On a technical ingredient of *Bifurcations of embedded curves & an extension of Taubes' Gromov invariant to CY 3-folds*

[Bai-Swaminathan, 2021, arXiv:2106.01206]

Paramjit Singh

December 6, 2021

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For such a CY 3-fold (X, ω), $g \geq 0$, $A \in H_2(X, \mathbb{Z})$, the moduli space $\overline{\mathcal{M}}_g(X, J, A)$ has virtual dimension 0 (by Riemann-Roch). Hence, one can "count" curves, obtaining Gromov Witten invariants $GW_{A,g}(X) \in \mathbb{Q}$, independent of J.

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Because of multiple covers and ghost components (which may have non-trivial automorphisms) these invariants are generally not $\mathbb{Z}$-valued.

Taubes' Gromov invariant

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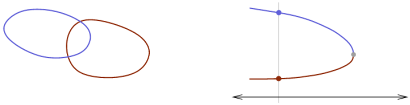

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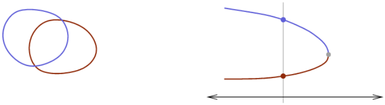

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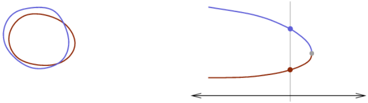

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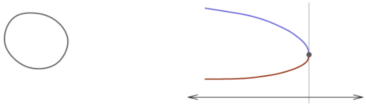

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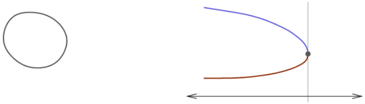

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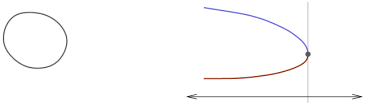

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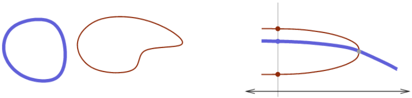

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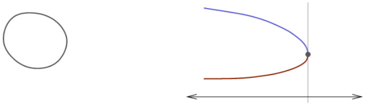

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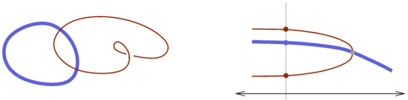

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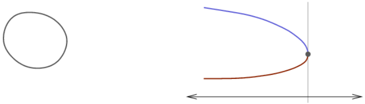

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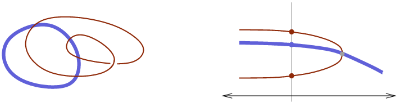

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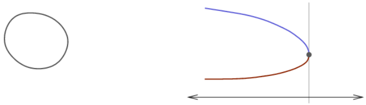

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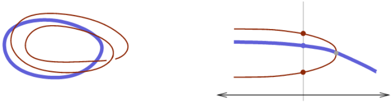

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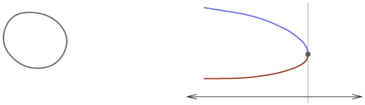

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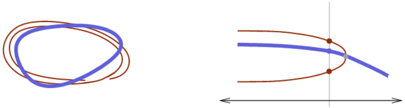

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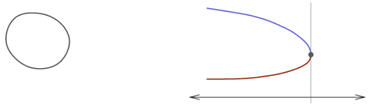

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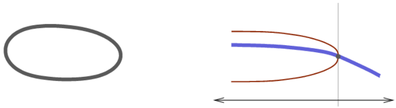

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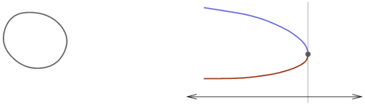

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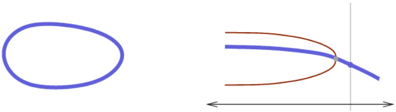

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We need correction terms to keep our count invariant under each bifurcation. This can be achieved by assigning signs $\varepsilon(u) = \pm 1$ to each embedded curve and for genus $g = 1$, a *suitable* weight $w(u, 2) \in \mathbb{Z}$. And then defining the Gromov invariant as a sum of signs for curves in class $2A$ plus correction weights for curves in class A.

