Differential Geometry III
Gauge Theory
Problem Set 3

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The purpose of this exercise sheet is to understand *fibre integration* and the *Haar volume form.*

(1) Let $p\colon X \to B$ be a fibre bundle with $\partial B = \varnothing$. Suppose that the fibres of p compact (but possibly with boundary) and of dimension d. Suppose the vector bundle $V_p\colon X$ is oriented; hence, the fibres of are oriented.

(a) Prove that there is a unique linear map $p_*\colon \Omega^\bullet(X) \to \Omega^{\bullet-d}(B)$ such that for every $\alpha \in \Omega^{d+k}(X)$, $b \in B$, and $\tilde v_1, \ldots, \tilde v_k \in \Gamma(TX|_{p^{-1}(b)})$ lifts of $v_1, \ldots, v_k \in T_bB$

$$(p_*\alpha)(v_1, \ldots, v_k) = \int_{p^{-1}(b)} i_{\tilde v_k} \cdots i_{\tilde v_1}\alpha|_{p^{-1}(b)}.$$

This map is the **fibre integration map**. Another intuitive notation for p_* is $\int_{X/B}$.

(b) Prove that for every $\alpha \in \Omega^\bullet(X)$ and $\beta \in \Omega^\bullet(B)$

$$p_*(\alpha \wedge p^*\beta) = p_*\alpha \wedge \beta.$$

(c) Suppose that B is oriented. Prove that for every $\alpha \in \Omega^\bullet(X)$

$$\int_X \alpha = \int_B p_*\alpha$$

(d) Set $\partial p := p|_{\partial X}\colon \partial X \to B$. Convince yourself that ∂p is a fibre bundle.

(e) Prove that for every $\alpha \in \Omega^\bullet(X)$

$$p_*\mathrm{d}\alpha - (-1)^d \mathrm{d}p_*\alpha = \partial p_*\alpha.$$

Remark. If $\partial X = \varnothing$, then p_* descends to de Rham cohomology $p_*\colon \mathrm{H}^\bullet_{\mathrm{dR}}(X) \to \mathrm{H}^{\bullet-d}_{\mathrm{dR}}(B)$. Set $K^\bullet := \ker p_*\colon \Omega^\bullet(X) \to \Omega^{\bullet-d}(B)$. The short exact sequence

$$0 \to K^\bullet \to \Omega^\bullet(X) \to \Omega^{\bullet-d}(B) \to 0$$

induces a long exact sequence

$$\cdots \to \mathrm{H}^k(K^\bullet) \to \mathrm{H}^k_{\mathrm{dR}}(X) \to \mathrm{H}^{k-d}_{\mathrm{dR}}(B) \xrightarrow{\delta} \mathrm{H}^{k+1}(K^\bullet) \cdots .$$

If p is an oriented sphere bundle, then the map $p^*\colon\ \mathrm{H}^k(B) \to \mathrm{H}^k(K^\bullet)$ is an isomorphism. The Bockstein homomorphism δ arises from taking the wedge product with the **Euler class** of p (more about that later in this course). This instance of the above elong exact sequence is the **Gysin sequence**. ♣

(2) Let G be a Lie group. Set $d := \dim G$.

(a) Prove that is a unique left-invariant volume form up to multiplication by a non-zero constant:

$$\dim \Omega^d(G)^L := \{\nu \in \Omega^d(G) : L_g^*\nu = \nu\} = 1.$$

A **Haar volume form** on G is a left-invariant volume form ν on G. ν is **normalised** if $\int_G \nu = 1$.

(b) Let ν be a Haar volume form on G. Prove that for every $g \in G$, $R_g^*\nu$ is a Haar volume form.

(c) The **modular function** of G is the function $\Delta \in C^\infty(G, \mathbf{R}^\times)$ defined by

$$\Delta(g) := \frac{R_g^*\nu}{\nu}.$$

Prove that $\Delta = 1$ if and only if G admits a right-invariant Haar measure. These groups are **unimodular**.

(d) Prove that $\Delta\colon\ G \to \mathbf{R}^\times$ is a Lie group homomorphism.

(e) Prove that $\Delta = 1$ if G is compact.

(f) Define $i\colon\ G \to G$ by $i(g) := g^{-1}$. Prove that $i^*\nu = \Delta\nu$.

(g) Consider the Lie group

$$G := \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0, y \in \mathbf{R} \right\}.$$

Compute the modular function of G.