,2}$. This is also the case for $W^{1,p}$.] The linear operator on the left-hand side of the above equations is an elliptic operator with trivial kernel. (See [Weh04] for a discussion of such operators.) Therefore, there is a constant $c = c(n, G) > 0$ such that

$$\|A\|_{W^{1,p}} \leqslant c\|F_A - \frac{1}{2}[A \wedge A]\|_{L^p} \leqslant c\|F_A\|_{L^p} + c\|A\|_{L^{2p}}^2.$$

Hölder's inequality and Sobolev embedding gives

$$\|A\|_{L^{2p}}^2 \leqslant c_S\|A\|_{W^{1,p}}^2.$$

(This is exactly the condition $p > n/2$). For $\varepsilon \ll 1$, the term $c_S\|A\|_{W^{1,p}}$ is at most $\frac{1}{2}$ and can be absorbed into the left-hand side. ■

To prove that $\mathscr{A}_\varepsilon^\star$ with $c = c(n, G)$ from above is an now application of the implicit function theorem (to find the gauge transformation, because the condition $\|A\|_{L^n} < cc_s\varepsilon_0$ is open). We omit details of the proof, but here is the crucial point. If A is in Uhlenbeck gauge and a is small, then $u = \exp(\xi)$ puts $A + a$ in Uhlenbeck gauge if and only if

$$\begin{aligned} \mathrm{d}^*\mathrm{d}\xi &= \mathrm{d}^*(e^{-\xi}(A + a)e^{\xi}) \\ \partial_r\xi &= 0 \quad \text{on the boundary.} \end{aligned}$$

The linearisation of the equation at $a = 0$, $\xi = 0$ is the LHS. By elliptic theory this is surjective, so one can apply the IFT. ■

Let ε as above.

Proposition 5.90. *There are constants $c_k > 0$ such that the following holds. If $A \in \mathscr{A}$ on $B_1(0)$ satisfies*

$$\begin{aligned} F_A^+ &= 0, \\ \|F_A\|_{L^p} &\leqslant \varepsilon, \end{aligned}$$

then there is a (smooth) gauge transformation u such that

$$\|u^*A\|_{W^{k,p}} \leqslant c_k$$

for all k.

Proof sketch. Theorem 5.84, we can assume that A is already in Uhlenbeck gauge (i.e. $u = \mathbf{1}$). The result then follows from elliptic theory applied to $A \mapsto (\mathrm{d}^+A, \mathrm{d}^*A, i(\partial_r)A)$ and a little elliptic bootstrapping. ■

By Rellich–Kondrachov, these $W^{k,p}$–bounds will give smooth convergence (on compact subsets). The question is now: where do we get the bound $\|F_A\|_{L^p} \leqslant \varepsilon$ from? At this stage we can only assume a global L^2 bound: $\|F_A\|_{L^2} = -\frac{1}{2}\sqrt{p_1(\mathrm{Ad}(P))}$.

5.14.2 ε–regularity

We continue with the situation on $B_1(0)$. The following result is essentially due to Uhlenbeck [Uhl82b, Theorem 3.5]

Theorem 5.91. *There are constants $c_{\mathrm{R}}, \varepsilon > 0$ such that the following holds. Let A be an anti-self-dual instanton on $B_1(0)$. If*

$$\|F_A\|_{L^2(B_1(0))} \leqslant \varepsilon,$$

then

$$\|F_A\|_{L^\infty(B_1(0))} \leqslant c_{\mathrm{R}}\|F_A\|_{L^2(B_1(0))}.$$

Proof. By the Weitzenböck formula implies that

$$\nabla_A^* \nabla_A F_A = (\mathrm{d}_A^* + \mathrm{d}_A)^2 F_A + \{F_A, F_A\} = \{F_A, F_A\}$$

with $\{-, -\}$ denoting a universal bilinear form. Therefore,

$$\Delta |F_A|^2 = 2\langle \nabla_A^* \nabla_A F_A, F_A \rangle - 2|\nabla_A F_A|^2 \leqslant c|F_A|^3.$$

This implies

$$\Delta |F_A| \leqslant c|F_A|^2.$$

It turns out that such an inequality automatically implies the above assertion. [This is discussed next.] ■