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A useful condition in dim 4 is that putting together the index formula with Riemann Hurwitz & the adjunction formula, one finds that a sequence of embedded 0-curves can never converge to a nodal curve, and it can converge to a multiple cover only in very specific situations. Likewise when the curves have genus 1, they end up converging to an unbranched cover of an embedded torus.

Attempt to generalize to dim 6

- It has been conjectured that there should be Gromov invariants in dim 6, that give integer-valued counts of embedded curves, together with contributions from their multiple covers that are determined by bifurcation analysis.

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- [Bai-Swaminathan] explain how to define such invariants in situations where the multiple covers making contributions have degree at most 2.

Genericity of embedded curves

$$\mathcal{W}_{\mathsf{emb}} := \left\{ J \in \mathcal{J}(X,\omega) \;\middle|\; \begin{array}{l}\exists \text{ a simple \& non-embedded curve or 2 simple} \\ \text{curves with distinct but intersecting images}\end{array} \right\}$$

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It is known that this *wall* $\mathcal{W}_{\mathsf{emb}}$ is of codim 2 in $\mathcal{J}(X,\omega)$. Thus, along generic paths $\gamma \subseteq \mathcal{J}(X,\omega)$, we may assume that all simple curves are embedded and pairwise disjoint by appealing to the Sard-Smale theorem (i.e., *paths can be chosen generically to avoid the walls*).

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The above then implies that we can assume v is embedded while u can be any arbirary non-constant stable map.

Genericity of super-rigidity

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$J \in \mathcal{J}(X, \omega)$ is called *super-rigid* if, for all non-constant J-stable maps $\Sigma' \xrightarrow{\varphi} \Sigma \subseteq X$, we have $\ker \varphi^* D^N_{\Sigma,J} = 0$, where $D^N_{\Sigma,J}$ is the normal CR operator of the embedded curve $\Sigma \subseteq X$.

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- If u is a super-rigid *curve*, then the only possible sequences that converge to a nodal branched cover of u consist of other covers of u. In other words, $\overline{\mathcal{M}}_h(d; u)$ is an open and closed subset of $\overline{\mathcal{M}}_h(dA, J)$.

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Theorem (Wendl '19)

Super-rigid almost complex structures are comeager in $\mathcal{J}(X, \omega)$ for $\dim X \geq 6$.

Counts of embedded curves of twice the primitive class

Call a homology class A primitive, if $\omega(A) > 0$ and we cannot write $A = kB$ for any $B \in H_2(X, \mathbb{Z})$ and integer $k \geq 2$. The main result of [Bai-Swaminathan] is to define a virtual count of embedded curves of twice the primitive homology class for all genera.

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Theorem (†)

Fix primitive $A \in H_2(X, \mathbb{Z})$ and integer $h \geq 0$. For super-rigid $J \in \mathcal{J}(X, \omega)$, define the virtual count of embedded genus h curves of class $2A$ to be the integer

$$\mathrm{Gr}_{2A,h}(X, \omega, J) = \sum_{C' : 2A, h} \mathrm{sgn}(C') + \sum_{g \leq h} \sum_{C : A, g} \mathrm{sgn}(C) \cdot w_{2,h}(D^N_{C,J})$$

The above is, infact, independent of the choice of super-rigid J and defines a symplectic invariant $\mathrm{Gr}_{2A,h}(X, \omega)$ of (X, ω).

Necessary condition for bifurcations

The first key ingredient in the proof of Theorem (†) is a *necessary* condition for the occurence of a bifurcation. Fix an even dimensional manifold X^{2n} (n arbitrary). Let $\mathcal{F} : V \to \mathcal{J}(X)$ be a smooth family of acs on X parametrized by a smooth finite dimensional manifold V. Given a curve C, denote by $\overline{\mathcal{M}}_h(C, k)$ the moduli space of genus h stable maps of degree k with target C.

