Differential Geometry 3
Winter Semester 2021/22

Thomas Walpuski

Purpose. The purpose of this course is to convey the foundations of *gauge theory*, discuss some of its relations to topology, Riemannian, symplectic, and complex geometry, and physics, and, finally, to equations arising from physics can be used to understand the topology of 4–manifolds.

Prerequisites. The prerequisites are a firm understanding of the theory of differentiable manifold (at the level of Differential Geometry 1 in winter semester 2020/21).

Topics. Here is an outline of the topics that I plan to cover.

(1) **Foundations**

   (a) (i) Lie groups
   (ii) the Lie algebra of a Lie group
   (iii) the exponential map
   (iv) Lie group actions on manifolds, the slice theorem, construction of quotients
   (v) de Rham cohomology of manifolds with Lie group actions; Lie algebra cohomology*
   (b) (i) fiber bundles
   (ii) principal fiber bundles
   (iii) associated bundles
   (iv) reduction of structure group
   (c) (i) Ehresmann connections
   (ii) parallel transport, holonomy
   (iii) curvature
   (iv) differential forms on fiber bundles
   (d) characteristic classes via Chern–Weil theory

(2) **Bridges**

   (a) *possibly:* Serre's spectral sequence in de Rham cohomology
   (b) Riemannian geometry through the lens of gauge theory
   (c) gauge theory in symplectic, Kähler, and hyperkähler geometry

(i) a review of/short introduction to symplectic, Kähler, and hyperkähler geometry
(ii) Gibbons–Hawking's construction of Ricci-flat/hyperkähler 4–manifolds
(iii) symplectic reduction, Kähler and hyperkähler quotients
(iv) variation of quotients, Duistermaat–Heckman formula, etc.

(d) gauge theory and physics
(i) the Yang–Mills equation
(ii) (anti-)self-duality: instantons, monopoles, Hitchin's equations/Higgs bundles
(iii) *possibly:* moduli spaces of these, the ADHM construction, the Nahm transform

(3) **Seiberg–Witten theory**

(a) spin/spin^c structures (*possibly* only in dimension 4)
(b) twisted Dirac operators
(c) the Seiberg–Witten equation
(d) the Lichnerowicz–Weitzenbock formula
(e) (sketch of) the construction of the Seiberg–Witten moduli space and the Seiberg–Witten invariant
(f) selected applications of Seiberg–Witten theory to the topology of 4–manifolds

Questions. Please, get in touch if you have further questions, requests for additional or alternative topics, etc.