Theorem 5.92 (ε–regularity). *Consider $B_1(0) \subset \mathbf{R}^4$. There are constants $c, \varepsilon > 0$ such that the following holds. If $f \in W^{1,2}(B_1(0)) \cap L^\infty(B_1(0))$, $f \geqslant 0$,*

$$\Delta f \leqslant f^2$$

holds weakly, and

$$\|f\|_{L^2(B_1(0))} \leqslant \varepsilon,$$

then

$$\|f\|_{L^\infty(B_{1/2}(0))} \leqslant c\|f\|_{L^2(B_1(0))}.$$

Proof using the mean value inequality. The mean value inequality implies that if $f \in C^\infty(B_r(0), [0, \infty))$ satisfies

$$\Delta f \leqslant \Lambda,$$

then

$$f(0) \leqslant c\big(r^{-2}\|f\|_{L^2(B_r(0))} + \Lambda r^2\big).$$

(The mean value inequality is very easy for $\mathbf{R}^n$. For non-flat backgrounds a proof is contained in [GT01, Theorem 9.20]; but see below.)

Define the auxiliary function $\phi\colon B_1(0) \to [0, \infty)$ by

$$\phi(x) := (1 - |x|)^2 f(x).$$

It suffices to prove that

$$\|\phi\|_{L^\infty} \leqslant c\varepsilon \quad \text{with} \quad \varepsilon = \|f\|_{L^2},$$

provided $\varepsilon \ll 1$.

Since ϕ vanishes on $\partial B_1(0)$, it achieves a maximum at some point $x_0 \in B_1(0)$. Set

$$r_0 := \frac{1}{2}(1 - |x_0|) \quad \text{and} \quad a_0 := f(x_0).$$

The task is then to prove that

$$r_0^2 a_0 \leqslant c\varepsilon.$$

For every $x \in B_{r_0}(x_0)$,

$$1 - |x| \geqslant 1 - |x_0| - r_0 = r_0.$$

[draw a picture of this] Therefore, since $\phi(x) \leqslant \phi(x_0)$,

$$f(x) = (1-|x|)^{-2}\phi(x) \leqslant (1-|x|)^{-2}\phi(x_0) = \left(\frac{1-|x_0|}{1-|x|}\right)^2 f(x_0) \leqslant 4a_0.$$

Therefore, by the mean value inequality for every $0 \leqslant r \leqslant r_0$

$$a_0 \leqslant c_0\big(r^{-2}\varepsilon + r^2 a_0^2\big),$$

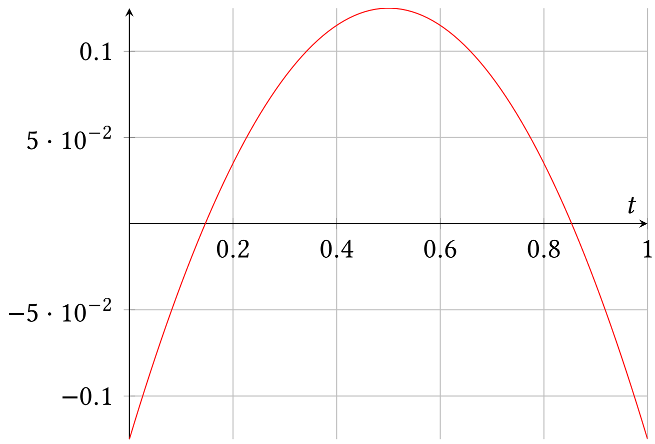
or, equivalently,

$$r^2 a_0 \leqslant c_0\big(\varepsilon + r^4 a_0^2\big).$$

That is for $t(r) := r^2 a_0$:

$$t(1 - c_0 t) - c_0\varepsilon \leqslant 0.$$

This inequality holds for every $r \in [0, r_0]$ and t is non-negative. An inspection of the graph of the polynomial on the left-hand side shows that, provided $\varepsilon \ll 1$, $t \leqslant 2c_0\varepsilon$. For $r = r_0$ this proves the assertion. ■



Remark 5.93. The polynomial

$$p(t) = t(1 - ct) - \varepsilon$$

has the roots

$$t_0 = \frac{1}{2c}\Big(1 - \sqrt{1 - 4c\varepsilon}\Big) \quad \text{and} \quad t_1 = \frac{1}{2c}\Big(\sqrt{1 - 4c\varepsilon} + 1\Big).$$