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Theorem (‡)

Let $x_\nu \to x$ be a convergent sequence in V, with $J_\nu := \mathcal{F}(x_\nu)$ for $\nu \geq 0$ and $J := \mathcal{F}(x)$. Let $h \geq 0$ be an integer and suppose that we have a sequence $(J_\nu, \varphi_\nu : \Sigma'_\nu \to X)$ of simple J_ν-curves of genus h converging, in the Gromov topology, to a stable map $(J, C' \xrightarrow{\varphi} C \subseteq X)$ with C being a smooth embedded J-curve of genus $g \leq h$ and $\varphi : C' \to C$ being an element of $\overline{\mathcal{M}}_h(C, k)$ for some integer $k \geq 1$. Then, exactly one of the following must be true:

- *We have $g = h$, $k = 1$ and $\varphi : C' \simeq C$.*
- *The natural pullback map*

$$\varphi^* : \ker D^N_{C,J} \to \ker \varphi^* D^N_{C,J} \qquad (\star)$$

  is injective but not surjective.

Consequence: Ruling out nodal degenerations

Theorem (‡) allows us to exclude the possibility of bifurcations from nodal curves (possibly with ghost components) into embedded curves once the path γ is chosen generically. When the homology class represented by $\varphi_k : \Sigma'_k \to X$ is primitive, the degree k is then necessarily equal to 1 and Theorem (‡) implies that the limit φ must be an isomorphism. In particular, the stable map $C' \xrightarrow{\varphi} C \subseteq X$ does not have any ghost irreducible component.

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The full power of the above is that ultimately we just need to worry about bifurcations from (branched) double covers in order to check the invariance of $\mathrm{Gr}_{2A,h}(X,\omega)$. In particular, if $A \in H_2(X,\mathbb{Z})$ is primitive, then for any path γ in

$$\mathcal{J}_{\mathsf{emb}}(X,\omega) := \left\{ J \in \mathcal{J}(X,\omega) \;\middle|\; \begin{array}{l} \text{all simple curves are unobstructed \& have index} \geq 0\text{, those with} \\ \text{index} < 2n-4 \text{ are embeddings; and any 2 such curves with} \\ \text{combined index} < 2n-4 \text{ either have disjoint images or} \\ \text{are related by reparametrization,} \end{array} \right\}$$

the moduli space $\mathcal{M}_h^{\mathsf{emb}}(X,\gamma,A)$ of embedded genus h curves of class A is compact.

Proof idea

- The proof is by a careful examination of the infinitesimal deformations and obstructions of the two moduli spaces

$$\overline{\mathcal{M}}(\mathcal{M}_g^{\mathsf{emb}}(X,\gamma,[C]),d) \subseteq \overline{\mathcal{M}}_h(X,\gamma,d[C])$$

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- The key assertion to prove is the following. The fibre of the normal bundle of the above inclusion, after suitably "thickening" both moduli spaces, at the point $(t, \Sigma \xrightarrow{\varphi} C \subseteq X)$ is canonically isomorphic to the vector space $\ker \varphi^* D^N_{C,J} / \ker D^N_{C,J}$.

Outline of the argument

Set $A = [C] \in H_2(X, \mathbb{Z})$ and consider the moduli spaces

$$\mathcal{M}_g^{\text{emb}}(X, \mathcal{F}, A) := \left\{ (y, \Sigma) \,\middle|\, \begin{array}{l} y \in V, \Sigma \subseteq X \text{ is an embedded } \mathcal{F}(y)\text{-curve} \\ \text{of genus } g \text{ in class } A \end{array} \right\}, \qquad (\spadesuit)$$

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$$\overline{\mathcal{M}}_h(\mathcal{M}_g^{\mathsf{emb}}(X,\mathcal{F},A),k) := \left\{ (y,\Sigma,\psi : \Sigma' \to \Sigma) \;\middle|\; \begin{array}{l} (y,\Sigma) \text{ lies in } (\spadesuit) \text{ and} \\ (\Sigma',\psi) \text{ lies in } \overline{\mathcal{M}}_h(\Sigma,k) \end{array} \right\}. \qquad (\blacklozenge)$$

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There is a natural inclusion of $(\blacklozenge)$ inside $(\clubsuit)$, given by

$$(y, \Sigma, \psi : \Sigma' \to \Sigma) \mapsto (y, \Sigma' \xrightarrow{\psi} \Sigma \subseteq X),$$

and the proof of Theorem (‡) will follow by showing that codimension of $(\blacklozenge)$ inside $(\clubsuit)$ at the point $(x, C' \xrightarrow{\varphi} C \subseteq X)$ is given by the dimension of the cokernel of $(\star)$.

