Differential Geometry III
Gauge Theory
Problem Set 6

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The purpose of this problem set is to understand the cocycle perspective on G–principal bundles. Throughout, let G be a Lie group and let B be a smooth manifold.

(1) Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of B. For $i_1, \ldots, i_k \in I$ set $U_{i_1,\ldots,i_k} := U_{i_1} \cap \cdots \cap U_{i_k}$. A k–**cochain** on $\mathcal{U}$ with values in the sheaf $C^\infty(\cdot, G)$ is an I^k–tuple $\underline{\phi} = (\phi_{i_1\cdots i_k} \in C^\infty(U_{i_1\cdots i_k}, G) : i_1, \ldots, i_k \in I)$. Denote the set of k–cochains by $\check{C}^k(\mathcal{U}, G)$. Define $\delta\colon \check{C}^1(\mathcal{U}, G) \to \check{C}^2(\mathcal{U}, G)$ by

$$(\delta\underline{\phi})_{ijk}(b) = \phi_{ik}(b)^{-1}\phi_{ij}(b)\phi_{jk}(b)$$

for every $b \in U_{ijk}$. Denote the set of 1–cocycles with $\check{Z}^1(\mathcal{U}, G) := \ker\delta$.

Let $\underline{\phi} \in \check{Z}^1(\mathcal{U}, G)$. Set

$$\tilde{P} := \coprod_{i\in I} U_i \times B.$$

and denote by $\sim$ the equivalence relation generated by

$$(i, b, g) \sim (j, b', h) \quad \Leftrightarrow \quad (b = b' \in U_{ij} \text{ and } h = \phi_{ij}(b)g).$$

Check that this really is an equivalence relation!

Set

$$P := \tilde{P}/\sim .$$

There is an obvious map $p\colon P \to B$ and an right obvious right action $R\colon P \times G \to P$ such that (p, R) is a G–principal bundle. This is the G–principal bundle associated with $\underline{\phi}$.

Prove up to isomorphism that every G–principal bundle over B arises from this construction for some choice of $\mathcal{U}$!

(2) Let $\underline{\phi}^1, \underline{\phi}^2 \in \check{Z}^1(\mathscr{U}, G)$. Consider the corresponding G–principal bundles (p_1, R_1) and (p_2, R_2). Suppose that $\psi\colon (p_1, R_1) \to (p_2, R_2)$ is an isomorphism.

Construct a 0*–chain* $\underline{\psi} \in \check{C}^0(\mathscr{U}, G)$ *such that*

$$\phi^2_{ij} = \psi_j \phi^1_{ij} \psi_i^{-1}!$$

Define a map $\tau\colon \check{C}^0(\mathscr{U}, G) \to \mathrm{Bij}(\check{C}^1(\mathscr{U}, G))$ by

$$\tau(\underline{\psi})(\underline{\phi})_{ij} := \psi_j \phi_{ij} \psi_i^{-1}.$$

Set

$$\check{\mathrm{H}}^1(\mathscr{U}, G) := \check{Z}^1(\mathscr{U}, G)/\tau(\check{C}^0(\mathscr{U}, G)) \quad \text{and} \quad \check{\mathrm{H}}^1(B, G) := \varprojlim_{\mathscr{U}} \check{\mathrm{H}}^1(\mathscr{U}, G).$$

The above exhibits the set of equivalence classes of G–principal bundles as the non-abelian Čech cohomology group $\check{\mathrm{H}}^1(B, G)$.

(3) What is $\check{\mathrm{H}}^1(B, G)$?

(4) Given $\underline{\phi} \in \check{Z}^1(\mathscr{U}, G)$ find a description of the gauge group $\mathscr{G}(p, R)$ of the associated G–principal bundle (p, R).

(5) Let $\underline{\phi} \in \check{Z}^1(\mathscr{U}, G)$ and construct (p, R). Find a description of $\mathscr{A}(p, R)$ and the right action of $\mathscr{G}(p, R)$ on $\mathscr{A}(p, R)$.