As long as $\varepsilon \leqslant 1/4c$, the roots are both real and positive. As long as $\varepsilon \ll_c 1$,

$$t_0 \leqslant 2\varepsilon$$

♣

5.14.3 Convergence away from finitely many points

Theorem 5.94. *Let (X, g) be a closed oriented Riemannian 4–manifold. Let $G < \mathrm{O}(N)$ be a Lie group (such that the embedding is compatible with minus the Killing form). Let $(p\colon P\colon X, R)$ be a principal G–bundle. Let (A_n) be a sequence in of ASD instantons on (p, R). After passing to a subsequence the following holds. There are*

(1) *finitely many points $x_1, \dots, x_k \in X$ and numbers $m_1, \dots, m_k \in \mathbf{N}$,*

(2) *an ASD instanton A on $(p, R)|_{X \setminus \{x_1, \dots, x_k\}}$, and*

(3) *a sequence of gauge transformations $u_k \in \mathscr{G}((p, R)|_{X \setminus \{x_1, \dots, x_k\}})$*

such that

(4) *The sequence of measures $|F_{A_n}|^2 \mathrm{vol}$ weakly converges to*

$$|F_A|^2 \mathrm{vol} + \sum_{a=1}^{k} 8\pi^2 m_a \delta_{x_k}.$$

(5) *For every compact subset $K \subset X \setminus \{x_1, \dots, x_k\}$, $u_n^* A_n|_K$ converges to $A|_K$ (in the C^∞ topology).*

Proof sketch. Conisder the sequence of measures $\mu_n := |F_{A_n}|^2 \mathrm{vol}$. The total mass of thesse measures is

$$c_{\mathrm{YM}} := -2\pi^2 p_1(\mathrm{Ad}(P));$$

in particular: it is uniformly bounded. Therefore, after passing to a subsequence μ_n weakly converges to a measure μ.

Let $x \in X$. Let ε be as in the ε–regularity theorem. If there exisits an $r > 0$ such that

$$\liminf_{n \to \infty} \int_{B_r(x)} |F_{A_n}|^2 < \varepsilon^2,$$

then on $B_r(x)$ a subsequence of A_n converges after gauge transformation. Therefore, it is crucial to understand the points for which this fails; that is: points with

$$\lim_{r \to 0} \liminf_{n \to \infty} \int_{B_r(x)} |F_{A_n}|^2 \geqslant \varepsilon^2,$$

If there are at least k points with this property, then

$$k\varepsilon^2 \leqslant \lim_{r \to 0} \liminf_{n \to \infty} \sum_{a=1}^{k} \int_{B_r(x_a)} |F_{A_n}|^2 \leqslant c_{\mathrm{YM}}.$$

This yields an apriori bound $k \leqslant c_{\mathrm{YM}}/\varepsilon^2$.

This identifies the points $\{x_1, \dots, x_k\}$. After passing to a subsequence, away from these points, (A_n) converges locally upto gauge transformations. These local gauge transformations can be patched together. (This is not trivial; see Donaldson and Kronheimer [DK90, §4.4.2] and Waldron [Wal19, §2.5].) This proves the convergence statement for A_n.

By Fatou's lemma

$$\delta = \mu - |F_A|^2 \mathrm{vol}$$

is a non-negative measure. It must be supported on $\{x_1, \ldots, x_k\}$. This proves the convergence statement for the measures with m_a non-necessarily integers.

To prove that m_a is an integer one first has to prove that A extends to all of X but a bundle which might be different from (p, R). [This *might* be proved in the next section.] Then, by construction for $r \ll 1$

$$\begin{aligned} m_a &= \frac{1}{8\pi^2} \lim_{n\to\infty} \int_{B_r(x_a)} |F_{A_n}|^2 - |F_A|^2 \\ &= \frac{1}{8\pi^2} \lim_{n\to\infty} \int_{B_r(x_a)} \langle F_{A_n} \wedge F_{A_n} \rangle - \langle F_A \wedge F_A \rangle. \end{aligned}$$

The last term can be rewritten as a Chern–Simons term and is known to be an integer. [The details are omitted.] ■

5.14.4 Uhlenbeck's removable singularities theorem

[discussed in problem session] sketch

(1) decay [Otway's unpublished argument]