Thickened moduli spaces

However, both (♣) and (♦) are not smooth in general, therefore the notion of codimension is not well-defined. To remedy the situation, we will need to thicken the moduli spaces (in the sense of [Pardon, 16]) compatibly so that the last inclusion becomes an inclusion of manifolds, at least near the point (x, φ), whose codimension can then be computed.

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- The approach involves an analysis of λ-thickened Cauchy-Riemann equations and applying implicit function theorem to them.

Concluding proof of Theorem (‡)

Proof by contradiction. Assume that $\dim\ker\varphi^* D^N_{C,J} = \dim\ker D^N_{C,J}$. Then the map of compatibility of thickenings (Lemma 3.7) is an injective continuous map between topological manifolds of the same dimension and thus, by Brouwer's Invariance of Domain, its image must cover an open neighborhood of our target base point. Restricting this over $V \times 0 \times 0 \subseteq V' \times E \times E'$ (V, V' are domains of parametrization, E, E' are domain vector spaces of thickening data), we conclude that

$$\overline{\mathcal{M}}_h(\mathcal{M}_g^{\mathsf{emb}}(X, \mathcal{F}, A), k) = \overline{\mathcal{M}}_h(X, \mathcal{F}, kA)$$

in a neighborhood of $(x, \varphi : C' \to C \subseteq X)$. This is a contradiction to the existence of the sequence of *simple* curves $(J_\nu, \varphi_\nu : \Sigma'_\nu \to X)$ unless $h = g$ and $k = 1$ (in which case φ must also be an isomorphism). □

Ruling out nodal degenerations

Consider a subset of $\mathcal{J}(X,\omega)$ of codim ≥ 3 (Lemma 5.20), involving

$$\mathcal{J}'_{A,g} := \left\{ J \in \mathcal{J}(X,\omega) \;\middle|\; \begin{array}{l} \text{there exists an embedded curve, a point } z \in C \text{ \&} \\ 0 \neq \sigma \in \ker D^N_{C,J} \text{ with } \sigma(z) = 0. \end{array} \right\}, \&$$

$$\mathcal{W}_{w,h}(\mathcal{J}_{A,g}) := \left\{ J \in \mathcal{J}(X,\omega) \;\middle|\; \begin{array}{l} \text{there exists an embedded curve, a holomorphic branched double} \\ \text{cover } \varphi : C' \to C \text{ \& } 0 \neq \sigma \in \ker D^{N}_{C,J}{}^{-}_{\varphi} \text{ such that the} \\ \text{corresponding } J_{D^N_{C,J}}\text{-holomorphic map } \hat{\sigma} : C' \to N_C \text{ defined} \\ \text{using Lemma 5.9 (association of } CR_{\mathbb{R}}\text{-ops with acs on} \\ \text{total space of } N\text{) is not an embedding.} \end{array} \right\}.$$

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Away from the above subset and away from walls (generically), for paths joining acs $J_\pm$ and $0 \leq g \leq h$, the following holds.

Ruling out nodal degenerations

Lemma

- *Among the tuples of data* $(t, \Sigma, \varphi : \Sigma' \xrightarrow{2:1} \Sigma)$, *the ones with*

$$\dim\ker D^N_{\Sigma,\gamma(t)} = 0, \qquad \dim\ker \varphi^* D^N_{\Sigma,\gamma(t)} = 1$$

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The proof takes a section in cokernel of $(\star)$ and considers it's associated $J_{D^N_{\Sigma,\gamma(t)}}$-holomorphic map. Then one looks at the irreducible components of Σ' on which φ is non-constant and behavior of sections in $\ker D^N_{\Sigma,\gamma(t)}$ on these components. One then enforces the desired degenerations by ruling out exceptional behavior as generically avoidable.

THE END.