Differential Geometry III
Gauge Theory
Problem Set 1

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1 Fibre bundles over $[0,1]$ are trivial

Let $p\colon X \to [0,1]$ be a fibre bundle. Prove that p is isomorphic to a trivial bundle $\mathrm{pr}_{[0,1]}\colon [0,1]\times F \to [0,1]$.

2 The Gauß–Manin connection

A brief review of de Rham cohomology. Let X be a smooth manifold. The **de Rham complex** is $(\Omega^\bullet(X), \mathrm{d})$. Its cohomology $\mathrm{H}^\bullet(\Omega^\bullet(X), \mathrm{d})$ is the **de Rham cohomology** of X:

$$\mathrm{H}^k_{\mathrm{dR}}(X) := \frac{\ker(\mathrm{d}\colon \Omega^k(X) \to \Omega^{k+1}(X))}{\mathrm{im}(\mathrm{d}\colon \Omega^{k-1}(X) \to \Omega^k(X))}.$$

If $f\colon X \to Y$ is a smooth map, then $f^*\colon \Omega^*(Y) \to \Omega^*(X)$ descends to

$$f^*\colon \mathrm{H}^k_{\mathrm{dR}}(Y) \to \mathrm{H}^k_{\mathrm{dR}}(X).$$

The latter depends only on the homotopy class of f. If f is a diffeomorphism, then f^* is an isomorphism.

Let $p\colon X \to B$ be a fibre bundle. Set

$$\mathscr{H}^k_{\mathrm{dR}}(p) := \coprod_{b\in B} \mathrm{H}^k_{\mathrm{dR}}(p^{-1}(b)).$$

Denote by $q\colon \mathscr{H}^k_{\mathrm{dR}}(p) \to B$ the canonical projection. This can be given the structure of a vector bundle and equipped with a flat covariant derivative, the **Gauß–Manin connection**.

(1) Using the local trivialisations of p construct a smooth structure on $\mathscr{H}^k_{\mathrm{dR}}(p)$

(2) Prove that $q\colon \mathscr{H}^k_{\mathrm{dR}}(p) \to B$ is a vector bundle.

(3) Construct a flat covariant derivative ∇ on $q\colon \mathscr{H}^k_{\mathrm{dR}}(p) \to B$.

(4) Let F be smooth manifold. Let $f \in \mathrm{Diff}(F)$. Denote X_f the **mapping torus** of f; that is:

$$X_f := ([0,1] \times F)/$$

with denoting the equivalence relation generated by $(0,x) \sim (1, f(x))$. X_f is a smooth manifold and the projection map $p\colon X \to S^1 = \mathbf{R}/\mathbf{Z}$ is a fibre bundle. Compute the monodromy of the ∇ in this example.

3 Connections from Riemannian metrics

Let $p\colon X \to B$ be a fibre bundle. Let g be a Riemannian manifold. Denote by H the g–orthogonal complement of V_p. This defines a connection A on p. Find a formula for the curvature of A.