(2) go to cylinder

(3) prove asymptotically flat

(4) gauge transform

(5) go back to ball

5.15 Digression: $\mathscr{H}^{\pm}(X, g)$ and product connections

Let (X, g) be a closed oriented Riemannian n–manifold. Consider the differential operator

$$\mathrm{d} + \mathrm{d}^* \colon \Omega^\bullet(X) \to \Omega^\bullet(X).$$

Denote by

$$\mathscr{H}^k(X, g) := \ker(\mathrm{d} + \mathrm{d}^*) \cap \Omega^k(X)$$

the space of harmonic k–forms on X. By Hodge theory,

$$\mathscr{H}^k(X, g) \cong \mathrm{H}^k_{\mathrm{dR}}(X).$$

The Hodge–$*$–operator induces an isomorphism

$$* \colon \mathscr{H}^k(X, g) \to \mathscr{H}^{n-k}(X, g).$$

If $n = 4k$, then $*$ has eigenvalues ± 1 on $\mathscr{H}^{2k}(X,g)$. The eigenspaces are denoted by

$$\mathscr{H}^{\pm}(X,g) \subset \mathscr{H}^{2k}(X,g).$$

Under the Hodge isomorphism, these correspond to the positive and negative definite subspace of the intersection form

$$Q\colon\ S^2\mathrm{H}^{2k}_{\mathrm{dR}}(X) \to \mathbf{R}$$

defined by

$$Q([\alpha],[\beta]) := \int_X \alpha \wedge \beta.$$

The **refined Betti number**

$$b^{\pm}(X) := \dim \mathscr{H}^{\pm}(X,g)$$

is independent of g.

Let us now specialise to $n = 4$. Consider the trivial bundle $(p\colon\ P := X \times G \to X, R)$ with the product (or trivial) connection A_0. According to Section 5.13, we can understand a neighborhood of $[A_0] \in \mathscr{M}$ as follows:

(1) If $u \in \mathscr{G} = C^\infty(X,G)$ fixes A_0, then u must be constant. Therefore, the isotropy group of A_0 is

$$\Gamma_{A_0} = G.$$

(2) The operator

$$\delta_{A_0} = (\mathrm{d}^*_{A_0}, \mathrm{d}^+_{A_0})\colon\ \Omega^1(X,\mathfrak{g}) \to \Omega^0(X,\mathfrak{g}) \oplus \Omega^+(X,\mathfrak{g})$$

is simply $\delta \otimes \mathrm{id}_{\mathfrak{g}}$ with

$$\delta := (\mathrm{d}^*, \mathrm{d}^+)\colon\ \Omega^1(X) \to \Omega^0(X) \oplus \Omega^+(X).$$

Evidently, $\mathscr{H}^1(X,g) \subset \ker\delta$. In fact, if $\delta\alpha = 0$, then $\mathrm{d}^*\alpha = 0$ and

$$0 = \int_X 2\langle \mathrm{d}^+\alpha, \mathrm{d}\alpha\rangle = \int_X 2\langle \mathrm{d}^*\mathrm{d}^+\alpha, \alpha\rangle = \int_X \langle \mathrm{d}^*\mathrm{d}\alpha, \alpha\rangle = \int_X |\mathrm{d}\alpha|^2.$$

Therefore, $\mathrm{d}\alpha = 0$. This shows that

$$\ker \delta_{A_0} = \mathscr{H}^1(X,g) \otimes \mathfrak{g}.$$

Moreover,

$$\operatorname{coker} \mathrm{d}^+_{A_0} = \mathscr{H}^+(X,g) \otimes \mathfrak{g}.$$

(3) Therefore, a neighborhood of $[A_0]$ is modelled on

$$f^{-1}(0)/G$$

for smooth map

$$f\colon\ \mathscr{H}^1(X,g) \otimes \mathfrak{g} \supset U \to \mathscr{H}^+(X,g) \otimes \mathfrak{g}.$$

In fact, $f = 0$. Thus the model is

$$\big(\mathscr{H}^1(X,g) \otimes \mathfrak{g}\big)/G = \big(\mathscr{H}^1(X,g) \otimes \mathfrak{t}\big)/W$$

with $\mathfrak{t}$ denoting a maximal abelian subalgebra and W denoting the Weyl group.

