Differential Geometry III
Gauge Theory
Problem Set 11

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The purpose of this problem session is to explain the Higgs mechanism. (If you want to learn more about this, the standard model, etc., please look at the wonderful book by Hamilton [Ham17].)

Geometric setting. Let G be a Lie group. Let $\langle\cdot,\cdot\rangle$ be an Ad–invariant inner product on $\mathfrak{g} := \mathrm{Lie}(G)$. Let $\rho\colon G \to \mathrm{O}(V)$ be a representation of G. Let $E \in C^\infty(X, [E_0,\infty))^G$. Let (X, g) be a Riemannian manifold. Let $(p\colon P \to X, R)$ be a G–principal bundle. Denote by $\mathbf{V} := P \times_\rho V$ the Euclidean vector bundle associated with ρ. For $A \in \mathscr{A}(p, R)$ and $\Phi \in \Gamma(\mathbf{V})$ consider the **Yang–Mills–Higgs functional**

$$\mathrm{YMH}(A,\Phi) := \frac{1}{2}\int_X |F_A|^2 + |\mathrm{d}_A\Phi|^2\,\mathrm{vol}_g + \int_X E(\Phi)\,\mathrm{vol}_g.$$

(1) Compute the Euler Lagrange equation of YMH.

A appears to have no *mass* in YMH: there is no term of the form $m|A|^2$ in YMH; in fact: such a term is utterly nonsensical (and most certainly not gauge invariant). However, though the Higgs mechanism a term with a very similar effect can appear.

Suppose that (A_0, Φ_0) is an absolute minimum of YMH. In fact, assume that

$$F_{A_0} = 0, \quad \mathrm{d}_{A_0}\Phi_0 = 0, \quad \text{and} \quad E(\Phi_0) = E_0.$$

Φ_0 is the **Higgs field**. Since Φ_0 is covariantly constant, it corresponds to an element of V. Denote by $H < G$ its stabiliser. (This is well-defined upto conjugation.) If $H \neq G$, then the symmetry is said to be **spontaneously broken** to H. In this case, $\Phi_0 \neq 0$.

(2) For $A = A_0 + a$ and $\Phi = \Phi_0 + \phi$ compute

$$\mathrm{YMH}(A,\Phi)$$

to leading order.

(3) Consider the map $R = R_{\Phi_0} \colon \mathfrak{g} \to V$ defined by

$$R(\xi) \coloneqq \rho(\xi)\Phi_0.$$

Prove that

$$\ker R = \mathfrak{h} \coloneqq \mathrm{Lie}(H).$$

In particular, there is a $m = m_{\Phi_0} > 0$ such that

$$|R\xi|^2 \geqslant m|\xi|^2$$

for every $\xi \in \mathfrak{h}^\perp$.

The term $\langle Ra\rangle^2$ features in the above perturbative expression for $\mathrm{YMH}(A, \Phi)$. It attaches a mass (of at least) to the components of a in $\mathfrak{h}^\perp$ (the *broken gauge bosons*). The compoents of a in $\mathfrak{h}$ (the *unbroken gauge bosons*) acquire no mass.

Suppose that E is non-degenerate; that is: E attains its minimum E_0 precisely along $G \cdot \Phi_0$ and $\mathrm{Hess}_{\Phi_0} E$ is positive defnite on $V_0^\perp$ with $V_0 \coloneqq \mathrm{Lie}(\rho)(\mathfrak{g})\Phi_0$. The latter term also features in the above perturbative expression for $\mathrm{YMH}(A, \Phi)$. This attaches a mass to the components of ϕ in $V_0^\perp$. (The components of ϕ in V_0 correspond to *Nambu–Goldstone bosons* and the components of ϕ in $V_0^\perp$ correspond to *Higgs bosons*.)

This is the **Higgs mechanism** in a nutshell.

(4) Come up with some choice of ρ and E and work out the above for those.

References

[Ham17] M. J. D. Hamilton. *Mathematical Gauge Theory. With Applications to the Standard Model of Particle Physics.* Universitext. Springer, 2017. DOI: 10.1007/978-3-319-68439-0. MR: 3837560. Zbl: 1390.81005 (cit. on p. 1)