5.16 A sketch of Taubes' gluing theorem

Uhlenbeck's compactness theorem suggests that ASD instantons can degenerate by concentrating at points. At these points one might expect BPST instantons to "bubble off". Taubes' gluing theorem is concerned with the question of whether one can construct such degenerating ASD instantons. The idea is to glue BPST instantons into a product (trivial) connection.

Throughout, suppose that (X, g) is an closed oriented Riemannian 4–manifold with

$$b^+(X) = 0.$$

Denote by A_0 the product connection on the trivial $\mathrm{Sp}(1)$–principal bundle over X. Recall that the BPST instanton is given by

$$A^{BPST} := \frac{\mathrm{Im}(\bar{q}\mathrm{d}q)}{|q|^2 + 1}$$

on the on the trivial $\mathrm{Sp}(1)$–principal bundle over $\mathbf{H} = \mathbf{R}^4$. The first task is to scale A^{BPST} down by $0 < \lambda \ll 1$ and glue it into A_0 to obtain an almost ASD instanton $\tilde{A}_\lambda$.

Identify a neighborhood of x_0 with $B_{2\varepsilon}(0)$. To simply our live, we will also assume that the metric g is the Euclidean metric on $B_\varepsilon(0)$. The local connection 1–form of A_0 simply vanishes. We would like to "glue" the scaled down version of A^{BPST} with A_0 over the annulus

$$B_{2\varepsilon}(0)\backslash\bar{B}_\varepsilon(0).$$

If $s_\lambda(q) := q/\lambda$, then

$$s_\lambda^* A^{BPST} := \frac{\mathrm{Im}(\bar{q}\mathrm{d}q)}{|q|^2 + \lambda^2}$$

Unfortunately, the restriction of this to $B_{2\varepsilon}(0)\backslash\bar{B}_\varepsilon(0)$ is not at all small. (This not unexpected: because otherwise we might be on our way to construct a non-flat ASD instanton on the trivial bundle—which is impossible.) Consider the gauge transformation $u(q) = q/|q|$ defined on $\mathbf{H}\backslash\{0\}$. A computation reveals that

$$u^* A^{BPST} = -\frac{\mathrm{Im}(\mathrm{d}q\bar{q})}{|q|^2(1 + |q|^2)}$$

and

$$s_\lambda^*(u^* A^{BPST}) = -\lambda^2 \frac{\mathrm{Im}(\mathrm{d}q\bar{q})}{|q|^2(\lambda^2 + |q|^2)}$$

The restriction of this to $B_{2\varepsilon}(0)\backslash\bar{B}_\varepsilon(0)$ is small if $\lambda \ll 1$.

Define a $\mathrm{Sp}(1)$–principal bundle over X by gluing the trivial $\mathrm{Sp}(1)$–principal bundles over $X\backslash\bar{B}_\varepsilon(0)$ and $B_{2\varepsilon}(0)$ over $B_{2\varepsilon}(0)\backslash\bar{B}_\varepsilon(0)$ via the gauge transformation $u(q) = q/|q|$. Choose a cut-off function $\chi\colon [0, 2) \to [0, 1]$ which is equal to one on $[0, 1]$ and has compact support. Define a connection $\tilde{A}_\lambda$ to agree with

$$\chi(|q|/\varepsilon) s_\lambda^* A^{BPST}$$

on $B_{2\varepsilon}(0)$ and with A_0 on $X\backslash\bar{B}_{2\varepsilon}(0)$. Since

$$u^*(\chi(|q|/\varepsilon) s_\lambda^* A^{BPST}) = -\chi(|q|/\varepsilon)\lambda^2 \frac{\mathrm{Im}(\mathrm{d}q\bar{q})}{|q|^2(\lambda^2 + |q|^2)},$$

this gives us the desired interpolation.

How small is $F^+_{\tilde{A}_\lambda}$? Certainly, $F^+_{\tilde{A}_\lambda}$ vanishes outside of the annulus $B_{2\varepsilon}(0)\backslash\bar{B}_\varepsilon(0)$. To simplify notation, set

$$a_\lambda := -\lambda^2\frac{\mathrm{Im}(\mathrm{d}q\bar{q})}{|q|^2(\lambda^2+|q|^2)}.$$

Observe that for $\varepsilon \leqslant |q| \leqslant 2\varepsilon$,

$$|a_\lambda| \leqslant c(\varepsilon)\cdot\lambda^2$$

[Note: while ε should be thought of as small, it is also fixed and λ is much smaller than ε.]

We compute

$$F_{\tilde{A}_\lambda} = \chi(|q|/\varepsilon)F_{u^*s^*_\lambda A^{BPST}} + \varepsilon^{-1}\chi'(|q|/\varepsilon)\mathrm{d}|q|\wedge a_\lambda - \frac{1}{2}(\chi(|q|/\varepsilon)^2-\chi(|q|/\varepsilon))[a_\lambda\wedge a_\lambda]$$

The first term is anti-self-dual. Therefore, it suffices to estimate the last two terms. Therefore,

$$\left\|F^+_{\tilde{A}_\lambda}\right\|_{L^\infty} \leqslant c(\varepsilon,\chi)\cdot\lambda^2.$$

The task at hand is now to find $a = a(\lambda)$ such that

$$F^+_{\tilde{A}_\lambda+a} = F^+_{\tilde{A}_\lambda} + \mathrm{d}^+_{\tilde{A}_\lambda}a + \frac{1}{2}[a\wedge a]^+ = 0.$$

To break the gauge symmetry it is customary to supplement this equation with

$$\mathrm{d}^*_{\tilde{A}_\lambda}a = 0.$$

The full system of equations is then

$$\delta_{\tilde{A}_\Lambda}a + \frac{1}{2}[\alpha\wedge a]^+ + F^+_{\tilde{A}_\lambda} = 0.$$

Remark 5.95. Schematically, this is of the form

$$Lx + \mathscr{N}(x) + E = 0.$$

with L linear, $\mathscr{N}$ non-linear, and E denoting the initial (pre-gluing) error. There is a standard approach towards solving such equations. Let us pretend that $\mathscr{N} = 0$ and that the problem is finite-dimensional. In this case, we can certainly always solve the equation provided that L is surjective. Indeed, if R is a right-inverse of L (that is: $LR = \mathbf{1}$), then

$$x = -RE$$

is the desired solution.

If $\mathscr{N}$ does not vanish, then the equation can still be rewritten as follows by setting $x = Ry$

$$y = -(\mathscr{N}(Ry) + E).$$

This is a fixed-point equation. It can be solved in $B_\rho(0)$ (uniquely) using Banach's fixed-point theorem provided $y \mapsto -(\mathscr{N}(Ry)+E)$ is a contraction for $|y| \leqslant \rho \ll 1$. ♣

The above scheme can be carried out in the situation at hand with

$$L_\lambda \coloneqq \delta_{\tilde{A}_\lambda}\colon\ W^{1,p}\Omega^1(X,\mathrm{Ad}(P)) \to L^p\Omega^0(X,\mathrm{Ad}(P)) \oplus L^p\Omega^+(X,\mathrm{Ad}(P))$$

and $p > 2$ (so that $W^{1,p} \hookrightarrow L^{2p}$ with a constant independent of λ by Kato). The hard part is to construct a right-inverse $R_\lambda\colon\ L^p \to W^{1,p}$ of L_λ with

$$\|R_\lambda\|_{\mathscr{L}} \leqslant c_1\lambda^{-\alpha}$$

with $0 \leqslant \alpha < 1$. (Here $\|-\|_{\mathscr{L}}$ denotes the operator norm.) (IMHO the best way to do this is to patch right-inverses for the models. For the trivial connection the right-inverse is easy to obtain. For the BPST instanton one has to think a bit.) The easy part is to observe that

$$\left\|F^+_{\tilde{A}_\lambda}\right\|_{L^p} \leqslant c_2\lambda^2$$

and that $\mathscr{N}(a) \coloneqq \frac{1}{2}[a \wedge a]^+$ satisfies

$$\|\mathscr{N}(a_1) - \mathscr{N}(a_2)\|_{L^p} \leqslant c_3(\|a_1\|_{W^{1,p}} + \|a_2\|_{W^{1,p}})\|a_1 - a_2\|_{W^{1,p}}.$$

Therefore,

$$a \mapsto -\left(\frac{1}{2}[R_\lambda a \wedge R_\lambda a]^+ + F^+_{\tilde{A}_\lambda}\right)$$

is a contraction on $\bar{B}_\rho(0) \subset L^p$ provided

$$c_3c_2^2\lambda^{-2\alpha}\rho < 1 \quad\text{and}\quad c_3c_2^2\lambda^{-2\alpha}\rho^2 + c_2\lambda^2 \leqslant \rho.$$

Since $\alpha < 1$, a suitable ρ can be found.

The upshot of all of this is that if $b^+(X) = 0$, then for every $x \in X$ and every $0 < \lambda \lll 1$ we can construct an ASD instanton $A_{\lambda,x}$ which is modelled (very closely) on a λ–scaled down BPST instanton in a neighborhood of x.

5.17 Donaldson's diagonalisation theorem

Theorem 5.96 (Donaldson [Don83, Theorem 1]). *Let X be a closed oriented* smooth 4*–manifold with $\pi_1(X) = \mathbf{1}$. If the intersection form $Q\colon\ \mathrm{H}^2(X,\mathbf{Z}) \otimes \mathrm{H}^2(X,\mathbf{Z}) \to \mathbf{Z}$ is positive or negative definite, then it is diagonalisable over $\mathbf{Z}$.*

This is a remarkable theorem. Over $\mathbf{Z}$ it is far from true that every symmetric bilinear form is diagonalisable. The quadratic form given by the matrix

$$E_8 = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & -1 & \\ & & & & -1 & 2 & & -1 \\ & & & & -1 & & 2 & \\ & & & & & -1 & & 2 \end{pmatrix}$$

is positive definite. However, E_8 is not diagonalisable over $\mathbf{Z}$. It is possible to construct a closed oriented *topological* 4–manifold X with $\pi_1(X) = \mathbf{1}$ with $Q = E_8$ (the "E_8 manifold"). Q is even and its signature $\sigma(X) = 8$. If X were smooth, then it would admit a spin structure Rohklin's theorem would imply that $\sigma(X)$ is divisible by 16. Therefore, X cannot be equipped with a smooth structure. Donaldson's theorem yields the same conclusion, of course.

The above theorem should be contrasted with the following.

Theorem 5.97 (Freedman [Fre82, Theorem 1.5]). *Let Q be an integral unimodular quadratic form. There is a closed topological 4–manifold X with $\pi_1(X)$ realising Q as its intersection form.*

Donaldson's theorem shows that many of Freedman's manifolds cannot be equipped with smooth structures.

Sketch of proof of Theorem 5.96. There is no loss in assuming that Q is negative definite; i.e.: $b^+(X) = 0$. Denote by $(p\colon P \to X, R)$ the $\mathrm{SU}(2)$–principal bundle with $c_2(P) = 1$. (By a Theorem of Dold and Whitney c_2 specifies (p, R) up to isomorphism.) Choose a generic metric on g in the sense of Freed–Uhlenbeck. Denote by $\bar{\mathscr{M}}$ the Uhlenbeck compactification of the moduli space of ASD instantons on (p, R) with respect to g. The proof is based on a detailed understanding of $\mathscr{M}$.

By the Freed–Uhlenbeck theorem, the subset $\mathscr{M}^*$ of irreducible ASD instantons carries the structure of a smooth manifold of dimension

$$\dim \mathscr{M}^* = 8c_2 + 3(b^1 - 1 - b^+) = 5.$$

We need to understand:

(1) The locus of reducible ASD instantons $\mathscr{M}^{\mathrm{red}} := \mathscr{M} \backslash \mathscr{M}^*$ and how it fits with $\mathscr{M}^*$.

(2) The locus of ideal ASD instantons $\partial\mathscr{M} := \overline{\mathscr{M}} \backslash \mathscr{M}$ and how it fits with $\mathscr{M}^*$.

The first part is by far the easier given the discussion in Section 5.13. The isotropy group Γ_A of a reducible ASD instanton A on (p, R) is either $\mathbf{1}$ or S^1. If $\Gamma_A = \mathbf{1}$, then A is flat; but (p, R) carries not flat connections (by Chern–Weil theory). If $\Gamma_A = S^1$, then as we already discussed, the Hermitian rank 2 vector bundle E associated with (p, R) splits as

$$E = L \oplus L^*$$

and A arises from an ASD instanton on L. Since $b_1(X) = 0$ and $b^+(X) = 0$, L admits unique ASD instanton up to gauge transformations. The bundle L must satisfy

$$1 = c_2(E) = -c_1(L)^2.$$

Indeed, the elements of $\mathscr{M}^{\mathrm{red}}$ precisely corresponds to the pairs $\pm x \in \mathrm{H}^2(X, \mathbf{Z})$ of solutions of

$$Q(x, x) = -1.$$

Denote the number of those solutions by

$$n(Q).$$

Describes $\mathscr{M}^{\mathrm{red}}$ as a finite set. How does it fit the rest of $\mathscr{M}^*$? The discussion in Section 5.13 says that a neigborhood of $[A] \in \mathscr{M}^{\mathrm{red}}$ is modelled on

$$f^{-1}(0)/\Gamma_A$$

with $f\colon \ker \delta_A \supset U \to \operatorname{coker} \mathrm{d}_A^+$ a smooth map with $f(0) = 0$ and $T_0 f = 0$. Here we restrict $\delta_A = (\mathrm{d}_A^*, \mathrm{d}_A^+)$ to $\Omega^1(X, L^2) \to \Omega^0(X, L^2) \oplus \Omega^+(X, L^2)$ Therefore, δ_A and d_A^+ are complex vector spaces and S^1 acts by multiplication with the square unit complex numbers. An application of the index theorem proves that

$$\dim_{\mathbf{C}} \ker \delta_A - \dim_{\mathbf{C}} \operatorname{coker} \mathrm{d}_A^+ = 3.$$

It follows ultimately from the Freed–Uhlenbeck theorem, that $\operatorname{coker} \mathrm{d}_A^+ = 0$. Therefore, a neighborhood of A is modelled on

$$\mathbf{C}^3/S^1 = \operatorname{cone}(\mathbf{C}P^2).$$

[DRAW PICTURE OF WHAT IS KNOWN SO FAR.]

The next task is to identify $\partial\mathscr{M}$. By the energy identity, every $[A] \in \mathscr{M}$ has $\mathrm{YM}(A) = 8\pi^2$. Therefore, if $[A_n]$ in $\mathscr{M}$ converges to $[A_0, \sum_{a=1}^k m_a x_a] \in \partial\mathscr{M}$, then A_0 must be flat, $k = 1$, and $m_1 = 1$ by the convergence statement about the measures. Since $\pi_1(X) = \mathbf{1}$, A_0 must be trivial. Therefore, $\partial\mathscr{M} \subset X$. Taubes' theorem proves that $\partial\mathscr{M} = X$. This suggests that $\overline{\mathscr{M}}\backslash\mathscr{M}^{\mathrm{red}}$ is a smooth manifold with boundary. While this is true, but requires some actual work to construct the charts on the boundary. (This is Donaldson's collar theorem.)

[UPDATE PICTURE.]

The upshot of the above analysis is that $\overline{\mathscr{M}}$ furnishes us with a compact cobordism between X and $n(Q)$ copies $\mathbf{C}P^2$. The proof can now be completed as follows.

Lemma 5.98. *If Q is a negative definite quadratic form over $\mathbf{Z}$, then $n(Q) \leqslant \operatorname{rk} Q$. Equality holds if and only if Q is diagonal.*

Proof. This proved by induction on $r := \operatorname{rk} Q$. If $Q(x) = -1$, then $\mathbf{Z}^r = \mathbf{Z}\alpha \perp (\mathbf{Z}\alpha)^\perp$ via

$$y \mapsto (\langle y, x\rangle \cdot x, y - \langle y, x\rangle \cdot x).$$

Of course, the new intersection form Q' on $(\mathbf{Z}\alpha)^\perp$ has $n(Q') = n(Q) - 1$ and $\operatorname{rk}(Q') = \operatorname{rk}(Q) - 1$. ∎

Since Q is negative definite, $\operatorname{rk}(Q) = \sigma(Q)$. The signature is invariant of oriented cobordisms. Therefore,

$$\operatorname{rk}(Q) = \sigma(Q) = \sum_{a=1}^{n(Q)} \varepsilon_a \sigma(\mathbf{C}P^2) = \sum_{a=1}^{n(Q)} \varepsilon_a \leqslant n(Q)$$

with $\varepsilon_a \in \{\pm 1\}$ according to the orientation the corresponding of $\mathbf{C}P^2$. It follows that

$$n(Q) = \operatorname{rk} Q$$

and, therefore, Q is diagonalisable. ∎

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